A METHODOLOGICAL GUIDE TO MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT

During the last few years, multiobjective optimization has received growing attention: the number of publications related to this subject between 1974 and 1979 exceeds 120. There are many approaches, techniques and tools related to multiobjective decision-making and optimization; however, not all approaches are equally developed, and the resulting tools are often applied because of certain traditions rather than their suitability for solving a given problem. Therefore, this paper is devoted to a comparative evaluation of various approaches and tools. This evaluation is based, however, first on a classification of problems of multiobjective decision making and optimization. Thereafter, the available approaches, methods, techniques and tools are shortly presented and evaluated in terms of suitability for various classes of problems.

The final part of the paper presents a broader description of a relatively new approach based on reference objective levels, not fully developed yet but applicable in many classes of problems. A new notion of extended threshold utility functions, other basic theoretical results, applicational examples and directions of further research related to this approach are presented.

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1. INTRODUCTION

Multiobjective decision making and optimization have many fields of application today. Their roots result from economic theory, but they are applied now in mathematical psychology, praxeology, in many environmental, sociological, and technical problems, in computer-aided design in engineering, and in control problems. The theory of multiobjective optimization is now a standard part of mathematical programming; however, its various applications still result in vexing methodological and theoretical questions.

An abstract problem of multiobjective optimization is well-defined for all but practical purposes, for its solution is a set rather than a single point. To choose a single point out of this set, additional information is required. Were this information explicit and readily available, the problem would not be a multiobjective one. The most important questions in multiobjective optimization are where, how, and in what form this additional information can be obtained. As to the question where, there is a universally accepted answer: from a "decision maker", that is, a person, an expert, a manager, or a group of them, or even an organization involved in decision making. The questions how and in what form are often answered in relation to the existing theoretical approaches and resulting techniques rather than, as they should be answered, in relation to the particular problem, to the needs of the decision maker or decision making organization.

2. PROBLEMS OF MULTIOBJECTIVE DECISION MAKING AND OPTIMIZATION

Three notions are basic for multiobjective problems: alternatives or alternative decisions, their attributes and related objectives, and a natural partial ordering of alternatives and attributes. Alternatives are just possible actions; alternative decisions are related to a choice of either an alternative, or a subset of alternatives described by additional constraints. For example, if alternatives are approvals of various nuclear power plant sites, alternative decisions might be related to the choice of sites for three power plants simultaneously, which is a different problem than choosing the site for just one of them. The generation of alternatives and alternative decisions is often a most difficult part of the problem, and requires additional information, knowledge and ingenuity.

Attributes are various common characteristics of the alternatives, pertinent to the problem. Sometimes, several attributes might be strongly correlated and can be aggregated into a composite one; some attributes might not be relevant for the problem at hand, some important attributes might be missing in the first description of the problem. The choice of a minimal set of pertinent attributes is again a most difficult part of the problem, highly judgemental and relying on sufficient information.

It is usually assumed that attributes are quantifiable, that is, they can be measured on some natural or artificial, numerical or descriptional scales. Quantified and, if necessary, aggregated attributes become objectives. A natural partial ordering of objectives and thus underlying attributes and alternative decisions is usually evident. For example, we can choose the scales for relevant attributes to represent the concept of multiobjective maximization, that is, such that an alternative decision is better than another one if all objectives attained under the first decision are not smaller and some are larger than those attained under the second one; the objectives represent then gains, profits, wins, etc. Clearly we could also choose the convention of multiobjective minimization by just inverting the scales, but for the purpose of a unified description the convention of maximization is adopted in this paper.

The quantification of attributes and the choice of reasonable scales again relies very much on available information. But the most crucial step in further eliciting the information pertinent to the problem is the stage of mathematical modelling, that is, describing the dependence of objectives on alternative decisions by functional relations.

There are many multiobjective problems where we cannot do it, where the attributes and objective levels attained under alternative decisions can be assessed only by experts, for example, when some of the attributes are of aesthetical or political nature. These problems shall be called the *multiobjective decision making* problems, since an actual optimization can be performed only implicitely in such a case. However, if a mathematical model of the relations between alternative decisions and all objectives can be built, then the available information is aggregated in a convenient form for further analysis and results in possibilities not only of *multiobjective optimization*, but also of automatic generation of other alternative decisions, etc. Naturally, the question of an *adequate and reliable* mathematical model of the given problem has a paramount importance here.

However, all the above questions and resulting classifications, although basic, do not express the essence of difficulties related to multiobjective optimization or decision making. Even if we eliminate by some procedure all *dominated* alternatives--that is, such alternatives that a better one can be found in the sense of the natural partial ordering of the objectives--the set of remaining *nondominated* alternatives is usually large and its elements are incomparable in the sense of the natural partial ordering. To choose between them, to introduce a *preference* ordering (a *complete* preordering, a complete ordering of equivalence classes) in the set of alternatives, additional information must be obtained from experts or decision makers. The central questions of multiobjective optimization are how and in what form to obtain this additional information. The right answer to these questions depends very much on the properties of the problem and the attitudes of decision makers, and should not be biased by available mathematical approaches and techniques.

The following classification is proposed here to express these properties and attitudes:

1°. *Aggregate preferences*. One of the basic questions in economic theory is how to represent a large number of decision makers ("economic agents") making independent but in a sense similar decisions. Another question, typical for mathematical psychology, is the description of a single decision maker making repetitively similar decisions under the assumption of non-varying preference pattern. The basic and intensively studied question is then how to represent the preference pattern, revealed by a large number of actual decisions, in an aggregated form, suitable for further analysis. Most of the extensive work on utility and value theory--see, e.g., [6], [10], [14]--is concerned with this question.

2°. *Unknown preferences in single decisions*. Very often, important and novel decisions--such as siting nuclear energy plants, or setting standards for some new type of pollution--are to be made by a body of decision makers who had not quite formulated a firm opinion on what type of decision they really like, had no clear opinion about the relevance of various attributes, simply because of lack of experience in such decisions or because of various uncertainties or even possible psychological biases related to such truly important and rare decision. The main task of a multiobjective decision theory in such cases (similar applications of multiobjective optimization are rather exceptional) is to help the decision makers to make up their minds. To do this a series of hypothetical alternative decisions can be constructed and evaluated by decision makers; the revealed preference patterns are helpful when making the actual decision.

Extensive research is related to this class of problems and resulted in several successful applications, see e.g. [2], [12], [14], [30].

3°. *Conscious preferences in single or repetitive decisions,* with possibly varying preference patterns. It is quite a common situation that decision makers have clear opinions about what they would like to achieve, have concious but possibly varying preferences. For example, planners in a planning office perform their task every year in varying economic situations. Very often in such cases mathematical models can be constructed and multiobjective optimization could be useful--provided that adequate procedures for the interaction between the decision maker and the model are developed. The model could be helpful to the decision-maker by generating new nondominated alternatives, meeting new requirements. However, there has been very little success in applying multiobjective optimization in such cases. There might be many reasons for this fact--see, e.g., [1]--but one of them is that the approaches and techniques useful for the classes 1° and 2° were usually adapted to solve problems of class 3°, where they cease to be useful. The decision makers, when certain that they could choose the best alternative provided they are presented with a reasonable set of them, do not like to waste time on hypothetical questions revealing their changing preference patterns; they simply would like to get new alternatives, closer to their changing requirements. This is a reason for the hypothesis that everyday single decisions are not made by maximization of utility functions but rather by establishing certain reference levels for objectives and trying to satisfy them [26]. This also motivated research looking for new approaches and tools of multiobjective optimization, alternative to the classical approaches and tools.

3. HISTORICAL PERSPECTIVE

Pareto's original work in 1896 [18] was motivated by economic problems. He introduced not only the basic notion of multiobjective optimality, but also that of preference; clearly, the problems he considered belong to the aggregated preference class.

He was also the first to use weighting coefficients in multiobjective optimization. In the classical basic theory of multiobjective optimality, weighting coefficients play a central role: necessary and sufficient conditions of multiobjective optimality, equilibria and trade-offs, and utility maximization are basically related to weighting coefficients.

Further work on economic theory was strongly related to the notion of preference and its representation by utility functions. In the foundations of general economic equilibrium theory, a consumer is assumed to maximize a utility function representing his preference ordering on commodity bundles. However, he is clearly an *average* consumer and aggregate preferences of large numbers of consumers are really of interest here. An *individual* consumer in his *everyday* decisions does not think in terms of maximizing utility, but in terms of goals, lists of things he is going to buy. The study of aggregate preferences and utility functions has reached a high mathematical level, with deep axiomatic basis, careful distinction between cardinal and ordinal utility functions, fine theorems on representations of preferences by utility functions, on aggregations of many utility functions, on revealing preference patterns in many decision, etc.--see e.g., Debreu 1959 [6], Fishburn 1970 [10]. This development stimulated a broad mathematical psychology research on individual decision maker's behavior--see e.g. Hogarth 1975 [13]. As long as repetitive decisions under nonvarying preferences (and thus, *aggregate preferences, averaged in time*) were studied, the use of utility functions has been proved successful. However, certain behavioral phenomena were found not to be quite consistent with the utility approach--given a status quo, or reference objective levels, individuals adopt different, asymmetrical attitudes to the possibilities of losses as compared to gains with respect to the status quo. This is another confirmation of the hypothesis that individuals in everyday decisions think rather in terms of goals, reference objective levels, than in terms of maximizing utility.

However, the assumption of individual utility maximization also guided other broad researches on individual and group decision making. Interpersonal utility comparisons, transforming utility functions in the presence of uncertainty or in probabilistic choice situations, interactive procedures of group decision making based on utility identification, etc., were studied extensively and even applied successfully--see, for example, Keeney and Raiffa 1975 [14], Bell, Keeney and Raiffa 1977 [2]. The successful applications of this broad theory are related, however, to the cases of *unknown preferences*, where the decision makers were willing to take part in psychometric experiments in order to learn about their own preferences. In other cases, where the decision makers knew their preferences better, attempts to apply this theory have failed--see, for example, Clarke 1979 [3].

The need for an alternative approach, particularly for multiobjective optimization problems and for the case of conscious but varying preferences, has been perceived for a long time. Attainable reference objective levels have been used by Dyer 1972 [7], Kornbluth 1973 [15] and others in so-called goal programming. Far unattainable aspirations objective levels have been used by Sakluvadze 1971 [20] Yu and Leitmann 1974 [27] as so-called utopia-type points. Wierzbicki 1975-1979 [22], [23], [25], [26], developed an alternative basic theory of multiobjective optimization where weighting coefficients and utility functions are replaced by *any* reference objective levels (attainable or not, utopia-type or not) and by related penalty scalarizing functions. Penalty scalarizing functions are in fact *ad hoc* constructed, only rough approximations of the preference patterns of a decision maker. However they depend heavily on and stress the importance of the information provided by him in the form of his desired objective levels. Therefore, they are particularly useful tools in case of varying preferences. These tools, although proved to be successful in several applications, have not been widely tested yet. Moreover, there are still further developments of this theory to be investigated, as, for example, the consideration of uncertainties and probabilistic situations. Nevertheless, this theory represents well many observations on everyday, individual decision-maker's behavior: his thinking in terms of desirable goals or reference objective levels, his unsymmetrical attitude towards losses and gains in respect to the reference objective levels, his readiness to vary reference objective levels in varying situations. The theory is not entirely separated from the classical theory: weighting coefficients might be determined *a posteriori* after a penalty scalarizing function has been used, and a penalty scalarizing function can be interpreted not only as an *ad hoc* approximation to a varying utility function, but even as a new type of utility function with stronger properties.

4. BASIC THEORY

Let E_0 be the set of alternative admissible decisions and G the space of objectives $x \in E_0$, $q \in G$. Let a mapping $f: E_0 \to G$ be given--either in the form of a mathematical model, for multiobjective optimization, or only implied by the specification of outcomes of any alternative decision, in a more general case. $Q_0 = f(E_0)$ is then called the set of attainable objectives. If $G = R^n$ and all objectives are to be maximized, the natural partial ordering in Q_0 is implied by the positive cone $R^+_n = \{q \in R^n : q_1 \geq 0, \ldots, q_n \geq 0\}$ and the strong positive cone $\tilde{R}^+_n = R^+_n \setminus \{0\}$.

$$q^2 \geqslant q^1 \Leftrightarrow q^2 - q^1 \in R^n_+ \quad ; \quad q^2 > q^1 \Leftrightarrow q^2 - q^1 \in \tilde{R}^n_+ \quad . \qquad (1)$$

A Pareto-maximal decision $\hat{x} \in E_0$ and objective $\hat{q} = f(\hat{x}) \in Q_0$, and the set $\hat{Q}_0$ of all Pareto-maximal objectives are then defined by:

$$\hat{q} = f(\hat{x}) \ , \ (\hat{q} + \tilde{R}^n_+) \cap Q_0 = \phi \quad ; \quad \hat{Q}_0 = \{\hat{q} \in Q_0 : (\hat{q} + \tilde{R}^n_+) \cap Q_0 = \phi\}. \qquad (2)$$

If G is more abstract space, for example, a Hilbert space of trajectories of a dynamic model of national economic growth, then the above definitions can be easily extended by substituting

R^n_+ by a chosen positive cone $D \subset G$, see, for example, [23]. However, the description in the paper is mostly restricted to the case $G = R^n$ and $D = R^n_+$.

Weighting coefficients

If the objectives $q_i = f_i(x)$ are simply added with weighting coefficients λ_i, a linear scalarizing function is obtained

$$s_\lambda(q) = \langle\lambda, q\rangle = \sum_{i=1}^{n} \lambda_i q_i \quad . \tag{3}$$

The weighting coefficients are assumed to be at least nonnegative, $\lambda \in R^n_+$; they are also usually normalized by requiring $\sum_{i=1}^{n} \lambda_i = 1$ or $\|\lambda\| = 1$ with any chosen norm in R^n.

Classical sufficient and necessary conditions of Pareto-optimality are usually stated in terms of weighting coefficients. If $\hat{q} \in \operatorname{Arg} \max_{q \in Q_0} \sum_{i=1}^{n} \lambda_i q_i$, $\hat{x} \in \operatorname{Arg} \max_{x \in E_0} \sum_{i=1}^{n} \lambda_i f_i(x)$ with $\lambda \in \overset{\circ}{R}{}^n_+ = \{\lambda \in R^n : \lambda_1 > 0, \ldots, \lambda_n > 0\}$, then $\hat{q} \in \hat{Q}_0$, $\hat{x}$ and $\hat{q} = f(\hat{x})$ are Pareto-maximal. If $\hat{q} = f(\hat{x})$ is Pareto-maximal and the set Q_0 is convex then there exists $\hat{\lambda} \in \tilde{R}^n_+$ such that $\hat{q} \in \operatorname{Arg} \max_{q \in Q_0} \sum_{i=1}^{n} \hat{\lambda}_i q_i$. See, e.g., [5], [11].

Observe, however, that if the necessary conditions of Pareto-maximality should be checked, we know that there should exist and appropriate $\hat{\lambda} \in \tilde{R}^n_+$ (provided the set Q_0 is convex) but it is difficult to find it. In fact, $\hat{q}$ is Pareto-optimal if Q_0 is convex and $(\hat{\lambda}, \hat{q}) \in \operatorname{Arg} \min_{\|\lambda\| = 1, \lambda \in R^n_+} \max_{q \in Q_0} \sum_{i=1}^{n} \lambda_i (q_i - \hat{q}_i)$. Even the sufficient condition of Pareto-maximality is not quite operational, if we parametrisize the Pareto-set $\hat{Q}_0$ by defining $\hat{q}(\lambda) = \operatorname{arg} \max_{q \in Q_0} \sum_{i=1}^{n} \lambda_i q_i$; the resulting $\hat{q}(\lambda)$ can be easily discontinuous even if the set Q_0 is convex, see Figure 1.

Therefore, the parametrization $\hat{q}(\lambda)$ is badly suited for scanning the Pareto set by changing λ.

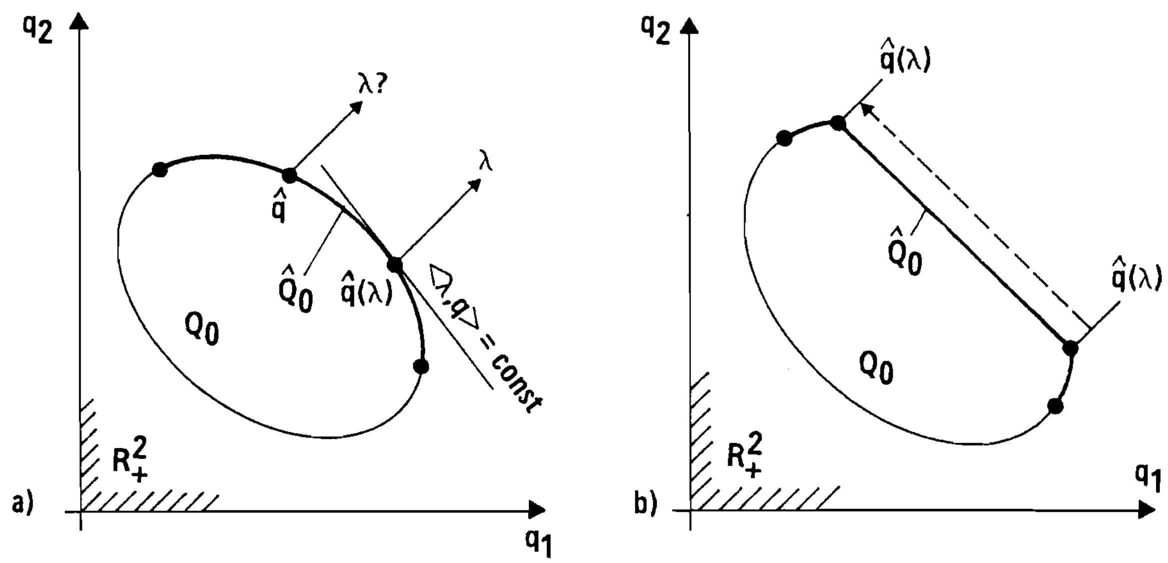

Figure 1. The use of weighting coefficients in multiobjective optimization:
a) the difficulty of finding the appropriate $\hat{\lambda}$ for a given Pareto-optimal $\hat{q}$;
b) the discountinuity of $\hat{q}(\lambda)$.

Moreover, the interpretations of weighting coefficients λ_i, though mathematically easy as the derivatives of a utility function, relative prices or trade-off coefficients (see next paragraph) are not readily intuitive: without knowing the set Q_0 well, it is difficult to say which $\hat{q}$ would correspond to a given λ. All the drawbacks of weighting coefficients are more of pragmatical than theoretical character; but they result in serious difficulties when actually applying weighting coefficients in multiobjective optimization.

Despite these pragmatical drawbacks, the theory of the multiobjective optimization based on weighting coefficients has been extensively developed. Weighting coefficients are, in fact, a type of Lagrange multipliers; all existing theory on separation of sets by linear functionals can be used here, infinite-dimensional cases, saddle-points and duality theorems can be investigated--see, e.g., [8], [31]. Much has been done in the use of weighting coefficients in multiobjective linear programming, including various theoretical investigations and computational algorithms--see, e.g., [4], [9], [19], [28], [30]. However, because of the pragmatical drawbacks mentioned above, it is not clear yet whether the algorithms based on weighting coefficients are the most practical ones. Some possibilities of alternative formulations will be presented in the next paragraphs.

By using hypothetical questions, "do you prefer the vector outcome q^1 to q^2, or vice-versa, or are you indifferent to the choice between them?", it is possible to establish indifference sets in the space G, see Figure 2. The indifference sets are ordered in increasing preference; under some additional assumptions, they could be represented as level-sets of a function called (*cardinal*) utility function. Naturally, the same level sets can correspond to many functions; the class of all functions having the same level sets coinciding with given indifference sets is called an *ordinal* utility function. The preference relation and utility function should be *consistent* with the natural partial ordering of the space G; in other words, the utility function u(q) should be order-preserving

$$q^2 \geqslant q^1 \Rightarrow u(q^2) \geq u(q^1) \quad , \tag{4}$$

or even strictly order preserving

$$q^2 > q^1 \Rightarrow u(q^2) > u(q^1) \quad . \tag{5}$$

The fundamental though simple consequence of strict order preservation is that each maximal point of a strictly order-preserving utility function in Q_0 is Pareto maximal, $\hat{q} \in \operatorname{Arg}\max_{q \in Q_0} u(q) \subset \hat{Q}_0$. Therefore, the indifference set corresponding to $\hat{q}$ is tangent to $\hat{Q}_0$ at $\hat{q}$, and can be weakly separated from Q_0 under additional convexity assumptions.

Clearly, if $\hat{q} = \arg\max_{q \in Q_0} u(q)$ and u is differentiable at $\hat{q}$, then

$$\hat{\lambda} = \frac{\partial u}{\partial q}(\hat{q}) / \| \frac{\partial u}{\partial q}(\hat{q}) \| \quad , \tag{6}$$

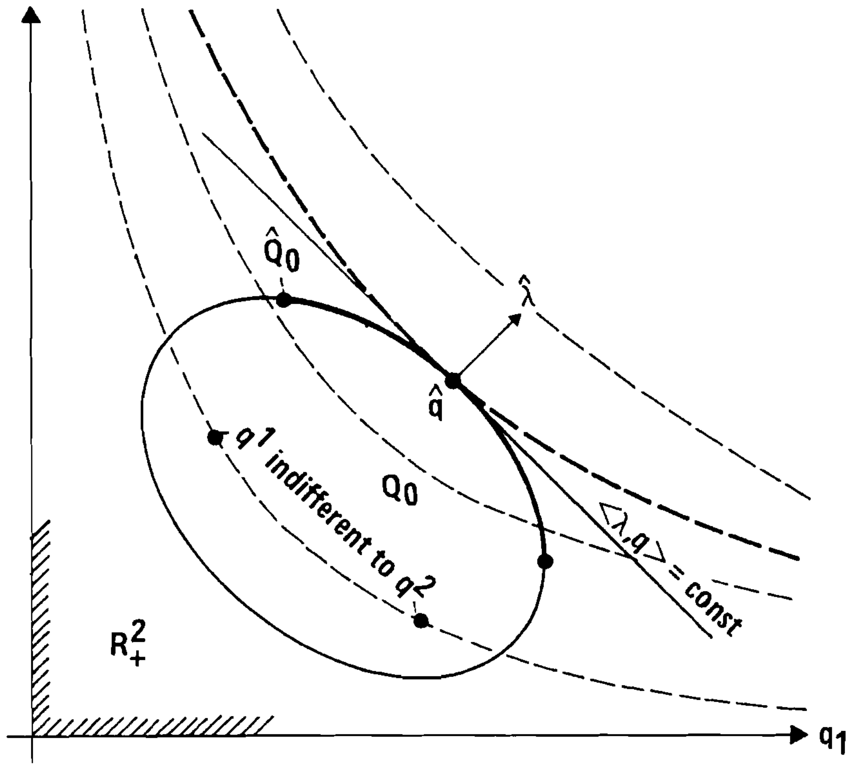

Figure 2. Indifference sets, utility function and Pareto-maximality.

is a weighting coefficient vector corresponding to $\hat{q}$. Thus, in Figure 2, $\hat{\lambda}$ expresses the optimal trade-off or marginal rates of substitution between q_1 and q_2 at $\hat{q}$. Moreover, $<\hat{\lambda},q-\hat{q}>+u(\hat{q})$ is a linear approximation at $\hat{q}$ to the ordinal utility function $u(q)$. It is usually required [6] that a utility function satisfies many additional axioms, related to its symmetry, convexity, etc. However, those properties of a utility function are not very pertinent for the purpose of this paper. Many further interesting questions, related to aggregating utility functions of many decision makers, including uncertainties, etc., [14], are also not discussed here. We should note only that the notion of a utility function, though powerful, is not fully operational in many questions of multiobjective optimization. The knowledge of a utility function corresponds to full information about the optimization problem and transforms it to a single-objective one. But suppose we have less information and would like only to know whether a given $\hat{q}$ is Pareto-maximal or not. Can we construct an *ad hoc* utility function that has maximum precisely at $\hat{q}$? The answer to this question, though positive, does not result from classical utility theory.

Extended threshold utility functions

If an individual decision maker behaves differently in situations when he cannot attain certain goals (the threshold of subsistence of a consumer gives an appropriate example here) than in situations when he can attain all goals and has to allocate surplus, then his utility function should express this behavior. In the maximization convention, given a threshold or reference objective level $\bar{q} \in R^n$, he does attain all goals if $q \in \bar{q} + R^n_+$; suppose a utility function $u((q - \bar{q})_+)$ is defined for this case, where $(q - \bar{q})_+ = (\max(0,q_2 - \bar{q}_1),\ldots,\max(0,q_n - \bar{q}_n))$; suppose $u((q - \bar{q})_+) \geq 0$ and $u((q - q)_+) = 0$ if some of the components of $(q - q)_+$ are zero. If $q \notin \bar{q}+R^n_+$, he just tries to attain his goals as closely as it is possible, that is, minimizes a norm $\|(\bar{q} - q)_+\|$. This extended utility function takes the form:

$$s(q-\bar{q}) = u((q-\bar{q})_+) - \rho\|(\bar{q}-q)_+\| \quad , \tag{7}$$

where $\rho > 0$ is a parameter. This function, which might be called an extended threshold utility function, is not only order-preserving, but possesses in fact a much stronger property than the classical forms of utility functions. This property might be called *strict order representation* and consists in the following relation:

$$S_0 \overset{df}{=} \{q \in R^n : s(q-\bar{q}) \geq 0\} = \bar{q} + R^n_+ \quad . \tag{8}$$

The strength of this property results from the following lemma:

Generalized necessary condition of Pareto-maximality. If a function $s: R^n \to R^1$ possesses the strict order representation property (8), and $\hat{q} \in Q_0$ is Pareto-maximal, then

$$\hat{q} = \arg\max_{q \in Q_0} s(q-\hat{q}) \quad ; \quad \max_{q \in Q_0} s(q-\hat{q}) = 0 \quad , \tag{9}$$

no matter whether the set Q_0 is convex or not. Moreover, if

$\bar{q} \notin Q_0 - R^+_n$, then $\max\limits_{q \in Q_0} s(q-\bar{q}) < 0$, and if $\bar{q} \in Q_0 - \mathring{R}^n_+$ (which implies $\bar{q} \notin \hat{Q}_0$), then $\max\limits_{q \in Q_0} s(q-\bar{q}) > 0$.

The proof is elementary: since $(\hat{q}+R^n_+) \cap Q_0 = \{\hat{q}\}$ due to (2), then $s(q-\hat{q}) < 0$ for all $q \in Q_0$, $q \neq \hat{q}$ due to (8). Clearly, $s(\hat{q}-\hat{q}) = 0$. If $\bar{q} \notin Q_0 - R^+_n$, then $\text{dist}(Q_0, \bar{q}+R^+_n) > 0$ and $s(q-\bar{q}) = -\|(\bar{q}-q)_+\| < 0$ for all $q \in Q_0$. If $\bar{q} \in Q_0 - \mathring{R}^+_n$, then there exists $q \in Q_0$ such that $q \in \bar{q}+\mathring{R}^+_n$ and $s(q-\bar{q}) = u((q-\bar{q})_+) > 0$.

This generalized necessary condition of Pareto-maximality, illustrated by Figure 3, is particularly operational: given any $\bar{q} \in R^n$, one can choose any utility function $u((q-\bar{q})_+)$ of desired properties--for example, $u((q-\bar{q})_+) = \prod_{i=1}^{n} (q_i-\bar{q}_i)_+$--and any norm--for example, $\|(\bar{q}-q)_+\| = \max\limits_i (\bar{q}_i-q_i)_+$--to formulate the extended utility function (8). By maximizing the function, the attainability and Pareto-maximality of $\bar{q}$ is easily checked. Since this function is order-preserving, its maximal points are Pareto-maximal except in some degeneratecases, see Figure 3.

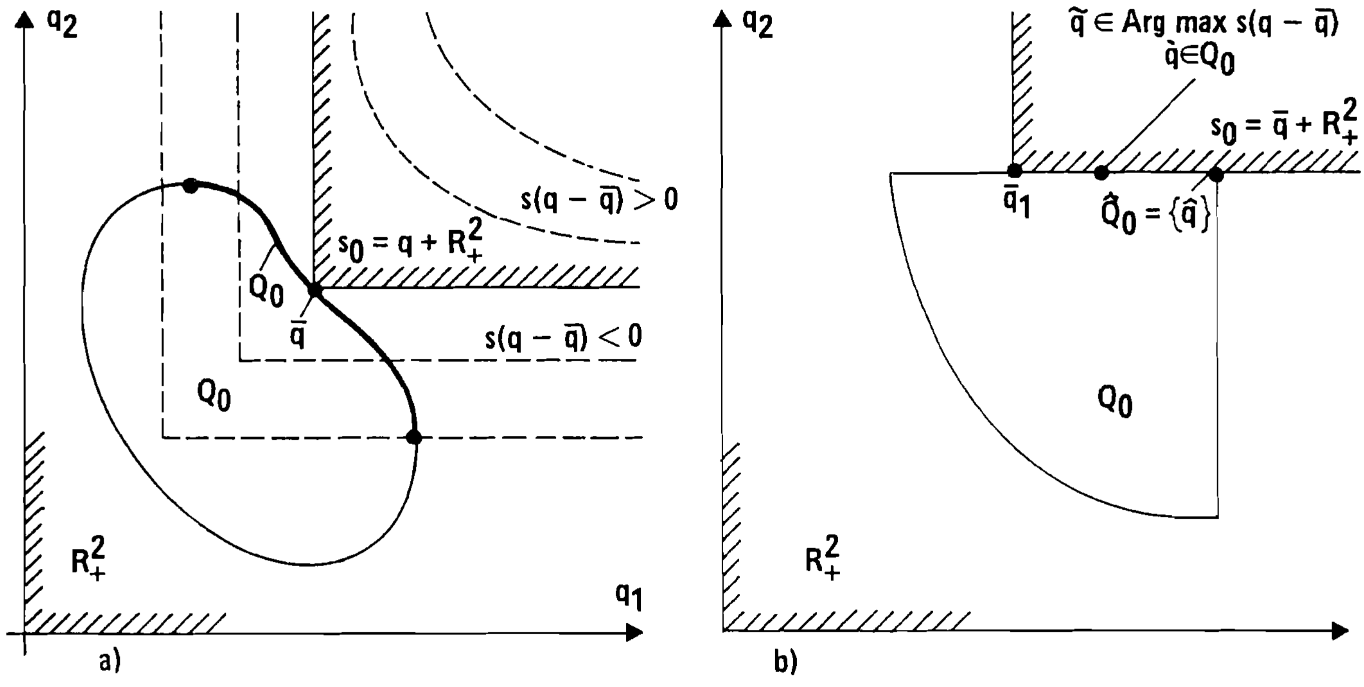

Figure 3. The use of an extended threshold utility function: a)necessary condition of Pareto-maximality for nonconvex problems, b)a degenerate case where a maximal point $\tilde{q}$ of an order-preserving function is not Pareto-maximal.

An extended threshold utility function can be treated as an cardinal utility function and be subject to psychometric identification, though its identification might be more difficult than that of classical utility functions (the basic question is then how to identify the threshold $\bar{q}$?). However, its main use is as an *ad hoc constructed* utility function, stressing the information contained in a threshold $\bar{q}$ specified by experts or decision makers. For the maximal points of $s(q-\bar{q})$ depend mainly on $\bar{q}$, although technically they depend also on the choice of $u((q-\bar{q})_+)$, ρ, and the norm.

Consider, for example, the dependence on the choice of norm. If the weighted sum of absolute values were chosen, $\|(\bar{q}-q)_+\| = \sum_{i=1}^{n} \lambda_i (\bar{q}_i - q_i)_+$, then, clearly, the choice of the norm would be (almost) equivalent to the choice of weighting coefficients, and would present some vexing problems. However, if the weighted maximum norm is chosen, $\|(\bar{q}-q)_+\| = \max_i \eta_i (\bar{q}_i - q_i)_+$, then the weighting coefficients $\eta_i > 0$ play quite a different role--they correspond to the choice of scales and not to the choice of trade-offs, and a reasonable choice of scales is a basic problem in all computations and measurements, much more typical to be solved intuitively than the choice of trade-offs. Similarly, in the weighted Euclidean norm $\|(\bar{q}-q)_+\| = \left(\sum_{i=1}^{n} \xi_i (\bar{q}_i - q_i)_+^2\right)^{1/2}$, the coefficients ξ_i correspond to the choice of scales though they imply *a posteriori* weighting coefficients $\hat{\lambda}_i$--since, if $s(q-\bar{q})$ is differentiable at its maximal point $\hat{q}$, the corresponding weighting coefficients $\hat{\lambda}$ can be determined as in (6) with s in place of u.

All these details--choice of the norm, of the penalty coefficient ρ, of the utility function u--play a truly technical role if the threshold or reference objective level $\bar{q}$ and the corresponding $\hat{q} = \arg\min_{q \in Q_0} s(q-\bar{q})$ are used as the main information exchanged between the decision maker and an optimization model in an interactive procedure generating nondominated alternatives corresponding to the requirements of the decision maker. The choice of the norm determines only the sense in which attainable Pareto-maximal $\hat{q}$ are close to unattainable reference levels $\bar{q}$. The choice of the utility function u determines only the sense in which surplus $\hat{q}-\bar{q}$ is allocated between various objectives, if the reference level is attainable. However if these details are chosen, a decision-maker in an interactive procedure learns quickly how to change his requirements $\bar{q}$ to obtain a desirable $\hat{q}$--see Figure 4. The reason for this is that $\bar{q}$ is formulated in terms much more readily understandable to the decision-maker (no weighting coefficients, no trade-offs, just desireable levels of objectives) and that the parametrization $\hat{q}(\bar{q}) = \arg\max_{q \in Q_0} s(q-\bar{q})$ is usually much more stable than the parametrization $\hat{q}(\lambda) = \arg\max_{q \in Q_0} \langle\lambda,q\rangle$.

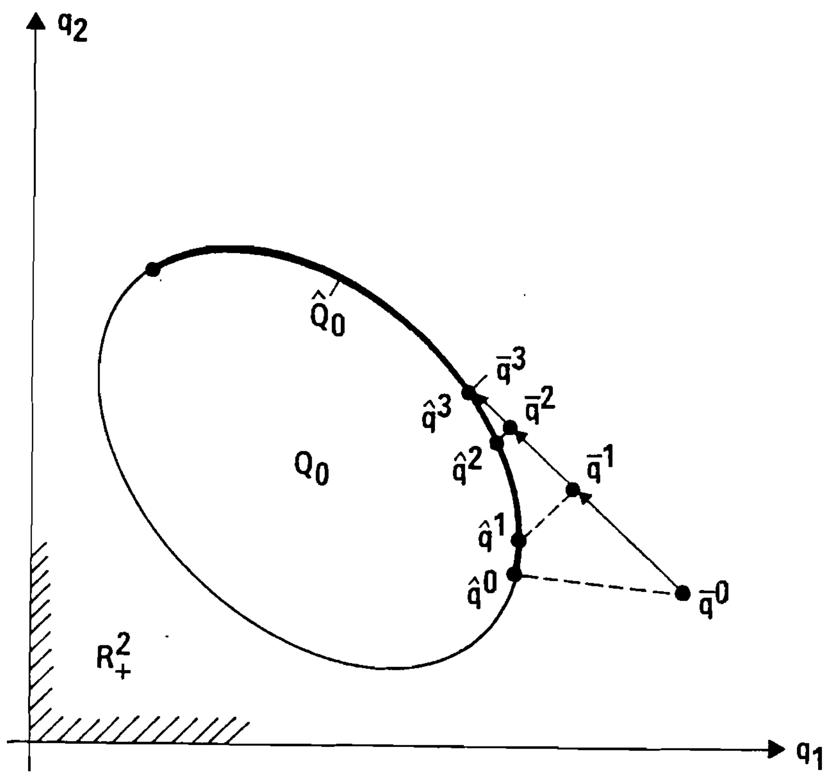

Figure 4. An iterative procedure generating nondominated alternatives $\hat{x}^i = \arg\max_{x \in X_0} s(f(x)-\bar{q}^i)$ and objectives $\hat{q}^i = f(\hat{x}^i)$ in response to varying requirements $\bar{q}^i$ of a decision maker.

Examples of some possible forms of extended threshold utility functions are as follows:

$$s(q-\bar{q}) = \prod_{i=1}^{n} (q_i-\bar{q}_i)_+ - \rho \max_{1\le i\le n} (\bar{q}_i-q_i)_+ \quad , \tag{10a}$$

$$s(q-\bar{q}) = \left(\prod_{i=1}^{n} (q_i-\bar{q}_i)_+\right)^2 - \rho \sum_{i=1}^{n} (\bar{q}_i-q_i)_+^2 \quad . \tag{10b}$$

The latter function is differentiable, though the underlying ordinal function is not. Both functions are convex, both use the unmodified multiplicative utility function. However, this function can also be modified, for example:

$$s(q-\bar{q}) = \min\Big(\rho \min_{1\le i\le n} (q_i-\bar{q}_i)_+ \;, \sqrt{\prod_{i=1}^{n} (q_i-\bar{q}_i)}\Big) - \rho \max_{1\le i\le n} (\bar{q}_i-q_i)_+ , \tag{10c}$$

to express the concept that the utility is related also to the smallest surplus $\min_{1\le i\le n} (q_i-\bar{q}_i)_+$; if the smallest surplus is less than the product of other surplusses multiplied by $1/\rho^2$, then $u(q-\bar{q}) = \rho \min_{1\le i\le n} (q_i-\bar{q}_i)$. When allocating the surplus, the smallest one has thus a "guaranteed" share. The indifference sets for the function (10c) are presented in Figure 5. Similar modification can be used also for linear utility functions, for two reasons: not only to "guarantee" a certain share of the smallest surplus, but also to modify the linear utility function continuously to zero if the smallest surplus becomes zero. This modified linear utility function results in the extended threshold function:

$$s(q-\bar{q}) = \min\Big(\rho \min_{1\le i\le n}(q_i-\bar{q}_i), \sum_{i=1}^{n} (q_i-\bar{q}_i)\Big) \quad ; \quad \rho > n \quad , \tag{10d}$$

which is particularly well suited for linear programming purposes, since its maximization is equivalent to a linear programming problem:

$$\text{maximize } y \;,\; q \in Q_0 \;,\; y \in Y_0(q-\bar{q}) = \tag{10e}$$

$$= \{y \in R^1 : y \le \rho(q_i - \bar{q}_i),\ i=1,\ldots,n;\ y \le \sum_{i=1}^{n}(q_i - \bar{q}_i)\} \quad .$$

After solving this problem, the weighting coefficients $\hat{\lambda}$ can be a posteriori determined from the dual program. All the functions (10a,...,d) are order preserving and strictly order representing, for arbitrary $\bar{q} \in R^n$.

Penalty scalarizing functions

The extended threshold utility functions actually form a subclass of a broader class of functions of the form $s(q-\bar{q})$, constructed in order to *ad hoc* approximate the preferences of a decision maker who has stated a desireable reference objective level $\bar{q} \in R^n$ (attainable or not). These functions should satisfy the following requirements:

1°. They should be order preserving and, if possible, strictly order preserving in q.
2°. They should be order representing or, at least, order approximating. Order approximation property is a relaxation of the requirements of order representation, expressed by the following relation:

$$\bar{q}+R^n_+ \subset S_0 \overset{\mathrm{df}}{=} \{q \in R^n : s(q-\bar{q}) \ge 0\} \subset \bar{q}+R^n_{+\varepsilon} \quad ; \tag{11}$$

$$R^n_{+\varepsilon} \overset{\mathrm{df}}{=} \{g \in R^n : \operatorname{dist}(q, R^n_+) \le \varepsilon \; \|q\| \} \quad .$$

In other words, the set S_0 should closely approximate $\bar{q}+R^n_+$ from above, where the closeness is expressed in terms of a conical neighborhood $R^n_{+\varepsilon}$ of the cone R^n_+. Observe that the cone $R^n_{+\varepsilon}$ can be used to define an ε-Pareto-maximality by the requirement $(\hat{q}+\tilde{R}^n_{+\varepsilon}) \cap Q_0 = \phi$, and that the generalized necessary condition of Pareto maximality from the previous paragraph can be easily restated in terms of the order approximation property (11) and the ε-Pareto maximality, see also Wierzbicki 1977 [23].

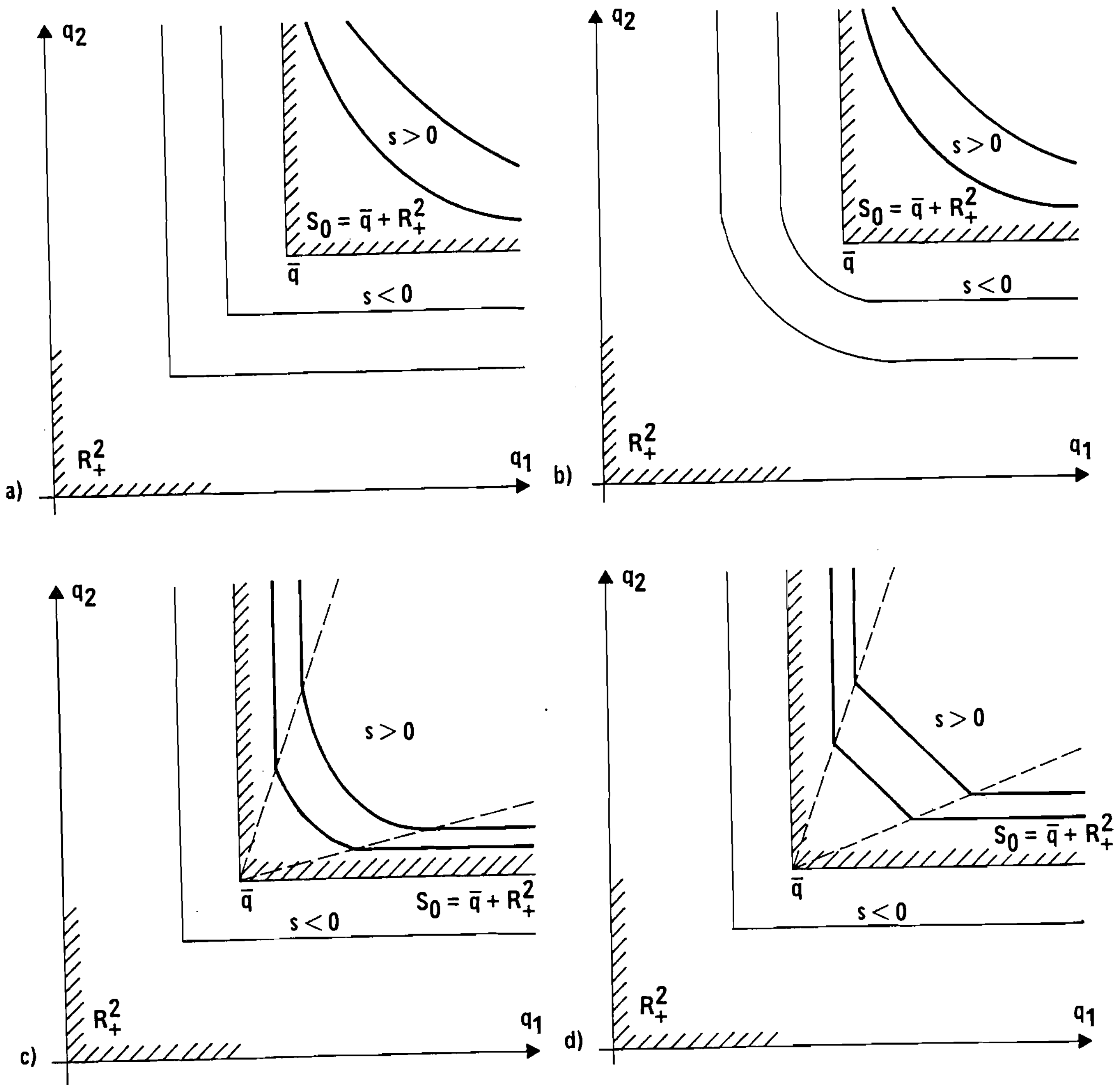

Figure 5. Indifference sets for extended threshold utility functions, a) the function (10a), b) the function (10b), c) the function (10c), d) the function (10d), resulting in a linear programming problem (10c).

3°. They should represent a concept of distance minimization between q and $\bar{q}$, if $q \notin \bar{q} + R^n_+$.
4°. They should represent either a concept of surplus allocation, or a concept of surplus maximization, if $q \in \bar{q} + R^n_+$.

Functions which satisfy the above requirements are called penalty scalarizing functions. While the first two requirements have strict mathematical meaning and result directly in sufficient and necessary conditions of Pareto-maximality in terms of penalty scalarizing function maximization, the last two requirements are merely guidelines to construct such functions. If we use a utility function in $\bar{q}+R^n_+$ to construct a penalty scalarizing function, we usually obtain an extended threshold utility function related to some concept of surplus allocation. But we can as well use other order-preserving functions in $q+R^n_+$, for example, a norm. This results in the basic form of a penalty scalarizing function:

$$s(q-\bar{q}) = ||q-\bar{q}|| - \rho||(\bar{q}-q)_+|| \quad , \quad \rho > 1 \quad . \tag{12}$$

The function is order-approximating with $\varepsilon \geq 1/\rho$ for arbitrary norm. If the norm is Euclidean, the function is strictly order preserving (hence, each maximal point is Pareto-maximal) but not even quasi-concave. If the norm is the sum of the absolute vales, the functions is strictly order-preserving and quasi-concave or even concave for $\rho > 2$.

It is, in fact, the simplest extension of the linear utility function in $\bar{q}+R^+_n$ to other linear forms if $q \notin \bar{q}+R^+_n$, see Figure 6b. Its maximization is equivalent to a linear programming problem of the form:

$$\sum_{i=1}^{n} y_i \quad , \quad q \in Q_0 \quad , \quad y \in Y(q-\bar{q}) \quad ; \tag{13}$$

$$Y(q-\bar{q}) = \{y \in R^n : y_i \leq q_i - \bar{q}_i ,\ y_i \leq (\rho-1)(q_i-\bar{q}_i),\ \text{all } i=1,\ldots,n\} \ .$$

The arbitrary choice of the weighting coefficients $\lambda_i = 1$ in the sum of absolute values norm has only a technical character here, since the solutions of (13) are usually at vertices of $Y(q-\bar{q})$ and the a posteriori determined weighting coefficients $\hat{\lambda}_i$, corresponding to a Pareto-maximal $\hat{q}$ and obtained from a dual program, are different than 1.

If the norm in (12) is maximum norm, the the functions is only order preserving and not quasi-concave, see its level sets in Figure 6c. However, it is a convenient function when the surplus $q-\bar{q}$ should be maximized in its norm subject to the soft constraint $q-\bar{q} \in R^n_+$, expressed by the penalty term $-\rho\|(\bar{q}-q)_+\|$. The function (12) extends and generalizes known goal programming [7], [15] and utopia point [21], [27] approaches. Morever, the function (12) can also be used if the objective space is infinite-dimensional, for example, a space of dynamic trajectories. In a Hilbert space, the vector $(\bar{q}-q)_+$ should be then understood--see [24]--as the projection of $\bar{q}-q$ on the dual cone $D^* = \{q^* \in G: \langle q^*, q\rangle \geq 0 \text{ for all } q \in D\}$, where D is the cone used instead of R^n_+ in the extended definition of Pareto-maximality; and additional condition $D \subseteq D^*$ should be also satisfied. More generally, in any linear lattice space, the function (12) takes a little more complicated form--see Wierzbicki 1977 [23].

Another group of penalty scalarizing functions is more closely related to the concept of goal programming [7], [15]. Suppose an objective q_1 is chosen to be maximized under constraints $q_2 \geq \bar{q}_2, \ldots, q_n \geq \bar{q}_n$. These constraints could be treated as soft ones and expressed by the penalty function:

$$s(q-\bar{q}) = q_1-\bar{q}_1-\rho\|(\bar{q}^r-q^r)_+\|_{R^{n-1}};\ q^r = (q_2,\ldots,q_n) \in R^{n-1} \quad , \tag{14}$$

where $\bar{q}_1$ does not influence the maximization of the function but is subtracted for the sake of complete presentation. In the original goal programming formulation, $\bar{q}^n = (\bar{q}_2,\ldots,\bar{q}_n)$ should be attainable reference objective levels. However, the function (14) is (strictly) order-preserving in q for any norm in R^{n-1} (not strictly, if the maximum norm is used), any $\rho > 0$ and *any, not necessarily attainable* $\bar{q} = (\bar{q}_1, \bar{q}_n)$--see Wierzbicki 1978 [25]. This function is also order-approximating with $\varepsilon \geq 1/\rho$. Therefore, it is a penalty scalarizing function expressing the concept of surplus maximization in one coordinate if $\bar{q}$ is attainable, and a concept of distance minimization if $\bar{q}$ is not attainable.

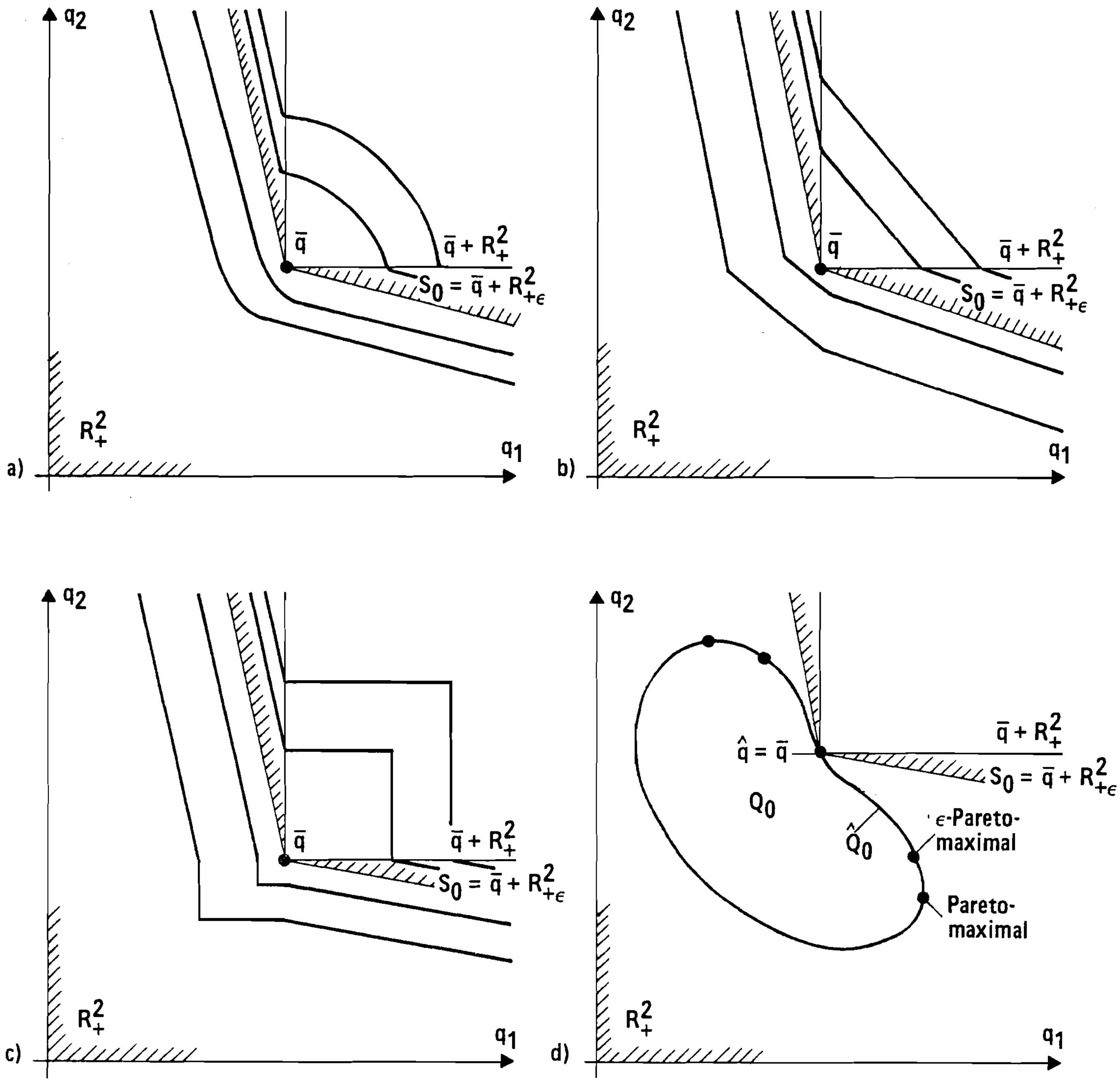

Figure 6. Level sets of penalty scalarizing functions: a) when the Euclidean norm is used, b) when the sum of absolute values norm is used, c) when the maximum norm is used; and d) an illustration of ε-Pareto maximality and of the weak separation of Q_0 and $\bar{q}+R^2_+$ by $S_0 = \bar{q}+R^2_{+\varepsilon}$.

The function (14) is concave; if either the sum of absolute values or the maximum norm is used, its maximization is equivalent to a linear programming problem. For example, if the maximum norm is used, the equivalent problem is:

$$\text{maximize } (q_1-\bar{q}_1-\rho y) \ , \ q \in Q_0 \ , \ y \in Y_0(q^r-\bar{q}^r) \ ; \qquad (15)$$

$$Y_0(q^r-\bar{q}^r) = \{y \in R^1 : y \geq 0, \ y \geq \bar{q}_i - q_i, \text{ all } i=2,\dots,n\} \ .$$

Again, after finding a solution $\hat{q}$ to this problem, the corresponing weighting coefficients $\hat{\lambda} = (1,\hat{\lambda}_2,\dots,\hat{\lambda}_n)$ can be obtained from the dual program.

The function (14) can be also generalized to an objective space $G = R^1 \times G^r$, where G^r is a Hilbert space, a linear lattice space, etc.

5. APPLICATION FIELDS OF REFERENCE OBJECTIVE SCALARIZATION

Analysis of multiobjective optimization models

When building a multiobjective optimization model, the analyst must experiment with it and scan the Pareto-set, that is, obtain a representation of it. Since the nature of the Pareto-set is, as a rule, not a priori known, an application of weighting coefficients to scan the set can lead to quite inconclusive results while the use of a penalty scalarizing function with changing reference objective levels gives reasonable representation of the set--see, for example, [22]. Experience in application of this method to some nonconvex problems of engineering design shows that the scalarizing penalty function of the form (14) is best suited for this purpose. By maximizing independently various objectives, approximate ranges of their change in the Pareto-set can be established. It is reasonable then to choose as q_1 the objective with the most uncertain range of change- for other objectives, a grid of reference levels can be constructed, and used consecutively in the function (14).

If the problem is nonlinear, it might be worthwhile to use instead of the function (14) its differentiable variant

$$s(q-\bar{q}) = q_1 - \bar{q}_1 - \frac{1}{2}\rho \| (\bar{q}^r - q^r)_+ \|^2_{E^{n-1}} , \qquad (16)$$

with the Euclidean norm. The function, though not quite order-approximating, is still strictly order-preserving for all $\rho > 0$. The maximal points of this function, although they usually violate the assumed reference levels slightly (depending on the choice of the penalty coefficient ρ) are Pareto-maximal points, hence neither an iterative increase of the penalty coefficient nor other iterative techniques of constrained optimization are needed here. Since the function (16) is differentiable, at each Pareto-maximal point $\hat{q}$ the corresponding vector of weighting coefficients $\hat{\lambda}$ can be computed from a formula analagous to (6).

If a multiobjective linear programming problem is investigated, a similar procedure based on the formula (15) can give more reasonable representation of the Pareto-set than a parametrization via weighting coefficients. The assumed reference objective levels are then precisely satisfied (as long as they are attainable and the obtained Pareto points are also ε-Pareto optimal with $\varepsilon = 1/\rho$), since the function (14) equivalently represented by (15) is an exact penalty function.

Interactive procedures of multiobjective optimization

The main strength of reference objective scalarization consists in the possibility of contructing efficient interactive procedures of multiobjective scalarization. There are many possible variants of such procedures, though all of them are based on the principle explained in Figure 4. The decision-maker specifies a reference objective point, and the optimization model responds with one or more Pareto-maximal alternatives, in a sense close to the decision maker requirements (or better, if the requirements are attainable). Then the decision-maker either chooses one of the alternatives, or modifies his reference objective point.

Various variants of such a procedure were described and analyzed in Wierzbicki 1979 [26] and Kallio and Lewandowski 1979 [16]. For example, given a reference objective point $\bar{q}^j$, the optimizaiton model determines first $\hat{q}^j = \arg\max_{q \in Q_0} s(q-\bar{q}^j)$, then $d_j = \| \bar{q}^j - \hat{q}^j \|$ and additional reference points $\bar{q}^{j,i} = \bar{q}^j + d_j e_i$ together with additional alternatives $\hat{q}^{j,i} = \arg\max_{q \in Q_0} s(q-\bar{q}^{j,i})$. Here $e_i = (0,\ldots,1_i,\ldots,0)$ is the i-th unit basis vector, and for each reference objective point $\bar{q}^j$ the procedure responds with n+1 alternative Pareto-maximal points $\hat{q}^j$, $\hat{q}^{j,i}$. If $d_j = \| \bar{q}^j - \hat{q}^j \|$ is large, at the beginning of the procedure, then the alternatives $\hat{q}^{j,i}$ are more widely spread. If the decision-maker moves his requirements $\bar{q}^j$ towards the Pareto-set, then d_j decreases and the procedure generates alternatives $\hat{q}^{j,i}$ more finely describing the Pareto-set in the region of decision makers' interests. Additional conditions which guarantee the convergence of this procedure are given in [26]. However, it is practically sufficient to ask the decision maker that he moves his requirements $\bar{q}^j$ generally in the direction of the Pareto-set (described to him by the alternatives $\hat{q}^j$, $\hat{q}^{j,i}$), and he usually terminates the procedure after a small number of iterations. Moreover, if we assume that the decision-maker has a preference relation described by a utility function, it is easy to show that the terminal point of this procedure does approximately maximize his utility, since the n+1 alternatives could be used to identify his preferences, and indifference sets. Such an interpretation is, however, not necessary: the decision maker is not asked about his preferences during this procedure, he modifies only his requirements in a natural and easily understandable fashion.

Trajectory optimization

In typical formulations of dynamic optimization, single or multiple objectives are obtained through aggregating the dynamic trajectories by integral functionals. This technique is motivated, however, by the traditional mathematical approaches to dynamic optimization, and not necessarily by the needs of the real world. Experienced analysts, economists and decision makers often evaluate intuitively entire trajectories, functions of time, better than aggregate integral indices. Adopting the viewpoint of the classical utility theory, we could say that they do have their own utility functionals, expressing their preferences over trajectories. However, how one can identify experimentally a utility function depending on an infinite number of objectives, or, after a discretization of time, even a utility function depending on a very large number of objectives? Clearly, we need here an ad hoc approximation of decision maker's preferences, constructed with the help of the best available information. Once the decision maker is experienced in evaluating trajectories, he can state his requirements in terms of a *reference trajectory* $\bar{q}(t)$, a scalar- or vector-valued function of time (for example, the gross national product and the inflation rate versus time, see [26]). Since the penalty scalarizing functions can be directly generalized to infinite-dimensional spaces, hence, if a dynamic model of the problem is available, it is possible to choose an appropriate objective space, to fomulate a penalty scalarizing functional, to apply any known dynamic optimization technique, and thus to construct an optimization model. The model responds to the decision-maker's requirements by (generalized) Pareto-optimal trajectories, in a sense close to the required if the latter are not attainable, and in a sense better than the required if the latter are attainable.

The simplest choice of the objective space is the space of square interable functions $L^2[0;T]$ where T is the time horizon, with the positive cone $D = \{q \in L^2[0;T] : q(t) \geq 0$ almost everywhere on $[0;T]\}$. The corresponding penalty scalarizing functional similar to (12) becomes then:

$$s(q-\bar{q}) = \int_0^T (q(t)-\bar{q}(t))^2 - \rho(\bar{q}(t)-q(t))_+^2 \, dt \quad . \tag{17}$$

If the time is discretized, then the sum replaces the integral; the problem becomes finite-dimensional, but it is still more convenient to think in terms of trajectories than in terms of separate objectives. Many other choices of the objective space, of the scaling of trajectories (for example, in terms of deflation rates), etc., are possible.

The concept of trajectory optimization via penalty function scalarization has been applied by Kallio and Lewandowski 1979 [16] in a study of alternative policies for the Finnish forestry industrial sector. The results of this study confirm the viewpoint that, in some cases, reference trajectories provide for a better information than aggregate scalar indices.

Semi-regularization of solutions of optimization models

If a single-objective optimization model possesses many comparable solutions, a standard technique of choosing between them is to find that one which is closest to a given reference point--not necessarily in the solution space, but in any space of chosen indices in which a reference point can be found from earlier experience and expertise. Denote the original objective by $q_1 = f_1(x)$ and the additional indices by $(q_2,\ldots,q_n) = q^r = f^r(x)$ and let the reference point $\bar{q}^r$ be given; we obtain thus, in fact, a multiobjective problem. The typical technique of choosing between various $x \in X_0$ nearly maximizing $f_1(x)$ is to maximize a penalty function, for example:

$$p(x,\rho) = f_1(x) - \tfrac{1}{2}\rho \| f^r(x) - \bar{q}^r \|^2_{E^{n-1}} \quad , \tag{18}$$

or, for linear problems:

$$p(x,\rho) = f_1(x) - \rho \max_{2 \le i \le n} |f_i^r(x) - \bar{q}_i^r| \quad , \tag{19}$$

for sufficiently small $\rho > 0$. This general technique is a case of Tikhonov regularization of solutions of badly determined problems. It also has various deep interpretations. For example, we are often sure that a mathematical model describes the reality sufficiently well for decisions and their outcomes known from experience. However, the optimal solutions for the model can be far from those known from experience, and we can doubt whether the model is sufficiently exact in this new region of decisions and outcomes. This question arises particularly often if linear programming models are used, and we consider only some vertices of the admissible set, corresponding either to optimal or nearly optimal solutions. Clearly, if there are many solutions which differ only a little in the objective function level $q_1 = f_1(x)$, we should choose the one that is in a sense closest to those known from experience.

Observe, however, that the penalty functions (18), (19) are special cases of the scalarizing functions (14), (15), (16) if we are sure that the reference point $\bar{q}^r$ is not attainable. If it is attainable, we do not always want only to be close to $\bar{q}^r$: we might as well like to exceed $\bar{q}^r$ in some or all of its components (examples of such indices might be some energy conservation indices, gross national product, etc.). Denote, therefore, $\bar{q}^r = (\bar{q}^s, \bar{q}^t)$ where $\bar{q}^s$ contains those components which are to be exceeded, if possible, and $\bar{q}^t$ those ones which should only be kept close to. Then the following penalty scalarizing function:

$$s(q-\bar{q}) = q_1 - \bar{q}_2 - \rho^s \| (q^s-\bar{q}^s)_+ \| - \rho^t \| q^t - \bar{q}^t \| \quad , \tag{20}$$

expresses the principle of *semiregularization*: keep $q^t = f^t(x)$ close to $\bar{q}^t$ and either keep close to or exceed $\bar{q}^s$ by $q^s = f^s(x)$. Other forms of this function, as in (15), (16), are also possible. Similarly as in equation (14), the reference level $\bar{q}_1$ does not have any meaning but of theoretical convenience in formulating an order approximation property. We can define a new positive cone D in R^n by $D = \{q \in R^n : q_1 \geq 0, q_i \geq 0$ for $i \in s$, $q_i = 0$ for $i \in t\}$ where s and t denote the sets of indices for q^s and q^t, and prove that the function (20) is D_ε-order approximating, that is if $S_0 \overset{df}{=} \{q\ R^n : s(q-\bar{q}) \geq 0\}$, then $\bar{q} + D \subset S_0 \subset \bar{q} + D_\varepsilon$, $D_\varepsilon = \{q \in R^n : \mathrm{dist}(q,D) \leq \varepsilon \|q\|\}$, $\varepsilon = \max(1/\rho^s, 1/\rho^t)$. The function (20) is also D-order preserving. These notions can be generalized as well to the case when q^s and q^t are elements of infinite dimensional spaces.

The semiregularization of solutions of optimization models is thus a special case of multiobjective optimization with appropriately defined positive cone, and the techniques of scanning the Pareto-set, interactive procedures or trajectory optimization techniques described in the above paragraphs can be used here.

Compromise-aiding procedures for cooperative games

If the decisions represented in a multiobjective optimization model can be made in reality by several distinct decision-makers, then the model is actually a game model: to accept a solution proposed by the model the decision makers have to agree about goals. It is natural, therefore, to construct compromise-aiding procedures where the decision makers bargain only in terms of reference objective levels and an optimization model provides them with various alternative Pareto-maximal solutions in response to their desired reference objective levels. Several variants of such a technique has been analyzed recently in Kallio and Lewandowski 1979 [16] and Wierzbicki 1979 [26]; the latter paper also contains a convergence analysis of such a procedure.

6. CONCLUSIONS AND POSSIBLE EXTENSIONS

An alternative approach to multiobjective optimization, based on the notion of reference objective levels rather than on weighting coefficients or utility functions, has many aspects. On one hand, it is a pragmatical approach: the information that is most likely to be obtained from the decision makers is used in order to construct rough, ad hoc approximations of their possibly varying preference patterns. However, this approach is also consistent with some practically observed behavioral properties of decision makers, namely, with the non-symmetrical attitude to the prospects of not attaining or to exceeding stated goals. On the other hand, the approach is well-founded mathematically: all basic theorems of multiobjective optimization, including sufficient and necessary conditions of Pareto-optimality, etc., can be equivalently or even more generally stated in terms of reference objective levels and penalty scalarizing functions than in terms of weighting coefficients and utility functions.

Although this approach is certainly not "the one" best suited to solve all classes of multiobjective decision-making and optimization problems, however some problems of repetitive decisions based on mathematical models, with varying preferences of decision makers, can be much more conveniently solved by this approach than by other known approaches. Much remains to be done, however, in a wider testing of this approach in many applicational fields. There are also important mathematical questions to be further investigated: the use of reference objective levels in stochastic optimization, in situations of uncertainty, a possible treatment of risks by this approach, etc.

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The following bibliography consists of two parts: first, the references quoted in this paper (part A) and then the results of an extensive, tough certainly not complete, bibliographical search for papers related to multiobjective optimization and decision making published in years 1974-79 (part B), excluding those from part A). Further references can be found in [29].

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