A MICROSCOPIC CONVEXITY PRINCIPLE FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

BAOJUN BIAN AND PENGFEI GUAN

1. Introduction

Caffarelli-Friedman [7] proved a constant rank theorem for convex solutions of semilinear elliptic equations in $\mathbb{R}^2$; a similar result was also discovered by Yau [28] at about the same time. Shortly thereafter, the result in [7] was generalized to $\mathbb{R}^n$ by Korevaar-Lewis [27]. This type of constant rank theorem is called a microscopic convexity principle. It is a powerful tool in the study of geometric properties of solutions of nonlinear differential equations and is particularly useful in producing convex solutions of differential equations via homotopic deformations. The great advantage of the microscopic convexity principle is that it can treat geometric nonlinear differential equations involving tensors on general manifolds. The proof of such a microscopic convexity principle for a σ_k type equation on the unit sphere S^n by Guan-Ma [17] is crucial in their study of the Christoffel-Minkowski problem. The microscopic convexity principle also provides some interesting geometric properties of solutions. For a symmetric Codazzi tensor, the microscopic convexity principle implies that the distribution of null space of the tensor is of constant dimension and is parallel.

The microscopic convexity principle has been validated for a variety of fully nonlinear differential equations involving the second fundamental form of hypersurfaces ([17, 16, 18, 8]). Understanding under what structural conditions the microscopic convexity principle is valid is central. Caffarelli-Guan-Ma [8] established such a principle for fully nonlinear equations of the form:

$$F(u_{ij}(x)) = \varphi(x, u(x), \nabla u(x)). \tag{1.1}$$

where $F(A)$ is symmetric and $F(A^{-1})$ is locally convex in A. Similar results were also proved for symmetric tensors on manifolds in [8]. Several interesting geometric applications were also given there. For applications, it is important to consider equations F involving other variables in addition to the hessian (u_{ij}). For example, it is desirable to include linear elliptic equations and quasilinear equations with variable coefficients. In many cases, a solution v to an equation

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may not be convex yet some transformation $u = h(v)$ of it may be convex (see e.g., [6, 7]). If v is a solution of equation (1.1), then $u = h(v)$ is a solution of equation

$$F(\nabla^2 u, \nabla u, u, x) = 0. \tag{1.2}$$

A similar situation also arises in the case of geometric flow for hypersurfaces.

In this paper, we study the microscopic convexity property for an equation of the general form (1.2) and related geometric nonlinear equations of elliptic and parabolic type. The core idea in the proof of a microscopic convexity principle is to establish a strong maximum principle for an appropriate auxiliary function. There have been significant contributions in the literature [7, 27, 17, 16, 18, 8] developing analytic techniques for this purpose. All of these methods break down for a general fully nonlinear elliptic equation of the form (1.2). The main contribution of this paper is the introduction of new analytic techniques involving quotients of elementary symmetric functions near the null set of $\det(u_{ij})$. The analysis is delicate as both symmetric functions in the quotient will vanish on the null set. This is a novel feature of this paper. It is another indication that these quotient functions of elementary symmetric functions are naturally embedded in the study of fully nonlinear equations. In a different context, the importance of quotient functions has been demonstrated in the beautiful work of Huisken-Sinestrari [22]. We believe our techniques will be useful in solving other problems in geometric analysis.

To illustrate our main results, we first consider equations in a flat domain. Let Ω be a domain in $\mathbb{R}^n$ and denote by $\mathcal{S}^n$ the space of real symmetric $n \times n$ matrices and $\mathcal{S}^n_+$ the space of positive definite real symmetric $n \times n$ matrices. Let $F = F(r, p, u, x)$ defined in $\mathcal{S}^n \times \mathbb{R}^n \times R \times \Omega$ be elliptic in the sense that

$$\left(\frac{\partial F}{\partial r_{\alpha\beta}}(\nabla^2 u, \nabla u, u, x)\right) > 0, \ \forall x \in \Omega\ . \tag{1.3}$$

Theorem 1.1. *Suppose $F = F(r, p, u, x) \in C^{2,1}(\mathcal{S}^n \times \mathbb{R}^n \times R \times \Omega)$ satisfies condition (1.3) and*

$$F(A^{-1}, p, u, x) \text{ is locally convex in } (A, u, x) \text{ for each } p\ . \tag{1.4}$$

If $u \in C^{2,1}(\Omega)$ is a convex solution of (1.2), then the rank of the hessian $(\nabla^2 u(x))$ is a constant l in Ω. For each $x_0 \in \Omega$, there exist a neighborhood U of x_0 and $(n-l)$ fixed directions $V_1, \ldots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \le j \le n-l$ and $x \in U$.

There is also a parabolic version.

Theorem 1.2. *Suppose $F = F(r, p, u, x, t) \in C^{2,1}(\mathcal{S}^n \times \mathbb{R}^n \times R \times \Omega \times [0, T))$ satisfies condition (1.3) and*

$$F(A^{-1}, p, u, x, t) \text{ is locally convex in } (A, u, x) \text{ for each pair } (p, t)\ . \tag{1.5}$$

Suppose $u \in C^{2,1}(\Omega \times [0,T))$ is a convex solution of the equation

$$\frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, x, t)\ . \tag{1.6}$$

For each $t \in (0,T)$, let $l(t)$ be the minimal rank of $(\nabla^2 u(x,t))$ in Ω, then the rank of $(\nabla^2 u(x,t))$ is constant $l(t)$ and $l(s) \le l(t)$ for all $s \le t < T$. For each $0 < t \le T$, $x_0 \in \Omega$ there exist a neighborhood $\mathcal{U}$ of x_0 and $(n-l(t))$ fixed directions $V_1, \dots, V_{n-l}(t)$ such that $\nabla^2 u(x,t)V_j = 0$ for all $1 \le j \le n-l(t)$ and $x \in \mathcal{U}$. Furthermore, for any $t_0 \in [0,T)$, there is a $\delta > 0$, such that the null space of $(\nabla^2 u(x,t))$ is parallel in (x,t) for all $x \in \Omega, t \in (t_0, t_0+\delta)$.

An immediate consequence of Theorem 1.1 is the proof of a conjecture raised by Korevaar-Lewis in [27] for convex solutions of mean curvature type elliptic equation

$$\sum_{i,j} a^{ij}(\nabla^2 u(x))u_{ij}(x) = f(x, u(x), \nabla u(x)) > 0. \tag{1.7}$$

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^n$ and suppose u is a convex solution of the elliptic equation (1.7). If*

$$f(x,u,p) \ \text{ is locally convex in } (x,u) \text{ for each } p, \tag{1.8}$$

then the hessian $(\nabla^2 u(x))$ is of constant rank in Ω.

Korevaar-Lewis [27] proved that the Hessian of any convex solution u of an elliptic equation (1.7) is of constant rank and u is constant in $n-l$ coordinate directions, provided that $\frac{1}{f(\cdot,p)}$ is strictly convex for any p fixed. They conjectured that the constant rank result still holds if $\frac{1}{f(\cdot,p)}$ is only assumed to be convex. They observed that when $n=2$, this can be deduced from the proofs of Caffarelli-Friedman in [7]. Set

$$F(\nabla^2 u, \nabla u, u, x) = -\frac{1}{\sum_{i,j} a^{ij}(\nabla^2 u(x))u_{ij}(x)} + \frac{1}{f(x,u(x),\nabla u(x))}$$

Then equation (1.7) is equivalent to $F(\nabla^2 u, \nabla u, u, x) = 0$. It is straightforward to check that F satisfies Conditions (1.3) and (1.4) under the assumptions in Corollary 1.3.

We now discuss some geometric equations on general manifolds. Preservation of convexity is an important issue for the geometric flows of hypersurfaces (see e.g., [21, 5] and the references therein). We have the following general result.

Theorem 1.4. *Suppose $F(A, X, \vec{n})$ is elliptic in A and $F(A^{-1}, X, \vec{n})$ is locally convex in (A, X) for each fixed $\vec{n} \in S^n$. Let $M(t) \subset \mathbb{R}^{n+1}$ be a compact hypersurface satisfying the geometric flow equation*

$$X_t = -F(g^{-1}h, X, \vec{n})\vec{n}, \quad t \in (0,T), \ M(0) = M_0\ , \tag{1.9}$$

where $X, \vec{n}, g, h$ are, respectively, the position vector, outer normal, induced metric and the second fundamental form of $M(t)$. If M_0 is convex, then $M(t)$ is strictly convex for all $t \in (0,T)$.

Alexandrov in [1, 3] studied existence and uniqueness of solutions of general nonlinear curvature equations,

$$F(g^{-1}h, X, \vec{n}(X)) = 0, \ \forall X \in M, \tag{1.10}$$

where X is the position function of M and $\vec{n}(X)$ is the unit normal of M at X. The following theorem addresses the convexity property of problems studied in [1, 3].

Theorem 1.5. *Suppose $F(A, X, \vec{n})$ is elliptic in A and $F(A^{-1}, X, \vec{n})$ is locally convex in (A, X) for each fixed $\vec{n} \in S^n$. Let M be an oriented immersed connected hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form h satisfying equation (1.10). Then h is of constant rank and its null space is parallel. In particular, if M is complete, then there is $0 \le l \le n$ such that $M = M^l \times \mathbb{R}^{n-l}$ for a strictly convex compact hypersurface M^l in $\mathbb{R}^{l+1}$ (if $l > 0$). If in addition M is compact, then M is the boundary of a strongly convex bounded domain in $\mathbb{R}^{n+1}$.*

Theorem 1.5 has similarities with the classical result of Hartman-Nirenberg in [20].

The microscopic convexity principle also can be used to prove some uniqueness theorems in differential geometry in the large. A surface immersed in $\mathbb{R}^3$ is called a Weingarten surface if its principle curvatures κ_1, κ_2 satisfy a relationship $F(\kappa_1, \kappa_2) = 0$ for some elliptic F (i.e, F satisfies condition (1.3)). Alexandrov [2] and Chern [12] proved that if M is a closed convex Weingarten surface in $\mathbb{R}^3$, then M is a sphere. In higher dimensions, there is an extensive literature (see e.g., [11, 13]) devoted to showing immersed hypersurfaces are spheres. We prove the following sphere theorem.

Theorem 1.6. *Suppose (M, g) is a compact connected Riemannian manifold of dimension n with nonnegative sectional curvature which is positive at one point. Suppose $F(A)$ is elliptic, and W is a Codazzi tensor on M satisfying the equation*

$$F(g^{-1}W) = 0 \ \text{on} \ M\,. \tag{1.11}$$

If either (1) $n = 2$, or
(2) $n \ge 3$, W is semi-positive definite and $F(A^{-1})$ is locally convex for $A > 0$,
then $W = cg$ for some constant $c \ge 0$.

Theorem 1.6 was proved by Ecker-Huisken in [13] under the assumption F is concave. Refer to Remark 5.7 for the relationship between concavity of $F(A)$ and the condition on F in case (2) of Theorem 1.6. Note that when $n = 2$, only the ellipticity assumption on F is needed in Theorem 1.6. Refer to [17, 18, 8] for other applications of the microscopic convexity principle in classical and conformal geometry and to [15] for applications in Kähler geometry.

A vast literature exists devoted to the study of the convexity of solutions of partial differential equations. There is a theory of macroscopic nature, where the problem is always considered in a convex domain in $\mathbb{R}^n$ with appropriate boundary conditions. In 1983, Korevaar made breakthroughs in [25, 26] where he obtained concavity maximum principles for a class of quasilinear elliptic equations. His results were improved by Kennington [24] and by Kawhol [23]. The theory was further developed to great generality by Alvarez-Lasry-Lions [4] in 1997. They established the existence of a convex solution of equation (1.2) for state constraint boundary values under conditions (1.3)-(1.4) assuming that F satisfies a comparison principle. Microscopic convexity implies macroscopic convexity if there is a deformation path (e.g., via the method of continuity or parabolic flow). Theorem 1.1 is the microscopic version of the macroscopic convexity principle in [4].

The rest of the paper is organized as follows. In section 2, we introduce a key auxiliary function $q(x) = \frac{\sigma_{l+2}(\nabla^2 u(x))}{\sigma_{l+1}(\nabla^2 u(x))}$ which is well defined by the Newton-Maclaurin inequalities. In Proposition 2.1 we demonstrate a key concavity inequality for $q(x)$ and in Corollary 2.2, we conclude that q has optimal $C^{1,1}$ regularity. In section 3, we establish a strong maximum principle for the function $\phi(x) = \sigma_{l+1}(\nabla^2 u(x)) + q(x)$ which is the main technical tool of the paper. In section 4, we discuss condition (1.4) and related results. The last section is devoted to geometric equations on manifolds.

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2. An Auxiliary function

$\nabla^2 u$ is of constant rank if and only if $\sigma_{l+1}(\nabla^2 u) \equiv 0$, where l is the minimum rank of $\nabla^2 u$. It was first shown by Caffarelli-Friedman in [7] that there is a strong maximum principle for $\sigma_{l+1}(\nabla^2 u)$ for any convex solution of $\Delta u = f$ when $\frac{1}{f}$ is locally convex (see also subsequential works [27, 17, 16, 18]). When F in (1.1) is a general symmetric function, such a maximum principle for $\sigma_{l+1}(\nabla^2 u)$ is difficult to prove. A major achievement in [8] is the establishment of a maximum principle for function $\sigma_{l+1}(\nabla^2 u) + A\sigma_{l+2}(\nabla^2 u)$ when $A > 0$ is sufficient large. For the general equation (1.2), we do not know how to prove the corresponding maximum principle for the previously known test functions. This lead us to search for a new auxiliary function. It turns out $\sigma_{l+1}(\nabla^2 u) + \frac{\sigma_{l+2}(\nabla^2 u)}{\sigma_{l+1}(\nabla^2 u)}$ is **the function**! The rest of this section is devoted to the analysis of this function near the null set $\mathcal{N} = \{\sigma_{l+1}(\nabla^2 u) = 0\}$.

With the assumptions of F and u in Theorem 1.1 and Theorem 1.2, u is automatically in $C^{3,1}$. This will be assumed in the rest of this paper. Let $W(x) = \nabla^2 u(x)$ and $l = \min_{x\in\Omega} \operatorname{rank}(\nabla^2 u(x))$. $l \leq n-1$ may also be assumed. Suppose $z_0 \in \Omega$ is a point where W is of minimal rank l.

Throughout this paper we assume that $\sigma_j(W) = 0$ if $j < 0$ or $j > n$. Define for $W = (u_{ij}) \in \mathcal{S}^n$

$$q(W) = \begin{cases} \frac{\sigma_{l+2}(W)}{\sigma_{l+1}(W)}, & \text{if } \ \sigma_{l+1}(W) > 0 \\ 0, & \text{if } \ \sigma_{l+1}(W) = 0 \end{cases} \tag{2.1}$$

For any symmetric function $f(W)$, we denote

$$f^{ij} = \frac{\partial f(W)}{\partial u_{ij}}, \quad f^{ij,km} = \frac{\partial^2 f(W)}{\partial u_{ij}\partial u_{km}}$$

For each $z_0 \in \Omega$ where W is of minimal rank l. We pick an open neighborhood $\mathcal{O}$ of z_0, for any $x \in \mathcal{O}$, let $\lambda_1(x) \leq \lambda_2(x)... \leq \lambda_n(x)$ be the eigenvalues of W at x. There is a positive constant $C > 0$ depending only on $\|u\|_{C^{3,1}}$, $W(z_0)$ and $\mathcal{O}$, such that $\lambda_n(x) \geq \lambda_{n-1}(x)... \geq \lambda_{n-l+1}(x) \geq C$ for all $x \in \mathcal{O}$. Let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices respectively. Let $\Lambda_G = (\lambda_{n-l+1}, ..., \lambda_n)$ be the "good" eigenvalues of W at x and $\Lambda_B = (\lambda_1, ..., \lambda_{n-l})$ be the "bad" eigenvalues of W at x. For the simplicity, write $G = \Lambda_G$, $B = \Lambda_B$ if there is no confusion. Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $\lambda_i(x) < \delta$ for all $i \in B$ and $x \in \mathcal{O}$.

Set

$$\phi = \sigma_{l+1}(W) + q(W) \tag{2.2}$$

where q defined as in (2.1). Use notation $h = O(f)$ if $|h(x)| \leq Cf(x)$ for $x \in \mathcal{O}$ with the positive constant C under control. It is clear that $\lambda_i = O(\phi)$ for all $i \in B$.

To get around $\sigma_{l+1}(W) = 0$, consider for $\epsilon > 0$ sufficient small,

$$q_\epsilon(W) = \frac{\sigma_{l+2}(W_\epsilon)}{\sigma_{l+1}(W_\epsilon)}, \quad \phi_\epsilon(W) = \sigma_{l+1}(W_\epsilon) + q_\epsilon(W), \tag{2.3}$$

where $W_\epsilon = W + \epsilon I$. We will also denote $G_\epsilon = (\lambda_{n-l+1} + \epsilon, ..., \lambda_n + \epsilon)$, $B_\epsilon = (\lambda_1 + \epsilon, ..., \lambda_{n-l} + \epsilon)$

We will work on q_ϵ to obtain a uniform C^2 estimate independent of ϵ. One may also work directly on q at the points where $\sigma_{l+1}(\nabla^2 u) \neq 0$ to obtained the same results in the rest of this section (with all relative constants independent of chosen point).

Set

$$v_\epsilon(x) = u(x) + \frac{\epsilon}{2}|x|^2, \tag{2.4}$$

then $W_\epsilon = (\nabla^2 v_\epsilon)$. To simplify the notation, we will write v for v_ϵ, q for q_ϵ, W for W_ϵ, G for G_ϵ and B for B_ϵ with the understanding that all the estimates will be independent of ϵ. In this setting, with $\mathcal{O}$ is small enough, there is $C > 0$ independent of ϵ such that

$$\sigma_{l+1}(W(x)) \geq C\epsilon, \quad \text{and} \quad \sigma_1(B(x)) \geq C\epsilon, \quad \text{for all } x \in \mathcal{O}. \tag{2.5}$$

Similarly write $h = O(f)$ if $|h(x)| \leq Cf(x)$ for $x \in \mathcal{O}$ with positive constant C under control independent of ϵ.

The importance of the function q is reflected in the following proposition. Set

$$V_{i\alpha} = v_{ii\alpha}\sigma_1(B) - v_{ii}\Big(\sum_{j\in B} v_{jj\alpha}\Big). \tag{2.6}$$

Proposition 2.1. *For each $z \in \mathcal{O}$ with $W(z)$ is diagonal, for any $\alpha, \beta \in \{1, \cdots, n\}$,*

$$\begin{aligned}\sum_{i,j,k,m} q^{ij,km} v_{ij\alpha} v_{km\beta} &= O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - 2\sum_{i\in B, j\in G} \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)\lambda_j} v_{ij\alpha} v_{ji\beta} \\ &\quad - \frac{\sum_{i\in B} V_{i\alpha} V_{i\beta}}{\sigma_1^3(B)} - \frac{\sum_{i,j\in B, i\neq j} v_{ij\alpha} v_{ji\beta}}{\sigma_1(B)}.\end{aligned} \tag{2.7}$$

The last two terms in (2.7) will play a key role in estimating linear terms of $v_{ij\alpha}$ $(i, j \in B)$ in our proof of Theorem 1.1 in the next section.

Corollary 2.2. *Let $u \in C^{3,1}(\Omega)$ be a convex function. $W(x) = (u_{ij}(x)), x \in \Omega$ and $l = \min_{x\in\Omega} \operatorname{rank}(W(x))$. Then the function $q(x) = q(W(x))$ defined in (2.1) is in $C^{1,1}(\Omega)$.*

The rest of this section is devoted to proving Proposition 2.1, and it involves some subtle analysis of the function q. The proof of Corollary 2.2 will be given at the end of this section. In preparation, several well known lemmas are listed. For the sake of completeness, proofs are provided. If W is any $n \times n$ diagonal matrix, denote by $(W|i)$ the $(n-1) \times (n-1)$ matrix with ith row and ith column deleted, and $(W|ij)$ the $(n-2) \times (n-2)$ matrix with i, jth rows and i, jth columns deleted.

Lemma 2.3. *Suppose W is diagonal. Then we have*

$$q^{ij} = \begin{cases} \frac{\sigma_{l+1}(W)\sigma_{l+1}(W|i) - \sigma_{l+2}(W)\sigma_l(W|i)}{\sigma_{l+1}^2(W)}, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(a). if $i = m, j = k, i \neq j$, then

$$q^{ij,km} = -\frac{\sigma_l(W|ij)}{\sigma_{l+1}(W)} + \frac{\sigma_{l+2}(W)\sigma_{l-1}(W|ij)}{\sigma_{l+1}^2(W)}$$

(b). if $i = j = k = m$, then

$$q^{ij,km} = -2\frac{\sigma_l(W|i)}{\sigma_{l+1}^3(W)}[\sigma_{l+1}(W)\sigma_{l+1}(W|i) - \sigma_l(W|i)\sigma_{l+2}(W)]$$

(c). if $i=j, k=m, i\neq k$, then

$$q^{ij,km}=\frac{\sigma_l(W|ik)}{\sigma_{l+1}(W)}-\frac{\sigma_{l+1}(W|i)\sigma_l(W|k)}{\sigma_{l+1}^2(W)}-\frac{\sigma_{l+1}(W|k)\sigma_l(W|i)}{\sigma_{l+1}^2(W)}$$

$$-\frac{\sigma_{l+2}(W)\sigma_{l-1}(W|ik)}{\sigma_{l+1}^2(W)}+2\frac{\sigma_{l+2}(W)\sigma_l(W|i)\sigma_l(W|k)}{\sigma_{l+1}^3(W)}$$

(d). otherwise

$$q^{ij,km}=0$$

Proof. Since W is diagonal, it follows from Proposition 2.2 in [17]

$$\frac{\partial\sigma_\gamma(W)}{\partial v_{ij}}=\begin{cases}\sigma_{\gamma-1}(W|i), & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases}$$

and

$$\frac{\partial^2\sigma_\gamma(W)}{\partial v_{ij}\partial v_{km}}=\begin{cases}\sigma_{\gamma-2}(W|ik), & \text{if } i=j, k=m, i\neq k\\ -\sigma_{\gamma-2}(W|ij), & \text{if } i=m, j=k, i\neq j\\ 0, & \text{otherwise}\end{cases}$$

for $1\leq\gamma\leq n$. We obtain thus

$$\sigma_{l+1}^{ij}=\frac{\partial\sigma_{l+1}}{\partial W_{ij}}=\begin{cases}\sigma_l(W|i), & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases}$$

and

$$\sigma_{l+1}^{ij,km}=\frac{\partial^2\sigma_{l+1}}{\partial W_{ij}\partial W_{km}}=\begin{cases}\sigma_{l-1}(W|ik), & \text{if } i=j, k=m, i\neq k\\ -\sigma_{l-1}(W|ij) & \text{if } i=m, j=k, i\neq j\\ 0 & \text{otherwise}\end{cases}\tag{2.8}$$

Direct computation yields

$$q^{ij}=\frac{1}{\sigma_{l+1}(W)}\frac{\partial\sigma_{l+2}(W)}{\partial v_{ij}}-\frac{\sigma_{l+2}(W)}{\sigma_{l+1}^2(W)}\frac{\partial\sigma_{l+1}(W)}{\partial v_{ij}}\tag{2.9}$$

and

$$q^{ij,km}=\frac{1}{\sigma_{l+1}(W)}\frac{\partial^2\sigma_{l+2}(W)}{\partial v_{ij}\partial v_{km}}-\frac{1}{\sigma_{l+1}^2(W)}\frac{\partial\sigma_{l+2}(W)}{\partial v_{ij}}\frac{\partial\sigma_{l+1}(W)}{\partial v_{km}}$$

$$-\frac{1}{\sigma_{l+1}^2(W)}\frac{\partial\sigma_{l+2}(W)}{\partial v_{km}}\frac{\partial\sigma_{l+1}(W)}{\partial v_{ij}}-\frac{\sigma_{l+2}(W)}{\sigma_{l+1}^2(W)}\frac{\partial^2\sigma_{l+1}(W)}{\partial v_{ij}\partial v_{km}}$$

$$+2\frac{\sigma_{l+2}(W)}{\sigma_{l+1}^3(W)}\frac{\partial\sigma_{l+1}(W)}{\partial v_{ij}}\frac{\partial\sigma_{l+1}(W)}{\partial v_{km}}\tag{2.10}$$

The lemma follows from (2.9) and (2.10). □

Lemma 2.4. *Suppose W is diagonal, then*

$$q^{ij} = \begin{cases} \frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)} + O(\phi), & \text{if } i=j\in B \\ O(\phi), & \text{if } i=j\in G \\ 0, & \text{if } i\neq j. \end{cases}$$

Furthermore $q^{ij,km}$ can be computed as follows:

(1) *If $i,j,k,m\in G$,*

$$q^{ij,km} = O(\phi)$$

(2) *If $j\in G, i\in B$,*

$$q^{ji,ij} = q^{ij,ji} = -\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)v_{jj}} + O(\phi)$$

(3) *If $i,j\in B, i\neq j$,*

$$q^{ij,ji} = -\frac{1}{\sigma_1(B)} + O(1)$$

(4) *If $i\in B$,*

$$q^{ii,ii} = -\frac{2}{\sigma_1^3(B)}(\sigma_1(B)\sigma_1(B|i)-\sigma_2(B)) + O(1)$$

(5) *If $i\in B, k\in G$,*

$$q^{kk,ii} = q^{ii,kk} = O(1)$$

(6) *If $i,k\in B, i\neq k$,*

$$q^{ii,kk} = \frac{2\sigma_2(B)-\sigma_1^2(B)+(v_{ii}+v_{kk})\sigma_1(B)}{\sigma_1^3(B)} + O(1)$$

(7) *otherwise*

$$q^{ij,km} = 0.$$

Proof. From [17], for $W=(G,B)$ and $\gamma\geq l$,

$$\sigma_\gamma(W) = \sum_{k=0}^{l}\sigma_k(G)\sigma_{\gamma-k}(B),$$

and

$$\sigma_\gamma(W|i) = \sum_{k=0}^{l}\sigma_k(G)\sigma_{\gamma-k}(B|i), \quad \text{for } i\in B;$$

$$\sigma_\gamma(W|i) = \sum_{k=0}^{l-1}\sigma_k(G|i)\sigma_{\gamma-k}(B), \quad \text{for } i\in G:$$

$$\sigma_\gamma(W|ij) = \sum_{k=0}^{l-2} \sigma_k(G|ij)\sigma_{\gamma-k}(B), \quad \text{for } i,j \in G;$$

$$\sigma_\gamma(W|ij) = \sum_{k=0}^{l-1} \sigma_k(G|i)\sigma_{\gamma-k}(B|j), \quad \text{for } i \in G, j \in B$$

$$\sigma_\gamma(W|ij) = \sum_{k=0}^{l} \sigma_k(G)\sigma_{\gamma-k}(B|ij), \quad \text{for } i,j \in B,$$

where $\sigma_{\gamma-k}(B) = 0$ if $\gamma - k > n - l$. The lemma follows directly from lemma 2.3 and above formulae. □

Next lemma provides an estimate for third order derivatives of convex functions.

Lemma 2.5. *Assume $v \in C^{3,1}(\Omega)$ is a convex function. Then there exists a positive constant C depending only on $dist\{\mathcal{O}, \partial\Omega\}$ and $\|v\|_{C^{3,1}(\Omega)}$ such that*

$$|v_{ij\alpha}(x)| \le C\Big(\sqrt{v_{ii}(x)} + \sqrt{v_{jj}(x)}\Big) \tag{2.11}$$

for all $x \in \mathcal{O}$ and $1 \le i, j, \alpha \le n$.

Proof. It follows from convexity of v that for any direction $\eta \in \mathbb{R}^n$ with $|\eta| = 1$

$$v_{\eta\eta}(x) \ge 0$$

for all $x \in \Omega$. It's well known that for any nonnegative $C^{1,1}$ function h, $|\nabla h(x)| \le Ch^{\frac{1}{2}}(x)$ for all $x \in \mathcal{O}$, where C depends only on $\|h\|_{C^{1,1}(\Omega)}$ and $dist\{\mathcal{O}, \partial\Omega\}$ (e.g., see [29]). Hence

$$|v_{\eta\eta\alpha}(x)| \le C\sqrt{v_{\eta\eta}(x)}.$$

where C is a positive constant depending only on $dist\{\mathcal{O}, \partial\Omega\}$ and $\|v_{\eta\eta}\|_{C^{1,1}(\Omega)}$ (which can be controlled by $\|u\|_{C^{3,1}(\Omega)}$). Now set $\eta = i$ if $i = j$ and $\eta = \frac{1}{\sqrt{2}}(e_i + e_j)$ if $i \ne j$. The proof of Lemma 2.5 is complete. □

Proof of Proposition 2.1. Let us divide $\sum_{i,j,k,m} q^{ij,km} v_{ij\alpha} v_{km\beta}$ into three parts according to Lemma 2.3:

$$\sum_{i,j,k,m} q^{ij,km}(W(z)) v_{ij\alpha} v_{km\beta} = I_{\alpha\beta} + II_{\alpha\beta} + III_{\alpha\beta}, \tag{2.12}$$

where

$$I_{\alpha\beta} = \sum_{i \ne j} q^{ij,ji} v_{ij\alpha} v_{ji\beta},$$

$$II_{\alpha\beta} = \sum_{i=1}^{n} q^{ii,ii} v_{ii\alpha} v_{ii\beta}$$

and

$$III_{\alpha\beta} = \sum_{i\neq k} q^{ii,kk} v_{ii\alpha} v_{kk\beta}.$$

Lemma 2.4 yields(using Lemma 2.5 and $\lambda_i = O(\phi)$)

$$\begin{aligned} I_{\alpha\beta} &= (\sum_{i,j\in G, i\neq j} + \sum_{i\in B, j\in G} + \sum_{j\in B, i\in G} + \sum_{i,j\in B, i\neq j}) q^{ij,ji} v_{ij\alpha} v_{ji\beta} \\ &= O(\phi) + O(\sum_{i,j\in B} |\nabla v_{ij}|) - \frac{1}{\sigma_1(B)} \sum_{i,j\in B, i\neq j} v_{ij\alpha} v_{ji\beta} \\ &\quad -2 \sum_{i\in B, j\in G} \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B) v_{jj}} v_{ij\alpha} v_{ji\beta}. \end{aligned} \tag{2.13}$$

Again from Lemma 2.4

$$\begin{aligned} II_{\alpha\beta} &= (\sum_{i\in G} + \sum_{i\in B}) q^{ii,ii} v_{ii\alpha} v_{ii\beta} \\ &= O(\phi) + O(\sum_{i,j\in B} |\nabla v_{ij}|) - 2\sum_{i\in B} \frac{\sigma_1(B)\sigma_1(B|i) - \sigma_2(B)}{\sigma_1^3(B)} v_{ii\alpha} v_{ii\beta} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} III_{\alpha\beta} &= (\sum_{i,j\in G, i\neq j} + \sum_{i\in B, j\in G} + \sum_{j\in B, i\in G} + \sum_{i,j\in B, i\neq j}) q^{ii,jj} v_{ii\alpha} v_{jj\beta} \\ &= O(\phi) + O(\sum_{i,j\in B} |\nabla v_{ij}|) + \sum_{i\neq j, i,j\in B} \frac{2\sigma_2(B) - \sigma_1^2(B) + (v_{ii} + v_{jj})\sigma_1(B)}{\sigma_1^3(B)} v_{ii\alpha} v_{jj\beta}. \end{aligned} \tag{2.15}$$

The algebraic identity

$$\begin{aligned} &\sum_{i,j\in B, i\neq j} [2\sigma_2(B) - \sigma_1^2(B) + (v_{ii} + v_{jj})\sigma_1(B)] v_{ii\alpha} v_{jj\beta} \\ &\quad -2\sum_{i\in B} [\sigma_1(B)\sigma_1(B|i) - \sigma_2(B|i)] v_{ii\alpha} v_{ii\beta} \\ &= -\sum_{i\in B} (\sigma_1(B) v_{ii\alpha} - v_{ii} \sum_{j\in B} v_{jj\alpha})(\sigma_1(B) v_{ii\beta} - v_{ii} \sum_{j\in B} v_{jj\beta}). \end{aligned} \tag{2.16}$$

implies

$$II_{\alpha\beta} + III_{\alpha\beta} = O(\phi) + O(\sum_{i,j\in B} |\nabla v_{ij}|) - \frac{\sum_{i\in B} V_{i\alpha} V_{i\beta}}{\sigma_1^3(B)}, \tag{2.17}$$

where $V_{i\alpha}$ defined in (2.6). □

Proof of Corollary 2.2. We only need to consider a small neighborhood $\mathcal{O}$ of these points in Ω where that the minimal rank is attained. For such fixed point $z \in \mathcal{O}$, we may assume $W(z)$

is diagonal by a rotation. Thus, for any fixed α and β

$$\frac{\partial^2 q(z)}{\partial x_\alpha \partial x_\beta} = \sum_{i,j} q^{ij}(W(z))u_{ij\alpha\beta} + \sum_{i,j,k,m} q^{ij,km}(W(z))u_{ij\alpha}u_{km\beta} \tag{2.18}$$

Since $0 \le \frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)} \le 1$, by Lemma 2.4

$$|q^{ij}(W(z))| \le C$$

for some constant C under control. This yields the estimate for the first term in (2.18)

$$\|q^{ij}(W(z))u_{ij\alpha\beta}\| \le C\|u\|_{C^{3,1}(\Omega)} \le C$$

Now treat the second term in (2.18). By Lemma 2.5, for $i, j \in B$

$$|u_{ij\alpha}| \le C(\sqrt{u_{ii}(x)} + \sqrt{u_{jj}(x)}) \le C\sqrt{\sigma_1(B)}. \tag{2.19}$$

Noting that $u_{jj} \ge C > 0, j \in G$ and $0 \le \frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)} \le 1$. From Proposition 2.1 it now follows that,

$$|\frac{\partial^2 q(W(z))}{\partial x_\alpha \partial x_\beta}| \le C$$

for all $z \in \mathcal{O}$. □

3. A strong maximum principle

In this section, we prove a strong maximum principle for ϕ defined in (2.2) for equation (1.2). The same result for equation (1.6) could be proved making Theorem 1.1 a corollary of Theorem 1.2. However we prefer to work on elliptic case first. With some minor modifications, the parabolic version will be proved at the end of next section.

Denote by $\mathcal{S}^n$ the set of all real symmetric $n \times n$ matrices, and denote by $\mathcal{S}^n_+ \subset \mathcal{S}^n$ to be the set of all positive definite symmetric $n \times n$ matrices. Let $\mathbb{O}_n$ be the space consisting all $n \times n$ orthogonal matrices. Define

$$\mathcal{S}_{n-1} = \{Q \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} Q^T \quad | \quad \forall Q \in \mathbb{O}_n, \forall B \in \mathcal{S}^{n-1} \},$$

and for given $Q \in \mathbb{O}_n$,

$$\mathcal{S}_{n-1}(Q) = \{Q \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} Q^T \quad | \quad \forall B \in \mathcal{S}^{n-1} \}.$$

Therefore $\mathcal{S}_{n-1}, \mathcal{S}_{n-1}(Q) \subset \mathcal{S}^n$. For any function $F(r,p,u,x)$, we denote

$$F^{\alpha\beta} = \frac{\partial F}{\partial r_{\alpha\beta}}, \quad F^u = \frac{\partial F}{\partial u}, \quad F^{x_i} = \frac{\partial F}{\partial x_i}, \quad F^{\alpha\beta,\gamma\eta} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial r_{\gamma\eta}}, \quad F^{\alpha\beta,u} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial u},$$

$$F^{\alpha\beta,x_k} = \frac{\partial^2 F}{\partial r_{\alpha\beta}\partial x_k}, \quad F^{u,u} = \frac{\partial^2 F}{\partial^2 u}, \quad F^{u,x_i} = \frac{\partial^2 F}{\partial u \partial x_i}, \quad F^{x_i,x_j} = \frac{\partial^2 F}{\partial x_i \partial x_j}. \tag{3.1}$$

For any p fixed and $Q \in \mathbb{O}_n$, $(A, u, x) \in \mathcal{S}_{n-1}(Q) \times \mathbb{R} \times \mathbb{R}^n$, we set

$$X_F^* = ((F^{\alpha\beta}(A, p, u, x)), -F^u(A, p, u, x), -F^{x_1}(A, p, u, x), \cdots, -F^{x_n}(A, p, u, x))$$

as a vector in $\mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n$. Set

$$\Gamma^{\perp}_{X_F^*} = \{\tilde{X} \in \mathcal{S}_{n-1}(Q) \times \mathbb{R} \times \mathbb{R}^n \quad | \quad < \tilde{X}, X_F^* >= 0\}, \tag{3.2}$$

Let $B \in \mathcal{S}_+^{n-1}, A = B^{-1}$ and

$$\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

For any given $Q \in \mathbb{O}_n$ and $\tilde{X} = ((X_{ij}), Y, Z_1, \cdots, Z_n) \in \mathcal{S}_{n-1}(Q) \times \mathbb{R} \times \mathbb{R}^n$, we define a quadratic form

$$\begin{aligned} Q^*(\tilde{X}, \tilde{X}) \quad = \quad & \sum_{i,j,k,l=1}^{n} F^{ij,kl} X_{ij} X_{kl} + 2 \sum_{i,j,k,l=1}^{n} F^{ij} (Q\tilde{A}Q^T)_{kl} X_{ik} X_{jl} + \sum_{i,j=1}^{n} F^{x_i,x_j} Z_i Z_j \\ & -2 \sum_{i,j=1}^{n} F^{ij,u} X_{ij} Y - 2 \sum_{i,j,k=1}^{n} F^{ij,x_k} X_{ij} Z_k + 2 \sum_{i=1}^{n} F^{u,x_i} Y Z_i + F^{u,u} Y^2, \end{aligned} \tag{3.3}$$

where functions $F^{ij,kl}, F^{ij}, F^{u,u}, F^{ij,u}, F^{ij,x_k}, F^{u,x_i}, F^{x_i,x_j}$ are evaluated at $(Q\tilde{B}Q^T, p, u, x)$.

We first state a lemma to be proven in next section (after Corollary 4.2).

Lemma 3.1. *If F satisfies condition (1.4), then for each $p \in \mathbb{R}^n$,*

$$Q^*(\tilde{X}, \tilde{X}) \geq 0, \forall \tilde{X} \in \Gamma^{\perp}_{X_F^*}. \tag{3.4}$$

Roughly speaking, the condition $Q^*(\tilde{X}, \tilde{X}) \geq 0, \forall \tilde{X} \in \Gamma^{\perp}_{X_F^*}$ is equivalent to the convexity of level set $\{(A, u, x) | \quad F(A^{-1}, p, u, x) = 0\}$ for each p fixed (implied in the proof of Lemma 4.1 in the next section). By restricting $A \in \mathcal{S}_{n-1}(Q)$, we reduce dimension requirement for A. This is useful in some applications, in particular when $n = 2$. We refer the next section for further discussions.

The following theorem is the core result of this paper. Theorem 1.1 is a direct consequence of Theorem 3.2 and Lemma 3.1.

Theorem 3.2. *Suppose that the function F satisfies conditions (1.3) and (3.4) and let $u \in C^{3,1}(\Omega)$ is a convex solution of (1.2). If $\nabla^2 u$ attains its minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of ϕ (defined in (2.2)), such that*

$$\sum_{\alpha,\beta} F^{\alpha\beta} \phi_{\alpha\beta}(x) \leq C(\phi(x) + |\nabla \phi(x)|), \quad \forall x \in \mathcal{O}. \tag{3.5}$$

In turn, $\nabla^2 u$ is of constant rank in $\mathcal{O}$. Moreover, for each $x_0 \in \Omega$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n-l)$ fixed directions $V_1, \cdots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \le j \le n-l$ and $x \in \mathcal{U}$.

Proof of Theorem 3.2. Let $u \in C^{3,1}(\Omega)$ be a convex solution of equation (1.2) and $W(x) = (u_{ij}(x))$. Let $z_0 \in \Omega$ be a point where $W = (\nabla^2 u)$ attains minimal rank l. We may assume $l \le n-1$, otherwise there is nothing to prove. As in the previous section, pick an open neighborhood $\mathcal{O}$ of z_0, for any $x \in \mathcal{O}$, let $G = \{n-l+1, n-l+2, ..., n\}$ and $B = \{1, ..., n-l\}$ be the "good" and "bad" sets of indices for eigenvalues of $\nabla^2 u(x)$ respectively.

Setting ϕ as (2.2), then we see from Corollary 2.2 that $\phi \in C^{1,1}(\mathcal{O})$,

$$\phi(x) \ge 0, \ \ \phi(z_0) = 0$$

and there is a constant $C > 0$ such that for all $x \in \mathcal{O}$,

$$\frac{1}{C}\sigma_1(B)(x) \le \phi(x) \le C\sigma_1(B)(x), \ \ \frac{1}{C}\sigma_1(B)(x) \le \sigma_{l+1}(W(x)) \le C\sigma_1(B)(x).$$

Fix a point $z \in \mathcal{O}$ and prove (3.5) at z. For each $z \in \mathcal{O}$ fixed, letting $\lambda_1 \le \lambda_2 ... \le \lambda_n$ be the eigenvalues of $W(z) = (u_{ij}(z))$ at z, one may assume $W(z) = (u_{ij}(z))$ is diagonal with proper choice of orthonormal coordinates, and $u_{ii}(z) = \lambda_i, i = 1, \cdots, n$.

Again, as in the previous section, we will avoid $\sigma_{l+1}(W) = 0$ by considering W_ϵ (defined in (2.3)) for $\epsilon > 0$ sufficient small, with $W_\epsilon = W + \epsilon I$, $G_\epsilon = (\lambda_{n-l+1} + \epsilon, ..., \lambda_n + \epsilon)$, $B_\epsilon = (\lambda_1 + \epsilon, ..., \lambda_{n-l} + \epsilon)$. Note that W_ϵ is the Hessian of function $u_\epsilon(x) = u(x) + \frac{\epsilon}{2}|x|^2$. This function $u_\epsilon(x)$ satisfies equation

$$F(\nabla^2 u_\epsilon, \nabla u_\epsilon, u_\epsilon, x) = R_\epsilon, \tag{3.6}$$

where $R_\epsilon(x) = F(\nabla^2 u_\epsilon, \nabla u_\epsilon, u_\epsilon, x) - F(\nabla^2 u, \nabla u, u, x)$. Since $u \in C^{3,1}$, we have

$$|R_\epsilon(x)| \le C\epsilon, \quad |\nabla R_\epsilon(x)| \le C\epsilon, \quad |\nabla^2 R_\epsilon(x)| \le C\epsilon, \quad \forall x \in \mathcal{O}. \tag{3.7}$$

We will work on equation (3.6) to obtain the differential inequality (3.5) for ϕ_ϵ defined in (2.3) with constant C_1, C_2 independent of ϵ. Theorem 3.2 would follow by letting $\epsilon \to 0$.

Set $v = u_\epsilon$, in the rest of this section. Write W for W_ϵ, G for G_ϵ, B for B_ϵ, q for q_ϵ and ϕ for ϕ_ϵ, with the understanding that all the estimates will be independent of ϵ. Note that (2.5) implies

$$\epsilon \le C\phi(x), \quad \text{for all } x \in \mathcal{O}, \tag{3.8}$$

and v satisfies the equation

$$F(\nabla^2 v, \nabla v, v, x) = R(x), \tag{3.9}$$

with $R(x)$ under control as follows:

$$|\nabla^j R(x)| \le C\phi(x), \quad \text{for all } j = 0, 1, 2, \quad \text{and for all } x \in \mathcal{O}. \tag{3.10}$$

Then

$$\phi_\alpha = \frac{\partial \phi}{\partial x_\alpha} = \phi^{ij} v_{ij\alpha}, \quad \phi_{\alpha\beta} = \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} = \phi^{ij} v_{ij\alpha\beta} + \phi^{ij,km} v_{ij\alpha} v_{km\beta}.$$

Differentiate equation (3.9) in x_i and then x_j and use (3.10) to obtain

$$\sum_{\alpha\beta} F^{\alpha\beta} v_{\alpha\beta i} + \sum_k F^{q_k} v_{ki} + F^v v_i + F^{x_i} = O(\phi), \tag{3.11}$$

$$\begin{aligned}
&\sum_{\alpha\beta} F^{\alpha\beta} v_{\alpha\beta ij} + \sum_{\alpha\beta} v_{\alpha\beta i}\Big(\sum_{\gamma\eta} F^{\alpha\beta,\gamma\eta} v_{\gamma\eta j} + \sum_k F^{\alpha\beta,q_k} v_{kj} + F^{\alpha\beta,v} v_j + F^{\alpha\beta,x_j}\Big) \\
&+ \sum_k F^{q_k} v_{kij} + \sum_k v_{ki}\Big(\sum_{\alpha\beta} F^{q_k,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{q_k,q_l} v_{lj} + F^{q_k,v} v_j + F^{q_k,x_j}\Big) \\
&+ F^v v_{ij} + v_i\Big(\sum_{\alpha\beta} F^{v,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{v,q_l} v_{lj} + F^{v,v} v_j + F^{v,x_j}\Big) \\
&+ \sum_{\alpha\beta} F^{x_i,\alpha\beta} v_{\alpha\beta j} + \sum_k F^{x_i,q_k} v_{kj} + F^{x_i,v} v_j + F^{x_i,x_j} = O(\phi).
\end{aligned} \tag{3.12}$$

As $v_{\alpha\beta ij} = v_{ij\alpha\beta}$ (this will have to be modified later by a commutator formula when we deal with symmetric curvature tensors on general manifolds), we get

$$\begin{aligned}
\sum F^{\alpha\beta} \phi_{\alpha\beta} &= \sum F^{\alpha\beta} \phi^{ij} v_{ij\alpha\beta} + \sum F^{\alpha\beta} \phi^{ij,km} v_{ij\alpha} v_{km\beta} \\
&= \sum F^{\alpha\beta} \phi^{ij,km} v_{ij\alpha} v_{km\beta} - \sum \phi^{ij} F^{q_k} v_{kij} - \sum \phi^{ij} [2 \sum F^{\alpha\beta,q_k} v_{\alpha\beta i} v_{kj} \\
&\quad + F^v v_{ij} + \sum F^{q_k,q_l} v_{ki} v_{lj} + 2 \sum F^{q_k,v} v_{ki} v_j + 2 \sum F^{q_k,x_j} v_{ki}] \\
&\quad - \sum \phi^{ij} [F^{\alpha\beta,\gamma\eta} v_{\alpha\beta i} v_{\gamma\eta j} + 2 \sum F^{\alpha\beta,v} v_{\alpha\beta i} v_j + 2 \sum F^{\alpha\beta,x_j} v_{\alpha\beta i} \\
&\quad + \sum F^{v,v} v_i v_j + \sum F^{v,x_i} v_j + \sum F^{x_i x_j}] + O(\phi)
\end{aligned} \tag{3.13}$$

We will estimate the terms in the right hand side of (3.13). The analysis will be devoted to those third order derivatives terms which have with at least two indices in B. Some of these are linear. Controlling these linear term is the main challenge. This is the place where the function q in (2.1) plays key role. The concavity results of q in last section will be used in crucial way. As for the remaining terms in (3.13), we will sort them out in a way such that condition (4.3) can be used to obtain appropriate control.

Note that since $W = (v_{ij})$ is diagonal at z, Lemma 2.3 and Lemma 2.4 imply,

$$\phi^{ij}(z) = \begin{cases} \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)} + O(\phi), & \text{if } i = j \in B \\ O(\phi), & \text{otherwise} \end{cases} \tag{3.14}$$

Hence at z

$$\sum_{i,j}\phi^{ij}[F^v v_{ij}+2\sum F^{\alpha\beta,q_k}v_{\alpha\beta i}v_{kj}+\sum F^{q_k,q_l}v_{ki}v_{lj}+2\sum(F^{q_k,v}v_{ki}v_j+F^{q_k,x_j}v_{ki})]$$

$$=\sum_{i=1}^n\phi^{ii}[F^v v_{ii}+2\sum F^{\alpha\beta,q_i}v_{\alpha\beta i}v_{ii}+F^{q_i,q_i}v_{ii}v_{ii}+2F^{q_i,v}v_{ii}v_i+2F^{q_i,x_i}v_{ii}]$$

$$=O(\phi)+\sum_{i\in B}\phi^{ii}[F^v+2\sum F^{\alpha\beta,q_i}v_{\alpha\beta i}+F^{q_i,q_i}v_{ii}+2F^{q_i,v}v_i+2F^{q_i,x_i}]v_{ii}$$

$$\leq O(\phi)+C\sum_{i\in B}(\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)})v_{ii}=O(\phi), \tag{3.15}$$

since $\lambda_i=O(\phi), i\in B$ and $\sigma_{l+1}(W)\geq\sigma_l(G)\sigma_1(B)$. This takes care of the third term on the right hand side of (3.13). For the second term we have

$$\sum\phi^{ij}F^{q_k}v_{kij}=O(\phi)+\sum_{i\in B}\phi^{ii}F^{q_k}v_{kii}=O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|) \tag{3.16}$$

For the third term in (3.13), by (3.14) we have,

$$\phi^{ij}[F^{\alpha\beta,\gamma\eta}v_{\alpha\beta i}v_{\gamma\eta j}+2F^{\alpha\beta,v}v_{\alpha\beta i}v_j+2F^{\alpha\beta,x_j}v_{\alpha\beta i}+F^{v,v}v_iv_j+2F^{v,x_i}v_j+F^{x_ix_j}]$$

$$=O(\phi)+\sum_{i\in B}\phi^{ii}[\sum F^{\alpha\beta,\gamma\eta}v_{\alpha\beta i}v_{\gamma\eta i}+2\sum F^{\alpha\beta,v}v_{\alpha\beta i}v_i$$

$$+2\sum F^{\alpha\beta,x_i}v_{\alpha\beta i}+F^{v,v}v_i^2+2F^{v,x_i}v_i+F^{x_ix_i}]$$

$$=O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)+\sum_{i\in B}(\sigma_l(G)+\frac{\sigma_1^2(B|i)-\sigma_2(B|i)}{\sigma_1^2(B)})$$

$$[\sum_{\alpha,\beta,\gamma,\eta\in G}F^{\alpha\beta,\gamma\eta}v_{i\alpha\beta}v_{i\gamma\eta}+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,v}v_{i\alpha\beta}v_i+2\sum_{\alpha,\beta\in G}F^{\alpha\beta,x_i}v_{i\alpha\beta}$$

$$+F^{v,v}v_i^2+2F^{v,x_i}v_i+F^{x_ix_i}]. \tag{3.17}$$

Now deal with the term $\sum F^{\alpha\beta}\phi^{ij,km}v_{ij\alpha}v_{km\beta}$ in (3.13). Note that

$$\phi^{ij,km}=\sigma_{l+1}^{ij,km}+q^{ij,km}.$$

Since $\sigma_{l-1}(W|ij)=O(\phi)$ for $i,j\in G, i\neq j$, for α,β fixed, by (2.8),

$$\begin{aligned}\sum\sigma_{l+1}^{ij,km}v_{ij\alpha}v_{km\beta} &= \sum_{i\neq k}\sigma_{l+1}^{ii,kk}v_{ii\alpha}v_{kk\beta}+\sum_{i\neq j}\sigma_{l+1}^{ij,ji}v_{ij\alpha}v_{ji\beta}\\ &= \sum_{i\neq k}\sigma_{l-1}(W|ik)v_{ii\alpha}v_{kk\beta}-\sum_{i\neq j}\sigma_{l-1}(W|ij)v_{ij\alpha}v_{ji\beta}\\ &= O(\phi+\sum_{i,j\in B}|\nabla v_{ij}|)-2\sum_{i\in B,j\in G}\sigma_{l-1}(G|j)v_{ij\alpha}v_{ij\beta}.\end{aligned}$$

As $\sigma_{l-1}(G|j) = \frac{\sigma_l(G)}{\lambda_j}, j \in G$, we have

$$\sigma_{l+1}^{ij,km} v_{ij\alpha} v_{km\beta} = O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - 2\sigma_l(G) \sum_{i\in B, j\in G} \frac{1}{\lambda_j} v_{ij\alpha} v_{ij\beta}.$$

By Proposition 2.1,

$$\begin{aligned}\sum_{i,j,k,m} q^{ij,km} v_{ij\alpha} v_{km\beta} &= O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - 2 \sum_{i\in B, j\in G} \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)\lambda_j} v_{ij\alpha} v_{ji\beta} \\ &\quad - \frac{\sum_{i\in B} V_{i\alpha} V_{i\beta}}{\sigma_1^3(B)} - \frac{1}{\sigma_1(B)} \sum_{i,j\in B, i\neq j} v_{ij\alpha} v_{ji\beta},\end{aligned}$$

where $V_{i\alpha}$ is defined in (2.6). We conclude that

$$\begin{aligned}\sum F^{\alpha\beta} \phi^{ij,km} v_{ij\alpha} v_{km\beta} &= O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - \sum_{\alpha,\beta} F^{\alpha\beta} [\frac{\sum_{i\in B} V_{i\alpha} V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} v_{ij\alpha} v_{ji\beta}}{\sigma_1(B)} \\ &\quad + 2\sum_{i\in B} (\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}) \frac{1}{\lambda_j} v_{ij\alpha} v_{ji\beta}]. \qquad (3.18)\end{aligned}$$

Combining (3.15)-(3.18), one reduces (3.13) to

$$\begin{aligned}\sum F^{\alpha\beta} \phi_{\alpha\beta} &= O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - \sum_{\alpha,\beta} F^{\alpha\beta} [\frac{\sum_{i\in B} V_{i\alpha} V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} v_{ij\alpha} v_{ji\beta}}{\sigma_1(B)}] \\ &\quad - \sum_{i\in B} [\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G} F^{\alpha\beta,\gamma\eta}(\Lambda) v_{i\alpha\beta} v_{i\gamma\eta} \\ &\quad + 2\sum_{\alpha\beta\in G} F^{\alpha\beta} \sum_{j\in G} \frac{1}{\lambda_j} v_{ij\alpha} v_{ij\beta} + 2\sum_{\alpha,\beta\in G} F^{\alpha\beta,v} v_{i\alpha\beta} v_i \\ &\quad + 2\sum_{\alpha,\beta\in G} F^{\alpha\beta,x_i} v_{i\alpha\beta} + F^{v,v} v_i^2 + 2F^{v,x_i} v_i + F^{x_i,x_i}]. \qquad (3.19)\end{aligned}$$

At this point, we have succeeded in regrouping the terms involving third order derivatives in terms of "B" and "G". First consider the last term on the right hand side of (3.19). For each $i \in B$, let

$$\begin{aligned}J_i &= [\sum_{\alpha,\beta,\gamma,\eta\in G} F^{\alpha\beta,\gamma\eta} v_{i\alpha\beta} v_{i\gamma\eta} + 2\sum_{\alpha,\beta\in G} F^{\alpha\beta} \sum_{j\in G} \frac{1}{\lambda_j} v_{ij\alpha} v_{ij\beta} \\ &\quad + 2\sum_{\alpha,\beta\in G} F^{\alpha\beta,v} v_{i\alpha\beta} v_i + 2\sum_{\alpha,\beta\in G} F^{\alpha\beta,x_i} v_{i\alpha\beta} + F^{v,v} v_i^2 + 2F^{v,x_i} v_i + F^{x_i,x_i}]. \qquad (3.20)\end{aligned}$$

By Condition (1.3), since $v \in C^{3,1}$(so $F^{\alpha\beta} \in C^{0,1}$) and $\bar{\mathcal{O}} \subset \Omega$, there exists a constant $\delta_0 > 0$, such that

$$(F^{\alpha\beta}) \geq \delta_0 I, \ \forall y \in \mathcal{O}. \qquad (3.21)$$

In particular $F^{nn} \geq \delta_0$. If $G \neq \emptyset$, so $n \in G$. Since $v_{ik} = \delta_{ik}\lambda_i$ at z, (3.11) implies, for $i \in B$

$$\sum_{\alpha,\beta\in G} F^{\alpha\beta} v_{\alpha\beta i} + F^v v_i + F^{x_i} = O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|).$$

If $G = \emptyset$, (3.11) also yields

$$F^{nn} v_{nni} + F^v v_i + F^{x_i} = O(\phi + \sum_{i,j\in B} |\nabla v_{ij}|),$$

In any case, set $X_{\alpha\beta} = 0$ if either $n-1 \geq \alpha \in B$ or $n-1 \geq \beta \in B$,

$$X_{nn} = v_{inn} - \frac{1}{F^{nn}}[\sum_{\alpha,\beta\in G} F^{\alpha\beta} v_{\alpha\beta i} + F^v v_i + F^{x_i}], \quad \text{if } G \neq \emptyset$$

$$X_{nn} = v_{inn} - \frac{1}{F^{nn}}[F^v v_i + F^{x_i}], \quad \text{if } G = \emptyset,$$

$X_{\alpha\beta} = v_{i\alpha\beta}$ otherwise, $Y = -v_i$ and $Z_k = -\delta_{ki}$. Thus $(X_{\alpha\beta}) \in \mathcal{S}_{n-1}$(identity matrix) and $\tilde{X} = ((X_{\alpha\beta}), Y, Z_1, \cdots, Z_n) \in \Gamma^{\perp}_{X_F^*}$. Condition (3.4) implies

$$J_i \geq -C(\phi + \sum_{i,j\in B} |\nabla v_{ij}|).$$

Since $C \geq \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)} \geq 0$, thus we obtain

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq C(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - \sum_{\alpha,\beta} F^{\alpha\beta}\left(\frac{\sum_{i\in B} V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} v_{ij\alpha}v_{ji\beta}}{\sigma_1(B)}\right). \tag{3.22}$$

The object of the final stage of the proof is to control the term $\sum_{i,j\in B} |\nabla v_{ij}|$ in (3.22) using the remaind terms on the right hand side.

By (3.21),

$$\sum_{\alpha,\beta} F^{\alpha\beta} V_{i\alpha}V_{i\beta} \geq \delta_0 \sum_{\alpha=1}^n V_{i\alpha}^2, \quad \sum_{\alpha,\beta} F^{\alpha\beta} v_{ij\alpha}v_{ij\beta} \geq \delta_0 \sum_{\alpha=1}^n v_{ij\alpha}^2.$$

Inserting above inequalities into (3.22), we then obtain

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq C(\phi + \sum_{i,j\in B} |\nabla v_{ij}|) - \delta_0 \sum_{\alpha=1}^n \left[\frac{\sum_{i\in B} V_{i\alpha}^2}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B\ i\neq j} |v_{ij\alpha}|^2}{\sigma_1(B)}\right]. \tag{3.23}$$

From Lemma 2.4, it follows that

$$\phi_\alpha = O(\phi) + \sum_{i\in B}\left(\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}\right) v_{ii\alpha}. \tag{3.24}$$

The key differential inequality (3.5) is the consequence of (3.23) and the following lemma.

Lemma 3.3. *Suppose $M \geq \lambda_i > 0, M \geq \gamma_i \geq \frac{1}{M}, \forall i = 1, \cdots, m$ for some $M > 0$, and suppose that $v_{ij\alpha} = v_{ji\alpha}, \forall i, j = 1, \cdots, m, \alpha = 1, \cdots, n$. Then there is a constant C depending only on n and M, such that for each α, for any $D > 0$, $\delta > 0$*

$$\sum_{i,j=1}^{m} |v_{ij\alpha}| \leq C(1 + \frac{2D}{\delta} + D)(\sigma_1(\lambda) + |\sum_{i=1}^{m} \gamma_i v_{ii\alpha}|) + \frac{\delta}{2D} \frac{\sum_{i\neq j}^{m} |v_{ij\alpha}|^2}{\sigma_1(\lambda)} + \frac{C}{D} \frac{\sum_{i=1}^{m} V_{i\alpha}^2}{\sigma_1^3(\lambda)}, \tag{3.25}$$

where $V_{i\alpha} = v_{ii\alpha}\sigma_1(\lambda) - \lambda_i \Big(\sum_{j=1}^{m} v_{jj\alpha} \Big)$.

Proof of Lemma 3.3. Use a trick devised in [14]. For each $\alpha = 1, \cdots, n$ fixed,

$$\sum_{i,j=1}^{m} |v_{ij\alpha}| = \sum_{i\neq j} |v_{ij\alpha}| + \sum_{i} |v_{ii\alpha}|$$

If $i \neq j$, for any $D > 0$, the Cauchy-Schwarz inequality yields

$$|v_{ij\alpha}| \leq \frac{D}{2}\delta^{-1}\sigma_1(\lambda) + \frac{\delta}{2D} \frac{|v_{ij\alpha}|^2}{\sigma_1(\lambda)} \tag{3.26}$$

The linear terms involving $v_{ii\alpha}$, $i = 1, \cdots, m$ still need to be controlled. Set

$$P = \{i| \ v_{ii\alpha} > 0\}, \ N = \{i| \ v_{ii\alpha} < 0\}, \ R = \{i| \ v_{ii\alpha} = 0\},$$

and consider two separate cases.

Case 1. Either $P = \emptyset$ or $N = \emptyset$. In this case, $v_{ii\alpha}$ has the same sign for all $i = 1, \cdots, m$. We derive easily

$$|v_{ii\alpha}| \leq C_1 |\sum_{i=1}^{m} \gamma_i v_{ii\alpha}|, \tag{3.27}$$

with C_1 under control.

Case 2. $P \neq \emptyset, \ N \neq \emptyset$. We may assume $\sum_{i\in P} v_{ii} \geq \sum_{j\in N} v_{jj}$ (changing $v_{ij\alpha}$ to $-v_{ij\alpha}$ if necessary). For $i \in P$,

$$v_{ii\alpha} \leq \sum_{k\in P} v_{kk\alpha} \leq C_2(|\sum_{i=1}^{m} \gamma_i v_{ii\alpha}| - \sum_{j\in N} v_{jj\alpha}), \tag{3.28}$$

for some positive constant C_2 under control. At this point, we have reduced the estimation of $v_{ii\alpha}$, $i \in P$ to the estimation of $-v_{jj\alpha}$, $j \in N$.

Claim: If $P \neq \emptyset, \ N \neq \emptyset, \ \sum_{i\in P} v_{ii} \geq \sum_{j\in N} v_{jj}$, then

$$\Big(\sum_{j\in N} v_{jj\alpha} \Big)^2 \leq \frac{4n^2}{\sigma_1^2(\lambda)} \sum_{i\in B} V_{i\alpha}^2.$$

Assuming the **Claim** is true, we get for all $k \in N$,

$$-v_{kk\alpha} \le -\sum_{j\in N} v_{jj\alpha} \le D\sigma_1(\lambda) + \frac{\left(\sum_{j\in N} v_{jj\alpha}\right)^2}{D\sigma_1(\lambda)} \le D\sigma_1(\lambda) + \frac{4n^2 \sum_{i\in B} V_{i\alpha}^2}{D\sigma_1^3(\lambda)}. \tag{3.29}$$

Consequently we also control terms involving $v_{ii\alpha},\ i \in P$ by (3.28).

We now validate the **Claim**.

Proof of **Claim**. First, by the Cauchy-Schwarz inequality

$$\Big(\sum_{i\in N} V_{i\alpha}\Big)^2 \le n^2 \sum_{i\in N} V_{i\alpha}^2 \le n^2 \sum_{i=1}^{m} V_{i\alpha}^2.$$

It follows from the definitions of the sets P, N, R and $V_{i\alpha}$ that

$$\begin{aligned} -\sum_{i\in N} V_{i\alpha} &= \sum_{i\in N}\Big(\lambda_i(\sum_{j\in N} v_{jj\alpha} + \sum_{k\in P} v_{kk\alpha}) - v_{ii\alpha}(\sum_{j\in N}\lambda_j + \sum_{j\in R}\lambda_j + \sum_{k\in P}\lambda_k)\Big) \\ &= \Big(\sum_{i\in N}\lambda_i\Big)\Big(\sum_{k\in P} v_{kk\alpha}\Big) - \Big(\sum_{k\in P\cup R}\lambda_k\Big)\Big(\sum_{i\in N} v_{ii\alpha}\Big) \end{aligned} \tag{3.30}$$

Since in this case

$$\sum_{i\in N}\lambda_i > 0, \sum_{k\in P} v_{kk\alpha} > 0, \sum_{j\in N} v_{jj\alpha} \le 0,$$

all the terms on the right hand side of (3.30) are nonnegative, hence

$$\Big(\sum_{i\in N} V_{i\alpha}\Big)^2 \ge \Big(\sum_{k\in P\cup R}\lambda_k\Big)^2\Big(\sum_{i\in N} v_{ii\alpha}\Big)^2 \ge \Big(\frac{1}{2}\sigma_1(\lambda)\Big)^2\Big(\sum_{i\in N} v_{ii\alpha}\Big)^2 = \frac{\sigma_1^2(\lambda)}{4}\Big(\sum_{i\in N} v_{ii\alpha}\Big)^2.$$

The lemma is proved. □

By Lemma 3.3 and (3.23), there exist positive constants C_1, C_2 independent of ϵ, such that

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \le C_1(\phi + |\nabla\phi|) - C_2 \sum_{i,j\in B} |\nabla v_{ij}|. \tag{3.31}$$

Taking $\epsilon \to 0$, (3.31) is proven with v replaced by u. By the Strong Maximum Principle, $\phi \equiv 0$ in $\mathcal{O}$. Since Ω is flat, following the arguments in [7, 27], for any $x_0 \in \Omega$, there is a neighborhood $\mathcal{U}$ and $(n-l)$ fixed directions $V_1, \cdots, V_{n-l}$ such that $\nabla^2 u(x) V_j = 0$ for all $1 \le j \le n-l$ and $x \in \mathcal{U}$. The proof of Theorem 3.2 is complete. □

Remark 3.4. The main step in the above proof is to control linear terms of $v_{ij\alpha}, i, j \in B$. If F is symmetric in (1.1), all terms involving $v_{ij\alpha}$ $(i, j \in B)$ are quadratic. In [8], a test function $\phi(x) = \sigma_{l+1}(\nabla^2 u(x)) + A\sigma_{l+2}(\nabla^2 u(x))$ was introduced. For $\tilde{q} = A\sigma_{l+2}(\nabla^2 u(x))$, it was proved in [8] that

$$\sum_{i,j,k,m} \tilde{q}^{ij,km} v_{ij\alpha} v_{km\beta} = O(\phi) - A \sum_{ij\in B} v_{ij\alpha} v_{ij\beta}. \tag{3.32}$$

The terms on the right hand side of (3.32) was used there to overcome quadratic terms of $v_{ij\alpha}$ $(i, j \in B)$. For general F in (1.2), we encounter linear terms of $v_{ij\alpha}, i, j \in B$. (3.32) is not good enough. The function q introduced in (2.1) produces (2.7) in Proposition 2.1 which was used in a crucial way in the proof here. It should also be noted that, with Lemma 2.5, the quadratic terms of $v_{ij\alpha}, i, j \in B$ can in fact be controlled by $\sigma_{l+1}(\nabla^2 u(x))$. Therefore, all the arguments in [8] can carry through for simpler test function $\phi(x) = \sigma_{l+1}(\nabla^2 u(x))$.

4. Condition (1.4) and discussions

We discuss the convexity condition (1.4) in this section. Write $A^{-1} = (A^{ij})$ for the inverse matrix A^{-1} of positive definite matrix A.

Lemma 4.1. *F satisfies Condition (1.4) if and only if*

$$\begin{aligned}&\sum_{i,j,k,l=1}^{n} F^{ij,kl}(A,p,u,x)X_{ij}X_{kl} + 2\sum_{i,j,k,l=1}^{n} F^{ij}(A,p,u,x)A^{kl}X_{ik}X_{jl} + F^{u,u}Y^2 \\ &-2\sum_{i,j=1}^{n} F^{ij,u}X_{ij}Y - 2\sum_{i,j,k=1}^{n} F^{ij,x_k}X_{ij}Z_k + 2\sum_{i=1}^{n} F^{u,x_i}YZ_i + \sum_{i,j=1}^{n} F^{x_i,x_j}Z_iZ_j \geq 0\end{aligned} \tag{4.1}$$

for every $X = (X_{ij}) \in \mathcal{S}^n$, $Y \in \mathbb{R}$ and $Z = (Z_i) \in \mathbb{R}^n$.

Proof. From the convexity of $\tilde{F}(B, u, x) = F(B^{-1}, p, u, x)$ (for each p fixed),

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma,\eta=1}^{n} \tilde{F}^{\alpha\beta,\gamma\eta}(B,u,x)\tilde{X}_{\alpha\beta}\tilde{X}_{\gamma\eta} + 2\sum_{\alpha,\beta=1}^{n} \tilde{F}^{\alpha\beta,u}\tilde{X}_{\alpha\beta}Y + \tilde{F}^{u,u}Y^2 \\ &+2\sum_{\alpha,\beta,k=1}^{n} \tilde{F}^{\alpha\beta,x_k}\tilde{X}_{\alpha\beta}Z_k + 2\sum_{k=1}^{n} \tilde{F}^{u,x_k}YZ_k + \sum_{i,j=1}^{n} \tilde{F}^{x_i,x_j}Z_iZ_j \geq 0\end{aligned} \tag{4.2}$$

for every $\tilde{X} \in \mathcal{S}^n$, $Y \in \mathbb{R}$, $Z = (Z_i) \in \mathbb{R}^n$ and $B \in \mathcal{S}^n_+$. A direct computation yields

$$\begin{aligned}\tilde{F}^{\alpha\beta}(B,u,x) &= -F^{ij}(B^{-1},p,u,x)B^{i\alpha}B^{j\beta}, \\ \tilde{F}^{\alpha\beta,u}(B,u,x) &= -F^{ij,u}(B^{-1},p,u,x)B^{i\alpha}B^{j\beta}, \\ \tilde{F}^{\alpha\beta,\gamma\eta}(B,u,x) &= F^{ij,kl}(B^{-1},p,u,x)B^{i\alpha}B^{j\beta}B^{k\gamma}B^{l\eta} \\ &+F^{ij}(B^{-1},p,u,x)(B^{i\gamma}B^{j\beta}B^{\eta\alpha} + B^{i\alpha}B^{j\eta}B^{\beta\gamma}).\end{aligned}$$

Other derivatives can be calculated in a similar way. Substituting these into (4.2), equation (4.1) follows directly. □

Let $Q \in \mathbb{O}_n$, define

$$\tilde{F}_Q(A,u,x) = F(Q\begin{pmatrix} 0 & 0 \\ 0 & A^{-1}\end{pmatrix}Q^T, p, u, x)$$

for $(A,u,x) \in \mathcal{S}_+^{n-1} \times \mathbb{R} \times \Omega$ and fixed p. Condition (1.4) implies the following condition

$$\tilde{F}_Q(A,u,x) \quad \text{is locally convex} \tag{4.3}$$

in $\mathcal{S}_+^{n-1} \times \mathbb{R} \times \Omega$ for any fixed $n \times n$ orthogonal matrix Q.

The approximation Lemma 4.1 yields

Corollary 4.2. *Let $Q \in \mathbb{O}_n$. Assume F satisfies condition (4.3), then*

$$Q^*(\tilde{X}, \tilde{X}) \geq 0, \tag{4.4}$$

for every $\tilde{X} = ((X_{ij}), Y, Z_1, \cdots, Z_n) \in \mathcal{S}_{n-1}(Q) \times \mathbb{R} \times \mathbb{R}^n$, where Q^* is defined in (3.3).

In particular, by Corollary 4.2, condition (4.3) implies (3.4). Since condition (1.4) implies (4.3), Lemma 3.1 is a consequence of Corollary 4.2.

Condition (4.3) is weaker than condition (1.4). In particular condition (4.3) is empty when $n = 1$. There is a wide class of functions which satisfy (4.4). The most important examples are σ_k and $\frac{\sigma_l}{\sigma_k}$ $(l > k)$. If g is non-decreasing and convex, $F_1, \cdots, F_m$ are in this class, then $F = g(F_1, \cdots, F_m)$ is also in this class. In particular, if $F_1 > 0$ and $F_2 > 0$ are in the class, so is $F = F_1^\alpha + F_2^\beta$ for any $\alpha \geq 1$, $\beta \geq 1$. Another property of condition (4.3) is the following

Corollary 4.3. *If F satisfies (4.4), then so does the function $G(A) = F(A + E)$ for any nonnegative definite matrix E.*

We also have the following lemma.

Lemma 4.4. *Suppose $n = 2$ and $F(A) \geq 0$ is symmetric and homogeneous of degree k. If either $k \leq 0$ or $k \geq 1$, then F satisfies (4.4).*

Proof. Since $n = 2$, condition (4.4) is equivalent to $F^{\lambda_2,\lambda_2} \geq 0$. By homogeneity, we have

$$\sum_{i,j=1}^{n} F^{\lambda_i,\lambda_j} \lambda_i \lambda_j = k(k-1)F.$$

$n = 2$ and $\lambda_1 = 0$ yields $F^{\lambda_2,\lambda_2}\lambda_2^2 = k(k-1)F(0,\lambda_2) \geq 0$. □

The simple example $u = \sum_{i=1}^n x_i^4$, $F(A) = \sigma_1(A)$ indicates that some condition is needed in Theorem 1.1. If F is independent of x, u, one may ask if the convexity assumption of $F(A^{-1}, p)$ for A in condition (1.4) (or condition (3.4)) is necessary for Theorem 1.1. As remarked earlier, when $n = 1$, this assumption is not necessary. For general $n \geq 2$, there is the following theorem.

Theorem 4.5. *Suppose $F(A,p)$ is elliptic and u is a convex solution of*

$$F(\nabla^2 u, \nabla u) = 0, \tag{4.5}$$

then $W = (\nabla^2 u)$ is either of constant rank, or its minimal rank is at least 2. In particular, if $n = 2$, then W is of constant rank.

Proof. The proof follows the same lines of proof as Theorem 3.2 with the following observations: condition (4.3) was only used to control J_i as defined in (3.20). Let l be the minimum rank of W. If $l = 0$, that is $G = \emptyset$, the proof of Theorem 3.2 works without any change since F is independent of (u, x) in our case. This leaves the case $l = 1$ i.e. $|G| = 1$ and we may assume $\alpha = n \in G$. Note that (3.19) still holds. Since $F(\nabla^2 u, \nabla u) = 0$, and

$$0 = \nabla_i F(\nabla^2 u, \nabla u) = F^{nn} u_{nni} + O(\phi + \sum_{i,j \in B} |\nabla u_{ij}|).$$

This gives

$$|u_{nni}| \leq C(\phi + \sum_{i,j \in B} |\nabla u_{ij}|).$$

Of course, the treatment of terms involving $u_{ij\beta}$ for $i, j \in B$ follows the same way as in the proof of Theorem 3.2. One may deduce that W is of constant rank. Finally, if $n = 2$, the only other case is $l = 2$. In this case, W is of full rank everywhere. □

Remark 4.6. The above proof of Theorem 4.5 indicates that if the minimal rank of W is either 0 or 1, then the rank of $(\nabla^2 u)$ is the same everywhere. There is no structure condition imposed on F except the ellipticity condition (1.3). This observation will be used in the proof of Theorem 1.6 in the next section. In general, for a nonlinear eigenvalue problem $F(\nabla^2 v) = \lambda v$, the function $u = -\log v$ satisfies equation (4.5) if F is of homogeneous degree of one. This is useful in the study of the log-concavity property (c.f. [6, 28, 10]) of nonlinear eigenvalue problem.

We conclude this section with the proof of Theorem 1.2. We have the following.

Proposition 4.7. *Let F and u as in Theorem 1.2. For each $0 < t_0 \leq T$, if $\nabla^2 u$ attains minimum rank l at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of ϕ (defined in (2.2)), such that for t close to t_0, $\sigma_l(u_{ij}(x, t)) > 0$ for $x \in \mathcal{O}$, and*

$$\sum_{\alpha,\beta} F^{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t(x, t) \leq C(\phi(x, t) + |\nabla \phi(x, t)|), \quad \forall x \in \mathcal{O}. \tag{4.6}$$

Proof of Proposition 4.7. The proof is similar to the proof of Theorem 3.2. Since $u \in C^3$, the assumptions on F automatically imply $u \in C^4$. Suppose $(\nabla^2 u(x, t_0))$ attains its minimal rank l at some point $x_0 \in \Omega$. We may assume $l \leq n - 1$, otherwise there is nothing to prove. By continuity, $\sigma_l(u_{ij}(x, t)) > 0$ in a neighborhood of (x_0, t_0). With $u_t = F(\nabla^2 u, \nabla u, u, x, t)$, using the same notations as in the proof of Theorem 3.2, equation (3.12) becomes

$$
\begin{aligned}
&\sum_{\alpha\beta} F^{\alpha\beta} v_{\alpha\beta ij} + \sum_{\alpha\beta} v_{\alpha\beta i}(\sum_{\gamma\eta} F^{\alpha\beta,\gamma\eta} v_{\gamma\eta j} + \sum_k F^{\alpha\beta,q_k} v_{kj} + F^{\alpha\beta,v} v_j + F^{\alpha\beta,x_j}) \\
&+\sum_k F^{q_k} v_{kij} + \sum_k v_{ki}(\sum_{\alpha\beta} F^{q_k,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{q_k,q_l} v_{lj} + F^{q_k,v} v_j + F^{q_k,x_j}) \\
&+F^v v_{ij} + v_i(\sum_{\alpha\beta} F^{v,\alpha\beta} v_{\alpha\beta j} + \sum_l F^{v,q_l} v_{lj} + F^{v,v} v_j + F^{v,x_j}) \\
&+\sum_{\alpha\beta} F^{x_i,\alpha\beta} v_{\alpha\beta j} + \sum_k F^{x_i,q_k} v_{kj} + F^{x_i,v} v_j + F^{x_i,x_j} = O(\phi) + v_{ij,t},
\end{aligned} \tag{4.7}
$$

and accordingly, equation (3.13) becomes

$$
\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} &= \sum F^{\alpha\beta}\phi^{ij} v_{ij\alpha\beta} + \sum F^{\alpha\beta}\phi^{ij,km} v_{ij\alpha} v_{km\beta} \\
&= \sum F^{\alpha\beta}\phi^{ij,km} v_{ij\alpha} v_{km\beta} - \sum \phi^{ij} F^{q_k} v_{kij} - \sum \phi^{ij}[2\sum F^{\alpha\beta,q_k} v_{\alpha\beta i} v_{kj} \\
&\quad +F^v v_{ij} + \sum F^{q_k,q_l} v_{ki} v_{lj} + 2\sum F^{q_k,v} v_{ki} v_j + 2\sum F^{q_k,x_j} v_{ki}] \\
&\quad -\sum \phi^{ij}[F^{\alpha\beta,\gamma\eta} v_{\alpha\beta i} v_{\gamma\eta j} + 2\sum F^{\alpha\beta,v} v_{\alpha\beta i} v_j + 2\sum F^{\alpha\beta,x_j} v_{\alpha\beta i} \\
&\quad +\sum F^{v,v} v_i v_j + \sum F^{v,x_i} v_j + \sum F^{x_i x_j}] + O(\phi) + \sum \phi^{ij} v_{ij,t}
\end{aligned} \tag{4.8}
$$

Note that since $\phi_t = \sum \phi^{ij} v_{ij,t}$, equation (4.8) can be written as

$$
\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t &= \sum F^{\alpha\beta}\phi^{ij,km} v_{ij\alpha} v_{km\beta} - \sum \phi^{ij} F^{q_k} v_{kij} \\
&\quad -\sum \phi^{ij}[F^v v_{ij} + 2\sum F^{\alpha\beta,q_k} v_{\alpha\beta i} v_{kj} + \sum F^{q_k,q_l} v_{ki} v_{lj} \\
&\quad +2\sum F^{q_k,v} v_{ki} v_j + 2\sum F^{q_k,x_j} v_{ki}] \\
&\quad -\sum \phi^{ij}[F^{\alpha\beta,\gamma\eta} v_{\alpha\beta i} v_{\gamma\eta j} + 2\sum F^{\alpha\beta,v} v_{\alpha\beta i} v_j + 2\sum F^{\alpha\beta,x_j} v_{\alpha\beta i} \\
&\quad +\sum F^{v,v} v_i v_j + \sum F^{v,x_i} v_j + \sum F^{x_i x_j}] + O(\phi)
\end{aligned} \tag{4.9}
$$

The right hand side of (4.9) is the same as the right hand side of (3.13). Using Corollary 4.2 in place of Lemma 3.1 in the proof of Theorem 3.2, the same analysis yields

$$
\sum F^{\alpha\beta}\phi_{\alpha\beta}(x,t) - \phi_t(x,t) \le C_1(\phi(x,t) + |\nabla\phi(x,t)|) - C_2 \sum_{i,j\in B} |\nabla v_{ij}|. \tag{4.10}
$$

□

Proof of Theorem 1.2. It follows from Proposition 4.7 and the Strong Maximum Principle for parabolic equations that $\phi \equiv 0$ locally. That is $\nabla^2 u(x,t)$ is of constant rank $l(t)$ for each $t > 0$. Since Ω is flat, by the arguments in [7, 27], for each $0 < t \le T$, $x_0 \in \Omega$, there exist a neighborhood $\mathcal{U}$ of x_0 and $(n - l(t))$ fixed directions $V_1, \cdots, V_{n-l(t)}$ such that $\nabla^2 u(x,t) V_j = 0$

for all $1 \le j \le n - l(t)$ and $x \in \mathcal{U}$. Going back to (4.10), we have $\sum_{i,j\in B} |\nabla u_{ij}(x,t)| \equiv 0$ and therefore the null space of $\nabla^2 u$ is parallel. □

Remark 4.8. Examining the proof of Theorem 1.1 shows that the local convexity condition in (1.4) is only needed near the set $\mathcal{N} = \{\det(\nabla^2 u) = 0\}$. $\forall x \in \mathcal{N}$, we let

$$\mathcal{D}_{u(x)} = \{r \text{ diagonal} | \quad r = Q(\nabla^2 u(x))Q^T \quad \text{for some } Q \in O(n)\}. \tag{4.11}$$

For each $\delta > 0$, set $I^\delta_{u(x)} = \{s| \quad |s - u(x)| \le \delta\}$, and

$$\tilde{D}^\delta_{u(x)} = \{A| \quad \|A^{-1} - r\| \le \delta, \text{ for some } r \in \mathcal{D}_{u(x)}\}.$$

The condition (1.4) in Theorem 1.1 can be replaced by: there is $\delta > 0$ and for $p = Q\nabla u(x)$ ($Q \in O(n)$),

$$F(A^{-1}, p, u, x) \quad \text{is locally convex in } (A, u, x) \text{ in } \tilde{D}^\delta_{u(x)} \times I^\delta_{u(x)} \times \mathcal{O}. \tag{4.12}$$

Similarly, condition (1.5) and condition (4.3) only need to be valid for (A, u, x) in $\tilde{D}^\delta_{u(x)} \times I^\delta_{u(x)} \times \mathcal{O}$ for each t. Note that the regularity assumptions on u and F in Theorem 1.2 and Theorem 4.7 can be reduced to C^2.

5. Geometric applications

We discuss geometric nonlinear differential equations in this section.

Proposition 5.1. *Suppose $F(A, X, \vec{n}, t)$ is elliptic in A and satisfies condition (4.4) for each fixed $\vec{n} \in \mathbb{S}^n$, $t \in [0, T]$ for some $T > 0$. Let $M(t)$ be an oriented immersed connected hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form $h(t)$ satisfying equation (1.9). Then $h(t)$ is of constant rank $l(t)$ for each $t \in (0, T]$ and $l(s) \le l(t)$ for all $0 < s \le t \le T$. Moreover the null space of h is parallel for each t.*

Proof. For $\epsilon > 0$, let $W = (g^{im}h_{mj} + \epsilon\delta_{ij})$, where $h = (h_{ij})$ is the second fundamental form of $M(t)$. Let $l(t)$ be the minimal rank of $h(t)$. For a fixed $t_0 \in (0, T)$, let $x_0 \in M$ such that $h(t_0)$ attains minimal rank at x_0. Set $\phi(x,t) = \sigma_{l+1}(W(x,t)) + \frac{\sigma_{l+2}}{\sigma_{l+1}}(W(x,t))$. By the results of section 2, ϕ is in $C^{1,1}$. The proposition will follow if we can establish that there are constants C_1, C_2 independent of ϵ such that

$$F^{ij}\phi_{ij} - \phi_t \le C_1\phi + C_2|\nabla\phi|, \quad \text{near } (x_0, t_0). \tag{5.1}$$

$X = (X^1, \cdots, X^{n+1})$ be the position vector and let $h^2 = (h^i_l h^l_j)$. We note that under (1.9), the Weingarten form $h^i_j = g^{im}h_{mj}$ satisfies the equation

$$\partial_t h^i_j = \nabla^i\nabla_j F + F(h^2)^i_j. \tag{5.2}$$

The same arguments used in the proof of Theorem 3.2 carry through with some modifications to prove a parabolic version of (3.12) using (5.2). In this case, W_{ijkm} and W_{kmij} may be different.

But as W is Codazzi, the commutator term can be controlled using the Ricci identity. Here $\vec{n}$ replaces p and the Gauss equation will be used. All these terms are controlled by CW_{ii}. Notice that $W_{ii} \le \phi$ for all $i \in B$, so we have the following formula corresponding to (3.19):

$$\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t = {} & O(\phi + \sum_{i,j\in B} |\nabla W_{ij}|) - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta} \sum_{i,j\in B, i\neq j} F^{\alpha\beta} W_{ij\alpha} W_{ij\beta} \\
& - \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta}\sum_{i\in B} F^{\alpha\beta}(W_{ii\alpha}\sigma_1(B) - W_{ii}\sum_{j\in B} W_{jj\alpha})(W_{ii\beta}\sigma_1(B) - W_{ii}\sum_{j\in B} W_{jj\beta}) \\
& - \sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G} F^{\alpha\beta,\gamma\eta}(\Lambda) W_{i\alpha\beta} W_{i\gamma\eta} + \sum_{\alpha} F^{X^\alpha} X^\alpha_{ii} \\
& + 2\sum_{\alpha\beta\in G} F^{\alpha\beta} \sum_{j\in G} \frac{1}{\lambda_j} W_{ij\alpha} W_{ij\beta} + 2\sum_{\alpha,\beta\in G}\sum_{\gamma=1}^{n+1} F^{\alpha\beta, X^\gamma} W_{i\alpha\beta} X_i^\gamma + \sum_{\gamma,\eta=1}^{n+1} F^{X^\gamma, X^\eta} X_i^\gamma X_i^\eta]. \qquad (5.3)
\end{aligned}$$

The term involving X_{ii} is controlled by Ch_{ii} (and in turn by CW_{ii}) using the Weingarten formula. We obtain

$$\begin{aligned}
\sum F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t = {} & O(\phi + \sum_{i,j\in B} |\nabla W_{ij}|) - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta} \sum_{i,j\in B, i\neq j} F^{\alpha\beta} W_{ij\alpha} W_{ij\beta} \\
& - \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta}\sum_{i\in B} F^{\alpha\beta}(W_{ii\alpha}\sigma_1(B) - W_{ii}\sum_{j\in B} W_{jj\alpha})(W_{ii\beta}\sigma_1(B) - W_{ii}\sum_{j\in B} W_{jj\beta}) \\
& - \sum_{i\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}][\sum_{\alpha,\beta,\gamma,\eta\in G} F^{\alpha\beta,\gamma\eta}(\Lambda) W_{i\alpha\beta} W_{i\gamma\eta} \\
& + 2\sum_{\alpha\beta\in G} F^{\alpha\beta} \sum_{j\in G} \frac{1}{\lambda_j} W_{ij\alpha} W_{ij\beta} + 2\sum_{\alpha,\beta\in G}\sum_{\gamma=1}^{n+1} F^{\alpha\beta, X^\gamma} W_{i\alpha\beta} X_i^\gamma + \sum_{\gamma,\eta=1}^{n+1} F^{X^\gamma, X^\eta} X_i^\gamma X_i^\eta]. \qquad (5.4)
\end{aligned}$$

The right hand side of (5.4) is the same as in (3.19) and the analysis in the proof of Theorem 3.2 can be used to show the right hand side of (5.4) can be controlled by $\phi + |\nabla\phi| - C\sum_{i,j\in B}|\nabla W_{ij}|$. The theorem follows by the same argument as in the end of the proof of Theorem 4.7. □

Note that Theorem 1.5 follows directly from Proposition 5.1 (since equation (1.10) is a special case of equation (1.9) by making M independent of t) and a splitting theorem for complete hypersurfaces in $\mathbb{R}^{n+1}$. We now prove Theorem 1.4. In fact, the local convexity condition on F in that theorem can be weakened to condition (4.4).

Theorem 5.2. *Suppose $F(A, X, \vec{n}, t)$ is elliptic in A and satisfies condition (4.4) for each fixed $\vec{n} \in \mathbb{S}^n$, $t \in [0, T]$ for some $T > 0$. Let $M(t) \subset \mathbb{R}^{n+1}$ be a compact hypersurface satisfying (1.9). If M_0 is convex, then $M(t)$ is strictly convex for all $t \in (0, T)$.*

Proof of Theorem 5.2. First, M_0 may be approximated by a strictly convex M_0^ϵ. By continuity, there is $\delta > 0$ (independent of ϵ), such that there is a solution $M^\epsilon(t)$ to (1.9) with

$M^\epsilon(0) = M_0^\epsilon$ for $t \in [0, \delta]$. We argue that $M^\epsilon(t)$ is strictly convex for $t \in [0, \delta]$. If not, there is $t_0 > 0$ so that $M^\epsilon(t)$ is strictly convex for $0 \le t < t_0$. But there is one point x_0 such that $(h_{ij}(x_0, t_0))$ is not of full rank, contradicting Proposition 5.1. Taking $\epsilon \to 0$, we conclude that $M(t)$ is convex for all $t \in [0, \delta]$. This implies that the set t where $M(t)$ is convex is open. It is obviously closed. Therefore, $M(t)$ is convex for all $t \in [0, T]$. Again, by Proposition 5.1, $M(t)$ is strictly convex for all $t \in (0, T]$. □

Remark 5.3. If $n = 2$, by Lemma 4.4, if $F(A)$ is homogeneous of degree k for either $k \ge 1$ or $k \le 0$, then F satisfies condition (4.4) automatically.

Let (M, g) be a Riemannian manifold (not necessary compact). A symmetric 2-tensor W is called a Codazzi tensor if w_{ijk} is symmetric with respect to indices i, j, k in local orthonormal frames. One of the important examples of the Codazzi tensor is the second fundamental form of hypersurfaces.

Theorem 5.4. *Let $F(A, x)$ is elliptic and $F(A^{-1}, x)$ is locally convex in (A, x). Suppose (M, g) is a connected Riemannian manifold with nonnegative sectional curvature, and W is a semi-positive definite Codazzi tensor on M satisfying equation*

$$F(g^{-1}W, x) = 0 \quad \text{on } M, \tag{5.5}$$

then W is of constant rank and its null space is parallel.

Proof. Since the proof is similar to the proof of Theorem 1.1 and we only indicate some necessary modifications.

We use the same notations as in the proof of Theorem 1.1. As before, we set $\phi(x) = \sigma_{l+1}(W(x)) + \frac{\sigma_{l+2}(W(x))}{\sigma_{l+1}(W(x))}$ as in (2.2). As before, we want to establish corresponding differential inequality (3.5) in this case for the Codazzi tensor W. We note that all the analysis in Section 3 carries through without any change if we use local orthonormal frames, except for the commutators of derivatives. Since W is Codazzi, we only need to take care of commutators of the form $W_{\alpha\alpha,\beta\beta} - W_{\beta\beta,\alpha\alpha}$. The Ricci identity states

$$W_{\alpha\alpha,\beta\beta} = W_{\beta\beta,\alpha\alpha} + R_{\alpha\beta\alpha\beta}(W_{\alpha\alpha} - W_{\beta\beta}), \tag{5.6}$$

where $R_{\alpha\beta\alpha\beta}$ are the sectional curvatures of (M, g). Following the same lines of the proof of Theorem 3.2, we have the corresponding differential inequality

$$\sum_{\alpha\beta} F^{\alpha\beta}\phi_{\alpha\beta}(x) \le C_1(\phi(x) + |\nabla\phi(x)|) - \sigma_l(G) \sum_{\alpha\in G, \beta\in B} F^{\alpha\alpha} R_{\alpha\beta\alpha\beta} W_{\alpha\alpha} - C_2 \sum_{i,j\in B} |\nabla W_{ij}|. \tag{5.7}$$

Since $R_{\alpha\beta\alpha\beta} \ge 0$, the strong maximum principle implies $\phi \equiv 0$ in M. Therefore W is of constant rank l. Again, by (5.7), $\sum_{i,j\in B} |\nabla W_{ij}| \equiv 0$, so the null space of W is parallel. □

Proof of Theorem 1.6. Deal with case (2) of the theorem first. Let $c = \min_{x\in M} W_s(x)$, where $W_s(x)$ is smallest eigenvalue of W at x. Set $\tilde{W} = g^{-1}(W - cg)$. Then $\tilde{W}$ is also a Codazzi tensor, it's rank is strictly less than n at some point, and it satisfies

$$\tilde{F}(\tilde{W}) = F(g^{-1}\tilde{W} + cI) = constant. \tag{5.8}$$

By our assumption, $c \geq 0$, it follows from Corollary 4.3 that $\tilde{F}$ satisfies condition (1.4). For $\phi(x) = \sigma_{l+1}(\tilde{W}(x)) + \frac{\sigma_{l+2}(\tilde{W})}{\sigma_{l+1}(\tilde{W}(x))}$, inequality (5.7) is valid. It follows from the proof of Theorem 3.2 that $\phi \equiv 0$ in M . This implies that the left hand side of (5.7) is identically 0, so is the right hand side. By assumption, $R_{\alpha\beta\alpha\beta} > 0$ at some point. It follows that G must be empty, that is $\tilde{W} \equiv 0$.

In case (1) we follow the arguments in the proof of Theorem 4.5 and Remark 4.6. Let $\tilde{W}$ defined as before (c may not be nonnegative in this case). Then $\tilde{W}$ is a semi-positive definite Codazzi tensor with minimal rank strictly less than 2 at some point, satisfying $\tilde{F}(\tilde{W}) = F(g^{-1}\tilde{W} + cI) = 0$, $\tilde{F}$ is elliptic. If $l = 0$, the proof for case (2) carries through without change. Assume $l = 1$, $|G| = 1$. At the given point, we may assume $\tilde{W}$ is diagonal and $n \in G$. Differentiate equation $\tilde{F}(\tilde{W}) = 0$, as in the proof of Theorem 4.5, to obtain

$$\nabla\tilde{W}_{nn} = O(\sum_{i,j\in B} \nabla\tilde{W}_{ij}).$$

Therefore, $\nabla\tilde{W}_{nn}$ can be controlled. It follows from the proof of Theorem 3.2 that inequality (5.7) is valid. In turn, we get $\phi \equiv 0$ in M. As in case (2), $R_{\alpha\beta\alpha\beta} > 0$ forces $\tilde{W} \equiv 0$. □

Remark 5.5. In spirit, our results are similar to Hamilton's strong maximum principle [19] for the tensor equation

$$W_t = \Delta W + \Phi(W), \tag{5.9}$$

under the assumption that $V^T\Phi(W)V \geq 0$ for any null direction of W. In our situation, the tensor equation for W is more complicated. For example, in the case of Theorem 4.7, $W = (\nabla^2 u)$ satisfies

$$W_t = F^{ij}\nabla_i\nabla_j W + \Phi(\nabla W, W, \nabla u, u, x, t), \tag{5.10}$$

where Φ involves $\nabla W, W, \nabla u, u, x, t$. Our main aim is to show that Φ is controlled by $\phi + |\nabla\phi|$ near the null set of ϕ.

Remark 5.6. Assume F in (1.9) is nonnegative and depends only on A. Set

$$\lambda_{min}(t) = \min_{x\in M(t)}\{\text{smallest eigenvalue of h(x,t)}\}, \quad W = (h^i_j(x,t)) - \lambda_{min}(s)I.$$

If W has zero eigenvalue at some time $t > s$, using Corollary 4.3 and (5.2), the above argument above can be used to show that

$$\sum_{\alpha\beta} F^{\alpha\beta}\phi_{\alpha\beta}(x) - \phi_t \leq C_1\phi(x) + C_2|\nabla\phi(x)| - \sigma_l(G) \sum_{\alpha\in G,\beta\in B} F^{\alpha\alpha}R_{\alpha\beta\alpha\beta}W_{\alpha\alpha}. \tag{5.11}$$

By Theorem 1.4, the sectional curvature of $M(t)$ is strictly positive and therefore the last term in (5.11) must vanish, that is $W \equiv 0$. In turn, Theorem 1.4 can be strengthened as follow:

$$\lambda_{min}(t) \geq \lambda_{min}(s), \quad \forall 0 \leq s \leq t \leq T,$$

and if equality holds for some $s < t_0$, then $(h^i_j(x,t)) = \lambda_{min}(s)I$ is constant for all $s \leq t$ and for all x, that is $M(t)$ is a sphere for all $t \geq s$.

Remark 5.7. Applying the same argument as in Remark 4.8, we can weaken the local convexity condition on F in Theorem 1.6 and Theorem 5.4. Let

$$\mathcal{D}_{W(x)} = \{r \text{ diagonal}| \quad r = Qg^{-1}(x)W(x)Q^T \text{for some } Q \in O(n)\},$$
$$\tilde{D}^{\delta}_{W(x)} = \{A| \quad \|A^{-1} - r\| \leq \delta, \text{ for some } r \in \mathcal{D}_{u(x)}\}.$$

In this case, we only need the condition: there is $\delta > 0$,

$$F(A^{-1}, x) \text{ is locally convex in } \tilde{D}^{\delta}_{W(x)} \times \mathcal{O}\ . \tag{5.12}$$

Note that when M is compact, for given Codazzi tensor W on M, there exists $\lambda > 0$ such that $\tilde{W} = \lambda g - W \geq 0$ everywhere. If $F(W)$ is concave in W, then $\tilde{F}(g^{-1}\tilde{W}) = -F(\lambda I - g^{-1}\tilde{W})$ satisfies condition (5.12).

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Department of mathematics, Tongji University, Shanghai 200092, People's Republic of China
E-mail address: bianbj@mail.tongji.edu.cn

Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada.
E-mail address: guan@math.mcgill.ca