Progress of Theoretical Physics, Vol. 49, No. 1, January 1973

# A Microscopic Theory of the So-Called "Two-Phonon" States in Even-Even Nuclei. I

## ——*Basic Ideas*——

Nobuo KANESAKI, Toshio MARUMORI,* Fumihiko SAKATA and Kenjiro TAKADA

*Department of Physics, Kyushu University, Fukuoka 812*
**Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo 188*

(Received June 2, 1972)

It is the main purpose of this series of papers to propose a new microscopic theory of describing the so-called "two-phonon" states in even-even nuclei, within the framework of the quasi-particle-new-Tamm-Dancoff method (including the ground-state correlations). The theory can clearly overcome the well-known difficulties of the higher-random-phase approximation (HRPA), which have so far arrested the further development of the essential merit of the HRPA superior to the boson expansion method. In addition, the spurious-state difficulty (due to the nucleon-number non-conservation of the quasi-particle representation) does not arise at all in the theory.

It is not the purpose of this paper, part I, to present a clear-cut formulation of the theory, but rather to put an emphasis on the explanation of the basic idea.

## § 1. Introduction

According to the simple "phonon" model of spherical even nuclei based on the "harmonic approximation" (i.e., the random-phase approximation (RPA)), the first excited state is described as the "one-phonon" state with $J^\pi=2^+$ and the second excited state consists of a degenerate "two-phonon" triplet ($0^+$, $2^+$, $4^+$), and both the $E2$ cross-over transition ($2_2{}^+\to 0_1{}^+$) and the $M1$ transition ($2_2{}^+\to 2_1{}^+$) are forbidden. In the simple phonon model, furthermore, any phonon state has no appreciable static quadrupole moment and the ratio of $B(E2;\,2_2{}^+\to 2_1{}^+)$ to $B(E2;\,2_1{}^+\to 0_1{}^+)$ simply becomes 2:1.

Experimental deviations from these simple regularities have become more and more of significance, and the special importance of the anharmonic effects in such finite quantal systems as nuclei has been recognized. Thus, various theoretical attempts to take into account the anharmonic effects, which have been neglected in the simple RPA, have so far been made. Among these, the boson expansion method[1] and the higher-random-phase approximation (HRPA)[2] are well known to be of the typical types of approaches.

In order to clarify the main motive for presenting this series of papers, we first start with a brief survey of the results of analyses of the anharmonic effects with the use of the boson expansion method by Yamamura, Tokunaga and two of the authors (T.M. and K.T.).[3] They firstly classified the anharmonic effects into two characteristic types; i) *dynamical effects*, i.e., effects due to the residual interaction $H_Y$ (which has been omitted in the RPA and is given by Eq. (2·6) and whose matrix elements are represented in Fig. 1 in § 2), and ii) *kinematical effects*, i.e., effects due to the Pauli-principle among the quasi-particles belonging to different quasi-particle pairs which are regarded as ideal bosons under the RPA. To check up the concept of the "two-phonon" state (i.e., the possibility of repeating the phonon excitation twice), they further decomposed the correction to the "two-phonon" state due to these anharmonic effects into two kinds of parts: One is of the same structure as the correction to the "one-phonon" state (and is called here the *first-kind correction*) and the other is a correction which newly appears only to the "two-phonon" state (and is called here the *second-kind correction*). If it is possible to keep the phonon picture for the first $2^+$ state by making a suitable self-consistent renormalization of the main anharmonic effects, the *first-kind correction* may also be renormalized into the new two-phonon concept because of its same structure as one to the "one-phonon" state. By definition, the *second-kind correction* cannot be renormalized into this new two-phonon concept, so that it destroys the possibility of repeating the excitations of phonon. After calculating such anharmonic effects (in the pairing-plus-quadrupole-force model) with the use of a perturbation theory based on the boson expansion method, they arrived at the following conclusion for the so-called two-phonon states:[3] i) The concept of the "two-phonon" states (i.e., the possibility of repeating the phonon excitation twice) is actually in breakdown in the sense that each *second-kind correction* due to both the *dynamical* and the *kinematical effects* becomes unexpectedly large (in its absolute value) when the "phonon" energy under the RPA becomes close to the actual experimental value. ii) When the energy of the "two-phonon" state under the RPA is sufficiently close to those of the non-collective two-quasi-particle states, there appears another important physical situation to destroy the concept of the two-phonon state. In this case, which is oftenest in actual nuclei, the coupling between the "two-phonon" modes and the non-collective two-quasi-particle-excitation modes due to the $H_Y$ interaction becomes too significant to be treated by the perturbation theory. Thus, we are forced to make a diagonalization of the coupling term, which leads to a strong mixing of the non-collective two-quasi-particle-excitation modes and the "two-phonon" modes, so that the concept of the two-phonon states is completely in breakdown.

From this conclusion, one may naturally expect that the HRPA (i.e., the second random-phase approximation) is promising in taking into account these significant anharmonic effects, because it does not admit the possibility of repeating the phonon excitation twice and introduces the following excitation operators[4] in the sense of the new-Tamm-Dancoff approximation (NTD):

$$C^\dagger = \sum \{\psi(\alpha\beta)\, a_\alpha{}^\dagger a_\beta{}^\dagger + \varphi(\alpha\beta)\, \tilde{a}_\alpha \tilde{a}_\beta$$

$$+\Psi(\alpha\beta\gamma\delta)\,a_\alpha{}^\dagger a_\beta{}^\dagger a_\gamma{}^\dagger a_\delta{}^\dagger+\Xi(\alpha\beta\gamma\delta)\,a_\alpha{}^\dagger a_\beta{}^\dagger \tilde{a}_\gamma\tilde{a}_\delta+\Phi(\alpha\beta\gamma\delta)\,\tilde{a}_\alpha\tilde{a}_\beta\tilde{a}_\gamma\tilde{a}_\delta\}, \qquad (1\cdot1)$$

where $a_\alpha{}^\dagger$ is the conventional quasi-particle creation operator and $\tilde{a}_\alpha\equiv(-)^{j_a-m_\alpha}a_{\bar{\alpha}}$.*) It is clear that in the HRPA the *kinematical effects* on the so-called "two-phonon" states due to the Pauli-principle *among the four quasi-particles* are fully taken into account. Furthermore, the *dynamical effects*, i.e., the coupling between the two-quasi-particle excitation modes and the "two-phonon" modes due to the $H_Y$ interaction are properly considered: Since both two-quasi-particle and four-quasi-particle amplitudes (in the sense of the NTD approximation) are taken into account in $C^\dagger$, the excitation energies of both the first and the second excited states (which roughly correspond to the "one-phonon" and the "two-phonon" states of the RPA, respectively) are simultaneously obtained through the equation of motion of $C^\dagger$.

Unfortunately, such a merit of this approach is merely of the formal logic. As is well known,[2] actually we encounter serious difficulties which are inherently connected with the nonhermiticity of the secular matrix coming from the linearized equations of motion of $C^\dagger$. Thus, in order to avoid the difficulties superficially in such a formal way as to lead us to the secular matrix in a simple symmetrical form, in the *conventional* HRPA[2] it is customary to use practically the following reduced excitation operators $C^\dagger_{\text{reduc}}$ instead of the operators $C^\dagger$ in (1·1):

$$C^\dagger_{\text{reduc}}=\sum\{\psi(\alpha\beta)\,a_\alpha{}^\dagger a_\beta{}^\dagger+\varphi(\alpha\beta)\,\tilde{a}_\alpha\tilde{a}_\beta$$
$$+\Psi(\alpha\beta\gamma\delta)\,a_\alpha{}^\dagger a_\beta{}^\dagger a_\gamma{}^\dagger a_\delta{}^\dagger+\Phi(\alpha\beta\gamma\delta)\,\tilde{a}_\alpha\tilde{a}_\beta\tilde{a}_\gamma\tilde{a}_\delta\}, \qquad (1\cdot2)$$

which have no $a^\dagger a^\dagger\tilde{a}\tilde{a}$ terms in comparison with the original operators $C^\dagger$. A decisive deficiency of the *conventional* HRPA[2] with the use of the reduced operators $C^\dagger_{\text{reduc}}$ is its essential inability to explain the large $E2(2_2{}^+\to2_1{}^+)$ transition, in contrast with the simple "phonon" model based on the RPA. As already mentioned by Tamura and Udagawa,[2] the origin of this situation is easily found by comparing the four-quasi-particle terms in $C^\dagger_{\text{reduc}}$ with the "two-phonon"-creation operators of the RPA:

$$X^\dagger X^\dagger=\sum_{\alpha\beta\gamma\delta}\{\psi^{(0)}(\alpha\beta)\,a_\alpha{}^\dagger a_\beta{}^\dagger+\varphi^{(0)}(\alpha\beta)\,\tilde{a}_\alpha\tilde{a}_\beta\}\,\{\psi^{(0)}(\gamma\delta)\,a_\gamma{}^\dagger a_\delta{}^\dagger+\varphi^{(0)}(\gamma\delta)\,\tilde{a}_\gamma\tilde{a}_\delta\}, \qquad (1\cdot3)$$

where $X^\dagger$ is the conventional "phonon"-creation operator of the RPA:

$$X^\dagger=\sum_{\alpha\beta}\{\psi^{(0)}(\alpha\beta)\,a_\alpha{}^\dagger a_\beta{}^\dagger+\varphi^{(0)}(\alpha\beta)\,\tilde{a}_\alpha\tilde{a}_\beta\}. \qquad (1\cdot4)$$

*) As a basis of single-particle states we adopt the spherically symmetric $j$-$j$ coupling shell model. The single-particle states are then characterized by the quantum numbers: The charge $q$ and $n, l, j, m$. The Greek letter $\alpha$ denotes the complete set of these quantum numbers $\alpha\equiv\{q, n, l, j, m\}$. We further use a Latin letter $a$ to indicate all the quantum numbers in $\alpha$ except for the magnetic quantum number $m$. In association with $\alpha$ we also use $\bar{\alpha}\equiv\{q, n, l, j, -m\}$. For a basis of stationary states, it is possible to build the entire treatment on real quantities if the phase convention is suitably chosen. Throughout this paper, we always assume this to be the case.

The "two-phonon" operators $X^\dagger X^\dagger$, which make the ratio $B(E2; 2_2^+ \to 2_1^+)/B(E2; 2_1^+ \to 0_1^+)$ equal to the large value 2, certainly contain the $a^\dagger a^\dagger \tilde{a}\tilde{a}$ terms, while the $C^\dagger_{\text{reduc}}$ operators which leads us to too small values of the ratio have no such terms. It is now clear that the addition of the important "two-phonon"-type collective nature to the $C^\dagger_{\text{reduc}}$ operators is nothing but denote the readopt of the original excitation operators $C^\dagger$ in (1·1), which inevitably bring the above-mentioned serious difficulties connected with the nonhermiticity of the secular matrix. Thus we may conclude that, without overcoming the difficulties not in superficies but in essence, we can never enjoy the above-mentioned essential merit of the HRPA in treating the anharmonic effects, so long as the importance of the $a^\dagger a^\dagger \tilde{a}\tilde{a}$ terms is quite evident. Nevertheless, any theories or methods to overcome the difficulties have not yet been developed.

The main purpose of this series of papers is to propose a new systematic theory which overcomes the difficulties in essence and treats both the kinematical and the dynamical anharmonic effects in a satisfactory way. Contrary to the HRPA, the underlying philosophy of our theory is not to intend a *direct, formal* diagonalization of the Hamiltonian in a subspace characterized by some excitation operators such as $C^\dagger$, but rather to start with extraction of the basic physical elements from the subspace. In the first step, we establish the concept of the correlated $n$-quasi-particle excitation modes (with $n=2, 4, 6\cdots$) including the corresponding ground-state correlations in the framework of the NTD approximation, i.e., the "dressed" $n$-quasi-particle modes. With the aid of these excitation modes, from the subspace under consideration we extract such a set of basis vectors as the (correlated) ground state, the "dressed" two-quasi-particle states and the "dressed" four-quasi-particle states, and construct a new subspace with this set of basis vectors. (If we want to discuss further the excited states corresponding to the "three-phonon" states of the RPA, then our new subspace should be extended so as to include the "dressed" six-quasi-particle states as the basis vectors.) The typical one of the dressed two-quasi-particle modes is well known as the "phonon". The dressed four-quasi-particle states correspond to the "two-phonon" states of the RPA, but the *kinematical effects* due to the Pauli-principle among the four quasi-particles are fully taken into account in these states. In the second step, we take account of the *dynamical effects* by diagonalizing the coupling between the dressed two-quasi-particle modes and the dressed four-quasi-particle modes due to the residual interaction $H_Y$. The eigenmode-creation operators thus obtained become formally of the same form as the original excitation operators $C^\dagger$ of the HRPA, when explicitly written in terms of the quasi-particle operators. However, in this case none of the above-mentioned difficulties inherent to the HRPA (which are essentially connected with the nonhermiticity of the secular matrix) appears because of our suitable choice of the physical subspace.

The outline of this two-step theory is presented in the next section together with the basic idea and the additional motive of our theory. In the remaining sections, the dressed four-quasi-particle modes are constructed as the essential basis of our theory and their physical properties are discussed.

In subsequent papers in a seriate form, we will make a precise and clear-cut formulation of the theory through the investigation of the structure of our physical subspace, and the transcription of the physical operators into the subspace and the details of the coupling between the different excitation modes will be discussed.

## § 2. Basic idea and outline of the theory

It is the purpose of the present section to give a first understanding of our theory and the details of its underlying basic idea.

### 2·1 *The Hamiltonian*

We start with the spherically symmetric *j-j* coupling shell model Hamiltonian with a general effective two-body interaction which is invariant under the space rotations and reflections and the time reversal. We shall use mainly the same notations as Baranger's.[5] The total Hamiltonian is then given by

$$H=\sum_\alpha (\epsilon_\alpha-\lambda)c_\alpha^\dagger c_\alpha+\sum_{\alpha\beta\gamma\delta}\mathcal{V}_{\alpha\beta\gamma\delta}c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma\,, \tag{2·1}$$

where $c_\alpha^\dagger$ and $c_\alpha$ are a creation and an annihilation operators of a nucleon in the single-particle state $\alpha$ with energy $\epsilon_\alpha$ and $\lambda$ represents the chemical potential. The matrix element of the two-body potential $\mathcal{V}_{\alpha\beta\gamma\delta}$ satisfies the antisymmetry relations

$$\mathcal{V}_{\alpha\beta\gamma\delta}=-\mathcal{V}_{\beta\alpha\gamma\delta}=-\mathcal{V}_{\alpha\beta\delta\gamma}=\mathcal{V}_{\beta\alpha\delta\gamma} \tag{2·2}$$

and can be expressed in the following forms:

$$\begin{aligned}\mathcal{V}_{\alpha\beta\gamma\delta}&=-\tfrac{1}{2}\sum_J G(abcdJ)\langle j_aj_bm_\alpha m_\beta|JM\rangle\langle j_cj_dm_\gamma m_\delta|JM\rangle\\&=-\tfrac{1}{2}\sum_{J'}F(acdbJ')(-)^{j_c-m_\gamma}\langle j_aj_cm_\alpha-m_\gamma|J'M'\rangle(-)^{j_b-m_\beta}\langle j_dj_bm_\delta-m_\beta|J'M'\rangle.^{*)}\end{aligned} \tag{2·3}$$

After the Bogoliubov transformation

$$\left.\begin{aligned}a_\alpha^\dagger&=u_ac_\alpha^\dagger-v_a\tilde{c}_\alpha\,,\\ \tilde{c}_\alpha&\equiv(-)^{j_a-m_\alpha}c_{\bar{\alpha}}\,,\quad u_a^2+v_a^2=1,\end{aligned}\right\} \tag{2·4}$$

---

*) The real quantities $G(abcdJ)$ and $F(abcdJ)$ have the following properties:

$$G(abcdJ)=G(cdabJ),$$
$$G(abcdJ)=-(-)^{j_a+j_b+J}G(bacdJ)=-(-)^{j_c+j_d+J}G(abdcJ)$$
$$=(-)^{j_a+j_b+j_c+j_d}G(badcJ),$$
$$F(acdbJ)=F(dbacJ)=(-)^{j_a+j_b+j_c+j_d}F(cabdJ),$$

which come from Eq. (2·2).

the Hamiltonian can be written in terms of the quasi-particle operators $a_\alpha^\dagger$ and $a_\alpha$ as

$$\left.\begin{aligned} &H=U+H_0+H_{\rm int}\,,\\ &H_0=\sum_\alpha E_a a_\alpha^\dagger a_\alpha\,,\quad H_{\rm int}=\sum_{\alpha\beta\gamma\delta}\mathcal{V}_{\alpha\beta\gamma\delta}:c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma:\,,\end{aligned}\right\}\tag{2·5}$$

where the first constant term $U$ denotes the BCS ground-state energy and is dropped hereafter, and $E_a$ is the quasi-particle energy determined as usual together with parameters $u_a$ and $v_a$. The symbol: : denotes the normal product with respect to the quasi-particle.

## 2·2 *A classification of various roles of the interaction*

In order to see the various roles of the interaction $H_{\rm int}$, we divide it in the following way:

$$\begin{aligned} &H_{\rm int}=\sum_{\alpha\beta\gamma\delta}\mathcal{V}_{\alpha\beta\gamma\delta}:c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma:=H_X+H_V+H_Y\,,\\ &H_X=\sum_{\alpha\beta\gamma\delta}\sum_J V_X(\alpha\beta\gamma\delta;JM)\,a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma\,,\\ &H_V=\sum_{\alpha\beta\gamma\delta}\sum_J V_V(\alpha\beta\gamma\delta;JM)\,\{a_\alpha^\dagger a_\beta^\dagger \tilde{a}_\delta^\dagger \tilde{a}_\gamma^\dagger+{\rm h.c.}\},\\ &H_Y=\sum_{\alpha\beta\gamma\delta}\sum_J V_Y(\alpha\beta\gamma\delta;JM)\,\{a_\alpha^\dagger a_\beta^\dagger \tilde{a}_\delta^\dagger a_\gamma+{\rm h.c.}\},\end{aligned}\tag{2·6}$$

where

$$\begin{aligned} V_X(\alpha\beta\gamma\delta;JM)&=V_X^{(1)}(\alpha\beta\gamma\delta;JM)+V_X^{(2)}(\alpha\beta\gamma\delta;JM),\\ V_X^{(1)}(\alpha\beta\gamma\delta;JM)&=-\tfrac{1}{2}\{F(\alpha\beta\gamma\delta;JM)\,(u_a v_b u_c v_d+v_a u_b v_c u_d)\\ &\qquad-F(\alpha\beta\delta\gamma;JM)\,(u_a v_b v_c u_d+v_a u_b u_c v_d)\},\\ V_X^{(2)}(\alpha\beta\gamma\delta;JM)&=-\tfrac{1}{2}G(\alpha\beta\gamma\delta;JM)\,(u_a u_b u_c u_d+v_a v_b v_c v_d),\\ V_V(\alpha\beta\gamma\delta;JM)&=\tfrac{1}{8}\{F(\alpha\beta\gamma\delta;JM)\,(u_a v_b v_c u_d+v_a u_b u_c v_d)\\ &\qquad-F(\alpha\beta\delta\gamma;JM)\,(u_a v_b u_c v_d+v_a u_b v_c u_d)\},\\ V_Y(\alpha\beta\gamma\delta;JM)&=\tfrac{1}{2}\{F(\alpha\beta\gamma\delta;JM)\,(u_a v_b u_c u_d-v_a u_b v_c v_d)\\ &\qquad-F(\alpha\beta\delta\gamma;JM)\,(v_a u_b u_c u_d-u_a v_b v_c v_d)\}\end{aligned}\tag{2·7}$$

and

$$\begin{aligned} F(\alpha\beta\gamma\delta;JM)&\equiv\langle j_a j_b m_\alpha m_\beta|JM\rangle\langle j_c j_d m_\gamma m_\delta|JM\rangle F(abcdJ),\\ G(\alpha\beta\gamma\delta;JM)&\equiv\langle j_a j_b m_\alpha m_\beta|JM\rangle\langle j_c j_d m_\gamma m_\delta|JM\rangle G(abcdJ),\end{aligned}\tag{2·8}$$

$F(abcdJ)$ and $G(abcdJ)$ being defined through Eq. (2·3). The matrix elements of each part are illustrated in Fig. 1.

The part $H_X$ conserves the numbers of quasi-particles and is the only one considered in the Tamm-Dancoff (TD) calculation for *a fixed number of quasi-particles.* In the TD method, all the collective correlations are *asymmetrically* attributed to the excited states exclusively. An improvement of the TD method is known to be the new Tamm-Dancoff (NTD) method, whose success in describing collective phenomena is essentially due to the *symmetrical* treatment of the collective correlations for both the excited states and the ground state. In this way the collective correlations, which has been obtained in the TD method, are properly incorporated in the ground state as the ground-state correlations. It is known that the part $H_V$ brings about these ground-state correlations. Therefore, the part $H_V$ plays an essential role in constructing the collective excitation modes, together with the part $H_X$. We call these two parts, $H_X$ and $H_V$, the *constructive force* of the collective excitation modes.

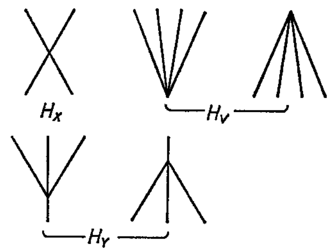


Fig. 1. Graphic representation of the interactions.

The part $H_Y$ does not contribute to the collective correlations obtained by the TD calculation for *a definite number of quasi-particles, i.e., n-quasi-particles.* We may therefore say that, in so far as the NTD method is adopted in describing a dressed $n$-quasi-particle mode (as the improvement of the TD method for $n$-quasi-particles), the part $H_Y$ does not play any important role, contrary to the *constructive force*, $H_X$ and $H_V$. The part $H_Y$ plays a decisive role as the essential coupling between the various excitation modes (in the NTD sense), and so we call it the *interactive force.*

It should be also noticed from Eq. (2·7) that the matrix elements of $H_Y$ contains the reduction $(u, v)$-factors which can be quite small in the middle of the shell, while the matrix elements of $H_X$ and of $H_V$ involve the enhancement $(u, v)$-factors which is close to unity for *low-lying states* in the middle of the shell.

A physical essence underlying the difference between the excitation modes $C^{\dagger}_{\text{reduc}}$ (in Eq. (1·2)) of the *conventional* HRPA[2] and the "two-phonon" excitations $X^{\dagger}X^{\dagger}$ (in Eq. (1·3)) of the RPA is now clearly recognized. In the modes $X^{\dagger}X^{\dagger}$ (although the important effect of the Pauli-principle among the quasi-particle pairs is not taken into account), the *constructive force*, $H_X$ and $H_V$, is so properly taken into account that the corresponding collective correlations are *symmetrically* treated for both the excited states and the ground state, while the $H_Y$ interaction does not play any role. On the contrary, in the modes $C^{\dagger}_{\text{reduc}}$ of the conventional HRPA, it turns out that the four-quasi-particle amplitudes $\Psi(\alpha\beta\gamma\delta)$ and $\Phi(\alpha\beta\gamma\delta)$ can connect with each other only through the two-quasi-particle amplitudes by the interaction $H_Y$. And the $H_V$ interaction, which is essential to introduce the collective ground-state correlations, does not play any role at all in connecting the four-quasi-particle amplitudes $\Psi(\alpha\beta\gamma\delta)$ and $\Phi(\alpha\beta\gamma\delta)$. In the modes $C^{\dagger}_{\text{reduc}}$, therefore, the four-quasi-particle correlations which have been obtained in the TD calculations are never properly incorporated into the ground state *sym-*

*metrically* as the ground-state correlations. Thus, the *conventional* HRPA[2] decisively spoils the essential merit of the NTD method. This is just the reason for the essential inability of the *conventional* HRPA[2] in explaining the large $E2(2_2^+ \to 2_1^+)$ transition, in contrast with the "phonon" model.

## 2·3 *Physical space in describing the excitations*

In the "phonon" model based on the RPA, the excitations of spherical even-even nuclei are described in an "ideal boson (phonon) space" which is formed with orthogonal basis vectors consisting of the ground state, the "one-phonon" states, the "two-phonon" states, the "three-phonon" states, $\cdots$, etc., and the deviations from the "phonon" model have been treated as the anharmonic effects *within the space.* In a marked contrast with the space of "phonon" model, our space (in describing the excitations) is formed with orthogonal basis vectors consisting of the (correlated) ground state, the *dressed* two-quasi-particle states, the *dressed* four-quasi-particle states, the *dressed* six-quasi-particle states, $\cdots$, etc., in the NTD sense. Hereafter we call this space the "quasi-particle-NTD space".

The basic physical idea underlying the introduction of the quasi-particle-NTD space is as follows. Remember at first that the use of the quasi-particles based on the BCS theory can be regarded as an attempt to classify both the ground state and the excited states in terms of the seniority number $v$,*) the value of which corresponds to the number of quasi-particles. Thus, the orthogonal basis vectors characterizing the quasi-particle representation are the BCS ground state (with $v=0$) and the independent quasi-particle states with fixed numbers of quasi-particles $n=v$ (where $v=2, 4, 6, \cdots$). Hereafter we call the space (which is formed with a set of these orthogonal bases) the "quasi-particle-TD space". The TD calculations *with the fixed number of quasi-particles* $n=v$ (diagonalizing the residual interaction $H_X$), therefore, are in accordance with the aim of the use of the quasi-particle representation, because all the "collectiveness" is obtained within the subspace (of the "quasi-particle-TD space") characterized by the seniority $v=n$. Now it is well known that, in spherical even-even nuclei, the ground-state correlations are particularly important (in describing the correlated excited states) as a collective predisposition which admits the correlated excited states to occur from it. Taking the special importance of both the seniority classification and the ground-state correlations into account simultaneously, we adopt our new "quasi-particle-NTD space" in which the orthogonal bases consist of the correlated ground state and the dressed $n(=v)$-quasi-particle states (with $n=2, 4, 6, \cdots$) in the sense of the NTD approximation (where all the collective correlations are symmetrically treated for both the excited states and the ground state).

It is important to notice that in our "quasi-particle-NTD space" the creation

*) Here $v \equiv \sum_a v_a$, where $v_a$ is the seniority number of the level $a$.

operators of the dressed $n$-quasi-particle states can be applied to the ground state *only once* and have never the vibrational character (i.e., the possibility of repeating the same excitation modes twice) such as the "phonon".

Since our main aim is to investigate the structure of the first and the second excited states in spherical even-even nuclei, we hereafter restrict ourselves only to a subspace in which the orthogonal set consists of the (correlated) ground state, the dressed two-quasi-particle states and the dressed four-quasi-particle states, as has been indicated in § 1.

### 2·4 *Dressed four-quasi-particle modes*

A serious formal difficulty in constructing the dressed $n$-quasi-particle modes in the framework of the NTD approximation (by using the *constructive force* $H_X$ and $H_V$) arises from the well-known spurious-state problem, which originates in the nucleon-number-non-conservation approximation of the quasi-particle representation. In the case of the dressed two-quasi-particle modes (i.e., the phonon modes), it is a well-known and major advantage of the NTD method that both the excited states and the (correlated) ground state are orthogonal to the spurious states within the framework of the NTD approximation. However, in the dressed $n$-quasi-particle modes (with $n>2$), the *literal* application of the NTD method never leads us to either the "physical" excited states or the "physical" ground states which are orthogonal to the spurious states, because the creation operators of the dressed $n$-quasi-particle modes themselves generally involve some components of the nucleon-number-fluctuation operators.[6]

In order to overcome this serious formal difficulty and to enjoy the proper advantage of the NTD method for the spurious-state problem, very recently Kuriyama, one of the authors (T.M.) and Matsuyanagi[6] have introduced a new concept to define precisely both the dressed $n$-quasi-particle states and the corresponding ground state as the "physical" states orthogonal to the spurious states. The concept is closely related to the transformation properties of the (corresponding) eigenmode-creation operators under the rotation in each quasi-spin space (belonging to each single-particle level $a$),[*] which has been introduced through the quasi-spin formalism[7] of the seniority coupling scheme.

Along this line, in this paper, we construct the dressed four-quasi-particle modes as the essential basis of our theory and discuss their various properties.

### 2·5 *Transcription of physical operators into the "quasi-particle-NTD space" and dynamical effects due to the $H_Y$ interaction*

Since the essence of our theory is to treat the (even-even nuclear) system within the "quasi-particle-NTD space", the orthogonal states of which are in com-

[*] It is well known that the Bogoliubov transformation simply corresponds to a rotation of the axes of reference in the quasi-spin space of level $a$ through an angle $\theta_a$ ($u_a \equiv \cos \theta_a/2$, $v_a \equiv \sin \theta_a/2$) about its $y$ axis.

plete one-to-one correspondence to those of the "quasi-particle-TD space", our next task is to find a method of transcription of the physical operators in the "quasi-particle-TD space" into our "quasi-particle-NTD space". The transcription should satisfy some self-consistency conditions with the framework of the (employed) NTD approximation under which the "quasi-particle NTD space" has been introduced. Details of the method of transcription will be discussed in the subsequent papers.

It can be seen that, after the transcription into the "quasi-particle-NTD space", the *interactive force* $H_Y$ (which has not played any role in constructing the dressed $n$-quasi-particle excitation modes) manifests itself as a coupling between the different excitation modes. In our theory, the dynamical effects on the first and the second excited states are then obtained by diagonalizing the transcribed $H_Y$ interaction within the subspace formed with the (correlated) ground state, the dressed two-quasi-particle states and the dressed four-quasi-particle states. As mentioned in the introduction, the eigenmode-creation operators thus obtained are formally of the same form as the original excitation operators $C^\dagger$ (Eq. (1·1) in the HRPA), when explicitly written in terms of the quasi-particle operators. In our theory, nevertheless, the difficulties (inherent to the HRPA), which have so far arrested the development of the merit of the HRPA, never appear because of our proper choice of the "quasi-particle-NTD space". In addition, the spurious-state difficulty (due to the nucleon-number-non-conservation of the quasi-particle representation) does not arise at all in our theory, since the system is always treated in the "quasi-particle-NTD space" which is formed with such orthogonal basis states that are orthogonal to any spurious states within the framework of the NTD approximation.

Interesting expressions of the transcribed electromagnetic moment operators will be also given in the subsequent papers.

### 2·6 *Use of the single j-shell model*

It is not the purpose of this paper, the part I, to go into concrete quantitative calculations, but rather to put an emphasis on the explanation of basic ideas. In order to illustrate the physical essence of the theory without unessential complications, therefore, we mainly develop the details of our theory with the use of the single $j$-shell model, except for the case in which we need to show explicitly that any difficulty does not arise at all in extending the essential idea discussed with the single $j$-shell model.

In the single $j$-shell model the single-particle states are characterized by a magnetic quantum number. Therefore, the Greek letter $\alpha$ in this case simply indicates $\alpha \equiv m_\alpha$. In association with $\alpha(=m_\alpha)$, the letter $\bar{\alpha}$ is used to denote $-m_\alpha$. It is also noticed that in this case the expressions (2·7) are reduced to

$$V_X(\alpha\beta\gamma\delta;JM) = -(1+(-)^J)F(\alpha\beta\gamma\delta;JM)(uv)^2 - \tfrac{1}{2}G(\alpha\beta\gamma\delta;JM)(u^4+v^4),$$

$$\left.\begin{aligned} V_V(\alpha\beta\gamma\delta;JM) &= \tfrac{1}{4}(1+(-)^J)F(\alpha\beta\gamma\delta;JM)(uv)^2, \\ V_Y(\alpha\beta\gamma\delta;JM) &= \tfrac{1}{2}(1+(-)^J)F(\alpha\beta\gamma\delta;JM)uv(u^2-v^2). \end{aligned}\right\} \tag{2·9}$$

## § 3. Construction of the dressed four-quasi-particle modes

In this section we construct the dressed four-quasi-particle modes (with the use of the single $j$-shell model), along the line developed by Kuriyama, one of authors (T.M.) and Matsuyanagi.[6]

### 3·1 *Quasi-spin formalism*

Let us define the conventional nucleon-pair operators of the orbit $j$ under consideration:

$$\left.\begin{aligned} \boldsymbol{A}_{JM}^\dagger &= \frac{1}{\sqrt{2}}\sum_{\alpha\beta}\langle jjm_\alpha m_\beta|JM\rangle c_\alpha{}^\dagger c_\beta{}^\dagger, \\ \boldsymbol{B}_{JM}^\dagger &= -\sum_{\alpha\beta}\langle jjm_\alpha m_\beta|JM\rangle c_\alpha{}^\dagger \tilde{c}_\beta\,. \end{aligned}\right\} \tag{3·1}$$

We can then easily see that three operators

$$\begin{aligned} &\hat{S}_+ = \Omega^{1/2}\boldsymbol{A}_{00}^\dagger\,, \qquad \hat{S}_- = \Omega^{1/2}\boldsymbol{A}_{00}\,, \\ &\hat{S}_0 = (\Omega/2)^{1/2}\{\boldsymbol{B}_{00}^\dagger - (\Omega/2)^{1/2}\} = \tfrac{1}{2}(\hat{N}-\Omega) \end{aligned} \tag{3·2}$$

satisfy the commutation properties of angular momentum operators

$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_0\,, \qquad [\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm\,, \tag{3·3}$$

so that we call them the quasi-spin operator. In Eq. (3·2), $\hat{N}\equiv\sum_\alpha c_\alpha{}^\dagger c_\alpha$ is the nucleon-number operator of the orbit $j$ and $\Omega\equiv j+1/2$. Now, let $S(S+1)$ and $S_0$ be the eigenvalues of the operators $\hat{S}^2=\hat{S}_+\hat{S}_-+\hat{S}_0(\hat{S}_0-1)$ and $\hat{S}_0$, respectively. The quantum numbers $S$ and $S_0$ are known to be related to the seniority $v$ and the nucleon number $N_0$ respectively through

$$S=\tfrac{1}{2}(\Omega-v)\,, \qquad S_0=\tfrac{1}{2}(N_0-\Omega)\,. \tag{3·4}$$

The quasi-spin operators $\hat{S}_\pm$, $\hat{S}_0$ are associated with the transformation properties of states and operators under rotations in the quasi-spin space of the orbit $j$. We can define quasi-spin-tensor operators $\boldsymbol{T}_{ss_0}$ of rank $s$ and its projection $s_0$ in the quasi-spin space (of the orbit $j$) as usual by the commutation relations

$$\left.\begin{aligned} &[\hat{S}_0, \boldsymbol{T}_{ss_0}] = s_0\boldsymbol{T}_{ss_0}\,, \\ &[\hat{S}_\pm, \boldsymbol{T}_{ss_0}] = \sqrt{(s\mp s_0)(s\pm s_0+1)}\,\boldsymbol{T}_{ss_0\pm 1}\,. \end{aligned}\right\} \tag{3·5}$$

The single nucleon operators $c_\alpha{}^\dagger$ and $\tilde{c}_\alpha$ are therefore regarded as spinors in the quasi-spin space (of the orbit $j$):

$$\boldsymbol{T}_{(1/2)(1/2)}(\alpha) \equiv c_\alpha{}^\dagger\,, \qquad \boldsymbol{T}_{(1/2)(-1/2)}(\alpha) = \tilde{c}_\alpha\,. \tag{3·6}$$

It is well known that the Bogoliubov transformation is simply a rotation of the axes of reference in the quasi-spin space (of the orbit $j$) through an angle $\theta$ about its $y$-axis:*)

$$\left.\begin{aligned} &U=\exp i\theta\hat{S}_y\,,\\ &\hat{S}_y=\frac{1}{2i}\{\hat{S}_+-\hat{S}_-\},\\ &u\equiv\cos(\theta/2),\qquad v\equiv\sin(\theta/2),\end{aligned}\right\} \tag{3·7}$$

so that we obtain the quasi-particles

$$\left.\begin{aligned} a_\alpha{}^\dagger &\equiv T_{(1/2)(1/2)}(\alpha)=UT_{(1/2)(1/2)}(\alpha)U^{-1}=uc_\alpha{}^\dagger-v\tilde{c}_\alpha\,,\\ \tilde{a}_\alpha &\equiv T_{(1/2)(-1/2)}(\alpha)=UT_{(1/2)(-1/2)}(\alpha)U^{-1}=u\tilde{c}_\alpha+vc_\alpha{}^\dagger.\end{aligned}\right\} \tag{3·8}$$

Generally the quasi-spin-tensor operators $T_{ss_0}$ in the quasi-particle representation are related to the original ones $\boldsymbol{T}_{ss_0}$, as usual, by $T_{ss_0}=U\boldsymbol{T}_{ss_0}U^{-1}=\sum_{s_0'}D^{s*}_{s_0's_0}(\phi=0,-\theta,\psi=0)\ \boldsymbol{T}_{ss_0'}$ where $D^s_{s_0's_0}(\phi,\theta,\psi)$ is the conventional $D$-functions for rotations.

The quasi-spin operators in the quasi-particle representation are given by

$$\left.\begin{aligned} \hat{S}_+ &= U\hat{\boldsymbol{S}}_+U^{-1}=\Omega^{1/2}A^\dagger_{00}\,,\\ \hat{S}_- &= U\hat{\boldsymbol{S}}_-U^{-1}=\Omega^{1/2}A_{00}\,,\\ \hat{S}_0 &= U\hat{\boldsymbol{S}}_0U^{-1}=(\Omega/2)^{1/2}\{B^\dagger_{00}-(\Omega/2)^{1/2}\}\\ &=\tfrac{1}{2}(\hat{n}-\Omega),\end{aligned}\right\} \tag{3·9}$$

where $A^\dagger_{JM}$ and $B^\dagger_{JM}$ are the quasi-particle-pair operators (of the orbit $j$) coupled to the angular momentum $JM$:

$$\begin{aligned} A^\dagger_{JM} &= \frac{1}{\sqrt{2}}\sum_{\alpha\beta}\langle jjm_\alpha m_\beta|JM\rangle a_\alpha{}^\dagger a_\beta{}^\dagger,\\ B^\dagger_{JM} &= -\sum_{\alpha\beta}\langle jjm_\alpha m_\beta|JM\rangle a_\alpha{}^\dagger\tilde{a}_\beta \end{aligned} \tag{3·10}$$

and $\hat{n}$ denotes the quasi-particle-number operator of the orbit $j$:

$$\hat{n}=\sum_\alpha a_\alpha{}^\dagger a_\alpha\,. \tag{3·11}$$

Since $\hat{S}^2=U\hat{\boldsymbol{S}}^2U^{-1}=\hat{\boldsymbol{S}}^2$, the quasi-spin quantum number $S$ has the *same physical meaning* as in Eq. (3·4):

$$S=\tfrac{1}{2}(\Omega-v). \tag{3·12a}$$

However, from Eq. (3·9) we can easily see that the physical meaning of the quantum number $S_0$ is now

*) In the case of actual many-shell configurations, the unitary operator of the Bogoliubov transformation $U$ becomes

$$U=\prod_a \exp i\theta_a\hat{S}_y(a)=\exp(i\sum_a\theta_a\hat{S}_y(a)),$$

where $\hat{S}_y(a)=(1/2i)\{\hat{S}_+(a)-\hat{S}_-(a)\}$ and $\hat{S}_\pm(a)$, $\hat{S}_0(a)$ denote the quasi-spin operators of the orbit $a$. Notice that the quasi-spin operators of different orbits commute with each other.

$$S_0=\frac{1}{2}(n-\Omega), \tag{3·12b}$$

where $n$ is the number of quasi-particles in the orbit $j$.

### 3·2 *Dressed four-quasi-particle modes*

In the same way as characterizes the conventional spherical tensor operators, the quasi-spin tensors $T_{ss_0}$ may be characterized by the amount of transferred quasi-spin $s$, i.e., by the amount of *transferred seniority* $\Delta v\equiv 2s$ they transfer to the states on which they operate. (The different $s_0$ components of the tensor have to possess the same intrinsic properties.) Now, as is well known, the "phonon"-creation operators are characterized by creations and destructions of two quasi-particles from the states on which they operate (i.e., by the change of quasi-particle number $\Delta n=2$). This means that the "phonon"-creation operators, i.e., the eigenmode operators of the dressed two-quasi-particle modes are characterized by the transferred seniority $\Delta v=2$, i.e., by the amount of the transferred quasi-spin $s=1$. From the standpoint of the quasi-spin formalism, therefore, the eigenmode operators of the dressed two-quasi-particle modes must be expressed in terms of the quasi-spin tensor $T_{s=1,s_0}$ (composed of two quasi-particle operators) with the transferred seniority $\Delta v\equiv 2s=2$.

From this consideration, we can precisely define the concept of dressed $n$-quasi-particle modes in the NTD sense, following Kuriyama, one of the authors (T.M.) and Matsuyanagi:[6] *The eigenmode operators of the dressed n-quasi-particle modes should be expressed in terms of the quasi-spin tensor* $T_{ss_0}$ *(composed of n-quasi-particle operators) with the transferred seniority* $\Delta v\equiv 2s=n$.

Eigenmode operators of our dressed four-quasi-particle modes may, therefore, be written as

$$\beta^\dagger_{nIK}=\sum_{\alpha\beta\gamma\delta}\sum_{s_0}\Psi_{nIK}(\alpha\beta\gamma\delta;s_0)T_{2s_0}(\alpha\beta\gamma\delta), \tag{3·13}$$

where $I$ and $K$ are the angular momentum and its projection respectively and $n$ denotes a set of additional quantum number to specify the mode. The eigenmode operators $\beta^\dagger_{nIK}$ transfer the quasi-spin $s=2$, i.e., $\Delta v=4$ to the state on which they operate, and the explicit form of the quasi-spin tensor $T_{2s_0}(\alpha\beta\gamma\delta)$ of rank $s=2$ in the quasi-particle representation is written as

$$\begin{aligned}
T_{22}(\alpha\beta\gamma\delta)&=a_\alpha{}^\dagger a_\beta{}^\dagger a_\gamma{}^\dagger a_\delta{}^\dagger,\\
T_{21}(\alpha\beta\gamma\delta)&=\frac{1}{\sqrt{4}}\{a_\alpha{}^\dagger a_\beta{}^\dagger a_\gamma{}^\dagger\tilde{a}_\delta+a_\alpha{}^\dagger a_\beta{}^\dagger\tilde{a}_\gamma a_\delta{}^\dagger+a_\alpha{}^\dagger\tilde{a}_\beta a_\gamma{}^\dagger a_\delta{}^\dagger+\tilde{a}_\alpha a_\beta{}^\dagger a_\gamma{}^\dagger a_\delta{}^\dagger\},\\
T_{20}(\alpha\beta\gamma\delta)&=\frac{1}{\sqrt{6}}\{a_\alpha{}^\dagger a_\beta{}^\dagger\tilde{a}_\gamma\tilde{a}_\delta+a_\alpha{}^\dagger\tilde{a}_\beta\tilde{a}_\gamma a_\delta{}^\dagger+a_\alpha{}^\dagger\tilde{a}_\beta a_\gamma{}^\dagger\tilde{a}_\delta\\
&\qquad+\tilde{a}_\alpha a_\beta{}^\dagger a_\gamma{}^\dagger\tilde{a}_\delta+\tilde{a}_\alpha a_\beta{}^\dagger\tilde{a}_\gamma a_\delta{}^\dagger+\tilde{a}_\alpha\tilde{a}_\beta a_\gamma{}^\dagger a_\delta{}^\dagger\},\\
T_{2-1}(\alpha\beta\gamma\delta)&=\frac{1}{\sqrt{4}}\{a_\alpha{}^\dagger\tilde{a}_\beta\tilde{a}_\gamma\tilde{a}_\delta+\tilde{a}_\alpha a_\beta{}^\dagger\tilde{a}_\gamma\tilde{a}_\delta+\tilde{a}_\alpha\tilde{a}_\beta a_\gamma{}^\dagger\tilde{a}_\delta+\tilde{a}_\alpha\tilde{a}_\beta\tilde{a}_\gamma a_\delta{}^\dagger\},\\
T_{2-2}(\alpha\beta\gamma\delta)&=\tilde{a}_\alpha\tilde{a}_\beta\tilde{a}_\gamma\tilde{a}_\delta\,.
\end{aligned} \tag{3·14}$$

Since $T_{2s_0}(\alpha\beta\gamma\delta)$ in Eq. (3·14) is antisymmetric with respect to $(\alpha, \beta, \gamma, \delta)$, the amplitude $\Psi_{nIK}(\alpha\beta\gamma\delta; s_0)$ in Eq. (3·13) also satisfies the same antisymmetry relation

$$P\Psi_{nIK}(\alpha\beta\gamma\delta; s_0) = \delta_P \Psi_{nIK}(\alpha\beta\gamma\delta; s_0), \tag{3·15}$$

where $P$ is the permutation operator with respect to $(\alpha, \beta, \gamma, \delta)$ and

$$\delta_P = \begin{cases} 1 & \text{for even permutations,} \\ -1 & \text{for odd permutations.} \end{cases} \tag{3·16}$$

3·3 *Elimination of spurious components from the eigenmode operators*

In the quasi-particle representation, the "phyiscal" states, which correspond to the actual states of nucleus (under consideration), are only the states orthogonal to the spurious states. Thus, in order that the eigenmode operators (3·13) are the "physical" ones which create the states orthogonal to the spurious states (within the framework of the NTD approximation), they are required never to contain any component of the nucleon-number fluctuation operator

$$\hat{N} - N_0 = (u^2 - v^2)(2\hat{S}_0 + \Omega) + 2uv(\hat{S}_+ + \hat{S}_-), \tag{3·17}$$

i.e., they are required never to contain the quasi-spin operators $\hat{S}_\pm$, $\hat{S}_0$. To make the eigenmode operator satisfy this requirement, we write the amplitudes in Eq. (3·13) as follows:

$$\Psi_{nIK}(\alpha\beta\gamma\delta; s_0) \equiv \sum_{\alpha'\beta'\gamma'\delta'} \Phi_{nIK}(\alpha'\beta'\gamma'\delta'; s_0) Q_I(\alpha'\beta'\gamma'\delta'; \alpha\beta\gamma\delta). \tag{3·18}$$

Then the eigenmode operator becomes

$$\beta^\dagger_{nIK} = \sum_{\alpha\beta\gamma\delta} \sum_{\alpha'\beta'\gamma'\delta'} \sum_{s_0} \Phi_{nIK}(\alpha'\beta'\gamma'\delta'; s_0) Q_I(\alpha'\beta'\gamma'\delta'; \alpha\beta\gamma\delta) T_{2s_0}(\alpha\beta\gamma\delta), \tag{3·19}$$

where $Q_I$ is the projection operator by which the quasi-spin operators $\hat{S}_\pm$, $\hat{S}_0$ are removed out of the eigenmode operator.

In order to get the explicit form of the projection operator, we rewrite Eq. (3·19) in the form

$$\beta^\dagger_{nIK} = \sum_{J_1J_2}{}' \sum_{s_0} \Psi_n(j^2(J_1)j^2(J_2)|\}j^4I; s_0) T_{2s_0}(j^2(J_1)j^2(J_2), IK), \tag{3·20}$$

where $\sum'_{J_1J_2}$ means the summation with respect to even values of $J_1$, $J_2$ and

$$T_{2s_0}(j^2(J_1)j^2(J_2), IK) \equiv \sum_{\alpha\beta\gamma\delta} \langle J_1J_2M_1M_2|IK\rangle\langle jjm_\alpha m_\beta|J_1M_1\rangle\langle jjm_\gamma m_\delta|J_2M_2\rangle \times T_{2s_0}(\alpha\beta\gamma\delta), \tag{3·21a}$$

$$\begin{aligned} \Psi_n(j^2(J_1)j^2(J_2)|\}j^4I; s_0) &\equiv \sum_{\alpha\beta\gamma\delta} \langle J_1J_2M_1M_2|IK\rangle\langle jjm_\alpha m_\beta|J_1M_1\rangle \\ &\qquad \times \langle jjm_\gamma m_\delta|J_2M_2\rangle \Psi_{nIK}(\alpha\beta\gamma\delta; s_0) \\ &= \sum_{\alpha\beta\gamma\delta} \langle J_1J_2M_1M_2|IK\rangle\langle jjm_\alpha m_\beta|J_1M_1\rangle\langle jjm_\gamma m_\delta|J_2M_2\rangle \\ &\qquad \times \{\sum_{\alpha'\beta'\gamma'\delta'} \Phi_{nIK}(\alpha'\beta'\gamma'\delta'; s_0) Q_I(\alpha'\beta'\gamma'\delta'; \alpha\beta\gamma\delta)\}. \end{aligned} \tag{3·21b}$$

From the antisymmetry relation (3·15), the amplitudes $\Psi_n(j^2(J_1)j^2(J_2)|\}j^4I;s_0)$ must satisfy the equation

$$\Psi_n(j^2(J_1)j^2(J_2)|\}j^4I;s_0)=-\sum_{J_1'J_2'}{}'\langle J_1J_2;I|J_1'J_2';I\rangle\Psi_n(j^2(J_1')j^2(J_2')|\}j^4I;s_0), \tag{3·22}$$

where $\langle J_1J_2;I|J_1'J_2';I\rangle$ is defined with 9-$j$ symbol by

$$\langle J_1J_2;I|J_1'J_2';I\rangle=\sqrt{(2J_1+1)(2J_2+1)(2J_1'+1)(2J_2'+1)}\begin{Bmatrix} j & j & J_1' \\ j & j & J_2' \\ J_1 & J_2 & I \end{Bmatrix}. \tag{3·23}$$

From the definition of the projection operator $Q_{I'}$ it is clear that the conditions

$$\Psi_n(j^2(0)j^2(J_2)|\}j^4I;s_0)=\Psi_n(j^2(J_1)j^2(0)|\}j^4I;s_0)=0 \tag{3·24}$$

must be fulfilled. It should be emphasized that Eq. (3·22) with the condition (3·24) is, in its form, precisely the same equation as what the coefficient of fractional parentage (c.f.p) with seniority $v=4$ for $j^4$-configurations must satisfy. Therefore, the solution $\Psi_n(j^2(J_1)j^2(J_2)|\}j^4I;s_0)$, which satisfies both Eqs. (3·22) and (3·24), can generally be written in the form

$$\Psi_n(j^2(J_1)j^2(J_2)|\}j^4I;s_0)=\sum_{J_1'J_2'}{}'Q_I(J_1J_2;J_1'J_2';v=4)\Phi_{nI}(J_1'J_2';s_0), \tag{3·25}$$

where

$$Q_I(J_1J_2,J_1'J_2';v=4)=\frac{1}{6}[\delta_{J_1J_1'}\delta_{J_2J_2'}+(-)^I\delta_{J_1J_2'}\delta_{J_2J_1'}-4\langle J_1J_2;I|J_1'J_2';I\rangle]$$
$$-\frac{1}{6}\left[1+\delta_{I0}-\frac{2}{\Omega}\right]^{-1}[\delta_{J_1I}\delta_{J_20}+(-)^I\delta_{J_10}\delta_{J_2I}-4\langle J_1J_2;I|I0;I\rangle]$$
$$\times[\delta_{J_1'I}\delta_{J_2'0}+(-)^I\delta_{J_1'0}\delta_{J_2'I}-4\langle J_1'J_2';I|I0;I\rangle]. \tag{3·26}$$

It can easily be shown that $Q_I(J_1J_2,J_1'J_2';v=4)$ possesses the following properties:

(i) $Q_I(J_1J_2,J_1'J_2';v=4)=Q_I(J_1'J_2',J_1J_2;v=4)$
$=(-)^IQ_I(J_1J_2,J_2'J_1';v=4)=(-)^IQ_I(J_2J_1,J_1'J_2';v=4)$,

(ii) $Q_I(J_10,J_1'J_2';v=4)=Q_I(J_1J_2,J_1'0;v=4)=0$,

(iii) $\sum_{J_1''J_2''}{}'Q_I(J_1J_2,J_1''J_2'';v=4)Q_I(J_1''J_2'',J_1'J_2';v=4)$
$=Q_I(J_1J_2,J_1'J_2';v=4)$,

(iv) $Q_I(J_1J_2,J_1'J_2';v=4)=-\sum_{J_1''J_2''}{}'\langle J_1J_2;I|J_1''J_2'';I\rangle Q_I(J_1''J_2'',J_1'J_2';v=4)$, (3·27)

so that the amplitudes $\Psi_n(j^2(J_1)j^2(J_2)|\}j^4I;s_0)$ with the form (3·25) automatically satisfy both Eqs. (3·22) and (3·24). According to Eqs. (3·21b) and (3·25), we can write

$$\Phi_{nIK}(\alpha\beta\gamma\delta; s_0) = \sum_{J_1J_2}{}' \langle J_1J_2M_1M_2|IK\rangle\langle jjm_\alpha m_\beta|J_1M_1\rangle\langle jjm_\gamma m_\delta|J_2M_2\rangle$$
$$\times \Phi_{nI}(J_1J_2; s_0), \tag{3·28a}$$

$$Q_I(\alpha\beta\gamma\delta; \alpha'\beta'\gamma'\delta') = \sum_{J_1J_2}{}' \sum_{J_1'J_2'}{}' \langle J_1J_2M_1M_2|IK\rangle\langle jjm_\alpha m_\beta|J_1M_1\rangle\langle jjm_\gamma m_\delta|J_2M_2\rangle$$
$$\times Q_I(J_1J_2, J_1'J_2'; v=4)\langle J_1'J_2'M_1'M_2'|IK\rangle\langle jjm_{\alpha'}m_{\beta'}|J_1'M_1'\rangle\langle jjm_{\gamma'}m_{\delta'}|J_2'M_2'\rangle. \tag{3·28b}$$

Thus the explicit form of the projection operator $Q_I$ has been given by Eqs. (3·28b) and (3·26).

The eigenmode operators of the physical dressed four-quasi-particle modes, which do not involve the quasi-spin operators $\hat{S}_\pm$, $\hat{S}_0$ at all, are finally given by

$$\beta^\dagger_{nIK} = \sum_{\alpha\beta\gamma\delta}\sum_{\alpha'\beta'\gamma'\delta'}\sum_{s_0}\Phi_{nIK}(\alpha\beta\gamma\delta; s_0)Q_I(\alpha\beta\gamma\delta; \alpha'\beta'\gamma'\delta')T_{2s_0}(\alpha'\beta'\gamma'\delta')$$
$$= \sum_{\alpha\beta\gamma\delta}\sum_{\alpha'\beta'\gamma'\delta'}\sum_{s_0}\Phi_{nIK}(\alpha\beta\gamma\delta; s_0)Q_I(\alpha\beta\gamma\delta; \alpha'\beta'\gamma'\delta'): T_{2s_0}(\alpha'\beta'\gamma'\delta'):, \tag{3·29}$$

where the symbol: : means the normal product with respect to the quasi-particles. In obtaining the last expression with the normal product, we have used the fact that the quantities $Q_I(\alpha\beta\gamma\delta; \alpha'\beta'\gamma'\delta')$ satisfy the following condition which is equivalent to the second property in Eq. (3·27):

$$\sum_{\alpha\beta\gamma\delta}\sum_{\alpha'\beta'\gamma'}\langle J_1J_2M_1M_2|IK\rangle\langle jjm_\alpha m_\beta|J_1M_1\rangle\langle jjm_\gamma m_\delta|J_2M_2\rangle$$
$$\times Q_I(\alpha\beta\gamma\delta; \alpha'\beta'\gamma'\bar{\gamma}')(-)^{j-m_{\gamma'}}\langle jjm_{\alpha'}m_{\beta'}|IK\rangle = 0. \tag{3·30}$$

## § 4. Properties of the eigenmode operators with $\varDelta v=4$

According to the discussions in § 2, our dressed four-quasi-particle modes should be constructed in terms of the *constructive force* $H_X$ and $H_V$. Thus, the correlation amplitudes $\Phi_{nIK}(\alpha\beta\gamma\delta; s_0)$ in the eigenmode operator (3·29) should be determined so that $\beta^\dagger_{nIK}$ becomes a "good" approximate eigenmode operator satisfying

$$[H_0 + H_X + H_V, \beta^\dagger_{nIK}] = \omega_{nI}\beta^\dagger_{nIK} - Z_{nIK}, \tag{4·1}$$

where "interaction" $Z_{nIK}$ is composed of quasi-spin-scalar constant terms (with $\varDelta v=0$), second-order normal products of bilinear quasi-spin tensor $T_{1s_0}$ with $\varDelta v=2$, tetralinear quasi-spin tensors $T_{1s_0}$ and $T_{00}$ with $\varDelta v=2$ and $\varDelta v=0$ respectively and sixth-order normal products. Thus, in our NTD approximation (in constructing the physical dressed four-quasi-particle modes with $\varDelta v=4$), the "interaction" $Z_{nIK}$ is neglected in the first step. With this approximation, Eq. (4·1) with Eq. (3·29) leads us to the following eigenvalue equation:

$$\omega_{nI}\sum_{\alpha'\beta'\gamma'\delta'}Q_I(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')\begin{pmatrix}\Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=2)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=1)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=0)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=-1)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=-2)\end{pmatrix}$$

$$=\sum_{\alpha'\beta'\gamma'\delta'}\begin{pmatrix}A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0 & -\frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0 & 0\\ 0 & \frac{1}{2}A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0 & -\frac{1}{2}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0\\ \frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}\\ 0 & \frac{1}{2}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0 & -\frac{1}{2}A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0\\ 0 & 0 & \frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0 & -A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}\end{pmatrix}$$

$$\times\begin{pmatrix}\Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=2)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=1)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=0)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=-1)\\ \Phi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=-2)\end{pmatrix}, \tag{4·2}$$

where

$$\left.\begin{aligned}A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}&\equiv\sum_{\alpha_1\beta_1\gamma_1\delta_1}\sum_{\alpha_2\beta_2\gamma_2\delta_2}Q_I(\alpha\beta\gamma\delta;\alpha_1\beta_1\gamma_1\delta_1)\mathcal{A}_{\alpha_1\beta_1\gamma_1\delta_1,\alpha_2\beta_2\gamma_2\delta_2}Q_I(\alpha_2\beta_2\gamma_2\delta_2;\alpha'\beta'\gamma'\delta'),\\ B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}&\equiv\sum_{\alpha_1\beta_1\gamma_1\delta_1}\sum_{\alpha_2\beta_2\gamma_2\delta_2}Q_I(\alpha\beta\gamma\delta;\alpha_1\beta_1\gamma_1\delta_1)\mathcal{B}_{\alpha_1\beta_1\gamma_1\delta_1,\alpha_2\beta_2\gamma_2\delta_2}Q_I(\alpha_2\beta_2\gamma_2\delta_2;\alpha'\beta'\gamma'\delta')\end{aligned}\right\} \tag{4·3}$$

and

$$\begin{aligned}\mathcal{A}_{\alpha_1\beta_1\gamma_1\delta_1,\alpha_2\beta_2\gamma_2\delta_2}\equiv&\frac{1}{4!}\sum_{P(\alpha_1\beta_1\gamma_1\delta_1)}\delta_P P\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}\delta_{\gamma_1\gamma_2}\delta_{\delta_1\delta_2}\cdot 4E_j\\ &+\frac{12}{4!}\sum_{P(\alpha_1\beta_1\gamma_1\delta_1)}\delta_P P\sum_J V_X(\alpha_1\beta_1\alpha_2\beta_2;JM)\delta_{\gamma_1\gamma_2}\delta_{\delta_1\delta_2},\end{aligned} \tag{4·4a}$$

$$\begin{aligned}\mathcal{B}_{\alpha_1\beta_1\gamma_1\delta_1,\alpha_2\beta_2\gamma_2\delta_2}\equiv&\frac{24}{4!}\sum_{P(\alpha_1\beta_1\gamma_1\delta_1)}\delta_P P\{\sum_{J'}V_Y(\alpha_1\beta_1\alpha_2\beta_2;JM)\\ &+2\sum_{J'}V_Y(\bar{\alpha}_1\beta_2\bar{\alpha}_2\beta_1;J'M')(-)^{j-m_{\alpha_1}}(-)^{j-m_{\alpha_2}}\}\delta_{\gamma_1\gamma_2}\delta_{\delta_1\delta_2}.\end{aligned} \tag{4·4b}$$

Here the matrix elements $V_X(\alpha\beta\gamma\delta;JM)$, $V_Y(\alpha\beta\gamma\delta;JM)$ are defined by Eq. (2·9) and $E_j$ is the single quasi-particle energy. The symbol $\sum_{P(\alpha_1\beta_1\gamma_1\delta_1)}$ in Eqs. (4·4a) and (4·4b) denotes the summation of all permutations of $(\alpha_1\beta_1\gamma_1\delta_1)$.

The eigenvalue equation (4·2) is simply reduced to

$$\omega_{nI}\begin{pmatrix}\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=2)\\ \Psi_{nIK}(\alpha\beta\gamma\delta;s_0=0)\\ \Psi_{nIK}(\alpha\beta\gamma\delta;s_0=-2)\end{pmatrix}$$

$$=\sum_{\alpha'\beta'\gamma'\delta'}\begin{pmatrix}A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & \frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0\\ \frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & 0 & -\frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}\\ 0 & -\frac{1}{\sqrt{6}}B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & -A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}\end{pmatrix}$$

$$\times\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\Psi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=2)\\ \Psi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=0)\\ \Psi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=-2)\end{pmatrix}, \tag{4·5}$$

$$\omega'_{nI}\begin{pmatrix}\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=1)\\ \Psi_{nIK}(\alpha\beta\gamma\delta;s_0=-1)\end{pmatrix}$$

$$=\frac{1}{2}\sum_{\alpha'\beta'\gamma'\delta'}\begin{pmatrix}A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & -B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}\\ B_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'} & -A_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}\end{pmatrix}\begin{pmatrix}\Psi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=1)\\ \Psi_{nIK}(\alpha'\beta'\gamma'\delta';s_0=-1)\end{pmatrix}, \tag{4·6}$$

where we have used Eq. (3·18) and the property $Q_I^2=Q_I$. Needless to say, the physical solutions corresponding to the dressed four-quasi-particle modes in the NTD sense are nothing but ones of Eq. (4·5). Therefore, the dressed four-quasi-particle eigenmode operators in Eq. (3·29) can now be written as

$$\beta^\dagger_{nIK}=\sum_{\alpha\beta\gamma\delta}\{\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=2):T_{22}(\alpha\beta\gamma\delta):$$
$$+\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=0):T_{20}(\alpha\beta\gamma\delta):$$
$$+\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=-2):T_{2-2}(\alpha\beta\gamma\delta):\}. \tag{4·7}$$

The form of Eq. (4·5) tells us that under the definition of inner product

$$(\Psi_{nIK}\cdot\Psi_{n'I'K'})\equiv\sum_{\alpha\beta\gamma\delta}(\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=2),\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=0),\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=-2))$$

$$\times\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\Psi_{n'I'K'}(\alpha\beta\gamma\delta;s_0=2)\\ \Psi_{n'I'K'}(\alpha\beta\gamma\delta;s_0=0)\\ \Psi_{n'I'K'}(\alpha\beta\gamma\delta;s_0=-2)\end{pmatrix}$$

$$=\sum_{\alpha\beta\gamma\delta}\{\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=2)\Psi_{n'I'K'}(\alpha\beta\gamma\delta;s_0=2)$$

$$-\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=0)\Psi_{n'I'K'}(\alpha\beta\gamma\delta;s_0=0)$$
$$+\Psi_{nIK}(\alpha\beta\gamma\delta;s_0=-2)\Psi_{n'I'K'}(\alpha\beta\gamma\delta;s_0=-2)\},\qquad(4\cdot8)$$

the correlation amplitudes satisfy the orthogonality relation in the sense that

$$(\boldsymbol{\Psi}_{nIK}\cdot\boldsymbol{\Psi}_{n'I'K'})=0\quad\text{if}\quad(nIK)\neq(n'I'K').\qquad(4\cdot9)$$

Let $\beta^\dagger_{nIK}$ with the positive eigenvalue of Eq. (4·6) $\omega_{n_+I}(>0)$ (which is reduced to $4E_j$ in the absense of the interaction) represent the creation operator $Y^\dagger_{nIK}$ of the mode under consideration:

$$\beta^\dagger_{n_+IK}\equiv Y^\dagger_{nIK},\qquad\omega_{n_+I}>0.\qquad(4\cdot10)$$

Then the corresponding annihilation operator $Y_{nIK}$ also satisfies Eq. (4·1) under our approximation (to neglect the "interaction" $Z_{nIK}$) with the negative eigenvalue $\omega_{n_-I}\equiv-\omega_{n_+I}<0$, so that

$$\beta^\dagger_{n_-IK}\equiv(-)^{I-K}Y_{nI\bar{K}},\qquad\omega_{n_-I}\equiv-\omega_{n_+I}<0.\qquad(4\cdot11)$$

We thus obtain

$$\beta^\dagger_{n_-IK}=(-)^{I-K}\beta_{n_+I\bar{K}},\qquad(4\cdot12)$$

which implies a condition for the correlation amplitudes

$$\begin{pmatrix}\Psi_{n_-IK}(\alpha\beta\gamma\delta;s_0=2)\\\Psi_{n_-IK}(\alpha\beta\gamma\delta;s_0=0)\\\Psi_{n_-IK}(\alpha\beta\gamma\delta;s_0=-2)\end{pmatrix}=(-)^I\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}\Psi_{n_+I\bar{K}}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};s_0=2)\\\Psi_{n_+I\bar{K}}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};s_0=0)\\\Psi_{n_+I\bar{K}}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};s_0=-2)\end{pmatrix},\qquad(4\cdot13)$$

which is consistent with Eq. (4·5).

## § 5. Physical meaning of the eigenmodes with $\Delta v=4$

For simplicity, we hereafter use the following notations:

$$\left.\begin{aligned}\Psi_{n_+IK}(\alpha\beta\gamma\delta;s_0=2)&\equiv\Psi_{nIK}(\alpha\beta\gamma\delta),\\\Psi_{n_+IK}(\alpha\beta\gamma\delta;s_0=0)&\equiv\Xi_{nIK}(\alpha\beta\gamma\delta),\\\Psi_{n_+IK}(\alpha\beta\gamma\delta;s_0=-2)&\equiv\Phi_{nIK}(\alpha\beta\gamma\delta).\end{aligned}\right\}\qquad(5\cdot1)$$

Relation (4·13) is then written as

$$\left.\begin{aligned}\Psi_{nIK}(\alpha\beta\gamma\delta)&=(-)^I\Psi_{n_-I\bar{K}}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};s_0=-2),\\\Xi_{nIK}(\alpha\beta\gamma\delta)&=(-)^I\Psi_{n_-I\bar{K}}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};s_0=0),\\\Phi_{nIK}(\alpha\beta\gamma\delta)&=(-)^I\Psi_{n_-I\bar{K}}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta};s_0=2)\end{aligned}\right\}\qquad(5\cdot2)$$

and the orthogonality relation (4·9) becomes of a simple form:

$$\sum_{\alpha\beta\gamma\delta}\{\Psi_{nIK}(\alpha\beta\gamma\delta)\Psi_{n'I'K'}(\alpha\beta\gamma\delta)-\Xi_{nIK}(\alpha\beta\gamma\delta)\Xi_{n'I'K'}(\alpha\beta\gamma\delta)$$
$$+\Phi_{nIK}(\alpha\beta\gamma\delta)\Phi_{n'I'K'}(\alpha\beta\gamma\delta)\}=N_n\delta_{nn'}\delta_{II'}\delta_{KK'},$$
$$(\text{from }(\boldsymbol{\Psi}_{n_+IK}\cdot\boldsymbol{\Psi}_{n_+'I'K'})=(\boldsymbol{\Psi}_{n_-IK}\cdot\boldsymbol{\Psi}_{n_-'I'K'})=N_n\delta_{nn'}\delta_{II'}\delta_{KK'})\qquad(5\cdot3a)$$

$$\sum_{\alpha\beta\gamma\delta}\{\Psi_{nIK}(\alpha\beta\gamma\delta)\Phi_{n'I'\bar{K}'}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta})-\Xi_{nIK}(\alpha\beta\gamma\delta)\Xi_{n'I'\bar{K}'}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta})$$

$$+\Phi_{nIK}(\alpha\beta\gamma\delta)\Psi_{n'I'\bar{K}'}(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta})\}=0\,,$$

$$(\text{from}\,(\boldsymbol{\Psi}_{n_+IK}\cdot\boldsymbol{\Psi}_{n_-'I'K'})=(\boldsymbol{\Psi}_{n_+'I'K'}\cdot\boldsymbol{\Psi}_{nIK})=0) \tag{5·3b}$$

where $N_n$ is the normalization constant.

The physical interpretation of the eigenmode operator

$$Y^\dagger_{nIK}=\sum_{\alpha\beta\gamma\delta}\{\Psi_{nIK}(\alpha\beta\gamma\delta):T_{22}(\alpha\beta\gamma\delta):$$

$$+\Xi_{nIK}(\alpha\beta\gamma\delta):T_{20}(\alpha\beta\gamma\delta):+\Phi_{nIK}(\alpha\beta\gamma\delta):T_{2-2}(\alpha\beta\gamma\delta):\} \tag{5·4}$$

is the following. The operator $Y^\dagger_{nIK}$ creates four-quasi-particles with *large* amplitudes $\Psi_{nIK}(\alpha\beta\gamma\delta)$ and annihilates four-quasi-particles with the *small* amplitudes $\Phi_{nIK}(\alpha\beta\gamma\delta)$, accompanying the two-quasi-particle creation and two-quasi-particle annihilation amplitudes $\Xi_{nIK}(\alpha\beta\gamma\delta)$. In the absence of ground-state correlations, $Y^\dagger_{nIK}$ becomes the operator which creates an exact four-quasi-particle eigenstates with the seniority $v=4$ in the sense of the TD method.

So far we have discussed only the eigenmode operators $Y^\dagger_{nIK}$ which have physical meaning. At this stage, it must be emphasized that the eigenvalue equation (4·2) has solutions which inevitably lead us to "special" eigenmode operators having the *largest* amplitudes $\Xi_{n_0IK}(\alpha\beta\gamma\delta)$:

$$Y^\dagger_{n_0IK}=\sum_{\alpha\beta\gamma\delta}\{\Psi_{n_0IK}(\alpha\beta\gamma\delta):T_{22}(\alpha\beta\gamma\delta):$$

$$+\Xi_{n_0IK}(\alpha\beta\gamma\delta):T_{20}(\alpha\beta\gamma\delta):+\Phi_{n_0IK}(\alpha\beta\gamma\delta):T_{2-2}(\alpha\beta\gamma\delta):\} \tag{5·5}$$

with

$$\left.\begin{aligned}\Psi_{n_0IK}(\alpha\beta\gamma\delta)&=\Psi_{n_0IK}(\alpha\beta\gamma\delta;s_0=2),\\ \Xi_{n_0IK}(\alpha\beta\gamma\delta)&=\Psi_{n_0IK}(\alpha\beta\gamma\delta;s_0=0),\\ \Phi_{n_0IK}(\alpha\beta\gamma\delta)&=\Psi_{n_0IK}(\alpha\beta\gamma\delta;s_0=-2).\end{aligned}\right\} \tag{5·6}$$

From the eigenvalue equation (4·2), we get the following orthogonality relations for the amplitudes in the "special" eigenmodes:

$$(\boldsymbol{\Psi}_{n_0IK}\cdot\boldsymbol{\Psi}_{n_0'I'K'})\equiv\sum_{\alpha\beta\gamma\delta}\{\Psi_{n_0IK}(\alpha\beta\gamma\delta)\Psi_{n_0'I'K'}(\alpha\beta\gamma\delta)$$

$$-\Xi_{n_0IK}(\alpha\beta\gamma\delta)\Xi_{n_0'I'K'}(\alpha\beta\gamma\delta)+\Phi_{n_0IK}(\alpha\beta\gamma\delta)\Phi_{n_0'I'K'}(\alpha\beta\gamma\delta)\}$$

$$=0\quad\text{if}\quad(n_0IK)\neq(n_0'I'K'), \tag{5·7a}$$

$$(\boldsymbol{\Psi}_{n_0IK}\cdot\boldsymbol{\Psi}_{nI'K'})=\sum_{\alpha\beta\gamma\delta}\{\Psi_{n_0IK}(\alpha\beta\gamma\delta)\Psi_{nI'K'}(\alpha\beta\gamma\delta)$$

$$-\Xi_{n_0IK}(\alpha\beta\gamma\delta)\Xi_{nI'K'}(\alpha\beta\gamma\delta)+\Phi_{n_0IK}(\alpha\beta\gamma\delta)\Phi_{nI'K'}(\alpha\beta\gamma\delta)\}$$

$$=0\,. \tag{5·7b}$$

The special eigenmodes have no physical meaning. In the absence of the ground-state correlations, which means $\Xi_{n_0IK}(\alpha\beta\gamma\delta)$ to vanish, the "special" eigenmodes do not appear. The outward appearance of this unphysical eigenmode $Y^{\dagger}_{n_0IK}$ is essentially based on a special situation of the ground-state correlations due to our dressed four-quasi-particle modes. The original *constructive force*, $H_X$ and $H_V$, responsible for the dressed four-quasi-particle mode (and so responsible for the ground-state correlations) is not a four-body interaction but the two-body interaction. Therefore, as is seen from Eq. (4·5), the four-quasi-particle creation amplitudes $\Psi_{nIK}(\alpha\beta\gamma\delta)$ and the four-quasi-particle annihilation amplitudes $\Phi_{nIK}(\alpha\beta\gamma\delta)$ can be coupled only via the two-quasi-particle creation and two-quasi-particle annihilation amplitudes $\Xi_{nIK}(\alpha\beta\gamma\delta)$. The existence of the amplitudes $\Xi_{nIK}(\alpha\beta\gamma\delta)$ inevitably leads to the appearance of the unphysical eigenmodes $Y^{\dagger}_{n_0IK}$ having the *largest* amplitudes $\Xi_{n_0IK}(\alpha\beta\gamma\delta)$.

In order to take the special importance of both the seniority classification of states and the ground-state correlations into account, we have introduced a new concept of "quasi-particle-NTD space" in describing the collective excitations. This quasi-particle-NTD space is, as presented in § 2, formed by the orthogonal basis vectors consisting of the correlated ground state and the dressed $n(=v)$-quasi-particle states ($n=2, 4, 6, \cdots$) in the sense of the NTD approximation. Hence any state vector with $n=4$ in this space should be able to be expanded *only* by the physical eigenmodes $Y^{\dagger}_{nIK}$ as

$$|\Phi\rangle=\sum_{nIK} C_{nIK} Y^{\dagger}_{nIK}|\Phi_0\rangle, \tag{5·8}$$

where $|\Phi_0\rangle$ is the correlated ground state. The possibility of this expansion means the completeness of our physical eigenmodes in the quasi-particle-NTD space.

Now, we should discuss the correlated ground state $|\Phi_0\rangle$. Equations

$$Y_{nIK}|\Phi_0\rangle=0\,, \qquad Y_{n_0IK}|\Phi_0\rangle=0 \tag{5·9}$$

provide us with the formal definition of the correlated ground state $|\Phi_0\rangle$. Therefore, characteristics of structure of the ground-state correlations (involved in $|\Phi_0\rangle$) should be determined *in principle* through properties of the fundamental equation (4·2) which defines the dressed four-quasi-particle modes with $\Delta v=4$ (i.e., $Y^{\dagger}_{nIK}$ and $Y^{\dagger}_{n_0IK}$) in the NTD sense. We can see that all matrix elements of the interaction in Eq. (4·2) consist of only the matrix elements of the *constructive force* $H_X$ and $H_V$ in Eq. (2·6). The diagrams considered in the correlated ground state $|\Phi_0\rangle$ are therefore closed diagrams which are composed by combining the matrix elements of $H_X$ and $H_V$ given in Fig. 1. It should be noticed that the matrix elements of $H_Y$ in Eq. (4·2) do not contribute at all to the ground-state correlations under consideration, so that the ground state $|\Phi_0\rangle$ may be generally written as a superposition of 0, 4, 8, 12, $\cdots$ quasi-particle states in the TD sense. It is generally written down as

$$|\Phi_0\rangle=C_0|\phi_0\rangle+\sum_{\alpha\beta\gamma\delta} C_1(\alpha\beta\gamma\delta)\, a_\alpha{}^{\dagger}a_\beta{}^{\dagger}a_\gamma{}^{\dagger}a_\delta{}^{\dagger}|\phi_0\rangle$$

$$+\sum_{\alpha\beta\gamma\delta\varepsilon\lambda\mu\nu} C_2(\alpha\beta\gamma\delta\varepsilon\lambda\mu\nu)\, a_\alpha{}^{\dagger}a_\beta{}^{\dagger}a_\gamma{}^{\dagger}a_\delta{}^{\dagger}a_\varepsilon{}^{\dagger}a_\lambda{}^{\dagger}a_\mu{}^{\dagger}a_\nu{}^{\dagger}|\phi_0\rangle$$

$$+\cdots, \tag{5·10}$$

where $|\phi_0\rangle$ is the BCS ground state and $C_0$ is the constant related to the normalization of $|\Phi_0\rangle$.

The coefficients $C_0, C_1, C_2, \cdots$ in Eq. (5·10) should be determined by solving Eqs. (5·9) in a way consistent with the framework of the approximation which we have used in obtaining the fundamental eigenvalue equation (4·2). Detailed discussion on this problem will be done together with a precise and clear-cut reformulation of our theory in a forthcoming paper.

## § 6. Generalization to the many-$j$-shell model

So far we have used the single $j$-shell model in order to illustrate the physical essence of our theory without fruitless complications. However, since the realistic case must be described by the many-$j$-shell model, the extension of our theory to the many-$j$-shell case becomes indispensable.

It is well known that the quasi-particle-TD approximation on the basis of the BCS theory can be regarded as an attempt to characterize both the ground state and the low-lying excited states by means of the seniority quantum numbers. The BCS ground state and the excited states in the TD approximation are given by

$$\left.\begin{aligned}&|\phi_0\rangle=|v_a=0,\ v_b=0,\ \cdots\rangle,\\&|\phi_{\text{excited}}\rangle=|v_a,\ v_b,\ \cdots;\ v=\sum_a v_a=n_0;\ \Gamma\rangle,\end{aligned}\right\} \tag{6·1}$$

where the quantities $v_a, v_b, \cdots$ are the seniority numbers belonging to the single-particle levels $a, b, \cdots$, respectively, and $v$ and $n_0$ are the total seniority and the number of quasi-particles, respectively. Here the notation $\Gamma$ stands for the additional quantum numbers characterizing the TD excited states.

In extension from the TD approximation to the NTD one, we introduce the dressed four-quasi-particle eigenmodes corresponding to the excited states in the TD approximation with $v=n_0=4$. According to the discussion in § 2, these eigenmodes should of course be constructed in terms of the constructive force $H_X$ and $H_V$. Therefore, the eigenmode operator can be written as

$$\begin{aligned}Y^\dagger_{nIK}=\sum_{\alpha\beta\gamma\delta}\{&\Psi^{(1)}_{nIK}(\alpha\beta\gamma\delta)\,a_\alpha{}^\dagger a_\beta{}^\dagger a_\gamma{}^\dagger a_\delta{}^\dagger+\Psi^{(2)}_{nIK}(\alpha\beta\gamma\delta)\,a_\alpha{}^\dagger a_\beta{}^\dagger\tilde{a}_\gamma\tilde{a}_\delta\\&+\Psi^{(3)}_{nIK}(\alpha\beta\gamma\delta)\,\tilde{a}_\alpha\tilde{a}_\beta\tilde{a}_\gamma\tilde{a}_\delta\}.\end{aligned} \tag{6·2}$$

Since the eigenmode operator (6·2) should be characterized by the transferred seniority $\Delta v=\sum_a\Delta v_a=4$, Eq. (6·2) is composed of products of quasi-spin tensors defined within each single-particle level and the sum of the transferred seniority of each quasi-spin tensor should be equal to 4. For example, in Eq. (6·2) the terms on the right-hand side, in which the single quasi-particle states $\alpha, \beta, \gamma, \delta$ belong to the same level, i.e., $a=b=c=d$, are composed of $T_{s(a)=2,s_0(a)}(\alpha\beta\gamma\delta;\,a=b=c=d)$ with the transferred seniority $\Delta v_a=2s(a)=4$, and the terms, in which

$\alpha, \beta, \gamma$ belong to the same level and $\delta$ belongs to different level, i.e., $a=b=c\neq d$, are composed of products of $T_{s(a)=3/2,s_0(a)}(\alpha\beta\gamma;\ a=b=c)$ with $\Delta v_a=3$ and $T_{s(d)=1/2,s_0(d)}(\delta)$ with $\Delta v_a=1$ (so that $\Delta v=\sum_a \Delta v_a=4$).

Now, we should remove the spurious components caused from the particle-number fluctuation in the BCS theory out of the eigenmode operator $Y^\dagger_{nIK}$. For this purpose, it is convenient to divide the summation $\sum_{\alpha\beta\gamma\delta}$ in Eq. (6·2) into five parts:

$$\sum_{\alpha\beta\gamma\delta} = \sum_{\alpha\beta\gamma\delta}^{(4)} + \sum_{\alpha\beta\gamma\delta}^{(31)} + \sum_{\alpha\beta\gamma\delta}^{(22)} + \sum_{\alpha\beta\gamma\delta}^{(211)} + \sum_{\alpha\beta\gamma\delta}^{(1111)}. \qquad (6\cdot3)$$

Here the symbols (4), (31), (22), (211) and (1111) denote the following five cases with respect to the single quasi-particle states $\alpha$, $\beta$, $\gamma$, $\delta$, respectively: (i) $a=b=c=d$, (ii) $a=b=c\neq d$, or $a=c=d\neq b$, (iii) $a=b, c=d, a\neq c$; or $a=c, b=d, a\neq b$, (iv) $a=b, a\neq c\neq d$; $c=d$, $a\neq b\neq c$; or $a=c$, $a\neq b\neq d$ and (v) $a\neq b\neq c\neq d$. The projection operator which removes the "quasi-spin" operators $S_\pm(a)$, $S_0(a)$ defined at each level $a$ from the eigenmode operator must be defined severally for each case mentioned above. Let these projection operators be $Q_I^{(4)}$, $Q_I^{(31)}$, $Q_I^{(22)}$, $Q_I^{(211)}$ and $Q_I^{(1111)}$, respectively. Using these, we should write the amplitudes in Eq. (6·2) as follows:

$$\Psi^{(k)}_{nIK}(\alpha\beta\gamma\delta)=\sum_{\alpha'\beta'\gamma'\delta'}^{(\sigma)} Q_I^{(\sigma)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')\Phi^{(k)}_{nIK}(\alpha'\beta'\gamma'\delta'),\quad k=1,2,3, \quad (6\cdot4)$$

where the symbol $\sum^{(\sigma)}_{\alpha'\beta'\gamma'\delta'}$ represents the summation in the subspace $(\sigma)$ which (is one of the five cases (4), (31), (22), (211) and (1111) and) is specified by the configuration of the single-particle states $(\alpha, \beta, \gamma, \delta)$. Then the eigenmode operator becomes

$$Y^\dagger_{nIK}=\sum_{\alpha\beta\gamma\delta}\sum_{\alpha'\beta'\gamma'\delta'}^{(\sigma)} Q_I^{(\sigma)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')\{\Phi^{(1)}_{nIK}(\alpha'\beta'\gamma'\delta')a_\alpha^\dagger a_\beta^\dagger a_\gamma^\dagger a_\delta^\dagger$$
$$+\Phi^{(2)}_{nIK}(\alpha'\beta'\gamma'\delta')a_\alpha^\dagger a_\beta^\dagger \tilde{a}_\gamma \tilde{a}_\delta+\Phi^{(3)}_{nIK}(\alpha'\beta'\gamma'\delta')\tilde{a}_\alpha\tilde{a}_\beta\tilde{a}_\gamma\tilde{a}_\delta\}. \qquad (6\cdot5)$$

Explicit forms of $Q_I^{(\sigma)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')$ can easily be obtained: From the definition, $Q_I^{(4)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')$ is nothing but Eq. (3·28b). The others are, for example,

$$Q_I^{(31)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')=\sum_J \langle Jj_dMm_\delta|IK\rangle Q_J^{(3)}(\alpha\beta\gamma;\alpha'\beta'\gamma')\langle Jj_dM'm_{\delta'}|IK\rangle$$
$$(m_\alpha+m_\beta+m_\gamma=M,\ m_{\alpha'}+m_{\beta'}+m_{\gamma'}=M') \text{ for } a=b=c\neq d, \qquad (6\cdot6a)$$

$$Q_I^{(22)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')=\sum_{J_1J_2}{}'\langle J_1J_2M_1M_2|IK\rangle Q^{(2)}_{J_1}(\alpha\beta;\alpha'\beta')Q^{(2)}_{J_2}(\gamma\delta;\gamma'\delta')$$
$$\times\langle J_1J_2M_1'M_2'|IK\rangle$$
$$(m_\alpha+m_\beta=M_1,\ m_{\alpha'}+m_{\beta'}=M_1',\ m_\gamma+m_\delta=M_2,\ m_{\gamma'}+m_{\delta'}=M_2') \text{ for } a=b,\ c=d,\ a\neq c, \qquad (6\cdot6b)$$

$$Q_I^{(211)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')=\sum_{J_1J_2}{}'\langle J_1J_2M_1M_2|IK\rangle\langle j_cj_dm_\gamma m_\delta|J_2M_2\rangle$$
$$\times Q^{(2)}_{J_1}(\alpha\beta;\alpha'\beta')\langle j_cj_dm_{\gamma'}m_{\delta'}|J_2M_2'\rangle\langle J_1J_2M_1'M_2'|IK\rangle$$
$$(m_\alpha+m_\beta=M_1,\ m_{\alpha'}+m_{\beta'}=M_1') \text{ for } a=b,\ a\neq c\neq d, \qquad (6\cdot6c)$$

$$Q_I^{(1111)}(\alpha\beta\gamma\delta;\alpha'\beta'\gamma'\delta')=\delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{\gamma\gamma'}\delta_{\delta\delta'}, \tag{6·6d}$$

where $Q_J^{(2)}$ and $Q_J^{(3)}$ in Eqs. (6·6a), (6·6b) and (6·6c) are projection operators which remove the spurious components out of two- and three-quasi-particle operators with angular momentum $J$, and project to the operators with seniority number 2 and 3, respectively. Those explicit forms can easily be obtained in a way similar to that in § 3·3 and they are

$$Q_J^{(2)}(\alpha\beta;\alpha'\beta')=\langle jjm_\alpha m_\beta|JM\rangle(1-\delta_{J0})\langle jjm_{\alpha'}m_{\beta'}|JM'\rangle,$$

$$(j_a=j_b\equiv j) \tag{6·7a}$$

$$Q_J^{(3)}(\alpha\beta\gamma;\alpha'\beta'\gamma')=\sum_{J_1J_1'}{}' \langle jjm_\alpha m_\beta|J_1M_1\rangle\langle J_1jM_1m_\gamma|JM\rangle Q_J^{(3)}(J_1,J_1';v=3)$$

$$\times\langle jjm_{\alpha'}m_{\beta'}|J_1'M_1'\rangle\langle J_1'jM_1'm_{\gamma'}|JM\rangle,\quad (j_a=j_b=j_c\equiv j) \tag{6·7b}$$

where

$$Q_J^{(3)}(J_1,J_1';v=3)=\frac{1}{3}\left[\delta_{J_1J_1'}+2\sqrt{(2J_1+1)(2J_1'+1)}\begin{Bmatrix} j & j & J_1 \\ j & J & J_1' \end{Bmatrix}\right]$$

$$-\frac{1}{3}\left[1-\frac{1}{\Omega}\delta_{Jj}\right]^{-1}\left[\delta_{J_10}-\frac{1}{\Omega}\sqrt{2J_1+1}\delta_{Jj}\right]\left[\delta_{J_1'0}-\frac{1}{\Omega}\sqrt{2J_1'+1}\delta_{Jj}\right]. \tag{6·8}$$

The eigenmode operator (6·5) thus formed possesses the total transferred seniority $\Delta v=4$ and never contains the "quasi-spin" operator $\hat{S}_\pm(a)$, $\hat{S}_0(a)$ defined in each level $a$.

## Acknowledgements

The authors are greatly indebted to Dr. A. Kuriyama, Dr. K. Matsuyanagi and members of the theoretical nuclear study group in Kyushu University for their fruitful and stimulating discussions.

### References

1) S. T. Beliaev and V. G. Zelevinsky, Nucl. Phys. **39** (1962), 582.
T. Marumori, M. Yamamura and A. Tokunaga, Prog. Theor. Phys. **31** (1964), 1009.
S. C. Pang, A. Klein and R. M. Dreizler, Ann. of Phys. **49** (1968), 477.
B. Sørensen, Nucl. Phys. **A97** (1967), 1.
D. Janssen, F. Dönau, S. Frauendorf and R. V. Jolos, Nucl. Phys. **A172** (1971), 145.
2) J. Sawicki, Phys. Rev. **126** (1962), 2231.
K. Ikeda, T. Marumori and K. Takada, Prog. Theor. Phys. **27** (1962), 1077.
D. Mitra and M. K. Pal, Phys. Letters **1** (1962), 153.
M. Savoia, J. Sawicki and A. Tomasini, Nuovo Cim. **32** (1964), 991.
T. Tamura and T. Udagawa, Nucl. Phys. **53** (1964), 33.
J. da Providencia, Nucl. Phys. **61** (1965), 87.
A. Mann, H. Nissimov and I. Unna, Nucl. Phys. **A139** (1969), 673.
3) T. Marumori, M. Yamamura, A. Tokunaga and K. Takada, Prog. Theor. Phys. **32** (1964), 726.
A. Tokunaga, Prog. Theor. Phys. **37** (1967), 315.
M. Yamamura, A. Tokunaga and T. Marumori, Prog. Theor. Phys. **37** (1967), 336.

4) A. M. Lane, *Nuclear Theory* (Benjamin, New York, 1964), p. 103.
5) M. Baranger, Phys. Rev. **120** (1960), 957.
6) A. Kuriyama, T. Marumori and K. Matsuyanagi, Prog. Theor. Phys. **45** (1971), 784.
7) Y. Wada, F. Takano and N. Fukuda, Prog. Theor. Phys. **19** (1958), 597.
P. W. Anderson, Phys. Rev. **112** (1958), 1900.
A. K. Kerman, Ann. of Phys. **12** (1961), 300.
K. Helmers, Nucl. Phys. **23** (1961), 594.
R. D. Lawson and M. H. Macfarlane, Nucl. Phys. **66** (1965), 80.
M. H. Macfarlane, *Lectures in Theoretical Physics*, Vol. VIIIC (The University of Colorado Press, 1966), p. 583.
M. Ichimura, *Progress in Nuclear Physics*, Vol. 10 (Pergamon Press, 1969).