A MILP model for an extended version of the Flexible Job Shop Problem[1]

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Ernesto G. Birgin[1], Paulo Feofiloff[1], Cristina G. Fernandes[1],
Everton L. de Melo[2], Marcio T. I. Oshiro[1], Débora P. Ronconi[2]

[1] Dept. of Computer Science, IME, University of São Paulo, Brazil
egbirgin@ime.usp.br, pf@ime.usp.br, cris@ime.usp.br, oshiro@ime.usp.br

[2] Dept. of Production Engineering, EPUSP, University of São Paulo, Brazil
everton.melo@usp.br, dronconi@usp.br

Abstract

A MILP model for an extended version of the Flexible Job Shop Scheduling problem is proposed. The extension allows the precedences between operations of a job to be given by an arbitrary directed acyclic graph rather than a linear order. The goal is the minimization of the makespan. Theoretical and practical advantages of the proposed model are discussed. Numerical experiments show the performance of a commercial exact solver when applied to the proposed model. The new model is also compared with a simple extension of the model described by Özgüven, Özbakır, and Yavuz (Mathematical models for job-shop scheduling problems with routing and process plan flexibility, *Applied Mathematical Modelling*, 34:1539–1548, 2010), using instances from the literature and instances inspired by real data from the printing industry.

1 Introduction

The Job Shop Scheduling (JS) problem can be stated as follows. Consider a set of machines and a set of jobs. Each job consists of a sequence of operations to be processed in a given order. Each operation must be processed individually on a specific machine, without preemption. The objective is to find a processing sequence for each machine that minimizes the makespan, which is the completion time of the last operation to be processed. The Flexible Job Shop Scheduling (FJS) problem is a generalization of the JS problem in which there may be several machines, not necessarily identical, capable of processing an operation. Specifically, for each operation, we are given the set of machines on which that operation can be processed. The goal is to decide on which machine each operation will be processed, and in what order the operations will be processed on each machine, so that the makespan is minimized.

The JS problem is known to be NP-hard [GJS76]. It is one of the most difficult combinatorial optimization problems according to Lawler *et al.* [LLRKS93]. Since the

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FJS problem is at least as difficult as the JS, it is also NP-hard. Many researchers use heuristic methods to deal with (the minimization of makespan of) the FJS problem. See for example [Bra93, CWC06, GGM11, MP89, VB08, ZSLG09]. In contrast, the number of publications concerned with the exact solution of the FJS problem is very small. Fattahi, Mehrabad, and Jolai [FMJ07] presented one of the most relevant papers in this direction. They proposed a mixed integer linear programming (MILP) model for the FJS problem and used it to solve a set of 20 instances of small and medium size with the LINGO software. The results were compared to those obtained by heuristic methods.

Özgüven, Özbakır, and Yavuz [OOY10] elaborated a more concise MILP model for the FJS problem, modifying an earlier one by Manne [Man60] to incorporate machine flexibility. We shall name their model ÖÖY. Özgüven *et al.* tested their model by solving the 20 instances mentioned above with the CPLEX solver. They obtained optimal solutions for the 10 small size instances faster than Fattahi *et al.* They also obtained optimal solutions for five of the medium size instances and better bounds for the other five, while Fattahi *et al.* did not find optimal solutions for any of those instances.

In the literature, each job in the FJS problem consists of a sequence of operations to be processed in a given order, just as in the ordinary JS problem. In an industrial environment, however, it is common to have jobs some of whose operations can be processed simultaneously. Moreover, some mutually independent sequences of operations may feed into an "assembling" operation. Similarly, there may be "disassembling" operations which split the sequence of subsequent operations into two or more mutually independent sequences. A representation of this kind of job is shown in Figure 1(a).

In a real problem from the printing industry [ZJL$^+$10], for example, some jobs consist of two independent sequences of operations followed by a third that puts together the results of the first two. A representation of this kind of job is shown in Figure 1(b). We say that such configuration is a *Y-job*, while the traditional type of job (a simple sequence of operations) is a *path-job*.

Vilcot and Billaut [VB08] have already considered a class of instances that includes Y-jobs and path-jobs. They describe an environment, coming from the printing and boarding industry, where each operation in a job can have more than one predecessor, but at most one direct successor. Alvarez-Valdés *et al.* [AVFT$^+$05] describe yet another environment, coming from a glass factory, that requires an even more general variant of the FJS problem.

In this paper, we extend the usual definition of the FJS problem to allow a job to be a set of operations with an arbitrary precedence relation, thus including general types of jobs like those in Figure 1. Then we propose a new MILP model for the extended FJS problem that focuses on the operations and the precedences among them, and leaves the jobs only implicitly defined.

Difficult problems such as JS and FJS require sophisticated techniques to produce reasonable results in acceptable time. It is usually preferable to get good sub-optimal results quickly than to wait for days to get an optimal solution. However, to evaluate the quality of the non-exact methods, it is desirable to compare their results with exact solutions (at least for some reference set of small and medium size instances). This is

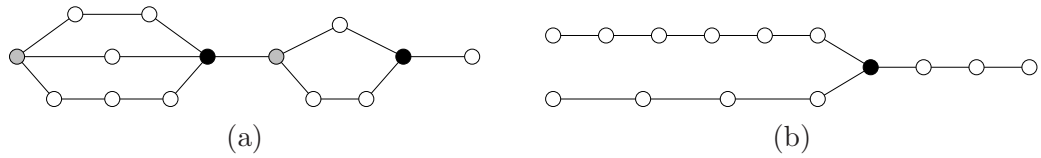

Figure 1: (a) A representation of a more general type of job. Each node represents an operation. The arcs represent precedence constraints and all arcs are directed from left to right. The black nodes are assembling operations and the gray nodes are disassembling operations. (b) A representation of a Y-job.

one of the main reasons for developing exact methods for the FJS problem. Good mixed integer programming models, combined with the availability of powerful MILP solvers, are capable of providing, if not an optimal solution, at least good bounds to many small and medium size instances. These bounds can then be used to evaluate the quality of heuristic methods.

The goal of this paper is to contribute to the development of exact models for the FJS problem, and to present a new set of instances to be used in comparisons between different MILP models and heuristics for the problem. Section 2 introduces some notation, while Section 3 presents the new model we propose for the problem. Section 3 also describes an adaptation of the ÖÖY model needed to address the more general types of jobs. Section 4 presents the results of a computational experiment involving the two models, using as benchmark the 20 instances of Fattahi *et al.*, the 15 instances of Brandimarte [Bra93], and a new set of 50 instances that contain jobs such as the ones depicted in Figure 1.

2 Notation

Let (V, A) be a dag, i.e. a directed acyclic graph. The vertices of the dag represent the *operations*. The arcs represent *precedence constraints*. (One can think of a *job* as a weakly connected component of the dag, but this concept is not needed for the models.) We are also given a set M of *machines* and a function F that associates a non-empty subset $F(v)$ of M with each operation v. The machines in $F(v)$ are the ones that can process operation v. Additionally, for each operation v and each machine k in $F(v)$, we are given a positive rational number $p_{v,k}$ representing the *processing time* of operation v on machine k.

A *machine assignment* is a function f that assigns a machine $f(v) \in F(v)$ with each operation v. Given a machine assignment f, let $p_v^f := p_{v,f(v)}$.

For each machine k, let V_k be the set of operations that can be processed on machine k, that is, $V_k = \{v \in V : k \in F(v)\}$. Let B_k be the set of all ordered pairs of distinct elements of V_k. The pairs (v, w) in B_k are designed to prevent v and w from using machine k at the same time. Let B denote the union of all B_k. Hence, $(v, w) \in B$ if and only if $v \neq w$ and $F(v) \cap F(w) \neq \emptyset$.

Given a machine assignment f, let B^f be the set of all ordered pairs of distinct opera-

tions to be processed on the same machine, that is, $B^f = \{(v, w) \in B : f(v) = f(w)\}$. A *selection* is any subset Y of B^f such that, for each (v, w) in B^f, exactly one of (v, w) and (w, v) is in Y. A selection corresponds to an ordering of the operations to be processed on the same machine. A selection Y is *admissible* if $(V, A \cup Y)$ is a dag. In other words, a selection is admissible if it conflicts neither with the given precedence constraints nor with itself.

Given a machine assignment f and an admissible selection Y, a *schedule* for $(V, A \cup Y, p^f)$ is a function s from V to the set of nonnegative rational numbers such that $s_v + p^f_v \leq s_w$ for each (v, w) in $A \cup Y$. (A schedule exists since $(V, A \cup Y)$ is a dag.) The number s_v is the *starting time* of operation v. (We write "s_v" instead of the more traditional "$s(v)$".) The *makespan* of a schedule s is the number $\text{mks}(s) := \max_{v \in V} (s_v + p^f_v)$. This definition does not preclude idle time in the schedule; the next one focuses on nondelay schedules.

The *length* of a (directed) path $(v_1, v_2, \ldots, v_l, v_{l+1})$ in the dag $(V, A \cup Y)$ is the number $p^f_{v_1} + p^f_{v_2} + \cdots + p^f_{v_l}$. (Note that $p^f_{v_{l+1}}$ is not part of the sum.) For any path P in $(V, A \cup Y)$ ending at v and any schedule s, the length of P is at most s_v. For each v in V, let s^*_v be the maximum of the lengths of all paths in $(V, A \cup Y)$ ending at v. The function s^* so defined is a schedule. We say that this is the *tight schedule* for $(V, A \cup Y, p^f)$. There is a simple dynamic programming algorithm that computes the tight schedule. Not surprisingly, the makespan of the tight schedule s^* is determined by long paths: there exists a path $P = (v_1, v_2, \ldots, v_l, v_{l+1})$ in $(V, A \cup Y)$ such that the length of P plus $p_{v_{l+1}}$ equals $\text{mks}(s^*)$. (Such P is known as a *critical* path.) It follows from the previous observations that the tight schedule has minimum makespan among all schedules for $(V, A \cup Y, p^f)$.

The *makespan* of an admissible selection Y for a given machine assignment f, denoted by $\text{mks}(Y)$, is the makespan of the tight schedule for $(V, A \cup Y, p^f)$. The FJS problem can now be stated as follows:

> FJS Problem (V, A, M, F, p): Find a machine assignment f and an admissible selection Y such that $\text{mks}(Y)$ is minimum.

3 Models

This section presents two models for the FJS problem. The first one is our new model while the second is an adapted version of the ÖÖY model by Özgüven, Özbakır, and Yavuz.

3.1 A new model for the FJS problem

In order to formulate the FJS problem as a MILP, we use a binary array x to represent the machine assignments and a binary array y to represent selections. The first array will have a component $x_{v,k}$ for each v in V and each k in $F(v)$. The second will have a component $y_{v,w}$ for each (v, w) in B. We also use two rational arrays s and p', and a rational number z, where s represents the starting times, p' represents the processing times corresponding to the machine assignment given by x, and z represents the makespan

of schedule s.

We need an upper bound L on the makespan of an optimal solution of the FJS problem. This can be the makespan of an arbitrary admissible selection or, alternatively, a global bound like $\sum_{v \in V} \max_{k \in F(v)} p_{v,k}$.

Our MILP can now be given as follows: find a rational number z, rational arrays s and p', and binary arrays x and y that

$$\begin{aligned}
&\text{minimize } z \\
&\text{subject to} \\
&\qquad s_v + p'_v \leq z \qquad \forall\, v \in V, && (1)\\
&\qquad \textstyle\sum_{k \in F(v)} x_{v,k} = 1 \qquad \forall\, v \in V, && (2)\\
&\qquad p'_v = \textstyle\sum_{k \in F(v)} p_{v,k} x_{v,k} \qquad \forall\, v \in V, && (3)\\
&\qquad y_{v,w} + y_{w,v} \geq x_{v,k} + x_{w,k} - 1 \qquad \forall\, k \in M \text{ and } \forall\, (v,w) \in B_k, && (4)\\
&\qquad s_v + p'_v \leq s_w \qquad \forall\, (v,w) \in A, && (5)\\
&\qquad s_v + p'_v - (1 - y_{v,w})L \leq s_w \qquad \forall\, (v,w) \in B, && (6)\\
&\qquad s_v \geq 0 \qquad \forall\, v \in V. && (7)
\end{aligned}$$

As x is binary, constraint (2) ensures that x is a machine assignment. Then constraint (3) makes array p' represent the processing times of operations. In fact, p' can be seen as an intermediate value, not a variable, that helps to simplify the presentation of the model. Since $p_{v,k} > 0$ for all v and k, thus $p'_v > 0$ and so constraint (6) makes sure that $y_{v,w}$ and $y_{w,v}$ are not both equal to 1. Hence, as y is binary, constraint (4) implies that y represents a selection. Indeed, if $x_{v,k} = x_{w,k} = 1$, which means v and w are assigned to machine k, then (4) forces y to decide whether v comes before or after w. Otherwise (i.e. if $x_{v,k}$ and $x_{w,k}$ are not both 1), constraint (4) is trivially satisfied. Once y is a selection and p' represents the processing times, constraints (5), (6), and (7) make s represent a schedule. Finally, the objective function and constraint (1) make sure z is the makespan of the schedule, and is as small as possible.

We show next that our MILP is equivalent to the FJS problem. Suppose (f, Y) is a feasible solution of an instance (V, A, M, F, p) of the problem, i.e. suppose f is a machine assignment and Y a corresponding admissible selection. Let s be the tight schedule for $(V, A \cup Y, p^f)$ and let $z := \text{mks}(s)$. For each v and each k in $F(v)$, let $x_{v,k} := 1$ if and only if $f(v) = k$. Let $y_{v,w} := 1$ for each (v,w) in Y and $y_{v,w} := 0$ for each (v,w) in $B \setminus Y$. Finally, let $p' := p^f$. Then, the tuple (s, x, p', y, z) is a feasible solution of our MILP and the value, z, of this solution is equal to $\text{mks}(s)$. Now consider the converse. Let (s, x, p', y, z) be a feasible solution of the MILP. For each v, let $f(v)$ be the unique k in $F(v)$ for which $x_{v,k} = 1$; such k exists because x is binary and constraint (2) holds. According to (3), $p' = p^f$. Let Y be the set of all pairs (v,w) in B such that $y_{v,w} = 1$. Constraint (4) makes sure that Y is a selection. Since $p_{v,k} > 0$ for all (v,k), constraints (5) to (7) make sure that the selection Y is admissible. Hence, (f, Y) is a feasible solution of the FJS problem. Moreover, s is a schedule for $(V, A \cup Y, p')$ and $\text{mks}(s) \leq z$ by virtue of (1). This discussion shows that every optimal solution of the FJS problem corresponds to an optimal solution of the MILP and conversely.

Let $\beta := \sum_{k\in M} |B_k|$ and $\varphi := \sum_{v\in V} |F(v)|$. Our MILP has $2|V| + |A| + |B| + \beta$ constraints, and $|V| + \varphi + |B|$ variables, of which $\varphi + |B|$ are binary.

3.2 The model of Özgüven, Özbakır, and Yavuz

The ÖÖY MILP model mentioned in the Introduction is designed to handle path-jobs only. In order to represent our more general job structure, a slight modification of that MILP is needed. (Thus, for example, the double index ij, representing operation j of job i, was replaced by a single index v representing an operation. Accordingly, the set O_i of operations in job i was replaced by our set A of precedence constraints combined with the set B_k. Finally, the role of the "big number" L was made precise.) We will refer to the modified version of the ÖÖY model as ÖÖY′. We introduce ÖÖY′ only because we are unable to compare our proposed model with ÖÖY directly. The modified model reduces to the original one when applied to instances with path-jobs only.

We shall use the following variables. First, a binary array x to represent machine assignments, with one component $x_{v,k}$ for each v in V and each k in $F(v)$. Second, a rational array s to represent the starting times, with one component $s_{v,k}$ for each v in V and each k in $F(v)$. Third, a rational array t to represent the completion times, with one component $t_{v,k}$ for each v in V and each k in $F(v)$. Finally, a binary array y to represent a selection, with one component $y_{v,w,k}$ for each k in M and (v, w) in B_k.

We also need an upper bound L on the makespan of an optimal solution. Again, this can be the makespan of some admissible selection or the sum $\sum_{v\in V} \max_{k\in F(v)} p_{v,k}$, for example. Finally, let $\hat{V}$ be the set of "terminal" vertices, i.e. the set of all v in V such that there is no (v, u) in A.

The ÖÖY′ model can be stated as follows: find a rational number z, rational arrays s and t and binary arrays x and y that

$$\begin{aligned}
&\text{minimize } z \\
&\text{subject to} \\
t_{v,k} &\le z && \forall\, v \in \hat{V} \text{ and } \forall\, k \in F(v), && (8)\\
\textstyle\sum_{k\in F(v)} x_{v,k} &= 1 && \forall\, v \in V, && (9)\\
s_{v,k} + t_{v,k} &\le 2x_{v,k}L && \forall\, v \in V \text{ and } \forall\, k \in F(v), && (10)\\
y_{v,w,k} + y_{w,v,k} &= 1 && \forall\, k \in M \text{ and } \forall\, (v, w) \in B_k, && (11)\\
s_{v,k} + p_{v,k} - (1 - x_{v,k})L &\le t_{v,k} && \forall\, v \in V \text{ and } \forall\, k \in F(v), && (12)\\
t_{v,k} - (1 - y_{v,w,k})L &\le s_{w,k} && \forall\, k \in M \text{ and } \forall\, (v, w) \in B_k, && (13)\\
\textstyle\sum_{k\in F(v)} t_{v,k} &\le \textstyle\sum_{k\in F(w)} s_{w,k} && \forall\, (v, w) \in A, && (14)\\
s_{v,k} &\ge 0 && \forall\, v \in V \text{ and } \forall\, k \in F(v), && (15)\\
t_{v,k} &\ge 0 && \forall\, v \in V \text{ and } \forall\, k \in F(v). && (16)
\end{aligned}$$

As x is binary, constraint (9) ensures that x is a machine assignment. If v is not assigned to machine k, constraint (10) makes $s_{v,k} = t_{v,k} = 0$. Thus, at most one term is nonzero in each sum of (14), and constraint (10) is trivial if $s_{v,k} = t_{v,k} = 0$. Given

a machine assignment x, the set of pairs (v, w) such that $y_{v,w,k} = 1$ and both v and w are assigned to machine k is a selection due to constraint (11). Now, constraints (12) to (16) make s store the starting times and t the completion times of a schedule. Indeed, (12) ensures that the starting and completion times of an operation are consistent, (13) ensures that two operations assigned to the same machine have starting and completion times consistent with the selection, and (14) ensures that the starting and completion times satisfy the precedences between operations. Finally, constraint (8) and the objective function ensure that z is the makespan of the schedule, and is as small as possible.

We show next that the ÖÖY′ MILP is equivalent to the FJS problem. Suppose (f, Y) is a feasible solution of an instance (V, A, M, F, p) of the problem. Let s^* be the tight schedule for $(V, A \cup Y, p^f)$ and $z := \text{mks}(s^*)$. For each v and each k in $F(v)$, let $x_{v,k} := 1$ if and only if $f(v) = k$. For each v and each k in $F(v)$, let $s_{v,k} := s^*_v$ if $f(v) = k$ and $s_{v,k} := 0$ otherwise. Similarly, let $t_{v,k} := s^*_v + p_{v,k}$ if $f(v) = k$ and $t_{v,k} := 0$ otherwise. Then s, t, x, and z satisfy constraints (8) to (10), (12), and (14) to (16). Now define y as follows. Let $v_1, v_2, \ldots, v_N$ be an arbitrary ordering of all the operations in V. For each k in M and each (v_i, v_j) in B_k, let $y_{v_i,v_j,k} := 1$ if and only if one of the following three conditions holds: $(v_i, v_j) \in Y$ and $f(v_i) = f(v_j) = k$; or $f(v_i) \neq k$ and $f(v_j) = k$; or $i > j$, $f(v_i) \neq k$ and $f(v_j) \neq k$. This definition of y satisfies constraints (11) and (13). Hence, the tuple (s, t, x, y, z) is a feasible solution of the MILP and the value, z, of this solution is equal to $\text{mks}(s^*)$. Now consider the converse. Let (s, t, x, y, z) be a feasible solution of the MILP. For each v, let $f(v)$ be the unique k in $F(v)$ for which $x_{v,k} = 1$; such k exists because x is binary and (9) holds. For each k, let Y_k be the set of all pairs (v, w) in B_k such that $x_{v,k} = 1$, $x_{w,k} = 1$, and $y_{v,w,k} = 1$. Let Y be the union of all Y_k, $k \in M$. According to (11), Y is a selection. According to (12), (13), and (14), Y is an admissible selection. Hence, (f, Y) is a feasible solution of the FJS problem. Finally, let $s'_v := s_{v,f(v)}$ and observe that s' is a schedule for $(V, A \cup Y, p^f)$ and $\text{mks}(s') \leq z$. This discussion shows that every optimal solution of the FJS problem corresponds to an optimal solution of the MILP and conversely.

As before, let $\beta := \sum_{k \in M} |B_k|$ and $\varphi := \sum_{v \in V} |F(v)|$. Additionally, let $\hat{\varphi} := \sum_{v \in \hat{V}} |F(v)|$. This MILP has $|V| + |A| + \hat{\varphi} + 2\varphi + 2\beta$ constraints, and $3\varphi + \beta$ variables, of which $\varphi + \beta$ are binary.

3.3 A brief comparison of the two models

In the next section, we compare the two models experimentally. Before that, we comment on some differences between the models.

The proposed model, presented in Section 3.1, is more compact than the ÖÖY′ model. The array variables that describe the selection and the starting times have one more dimension in ÖÖY′, since they are also indexed by the machines. Moreover, ÖÖY′ has variables for the completion times that depend not only on the operations but also on the machines. Therefore, the ÖÖY′ model uses significantly more variables. Concretely, since $\varphi \geq |V|$ and $\beta \geq |B|$, the ÖÖY′ model uses at least φ more variables, at least as many binary variables, and has at least $\hat{\varphi} + \varphi$ more constraints than our model.

Another difference between the two models comes from the following observations. Consider the linear relaxation of the proposed MILP, i.e. drop the requirement that x and y be integer. Let $(\dot{s}, \dot{x}, \dot{p}', \dot{y}, \dot{z})$ be an optimal solution of the resulting LP. As (5) enforces the precedence constraints in A, considering the processing times given by $\dot{p}'$, the starting times given by $\dot{s}$ are a schedule for $(V, A, \dot{p}')$. Moreover, by (1), $\mathrm{mks}(\dot{s}) \leq \dot{z}$. Now let s^* be the tight schedule for $(V, A, \dot{p}')$. Then of course $\mathrm{mks}(s^*) \leq \mathrm{mks}(\dot{s}) \leq \dot{z}$. Hence, the optimal value, $\dot{z}$, of the relaxed MILP is not smaller than the makespan of the tight schedule for $(V, A, \dot{p}')$. (In fact, $\dot{z}$ is also not smaller than the makespan of the tight schedule for (V, A, p''), where $p''_v = \min_{k \in F(v)} p_{v,k}$.)

Similarly, consider the linear relaxation of the ÖÖY′ MILP, i.e. drop the requirement that x and y be integer. Now take any instance of the FJS problem in which

- $p_{v,k} \leq L/2$ for each v and each k and
- $|V_k| \geq 2$ for each k in M,

where $V_k := \{v \in V : k \in F(v)\}$. (We believe many instances of the FJS problem do possess these properties.) Then the linear relaxation of the ÖÖY′ MILP has an optimal solution $(\dot{s}, \dot{t}, \dot{x}, \dot{y}, \dot{z})$ with $\dot{s} = 0$, $\dot{t} = 0$, $\dot{x}_{v,k} = 1/|V_k|$, $\dot{y}_{v,w,k} = 1/2$, and $\dot{z} = 0$. That is, for these instances, the optimal value of the relaxation of the ÖÖY′ MILP is potentially much smaller than the optimal value of the relaxation of the proposed MILP. Moreover, the integrality gap of the ÖÖY′ MILP is unbounded. This seems to be an undesirable property of this MILP, a property not shared by the proposed MILP.

4 Computational experiments

This section presents the results of a computational experiment involving the two MILP models. We used as benchmarks the 20 instances of Fattahi *et al.* [FMJ07], the 15 instances of Brandimarte [Bra93], and a new set of 50 instances that contain jobs as the ones depicted in Figure 1.

In an application of the FJS problem coming from the printing industry [ZJL+10], each job represents an order for a certain number of printed copies of some object. The way to process each order depends on the type of object to be printed. Some of the orders correspond to a path-job. Some others, like the printing of a book, correspond to a Y-job as in Figure 1(b) with the following parts: the printing of the book cover, the printing of the text blocks (book pages), and the binding. There is more than one choice of machine to process some of the operations.

To analyze the performance of the MILP models on this kind of instances, we developed a generator of random instances of Y-jobs. The generator has four integers as parameters: the number n of jobs, the number o of operations per job, the number m of machines, and the maximum number q of machines that can process the same operation. First, the dag representing the jobs is created. Each job is initially a path-job with operations 1, 2, ..., o. Then, for each job, two operations i and j are chosen independently at random. If $i = 1$, $j = 1$, or $i = j$, the job is not changed. Else, assuming $i < j$, arc $(i-1, i)$ is replaced by arc $(i-1, j)$. Next, for each operation, a set of at most q machines is selected

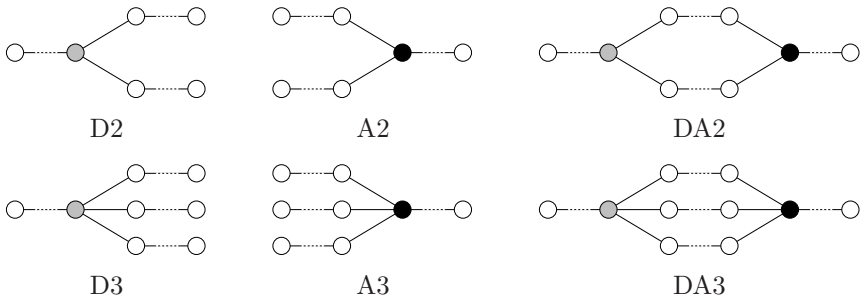

Figure 2: The six types of dags — D2, D3, A2, A3, DA2, and DA3 — produced by our second generator of instances. Each node represents an operation. The arcs represent precedence constraints and are all directed from left to right. In each dag, all maximal paths have the same length. (The black nodes are assembling operations and the gray nodes are disassembling operations.)

at random and for each of these machines a processing time is chosen at random in the set $\{20, \ldots, 200\}$. (All random choices use a uniform distribution.) A set of 20 instances of this type, with n and o in $\{4, \ldots, 17\}$, m in $\{7, \ldots, 26\}$, and q in $\{3, 5, 8\}$, was generated. The instances were named YJS01 through YFJS20.

We also developed a generator of random instances of a more general kind. The generator has two integer parameters: the number n of jobs and the number m of machines. Each job is represented by a dag of one of the six possible types — D2, D3, A2, A3, DA2, and DA3 — indicated in Figure 2. In each dag, all maximal paths have the same length. (Hence, in DA3, for example, the three "parallel" paths in the middle section have the same number of operations.) For each job, one of the six types of dag is chosen at random. Then, the length of the maximal paths through the dag is chosen at random from the set $\{\lceil m/2 \rceil, \ldots, m\}$. The lengths of the sections of the dag (left and middle sections in case of DA and left in case of D and A) are also chosen at random. For each operation v, the size of the set $F(v)$ is chosen at random in $\{\lceil 0.3m \rceil, \ldots, \lceil 0.7m \rceil\}$ and then the elements of $F(v)$ are chosen at random from $\{1, 2, \ldots, m\}$. The processing time of v on one of the machines in $F(v)$ is set to a random number p in $\{1, \ldots, 99\}$. The processing times of v on each of the remaining machines in $F(v)$ is a random number in $\{p, \ldots, \min(3p, 99)\}$. (All random choices use uniform distribution.) A set of 30 instances was generated in the manner just described, with n in $\{4, \ldots, 12\}$ and m in $\{5, \ldots, 10\}$. The instances were named DAFJS01 through DAFJS30 and used in the computational experiments presented next.

To solve the MILPs, we used the IBM ILOG CPLEX 12.1 solver with the following settings: 3600s for time limit, 1 for maximum number of threads, and 2048MB for working memory. All other parameters were left at their default values. The computer used in the experiments has an Intel Xeon E5440 2.83GHz processor.

The computational experiments compare the ÖÖY′ model with our proposed model. In a set of prelimary experiments, CPLEX was not able to find a feasible solution to a few of the instances within the time limit, regardless of the model used. To remedy this,

Instance	Size	EST	ÖÖY′ model		new model	
			mks	CPU(s)	mks	CPU(s)
SFJS01	2, 2, 2	66	66	0.01	66	0.00
SFJS02	2, 2, 2	107	107	0.01	107	0.01
SFJS03	3, 2, 2	255	221	0.02	221	0.05
SFJS04	3, 2, 2	367	355	0.02	355	0.02
SFJS05	3, 2, 2	143	119	0.04	119	0.04
SFJS06	3, 3, 2	360	320	0.03	320	0.01
SFJS07	3, 3, 5	407	397	0.01	397	0.00
SFJS08	3, 3, 4	273	253	0.03	253	0.04
SFJS09	3, 3, 3	230	210	0.02	210	0.01
SFJS10	4, 3, 5	608	516	0.01	516	0.02
MFJS01	5, 3, 6	526	468	0.53	468	0.26
MFJS02	5, 3, 7	540	446	1.20	446	0.87
MFJS03	6, 3, 7	655	466	4.09	466	1.66
MFJS04	7, 3, 7	690	554	126	554	27.43
MFJS05	7, 3, 7	690	514	5.70	514	4.55
MFJS06	8, 3, 7	838	634	1739	634	52.48
MFJS07	8, 4, 7	1130	[859;881] 2.50%	3600	879	1890
MFJS08	9, 4, 8	1129	[764;884] 13.57%	3600	[775;884] 12.33%	3600
MFJS09	11, 4, 8	1343	[845.26;1104] 23.44%	3600	[809.03;1137] 28.84%	3600
MFJS10	12, 4, 8	1559	[951.30;1263] 24.68%	3600	[944.80;1251] 24.48%	3600

Table 1: Solving the Fattahi *et al.* [FMJ07] instances with the ÖÖY′ and the new model. Both MILPs were solved using CPLEX, with initial feasible solution provided by the EST heuristic. The size of an instance is a triple (n, o, m), where n is the number of jobs, o is the the number of operations per job, and m is the number of machines. All operations have integer processing times. The smallest CPU times and gaps are highlighted.

we gave CPLEX an initial feasible solution produced by a constructive heuristic we call EST, for "earliest starting time". (The makespan of the schedule found by EST was also used to set the value of parameter L.)

The EST heuristic produces a permutation $(v_1, v_2, \ldots, v_N)$ of the set V of operations such that $i < j$ whenever $(v_i, v_j) \in A$, and a corresponding sequence $(f_1, f_2, \ldots, f_N)$ of machines. Of course $f_i \in F(v_i)$ for each i, and v_i is to be processed on machine f_i. This pair of sequences defines an admissible selection: just take the set of all ordered pairs (v_i, v_j) in $V \times V$ such that $i < j$ and $f_i = f_j$.

Each iteration of EST starts with sequences $(v_1, \ldots, v_{q-1})$ and $(f_1, \ldots, f_{q-1})$ such that no arc in A enters the set $U := \{v_1, \ldots, v_{q-1}\}$. Let $A_U := A \cap (U \times U)$, and let Y_U be the set of all pairs (v_i, v_j) in $U \times U$ such that $i < j$ and $f_i = f_j$. Let s^* be the tight schedule for $(U, A_U \cup Y_U, p')$, where $p'_{v_i} := p_{v_i, f_i}$. Then the completion time of v_i will be $c_i := s^*_{v_i} + p'_{v_i}$ and each machine k will become available at time

$$\max \{c_i : 1 \leq i < q \text{ and } f_i = k\}$$

(or 0 if there is no i such that $f_i = k$). From this information, the heuristic chooses v_q in $V \setminus U$ and f_q in $F(v_q)$. The idea is to choose a pair (v_q, f_q) whose execution can start the earliest. This rule is, in general, satisfied by several pairs. Experience shows that an additional tie-breaking rule can significantly improve the heuristic. Let $\bar{p}$ be the mean

Instance	Size	EST	ÖÖY′ model		new model	
			mks	CPU(s)	mks	CPU(s)
MK01	10, 5-6, 6	49	40	1621	[39;40] 2.5%	3600
MK02	10, 5-6, 6	41	[23;30] 23.33%	3600	[25;29] 13.79%	3600
MK03	15, 10, 8	204	[63;204] 69.12%	3600	[92;204] 54.90%	3600
MK04	15, 3-9, 8	73	[38;67] 43.28%	3600	[41.09;65] 36.78%	3600
MK05	15, 5-9, 4	186	[58.85;186] 68.36%	3600	[66.2;184] 64.02%	3600
MK06	10, 15, 15	98	[33;98] 66.33%	3600	[37;98] 62.24%	3600
MK07	20, 5, 5	214	[62;174] 64.37%	3600	[61.17;192] 68.14%	3600
MK08	20, 10-14, 10	523	[181.25;523] 65.34%	3600	[181;523] 65.39%	3600
MK09	20, 10-14, 10	336	[140.32;336] 58.24%	3600	[146;336] 56.55%	3600
MK10	20, 10-14, 15	274	[104;274] 62.04%	3600	[119;274] 56.57%	3600
MK11	30, 5-7, 5	698	[158.88;695] 77.14%	3600	[152;698] 78.22%	3600
MK12	30, 5-9, 10	566	[160;524] 69.47%	3600	[180;546] 67.03%	3600
MK13	30, 5-9, 10	500	[153;482] 68.26%	3600	[157;500] 68.60%	3600
MK14	30, 8-11, 15	719	[228.97;719] 68.15%	3600	[232;719] 67.73%	3600
MK15	30, 8-11, 15	443	[154;443] 65.24%	3600	[190;443] 57.11%	3600

Table 2: Solving the Brandimarte [Bra93] instances through the ÖÖY′ and the new model. Both MILPs were solved by CPLEX, with initial feasible solution provided by the EST heuristic. All operations have integer processing times. The smallest gaps are highlighted.

processing time, i.e. $\bar{p}_v := (\sum_{k \in F(v)} p_{v,k})/|F(v)|$ for each operation v. The tie-breaking rule can be stated as follows. If there are several candidate pairs (w, k) for the role of (v_q, f_q), choose a pair that maximizes the largest sum of the form $\bar{p}_w + \bar{p}_{z_1} + \bar{p}_{z_2} + \cdots + \bar{p}_{z_l}$, where $(w, z_1, z_2, \ldots, z_l)$ is a path in (V, A). This heuristic takes time $O(|V||A| + |V|^2|M|)$.

Table 1 shows the results for the instances of Fattahi *et al.* [FMJ07] and Table 2 for the instances of Brandimarte [Bra93]. These two sets of instances contain only path-jobs. Tables 3 and 4 show the results for the two new sets of instances, generated as described above. The Instance column records the names of the instances. The Size column contains the number of jobs, the number of operations per job (or an interval *a*-*b*, when the jobs have between a and b operations), and the number of machines. Column EST shows the makespan of the schedule produced by the heuristic. For each model, the mks column records the optimal makespan or the lower and upper bounds found by CPLEX, followed by the relative gap (the difference between upper and lower bounds, divided by the upper bound). The CPU column records the CPU time, in seconds, taken by CPLEX to solve the MILP (or 3600, if CPLEX did not solve the MILP in one hour). In all cases, the EST heuristic used no more than a hundredth of a second.

In Table 1 we observe that, for most instances, CPLEX ran fastest under the new model. Moreover, CPLEX found an optimal solution for instance MFJS07 with the new model, while it did not find one (within the time limit) with ÖÖY′. On the other hand, for instance MFJS09, CPLEX obtained a solution with better makespan and better lower bound using ÖÖY′ than using the new model. Recall that model ÖÖY′ reduces to the original ÖÖY model when applied to the instances considered in Table 1 and that, moreover, numerical experiments of Özgüven *et al.* [OOY10] show that their model is several orders of magnitude faster than the model introduced in Fattahi *et al.* [FMJ07].

Instance	Size	EST	ÖÖY′ model		new model	
			mks	CPU(s)	mks	CPU(s)
YFJS01	4, 10, 7	1318	773	68.37	773	11.5
YFJS02	4, 10, 7	1243	825	6.03	825	9.88
YFJS03	6, 4, 7	439	347	11.11	347	3.72
YFJS04	7, 4, 7	569	390	22.63	390	7.82
YFJS05	8, 4, 7	566	445	660.79	445	357.55
YFJS06	9, 4, 7	633	[378.90;452] 16.17%	3600	[425.29;449] 5.28%	3600
YFJS07	9, 4, 7	628	[439;460] 4.57%	3600	444	1392
YFJS08	9, 4, 12	531	353	3.26	353	0.67
YFJS09	9, 4, 12	506	[238;242] 1.65%	3600	242	14.03
YFJS10	10, 4, 12	541	399	22.21	399	4.03
YFJS11	10, 5, 10	740	526	1699.64	526	177.43
YFJS12	10, 5, 10	813	[465;541] 14.05%	3600	512	3218.89
YFJS13	10, 5, 10	717	[376;405] 7.16%	3600	405	1624.66
YFJS14	13, 17, 26	2055	[1317;1461] 9.86%	3600	1317	3293.58
YFJS15	13, 17, 26	2296	[1239;1260] 1.67%	3600	[1239;1244] 0.40%	3600
YFJS16	13, 17, 26	2006	[1189;1432] 16.97%	3600	[1200;1245] 3.61%	3600
YFJS17	17, 17, 26	2408	[1133;1832] 38.16%	3600	[1133;2379] 52.37%	3600
YFJS18	17, 17, 26	2082	[1220;1772] 31.15%	3600	[1220;2082] 41.40%	3600
YFJS19	17, 17, 26	2038	[862;1897] 54.56%	3600	[926;1581] 41.43%	3600
YFJS20	17, 17, 26	2369	[705;1686] 58.19%	3600	[968;2312] 58.13%	3600

Table 3: Solving FJS instances that consist of Y-jobs with the ÖÖY′ and the new model. Both MILPs were solved by CPLEX, with initial feasible solution provided by the EST heuristic. All makespans are integer since all processing times are integer. (Curiously, many of the lower bounds are also integer.) The smallest CPU times and gaps are highlighted.

The instances in Table 2 are larger, in terms of total number of operations, and clearly more difficult. Under the new model, CPLEX obtained better bounds for most of them. For the instance MK01, it found an optimal solution in just 60 seconds but could not raise the lower bound to prove the optimality of the solution. Under the ÖÖY′ model, CPLEX found an optimal solution in 1621 seconds.

Table 3 shows that, on instances with Y-jobs, CPLEX performed better with the new model, running significantly faster and obtaining better bounds, except on instances YFJS17 and YFJS18. Most of the instances in Table 4 seem to be more difficult. CPLEX found an optimal solution for instances DAFJS01 and DAFJS02 with the new model, while it could not find one (within the time limit) with ÖÖY′. For instances DAFJS03 and DAFJS04, CPLEX found an optimal solution under the new model considerably faster than it did under ÖÖY′. For the other 26 instances, CPLEX did not find an optimal solution with either model. For only two of these 26 instances, CPLEX achieved better bounds when using the ÖÖY′ model. For eight of the 26 instances, CPLEX ended with the same bounds for both models, and for 16 of the 26 instances, CPLEX achieved better bounds when using the new model.

Here is a summary of our results. For 34 of the 85 instances considered, CPLEX found an optimal solution and proved its optimality when using the new model. Using the ÖÖY′ model, CPLEX did so for 27 of the 85 instances. For all but 7 of the 85 instances, CPLEX produced at least as good a lower bound under the new model as under ÖÖY′. For 27

Instance	Size	EST	ÖÖY′ model		new model	
			mks	CPU(s)	mks	CPU(s)
DAFJS01	4, 5-9, 5	327	[255.0;257] 0.78%	3600	257	78.93
DAFJS02	4, 5-7, 5	382	[270.0;297] 9.09%	3600	289	1271.7
DAFJS03	4, 10-17, 10	710	576	371.39	576	15.8
DAFJS04	4, 9-14, 10	653	606	18.2	606	1.22
DAFJS05	6, 5-13, 5	482	[355.01;411] 13.62%	3600	[347.53;403] 13.76%	3600
DAFJS06	6, 5-13, 5	489	[326;446] 26.91%	3600	[326;435] 25.06%	3600
DAFJS07	6, 7-23, 10	717	[491.11;717] 31.5%	3600	[497;562] 11.57%	3600
DAFJS08	6, 6-23, 10	847	[517;690] 25.07%	3600	[628;631] 0.48%	3600
DAFJS09	8, 4-9, 5	535	[315;497] 36.62%	3600	[315;475] 33.68%	3600
DAFJS10	8, 4-11, 5	629	[336;567] 40.74%	3600	[336;575] 41.57%	3600
DAFJS11	8, 10-23, 10	708	[658;708] 7.06%	3600	[658;708] 7.06%	3600
DAFJS12	8, 9-22, 10	720	[530;720] 26.39%	3600	[530;720] 26.39%	3600
DAFJS13	10, 5-11, 5	766	[252;751] 66.44%	3600	[304;718] 57.66%	3600
DAFJS14	10, 4-10, 5	871	[313;866] 63.86%	3600	[358.95;860] 58.26%	3600
DAFJS15	10, 8-19, 10	818	[497;818] 39.24%	3600	[512;818] 37.41%	3600
DAFJS16	10, 6-20, 10	831	[462;831] 44.4%	3600	[640;819] 21.86%	3600
DAFJS17	12, 4-11, 5	910	[300;910] 67.03%	3600	[300;909] 67.0%	3600
DAFJS18	12, 5-9, 5	951	[322;951] 66.14%	3600	[322;951] 66.14%	3600
DAFJS19	8, 7-13, 7	601	[512;601] 14.81%	3600	[512;592] 13.51%	3600
DAFJS20	10, 6-17, 7	815	[399;815] 51.04%	3600	[434;815] 46.75%	3600
DAFJS21	12, 5-16, 7	965	[504;965] 47.77%	3600	[504;965] 47.77%	3600
DAFJS22	12, 5-17, 7	902	[464;902] 48.56%	3600	[464;902] 48.56%	3600
DAFJS23	8, 6-17, 9	632	[450;605] 25.62%	3600	[450;538] 16.36%	3600
DAFJS24	8, 6-25, 9	674	[476;674] 29.38%	3600	[476;666] 28.53%	3600
DAFJS25	10, 9-19, 9	897	[584;897] 34.89%	3600	[584;897] 34.89%	3600
DAFJS26	10, 8-17, 9	903	[565;903] 37.43%	3600	[565;903] 37.43%	3600
DAFJS27	12, 7-22, 9	981	[503;981] 48.73%	3600	[503;981] 48.73%	3600
DAFJS28	8, 8-15, 10	703	[535;695] 23.02%	3600	[535;671] 20.27%	3600
DAFJS29	8, 7-19, 10	760	[609;753] 19.12%	3600	[609;726] 16.12%	3600
DAFJS30	10, 8-19, 10	657	[401;657] 38.96%	3600	[467;656] 28.81%	3600

Table 4: Solving the more general FJS instances through the ÖÖY′ and the new model. Both MILPs were solved using CPLEX, with initial feasible solution provided by the EST heuristic. All makespans are integer since all processing times are integer. The smallest CPU times and gaps are highlighted.

of the 85 instances, CPLEX was strictly faster under the new model. For 11 of the 85 instances, CPLEX was faster under the ÖÖY′ model, and for 49 instances, under both models, CPLEX ran for one hour without finding an optimal solution. On the instances with Y-jobs and the more general ones, CPLEX performed especially well under the new model. Taking all this into account, we conclude that, in general, CPLEX produced better results with the new model than with ÖÖY′.

5 Conclusion

The FJS problem is a generalization of the JS problem in which there may be several machines, not necessarily identical, capable of processing an operation. In the literature

on the problem, each job consists of a sequence of operations to be processed in a given order, as in the ordinary JS problem. In the present paper, we extended the definition of the FJS problem to allow an arbitrary precedence relation over the set of operations and we presented a new MILP model for the extended problem. We also presented computational experiments indicating that the proposed model is better than that of Özgüven, Özbakır, and Yavuz [OOY10]. Some of our experiments were done on a new set of instances, inspired by a real application. This set can be used as benchmark in future computational experiments on the FJS problem.

For benchmarking purposes, and to allow reproduction of the results presented in this paper, the C/C++ code for the MILP models (using the IBM ILOG CPLEX Concert Technology, version 12.1), as well as the code of the EST heuristic, the code of the two generators, and the four sets of instances used in the experiments are available for download at `http://www.ime.usp.br/~cris/fjs/`.

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