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A MINIMAL PARTIAL DEGREE $\leqq 0'$

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ABSTRACT. We construct, recursively in $\mathbf{0}'$, a minimal element in the upper semilattice, excluding least element, of Turing degrees of partial functions.

We are concerned here with partial Turing degrees in the sense of [**4**]. By a *function* we mean a unary function with domain $\subseteq \omega$, the set of natural numbers, and range $\subseteq \{0, 1\}$. A function f is Turing reducible to a function $g(f \leqq_T g)$ if f can be computed by a Turing machine which has access to the values of g where g is defined. A machine computation is considered divergent if it calls for a value of g where g is undefined. The purpose of this restriction is to ensure the single-valuedness of machine computations (cf. [**4**]). Equivalently, $f \leqq_T g$ if f is in the closure of g and the usual initial functions (successor, constants, and projections) under composition, primitive recursion, and minimalization.

A *partial degree* is an equivalence class under Turing interreducibility. The degree of f is denoted by $\mathbf{f}$. In [**4**] we show that the set $\mathscr{D}$ of degrees of partial functions with the naturally induced partial order is an upper semilattice with least element, the degree $\mathbf{0}$ of the partial recursive functions, and that the set $\mathscr{T}$ of degrees which contain a total function, a function with domain ω, is a proper subset of $\mathscr{D}$. In [**4**] we construct a degree minimal in $\mathscr{D}-\{\mathbf{0}\}$ but leave open the problem of carrying out such a construction effectively in $\mathbf{0}'$, the degree of the halting problem for the partial recursive functions. Below we present a construction which is effective in a somewhat stronger sense.

We assume an effective indexing of Turing machines and let $\Phi_e(x, F)$ denote the functional from $\omega \times \{\text{partial functions}\}$ to ω induced by the eth machine. Hence for a partial function f, $\Phi_e(x, f)$ is the partial function computed by the eth machine given access to the values of f.

Let the (*semi*) *characteristic* function (S_α) C_α of a set $\alpha \subseteq \omega$ be the function whose value is 0 on α and (undefined) 1 elsewhere. We use $\Phi_e(x)$ to denote $\Phi_e(x, S_\omega)$. Observe that if $\alpha \subseteq \beta \subseteq \omega$ then $\Phi_e(x, S_\beta)$ is an extension

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of $\Phi_e(x, S_\alpha)$ for all e. In particular if $\Phi_e(x, S_\alpha)$ is total then $\Phi_e(x, S_\alpha)=\Phi_e(x)$ and hence is recursive.

A *semicharacteristic degree* is a degree containing a semicharacteristic function. Note that if f is not partial recursive then at least one of $\{n:f(n)=0\}$ and $\{n:f(n)=1\}$ is not recursively enumerable (r.e.). Hence if $\mathbf{0}<\boldsymbol{f}$ then $\mathbf{0}<\boldsymbol{S_\alpha}\leqq\boldsymbol{f}$ for some $\alpha\subseteq\omega$ and degrees minimal in $\mathscr{D}-\{0\}$ must be semicharacteristic. Since by preceding observations, the only total, semicharacteristic degree is $\mathbf{0}$ the solutions to the minimal degree and minimal degree $\leqq\mathbf{0}'$ problems for $\mathscr{T}-\{\mathbf{0}\}$, due to Spector [**6**] and Sacks [**2**] respectively, do not carry over to $\mathscr{D}-\{\mathbf{0}\}$.

By an *array* we mean an r.e. by canonical index set of finite sets where the *canonical index* $|\{n_1, \cdots, n_k\}|$ of $\{n_1, \cdots, n_k\}$ is $2^{n_1}+\cdots+2^{n_k}$ and $|\varnothing|=0$. We let A_e denote the array given by the range of $\Phi_e(x)$. The nth *cell* $A_{e,n}$ of A_e, when defined, is the finite set whose canonical index is $\Phi_e(n)$. For $\alpha\subseteq\omega$, let $A_e^\alpha=\{n: A_{e,n}$ is defined and $\subseteq\alpha\}$. Observe for $\alpha,\beta\subseteq\omega$ that $S_\alpha \leqq_T S_\beta$ iff $\alpha=A_e^\beta$ for some e. To see this when $S_\alpha=\Phi_f(x, S_\beta)$ let A_e be the array whose nth cell, when defined, is the smallest by $\subseteq$ finite set γ such that $\Phi_f(n, S_\gamma)$ is defined. Such an array may be recursively enumerated by following the computations of the fth machine given access to S_ω. Conversely, for a given A_e, consider the machine which, for argument n, converges to 0 if $A_{e,n}$ is defined and if given access to some function g extending $S_{A_{e,n}}$ and which diverges otherwise. Hence it is clear that the search for a degree $\leqq\mathbf{0}'$ minimal in $\mathscr{D}-\{\mathbf{0}\}$ is a search for a non-r.e. $\alpha\subseteq\omega$ such that $\boldsymbol{S_\alpha}\leqq\mathbf{0}'$ and for each e either A_e^α is r.e. or $S_\alpha \leqq_T S_{A_e^\alpha}$.

THEOREM. *There is a co-r.e., nonrecursive set $\alpha\subseteq\omega$ such that for any e either A_e^α is r.e. or $S_\alpha \leqq_T S_{A_e^\alpha}$.*

PROOF. We use an e-state type priority construction to enumerate the complement $\bar{\alpha}$ of α. We assume an effective, 1-1 enumeration of all pairs (e_i, n_i) for which A_{e_i,n_i} is defined and a simultaneous enumeration of the r.e. sets $W_0, W_1, \cdots$ where W_e is the domain of $\Phi_e(x)$. The construction is in stages. A set β is said to be fixed at stage t if either $\beta\subseteq\alpha$ or some member of β has been enumerated in $\bar{\alpha}$ by stage t. In order to make A_e^α r.e. we hope to ensure that there is a stage t such that every cell of A_e which appears after stage t is fixed when it appears. In the other case, in order to make $S_\alpha \leqq_T S_{A_e^\alpha}$, we hope to ensure that by following the construction we can for all but finitely many n either decide whether n is in α or find a cell $A_{e,m}$ such that $n\in\alpha$ iff $A_{e,m}\subseteq\alpha$. Simultaneously, we ensure that α is not r.e. by trying with the use of followers to make $\bar{\alpha}$ touch every r.e. set.

At stage t of the construction a number may be acted upon by being permanently *frozen* in α, permanently *removed* from α, *assigned* or *canceled* as a follower of an index, *attached* to $|A_{e_t, n_t}|$, or *associated* in an equivalence relation with other numbers. Only finitely many numbers will be acted upon at any stage. A number not yet acted upon is said to be *free*. A number which has been removed or frozen is said to be *decided*.

If a number m is not free then X_m (possibly $=\{m\}$) denotes the equivalence class or *association* containing m. The *rank* ρX of an association X is the least index to which a member of X was ever assigned as a follower. A number m is said to be *e-covered* if some member of X_m is attached to a canonical index of a cell of A_e. The *state* σX of an association X is $\{e \leqq \rho X$: some member of X is e-covered$\}$.

In order to ensure that $S_\alpha \leqq_T S_{A_e^\alpha}$ when A_e^α is not r.e. we need to make use of the attachments and associations set up during the construction. Hence it is necessary to keep the construction "aligned" at each stage. The construction is said to be *aligned* at stage t if, based on the information available at the end of stage t, the following two conditions hold:

(1) If n is attached to $|A_{e,m}|$ then $n \in \alpha$ iff $A_{e,m} \subseteq \alpha$.

(2) If n and m are associated then $n \in \alpha$ iff $m \in \alpha$.

A number n is said to *impinge* on a number m (at a given point in the construction) if $n \in \alpha$, $m \in \alpha$, and removing n and aligning would force the removal of m.

Stage t of the construction is the application of the following four clauses based on the information available prior to the application of each clause.

Clause 1. Let r be the least (if any) rank $\geqq e_t$ and X the least, by $|X|$, undecided association with $r = \rho X$, $e_t \notin \sigma X$, and no members impinging on a follower $k \in \alpha$ of any index $\leqq r$. If $A_{e_t, n_t} \subseteq \alpha$ and has an undecided member m such that (i) m is free or $r < \rho X_m$, (ii) m does not impinge on a follower $k \in \alpha$ of an index $\leqq r$, and (iii) no $m' \neq m$ is attached to $|A_{e_t, n_t}|$, then (iv) attach m to $|A_{e_t, n_t}|$, (v) make $X \cup X_m$ an association, and (vi) cancel all followers of indices $> r$. Otherwise do nothing under this clause.

Clause 2. Let e be the least (if any) index with a follower $k \in W_e \cap \alpha$. Cancel all followers of indices $> e$, remove k, and align the construction.

Clause 3. Let n be the largest number not free or in some cell A_{e_i, n_i} for $i \leqq t$ and freeze all free $m < n$.

Clause 4. Assign the least free n to the least index e without a follower.

Observe by cancellation in Clauses 1–2 and assignment in Clause 4 that the indices with followers form an initial segment; that an index has at most one follower at a time; and that an association contains at most one follower. Observe by Clause 3 that the nonfree numbers form an initial segment at the end of each stage and, by Clauses 2–4, that the construction is aligned at the end of each stage. Finally, observe that for any n with X_n defined ρX_n is defined unless n is frozen when X_n is defined,

ρX_n never rises and, by Clause 1, if X_n grows then σX_n or σX for some X with $\rho X \leqq \rho X_n$ is enlarged and, in the latter case, ρX_n drops to ρX because X_n grows to $X \cup X_n$. Hence associations eventually stop growing. We call an association X complete at t if X has stopped growing by stage t. Of course, we cannot in general tell when an association is complete. Now the following five lemmas establish the theorem.

LEMMA 1. *α is a co-r.e.*

PROOF. The construction is effective and $\bar{\alpha}$ consists of those numbers which are removed under Clause 2 at some stage of the construction.

LEMMA 2. *Every index gets a permanent follower.*

PROOF. By Clauses 1–2 the follower of 0 assigned at stage 0 cannot be canceled. Suppose all $i \leqq e$ have permanent followers assigned by stage t and if the follower n of e is removed this is done by stage t. Then any association X of eventual rank $\leqq e$ has a member m with X_m defined by stage t. Hence there are only finitely many associations with eventual rank $\leqq e$. Suppose all such associations are complete by stage $t' \geqq t$. Then, by Clauses 1–2, no follower of $e+1$ can be canceled after stage t' and, by Clause 4, $e+1$ gets a permanent follower.

LEMMA 3. *If u, $v \in \alpha$ follow e, f respectively and u impinges on v then $e \leqq f$.*

PROOF. Suppose u, $v \in \alpha$ follow e, f respectively and $f < e$. Then the assignment of v to follow f and any Clause 1 action involving X_v must take place before u is assigned to follow e. Since by Clause 4, u is free prior to its assignment to follow e, any impingement of u on v must be introduced through Clause 1 after u and v are assigned. In the notation of Clause 1 impingement may be introduced through m by (iv) or through X by (v). But if m or a member of X impinges on v and Clause 1 applies then $\rho X = r \leqq f$. In this case u following $e > f$ is canceled.

LEMMA 4. *α is not r.e.*

PROOF. Let n be the permanent follower of e. Then n is not removed by Clause 2 for some $f < e$ else n would be canceled and n is not removed by Clause 2 for some $f > e$ by Lemma 3. Hence n remains in α iff n never appears in W_e. Thus $\alpha \neq W_e$.

From the proof of Lemma 2 we know that we may define t_e as the least stage by which all associations of eventual rank $\leqq e$ are complete and fixed with respect to α. Of course, t_e cannot be found effectively. We say a number n is e-decided if at some stage $t > t_e$ n is frozen, removed, or impinges on some $m \in \alpha$ with $\rho X_m \leqq e$. Observe that once n is e-decided n

is fixed with respect to α since freezing and removal are permanent and impingement of n on m with X_m fixed $\subseteq \alpha$ forces n to remain fixed.

LEMMA 5. *For any e either A_e^α is r.e. or any n free at t_e is eventually e-covered or e-decided.*

PROOF. Let e be given and suppose some n free at t_e is never e-covered nor e-decided. Then there is a least rank $r > e$ and a least, by $|X|$, association X of rank r such that no member of X is ever e-covered or e-decided. Then at any stage $t > t_r$ with $e_t = e$, A_{e_t, n_t} must be ineligible for use in Clause 1 since r, X satisfy the hypotheses of that clause. In this case A_{e_t, n_t} is fixed with respect to containment in α. If $A_{e_t, n_t} \nsubseteq \alpha$ this is clear. If $m' \in A_{e_t, n_t}$ is attached to $|A_{e_t, n_t}|$ and $\rho X_{m'} \leqq r$ then m' is fixed in α and by alignment A_{e_t, n_t} is fixed $\subseteq \alpha$. If every unfrozen $m \in A_{e_t, n_t}$ impinges on a follower of some index $\leqq r$ or has $\rho X_m \leqq r$, then every such m is fixed in α. Hence, modulo $\{n_t : t \leqq t_r\}$, A_e^α is r.e.

The theorem now follows immediately for, in case A_e^α is not r.e. S_α may be reduced to $S_{A_e^\alpha}$ by following the construction after t_e and waiting for those n free at t_e to be e-decided or e-covered. ∎

The following questions concerning minimal partial degrees suggest themselves. One would like to know which total degrees have minimal predecessors in the partial degrees (and in particular whether all total degrees do) and which total degrees are candidates for C_α when S_α is minimal (especially when $\bar{\alpha}$ is r.e.). In this context it would be nice to have a simpler characterization of sets α with S_α minimal. It is then natural to ask what sorts of co-r.e. sets fit this characterization. (It can be shown that $\bar{\alpha}$ cannot be maximal and $\bar{\alpha}$ of the theorem is clearly not simple.) Finally one would like to know whether a minimal partial degree can have arbitrary jump (cf. **[1]**) and whether it can be made to satisfy various incomparability conditions (cf. **[3]**, **[5]**).

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