A minimum problem with free boundary for a degenerate quasilinear operator

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Abstract

In this paper we prove $C^{1,\alpha}$ regularity (near flat points) of the free boundary $\partial\{u>0\}\cap\Omega$ in the Alt-Caffarelli type minimum problem for the p-Laplace operator:

$$J(u)=\int_\Omega\left(|\nabla u|^p+\lambda^p\chi_{\{u>0\}}\right)dx\to\min\qquad(1<p<\infty.)$$

1 Introduction

For a given domain Ω in $\mathbf{R}^n$ consider the problem of minimizing the energy functional

$$J(u)=\int_\Omega\left(|\nabla u|^p+\lambda^p\chi_{\{u>0\}}\right)dx\qquad(1<p<\infty)\tag{1.1}$$

among all functions $u\in W^{1,p}(\Omega)$ with $u-u_0\in W_0^{1,p}(\Omega)$ for a prescribed $u_0\geq 0$. We assume that λ is a positive constant. Then the minimizer u satisfies (in a certain weak sense, see Theorem 2.1) the following overdetermined system

$$\Delta_p u=0\quad\text{in }\{u>0\},\qquad u=0,\ |\nabla u|=c\quad\text{on }\partial\{u>0\}\cap\Omega\tag{1.2}$$

with $c=\lambda/(p-1)^{1/p}$. Here $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. We are interested in regularity properties of the so-called *free boundary*

$$\Gamma=\partial\{u>0\}\cap\Omega.$$

Problems of this kind, known as *Bernoulli-type problems*, appear for instance in two dimensional flows [5], heat flows [1], electrochemical machining [13], etc. Also, the same free boundary condition appears under simplifying assumptions in the limit of high activation energy in combustion theory, see e.g. [4].

For $p=2$ the problem was studied in the by now classical paper of Alt and Caffarelli [2]. Our objective in this paper is to prove the regularity of

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the free boundary for any $1 < p < \infty$. The difficulty of the problem and its main difference from [2] is that the governing operator, the p-Laplacian, is not uniformly elliptic (degenerate for $p > 2$ and singular for $1 < p < 2$.) The case of uniformly elliptic quasilinear equations has been treated by Alt, Caffarelli and Friedman [3]. The main result of [3], similar to the one of [2], states that near *flat* free boundary points the free boundary is $C^{1,\alpha}$ regular. On the other hand, the regularity of the free boundary in (1.2) will imply nondegeneracy of $|\nabla u|$ near the free boundary, which will make the p-Laplacian uniformly elliptic. Hoverer, it appears not to be easy to establish the nondegeneracy up to the free boundary without the regularity. We overcome this difficulty by proving both properties, nondegeneracy of the gradient and regularity of the free boundary, simultaneously.

Finally, we have to mention that under suitable convexity assumptions, one can establish the existence and uniqueness of classical solutions to the free boundary problem (1.2) by using a Perron-type method, see [10], [11]. For general configurations, certain weak solutions can be obtained in the limit of a singular perturbation problem related to combustion theory

$$\Delta_p u^\varepsilon = \frac{1}{\varepsilon}\,\beta\left(\frac{u^\varepsilon}{\varepsilon}\right)$$

as $\varepsilon \to 0$, where $\beta \geq 0$ and $\operatorname{supp}\beta = [0,1]$, see [6].

The structure of the paper is as follows. In Sections 3 and 4 we establish the uniform Lipschitz continuity and a certain nondegeneracy of the minimizers at any free boundary point. As a consequence, we obtain in Section 5 that the free boundary has locally finite perimeter. Section 6 contains the key flatness-nondegeneracy theorems (Theorems 6.3 and 6.4), which together with Sections 7– 8 imply our main result, Theorem 9.1, that the free boundary is $C^{1,\alpha}$ regular near flat free boundary points.

2 Preliminaries

The existence of minimizers of (1.1) as well as their relation with the free boundary problem (1.2) can be established precisely as in [2].

Theorem 2.1 *If $J(u_0) < \infty$, then there exists an* absolute minimizer *u of J in the class $\mathcal{K} = \{v \in W^{1,p}(\Omega) : v - u_0 \in W_0^{1,p}(\Omega)\}$, i.e. a function $u \in \mathcal{K}$ such that*

$$J(u) \leq J(v) \quad \text{for any } v \in \mathcal{K}.$$

Any absolute minimizer u is nonnegative, $\Delta_p u = 0$ in $\{u > 0\}$ and, moreover, it satisfies the free boundary condition (1.2) in the following very weak sense:

$$\lim_{\varepsilon \searrow 0} \int_{\partial\{u>\varepsilon\}\cap\Omega} ((p-1)|\nabla u|^p - \lambda^p)\eta \cdot \nu = 0$$

for any $\eta \in W_0^{1,p}(\Omega)^n$ and where ν is the outward normal. □

Note that every absolute minimizer u is p-subharmonic in Ω, since $J(u) \leq J(u - \varepsilon\eta)$ and $\chi_{\{u-\varepsilon\eta>0\}} \leq \chi_{\{u>0\}}$ for any nonnegative $\eta \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. In particular, $\Delta_p u$ is a nonnegative Radon measure with support in $\Omega \cap \partial\{u > 0\}$. Also observe that, if $u_0 \geq 0$ is uniformly bounded in Ω, then

$$0 \leq \inf_\Omega u \leq \sup_\Omega u \leq \sup_\Omega u_0$$

(here inf and sup are understood in the essential sense.)

Most of the results in this paper are proved also for so-called *local minimizers* $u \in \mathcal{K}$ such that $J(u) \leq J(v)$ for any $v \in \mathcal{K}$ with

$$\|\nabla u - \nabla v\|_{L^p(\Omega)} + \|\chi_{\{u>0\}} - \chi_{\{v>0\}}\|_{L^1(\Omega)} < \varepsilon$$

for some $\varepsilon > 0$.

Next, we remark that if u is a (local) minimizer of J in Ω, then the rescaling of u around x_0 by a factor of r

$$u_r(x) := \frac{1}{r}\, u(rx + x_0)$$

is a (local) minimizer in $\Omega_r := \{(x - x_0)/r : x \in \Omega\}$. Rescalings are especially useful since, as we prove in Section 3, the (local) minimizers are uniformly Lipschitz continuous and therefore we can extract a subsequence u_r converging as $r \to 0$ to a function u_0 in $\mathbf{R}^n$. The latter process is referred to as *blow-up*.

Finally, throughout the paper, without loss of generality, we assume that

$$\lambda = \lambda_p := (p-1)^{1/p}$$

in the functional (1.1) so that we have $c = 1$ in (1.2).

3 Lipschitz continuity

In this section we establish the Lipschitz continuity of minimizers.

Let u be an absolute (local) minimizer of J in Ω. Hence u is p-subharmonic and we can assume that u is upper semicontinuous. For any (small) $B = B_r(y) \subset\subset \Omega$ denote by $v = v_B$ the solution of the Dirichlet problem

$$\Delta_p v = 0 \text{ in } B, \quad v - u \in W_0^{1,p}(B).$$

From the minimality of u in B we have

$$\int_B \big(|\nabla u|^p + \lambda^p \chi_{\{u>0\}}\big) \leq \int_B (|\nabla v|^p + \lambda^p),$$

or

$$\int_B (|\nabla u|^p - |\nabla v|^p) \leq \lambda^p \int_B \chi_{\{u=0\}}.$$

Set now

$$u^s(x) = s\, u(x) + (1-s) v(x), \quad 0 \leq s \leq 1.$$

Clearly $u^0 = v$ and $u^1 = u$. We thus obtain

$$\begin{aligned}\int_B (|\nabla u|^p - |\nabla v|^p) &= p\int_0^1 ds \int_B |\nabla u^s|^{p-2}\nabla u^s \cdot \nabla(u-v)\\ &= p\int_0^1 ds \int_B (|\nabla u^s|^{p-2}\nabla u^s - |\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)\\ &= p\int_0^1 \frac{ds}{s}\int_B (|\nabla u^s|^{p-2}\nabla u^s - |\nabla v|^{p-2}\nabla v)\cdot\nabla(u^s-v),\end{aligned}$$

where in the second step we used that $\int_B |\nabla v|^{p-2}\nabla v\cdot\nabla(u-v) = 0$.

Next, we apply the well-known inequality

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \geq \gamma \begin{cases} |\xi-\eta|^2(|\xi|+|\eta|)^{p-2}, & 1<p\leq 2\\ |\xi-\eta|^p & p\geq 2\end{cases}$$

for any nonzero $\xi,\eta\in\mathbf{R}^n$ and a constant $\gamma=\gamma(n,p)>0$.

For $p\geq 2$ we obtain

$$\int_B (|\nabla u|^p - |\nabla v|^p) \geq \gamma p\int_0^1 \frac{ds}{s}\int_B |\nabla(u^s-v)|^p = \gamma p\int_0^1 s^{p-1}ds\int_B |\nabla(u-v)|^p$$

and consequently

$$\int_B |\nabla(u-v)|^p \leq C\int_B \chi_{\{u=0\}}. \tag{3.1}$$

In the case $1<p\leq 2$ we have

$$\begin{aligned}\int_B (|\nabla u|^p - |\nabla v|^p) &\geq \gamma p\int_0^1 \frac{ds}{s}\int_B |\nabla(u^s-v)|^2(|\nabla u^s|+|\nabla v|)^{p-2}\\ &\geq c\int_0^1 s\,ds\int_B |\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p-2}\\ &\geq c\int_B |\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p-2}.\end{aligned}$$

On the other hand, using the Hölder inequality, we have

$$\int_B |\nabla(u-v)|^p \leq \left(\int_B |\nabla(u-v)|^2(|\nabla u|+|\nabla v|)^{p-2}\right)^{p/2}\left(\int_B (|\nabla u|+|\nabla v|)^p\right)^{1-p/2},$$

hence

$$\int_B |\nabla(u-v)|^p \leq C\left(\int_B \chi_{\{u=0\}}\right)^{p/2}\left(\int_B |\nabla u|^p\right)^{1-p/2}, \tag{3.2}$$

where $C=C(n,p)>0$ and we have used that $\int_B |\nabla v|^p \leq \int_B |\nabla u|^p$.

We are now ready to prove the first result on the regularity of minimizers.

Lemma 3.1 *Let u be a bounded absolute minimizer in B_1. Then u is C^α regular in $B_{7/8}$ for some $\alpha=\alpha(n,p)\in(0,1)$ and*

$$\|u\|_{C^\alpha(B_{7/8})} \leq C(n,p,\|u\|_{L^\infty(B_1)}).$$

Proof. Let $M = \|u\|_{L^\infty(B_1)}$ and $B = B_r(y)$ with $y \in B_{7/8}$ and $0 < r \le 1/16$. Since u is a p-subsolution, a Caccioppoli type inequality (see [9], Lemma 3.27) implies that

$$\int_B |\nabla u|^p \le \frac{C}{r^p} \int_{2B} u^p \le C\, M^p\, r^{n-p},$$

where $\alpha B = B_{\alpha r}(y)$. On the other hand, if $v = v_B$ is the p-harmonic function with $u - v \in W_0^{1,p}(B)$, we have

$$\sup_{\frac{1}{2}B} |\nabla v| \le \left(\frac{C}{r^n} \int_B |\nabla v|^p \right)^{1/p} \le \frac{C\, M}{r}$$

Now, let us take a small $\varepsilon > 0$, to be specified below, and $0 < r \le r_0(\varepsilon)$ such that $r^\varepsilon \le 1/2$. Then

$$\begin{aligned} \|\nabla u\|_{L^p(B_{r^{1+\varepsilon}}(y))} &\le \|\nabla(u-v)\|_{L^p(B_{r^{1+\varepsilon}}(y))} + \|\nabla v\|_{L^p(B_{r^{1+\varepsilon}}(y))} \\ &\le \|\nabla(u-v)\|_{L^p(B_r(y))} + C r^{(1+\varepsilon)(n/p)} \|\nabla v\|_{L^\infty(B_{r/2}(y))} \\ &\le C(M,n,p) \begin{cases} r^{n/p-(1-p/2)} + r^{(1+\varepsilon)(n/p)-1}, & \text{for } 1 < p \le 2 \\ r^{n/p} + r^{(1+\varepsilon)(n/p)-1}, & \text{for } p \ge 2 \end{cases}. \end{aligned}$$

Thus, for $\rho = r^{1+\varepsilon}$, we have

$$\|\nabla u\|_{L^p(B_\rho(y))} \le C(M,n,p) \rho^{n/p-(1-\alpha)}, \tag{3.3}$$

where $\alpha = \alpha(n,p) > 0$, if we take $\varepsilon = \varepsilon(n,p) > 0$ sufficiently small. Applying Morrey's theorem, see e.g. [14], Theorem 1.53, we conclude the proof of the lemma. □

The next lemma is the main step in proving the Lipschitz regularity of minimizers.

Lemma 3.2 *Let u be a bounded absolute minimizer in B_1 and $u(0) = 0$. Then there exists a constant $C = C(n,p) > 0$ such that*

$$\|u\|_{L^\infty(B_{1/4})} \le C.$$

Proof. Indeed, assume the contrary. Then there exists a sequence of bounded absolute minimizers u_k in B_1, $k = 1, 2, \ldots$, such that

$$\max_{\overline{B}_{1/4}} u_k(x) > k.$$

Set

$$d_k(x) = \operatorname{dist}(x, \{u_k = 0\}) \quad \text{in } B_1$$

and define

$$\mathcal{O}_k = \{x \in B_1 : d_k(x) \le (1 - |x|)/3\}.$$

Observe that $\overline{B}_{1/4} \subset \mathcal{O}_k$. In particular

$$m_k := \sup_{\mathcal{O}_k} (1 - |x|)\, u_k(x) \ge \frac{3}{4} \max_{\overline{B}_{1/4}} u_k(x) > \frac{3}{4}\, k.$$

Since $u_k(x)$ is bounded (for fixed k), we will have $(1-|x|)\,u_k(x)\to 0$ as $|x|\to 1$, and therefore m_k will be attained at some point $x_k\in\mathcal{O}_k$:

$$(1-|x_k|)\,u_k(x_k)=\max_{\mathcal{O}_k}\,(1-|x|)\,u_k(x). \tag{3.4}$$

Clearly,

$$u_k(x_k)=\frac{m_k}{1-|x_k|}\geq m_k>\frac{3}{4}\,k.$$

Since $x_k\in\mathcal{O}_k$, by the definition we will have

$$d_k:=d_k(x_k)\leq(1-|x_k|)/3. \tag{3.5}$$

Let now $y_k\in\partial\{u_k>0\}\cap B_1$ be such that

$$|y_k-x_k|=d_k. \tag{3.6}$$

The two inclusions

$$B_{2d_k}(y_k)\subset B_1\qquad\text{and}\qquad B_{d_k/2}(y_k)\subset\mathcal{O}_k,$$

both follow from (3.5)-(3.6). In particular, for $z\in B_{d_k/2}(y_k)$

$$(1-|z|)\geq(1-|x_k|)-|x_k-z|\geq(1-|x_k|)-(3/2)d_k\geq(1-|x_k|)/2.$$

This, in conjunction with (3.4), implies that

$$\max_{\overline{B}_{d_k/2}(y_k)}u_k\leq 2u_k(x_k).$$

Next, since $B_{d_k}(x_k)\subset\{u_k>0\}$, u_k satisfies $\Delta_p u_k=0$ in $B_{d_k}(x_k)$. By the Harnack inequality for p-harmonic functions there is a constant $c=c(n,p)>0$ such that

$$\min_{\overline{B}_{3d_k/4}(x_k)}u_k\geq cu_k(x_k).$$

In particular,

$$\max_{\overline{B}_{d_k/4}(y_k)}u_k\geq cu_k(x_k).$$

Consider now

$$w_k(x)=\frac{u_k(y_k+(d_k/2)x)}{u_k(x_k)}.$$

From the properties of u_k above, we obtain

$$\max_{\overline{B}_1}w_k\leq 2,\quad \max_{\overline{B}_{1/2}}w_k\geq c>0,\quad w_k(0)=0. \tag{3.7}$$

We will also have that w_k is an absolute minimizer of

$$J_k(w)=\int|\nabla w|^p+\lambda_k^p\chi_{\{w>0\}},\quad \lambda_k^p=\frac{d_k\lambda^p}{2u_k(x_k)}\to 0,$$

in B_1. Let now v_k be such that $v_k - w_k \in W_0^{1,p}(B_{3/4})$ and $\Delta_p v_k = 0$ in $B_{3/4}$. From the minimality of w_k,

$$\int_{B_{3/4}} \left(|\nabla w_k|^p + \lambda_k^p \chi_{\{w_k>0\}}\right) \le \int_{B_{3/4}} (|\nabla v_k|^p + \lambda_k^p). \tag{3.8}$$

Arguing as in the proof of (3.1)–(3.2) and Lemma 3.1, we obtain that

$$\int_{B_{3/4}} |\nabla(w_k - v_k)|^p \le C(\lambda_k^p) \to 0 \tag{3.9}$$

and that w_k and v_k are uniformly C^α in $B_{5/8}$. Thus we can extract subsequences (still denoted by w_k and v_k) such that $w_k \to w_0$ and $v_k \to v_0$ in uniformly on $B_{5/8}$. Observe that $\Delta_p v_0 = 0$ in $B_{5/8}$ and that (3.9) implies that $w_0 \equiv v_0 + c$. Hence $\Delta_p w_0 = 0$ in $B_{5/8}$. By the strong minimum principle $w_0 = 0$ in $B_{5/8}$, since $w_0 \ge 0$ and $w_0(0) = 0$. On the other hand, (3.7) implies

$$\max_{\overline{B}_{1/2}} w_0 \ge c > 0,$$

a contradiction.

The lemma is proved. □

Theorem 3.3 *If u is a local minimizer in Ω, then $u \in \mathrm{Lip}\,(\Omega)$. Moreover, for every $K \subset\subset \Omega$ such that $K \cap \partial\{u > 0\} \neq \emptyset$ there exist a constant $C = C(n, p, \mathrm{dist}\,(K, \partial\Omega)) > 0$ such that*

$$\|\nabla u\|_{L^\infty(K)} \le C.$$

Proof. The statement follows easily from Lemma 3.2. We refer to the proof of Theorem 2.3 in [3] for more details. See also the proof of Theorem 2.1 in [6]. □

4 Nondegeneracy

As a simple corollary of Theorem 3.3 we obtain the following statement.

Lemma 4.1 *For every $K \subset\subset \Omega$ there exists a constant $C = C(n, p, \mathrm{dist}\,(K, \partial\Omega)) > 0$ such that for any absolute (local) minimizer u*

$$\frac{1}{r} \fint_{\partial B_r} u > C \quad \textit{implies} \quad u > 0 \ \textit{in}\ B_r$$

for any (small) ball $B_r \subset K$.

Proof. Indeed, otherwise we will have that $B_r \subset K$ contains a free boundary point and therefore, by Theorem 3.3, $u \le Cr$ on ∂B_r, a contradiction. □

Next we prove a key nondegeneracy lemma.

Lemma 4.2 *For any $\gamma > p-1$ and $0 < \kappa < 1$ there exists a constant $c = c(\kappa, \gamma, n, p) > 0$ such that for every absolute (local) minimizer u and for any (small) ball $B_r \subset \Omega$*

$$\frac{1}{r}\left(\fint_{B_r} u^\gamma\right)^{1/\gamma} < c \quad \textit{implies} \quad u = 0 \textit{ in } B_{\kappa r}.$$

Proof. Without loss of generality we may consider the case $r = 1$. Set

$$\varepsilon = \frac{1}{\sqrt{\kappa}} \sup_{B_{\sqrt{\kappa}}} u.$$

By the Harnack inequality for p-subharmonic functions (see Theorem 1.3 in [15])

$$\varepsilon \le C \left(\fint_{B_1} u^\gamma\right)^{1/\gamma}.$$

Let $\phi(x) = \phi_\kappa(|x|)$ be the solution of

$$\Delta_p \phi = 0 \text{ in } B_{\sqrt{\kappa}} \setminus B_\kappa, \quad \phi = 0 \text{ on } \partial B_\kappa, \quad \phi = 1 \text{ on } \partial B_{\sqrt{\kappa}}$$

and put $\phi = 0$ on B_κ. Set $v = \varepsilon\sqrt{\kappa}\phi$ in $B_{\sqrt{\kappa}}$. Then $v \ge u$ on $\partial B_{\sqrt{\kappa}}$ and $w = \min(u, v)$ is an admissible function. Therefore $J(u) \le J(w)$, or equivalently

$$\int_{B_{\sqrt{\kappa}}} \left(|\nabla u|^p + \lambda^p \chi_{\{u>0\}}\right) \le \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} \left(|\nabla w|^p + \lambda^p \chi_{\{w>0\}}\right).$$

Hence

$$\begin{aligned}
\int_{B_\kappa} \left(|\nabla u|^p + \lambda^p \chi_{\{u>0\}}\right) &\le \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} (|\nabla w|^p - |\nabla u|^p) \\
&\le p \int_{B_{\sqrt{\kappa}} \setminus B_\kappa} |\nabla w|^{p-2} \nabla w \cdot \nabla(w - u) \\
&= -p \int_{\partial B_k} |\nabla w|^{p-2} (w - u) \nabla w \cdot \nu \\
&= p \int_{\partial B_\kappa} u |\nabla v|^{p-2} \nabla v \cdot \nu.
\end{aligned}$$

Since also $|\nabla v| \le C\varepsilon$ on ∂B_κ we find that

$$\int_{B_\kappa} \left(|\nabla u|^p + \lambda^p \chi_{\{u>0\}}\right) \le C\varepsilon^{p-1} \int_{\partial B_\kappa} u.$$

On the other hand

$$\begin{aligned}
\int_{\partial B_\kappa} u &\le C(n, \kappa) \left(\int_{B_\kappa} u + \int_{B_\kappa} |\nabla u|\right) \\
&\le C(n, \kappa, p) \left(\int_{B_\kappa} \varepsilon\, \lambda^p \chi_{\{u>0\}} + \int_{B_\kappa} \left(|\nabla u|^p + \lambda^p \chi_{\{u>0\}}\right)\right) \\
&\le C(n, \kappa, p)(1 + \varepsilon) \int_{B_\kappa} \left(|\nabla u|^p + \lambda^p \chi_{\{u>0\}}\right).
\end{aligned}$$

Therefore, if ε is small enough, we obtain $u = 0$ in B_κ. □

Corollary 4.3 *For any $K \subset\subset \Omega$ there exist constants c, $C > 0$ such that if $B_r(x) \subset K \cap \{u > 0\}$ touches $\partial\{u > 0\}$ then*

$$cr \le u(x) \le Cr.$$

□

Theorem 4.4 *For any $K \subset\subset \Omega$ there exist a constant $c = c(n, p, K, \Omega)$, $0 < c < 1$ such that for any absolute (local) minimizer u and for any (small) ball $B_r = B_r(x) \subset K$ with $x \in \partial\{u > 0\}$,*

$$c < \frac{\mathcal{L}^n\left(B_r \cap \{u > 0\}\right)}{\mathcal{L}^n(B_r)} < 1 - c.$$

Proof. By Lemma 4.2 there exists $y \in B_{r/2}$ such that $u(y) \ge cr > 0$. By Lipschitz continuity, $u > 0$ in $B_{\kappa r}(y)$, for a small $\kappa > 0$, and thus the estimate from below follows.

To prove the estimate from above, it is enough to consider the case $r = 1$. Assume the contrary. Then there exists a sequence of absolute minimizers u_k in $B_1(0)$, such that $0 \in \partial\{u_k > 0\}$ and

$$\mathcal{L}^n(\{u_k = 0\}) =: \varepsilon_k \to 0.$$

Let v_k be such that $v_k - u_k \in W_0^{1,p}(B_{1/2})$ and $\Delta_p v_k = 0$ in $B_{1/2}$. Arguing as in the proof (3.1)– (3.2), we obtain that

$$\int_{B_{1/2}} |\nabla v_k - \nabla u_k|^p \le C(\varepsilon_k) \to 0.$$

Since u_k and v_k are uniformly Lipschitz continuous in $\overline{B}_{1/4}$ we may assume that $u_k \to u_0$ and $v_k \to v_0$ uniformly in $B_{1/4}$. Observe that $\Delta_p v_0 = 0$ and that the estimate above implies that $u_0 = v_0 + c$. Hence $\Delta_p u_0 = 0$ in $B_{1/4}$ and from the strong minimum principle it follows that $u_0 \equiv 0$ in $B_{1/4}$, since $u_0 \ge 0$ and $u_0(0) = 0$. On the other hand we know

$$\left(\fint_{B_{1/4}} u_k^\gamma\right)^{1/\gamma} \ge c > 0, \quad \text{for } \gamma > p - 1,$$

which implies a similar inequality for u_0, a contradiction.

The theorem is proved. □

Remark 4.5 Theorem 4.4 implies that the free boundary $\partial\{u > 0\}$ has Lebesgue measure zero for every local minimizer. Moreover, it implies that for every $K \subset\subset \Omega$, the intersection $\partial\{u > 0\} \cap K$ has Hausdorff dimension less than n. In fact, to prove these statements, it is enough to use only the left-hand side estimate in Theorem 4.4.

5 The measure $\Lambda = \Delta_p u$

The main objective of this section is to show that for any absolute (local) minimizer u the free boundary is locally of finite perimeter. For that purpose, set

$$\Lambda = \Delta_p u.$$

Then Λ is a nonnegative Radon measure.

Theorem 5.1 *For any $K \subset\subset \Omega$ there exist constants c, $C > 0$ such that for any (local) minimizer u*

$$cr^{n-1} \le \int_{B_r} d\Lambda \le Cr^{n-1}$$

for any (small) ball $B_r = B_r(x) \subset K$ with $x \in \partial\{u > 0\}$.

Proof. Let $\zeta \in C^\infty(\Omega)$, $\zeta \ge 0$, be a test function. Then

$$\int_{B_r} \zeta d\Lambda = -\int_{B_r} |\nabla u|^{p-2}\nabla u \cdot \nabla\zeta.$$

Approximating χ_{B_r} by suitable test functions ζ we get that (for almost all $r > 0$)

$$\int_{B_r} d\Lambda = \int_{\partial B_r} |\nabla u|^{p-2}\nabla u \cdot \nu \le Cr^{n-1},$$

where in the last step we used that u is Lipschitz continuous.

To prove the estimate from below, is enough to consider the case $r = 1$. Assume the contrary. Thus there exists a sequence of absolute minimizers u_k in the unit ball $B_1(0)$ such that $0 \in \partial\{u_k > 0\}$ and for the measures $\Lambda_k = \Delta_p u_k$

$$\int_{B_1} d\Lambda_k =: \varepsilon_k \to 0.$$

Since the functions u_k are uniformly Lipschitz continuous, we may assume that $u_k \to u_0$ uniformly on $B_{1/2}$, where u_0 is Lipschitz continuous as well. Consider then the uniformly bounded sequence $g_k = |\nabla u_k|^{p-2}\nabla u_k$. We may extract a subsequence (still denoted by g_k) such that $g_k \to g_0$ star-weakly in $B_{1/2}$. We claim that $g_0 = |\nabla u_0|^{p-2}\nabla u_0$. Indeed, if $B_\rho = B_\rho(y) \subset \{u_0 > 0\}$ then one can extract a subsequence of u_k locally converging to u_0 in $C^{1,\alpha}(B_\rho)$. Hence $g_0 = |\nabla u_0|^{p-2}\nabla u_0$ in B_ρ. Next, if $B_\rho \subset \{u_0 = 0\}$, $u_k = 0$ in $B_{\rho(1-\delta)}$ for sufficiently large $k \ge k(\delta)$; otherwise we will have $\int_{B_\rho} u_0^\gamma \ge c(\delta)$ for $\gamma > p - 1$. Thus, $g_0 = 0 = |\nabla u_0|^{p-2}\nabla u_0$ in B_ρ in this case as well. Finally, we show that $\partial\{u_0 > 0\} \cap B_{1/2}$ has vanishing Lebesgue measure. Indeed, every point $x_0 \in \partial\{u_0 > 0\} \cap B_{1/2}$ is a limit of a sequence $x_k \in \partial\{u_k > 0\} \cap B_{1/2}$. Using this we can prove that u_0 satisfies the nondegeneracy condition

$$\left(\int_{B_r(x_0)} u_0^\gamma\right)^{1/\gamma} \ge cr^n$$

for any ball $B_r(x_0) \subset B_{1/2}$. Along with the Lipschitz continuity this is enough to prove that $\mathcal{L}^n\left(B_r(x_0) \cap \{u_0 > 0\}\right) \geq c\,\mathcal{L}^n\left(B_r(x_0)\right)$ for $c > 0$. This implies that $\mathcal{L}^n(\partial\{u_0 > 0\}) = 0$, see Remark 4.5.

Now, recall that $|\nabla u_k|^{p-2}\nabla u_k$ converges star-weakly to $|\nabla u_0|^{p-2}\nabla u_0$ in $B_{1/2}$. Hence for every $\zeta \in C_0^\infty(B_{1/2})$, $\zeta \geq 0$, one has

$$\int_{B_{1/2}} |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\zeta = -\lim_{k\to\infty} \int_{B_{1/2}} |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\zeta$$

or

$$\int_{B_{1/2}} \zeta d\Lambda_0 = \lim_{k\to\infty} \int_{B_{1/2}} \zeta d\Lambda_k \leq \|\zeta\|_{L^\infty(B_{1/2})} \lim_{k\to\infty} \varepsilon_k = 0$$

Thus, $\Lambda_0 = 0$ in $B_{1/2}$, which means that u_0 is p-harmonic. From the facts $u_0 \geq 0$, $u_0(0) = 0$, and the strong minimum principle, we infer $u_0 = 0$ in $B_{1/2}$. On the other hand, since $0 \in \partial\{u_k > 0\}$, by nondegeneracy we have $\left(\fint_{B_{1/4}} u_k^\gamma\right)^{1/\gamma} \geq c > 0$ and $\gamma > p - 1$, and therefore a similar inequality holds for u_0. Hence, we have reached a contradiction.

The theorem is proved. □

Theorem 5.2 *Let u be a local minimizer in Ω. Then*

(i) $\mathcal{H}^{n-1}(K \cap \partial\{u > 0\}) < \infty$ *for every* $K \subset\subset \Omega$.

(ii) *There is a Borel function q_u such that*

$$\Delta_p u = q_u \mathcal{H}^{n-1} \lfloor \partial\{u > 0\},$$

that is for every $\zeta \in C_0^\infty(\Omega)$

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\zeta = \int_{\Omega\cap\partial\{u>0\}} \zeta\, q_u d\mathcal{H}^{n-1}.$$

(iii) *For any $K \subset\subset \Omega$ there exist constants c, $C > 0$ such that*

$$c \leq q_u(x) \leq C, \quad cr^{n-1} \leq \mathcal{H}^{n-1}(B_r(x) \cap \partial\{u > 0\}) \leq Cr^{n-1}$$

for every ball $B_r(x) \subset K$ with $x \in \partial\{u > 0\}$.

Proof. This follows easily from Theorem 5.1. For more details see the proof of Theorem 4.5 in [2]. □

From (i) in Theorem 5.2 it follows that the set $A = \Omega \cap \{u > 0\}$ has finite perimeter locally in Ω, see [7], Chapter 4, in the sense that

$$\mu_u = -\nabla\chi_A$$

is a Borel measure and the total variation $|\mu_u|$ is a Radon measure. We define the reduced boundary of A by

$$\partial_{\text{red}} A = \{x \in \Omega : |\nu_u(x)| = 1\},$$

where $\nu_u(x)$ is the unique unit vector with

$$\int_{B_r(x)} \left|\chi_A - \chi_{\{y:(y-x)\cdot\nu_u(x)<0\}}\right| = o(r^n),$$

if such vector exists, and $\nu_u(x) = 0$ otherwise. In what follows, we will use some results about sets of finite perimeter, mainly from [7], Chapter 4, such as

$$\mu_u = \nu_u \mathcal{H}^{n-1} \lfloor \partial_{\mathrm{red}}\{u > 0\}.$$

To proceed we will need the some properties of so-called blow-up limits. Let u be a local minimizer in Ω, $K \subset\subset \Omega$ and $B_{\rho_k}(x_k) \subset K$ be a sequence of balls with $\rho_k \to 0$, $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$. Consider then the blow-up sequence

$$u_k(x) = \frac{1}{\rho_k} u(x_k + \rho_k x). \tag{5.1}$$

Since u_k are uniformly Lipschitz continuous, for a subsequence,

$$u_k \to u_0 \quad \text{in } C^\alpha_{\mathrm{loc}}(\mathbf{R}^n) \text{ for every } 0 < \alpha < 1, \tag{5.2}$$

$$\nabla u_k \to \nabla u_0 \quad \text{star-weakly in } L^\infty_{\mathrm{loc}}(\mathbf{R}^n), \tag{5.3}$$

$$\partial\{u_k > 0\} \to \partial\{u_0 > 0\} \quad \text{locally in the Hausdorff distance}, \tag{5.4}$$

$$\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}} \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^n). \tag{5.5}$$

Moreover, if $x_k \in \partial\{u_k > 0\}$ then $x_0 \in \partial\{u_0 > 0\}$. For the proof we refer to Section 4.7 in [2] and pp. 19–20 in [3]; see also our proof of Theorem 5.1.

The following lemma is an analogue of Lemma 3.3 in [3], with the same proof.

Lemma 5.3 *The limit u_0 of a blow-up sequence of u with respect to balls $B_{\rho_k}(x_k)$, $u(x_k) = 0$, is an absolute minimizer of J in any ball.* □

Lemma 5.4 *Let u be a local minimizer and $x_0 \in \partial\{u > 0\}$. Then*

$$\limsup_{x \to x_0, u(x)>0} |\nabla u(x)| = 1.$$

Proof. Denote the lim sup by ℓ and let the sequence $y_k \to x_0$, $u(y_k) > 0$, be such that $|\nabla u(y_k)| \to \ell$. Set $\rho_k = \mathrm{dist}(y_k, \partial\{u > 0\})$ and let $x_k \in \partial B_{\rho_k}(y_k) \cap \partial\{u > 0\}$. Consider then the blow-up sequence (5.1) and assume that (5.2)–(5.5) hold. Also, assume that $e_k := (x_k - y_k)/|x_k - y_k|$ converges to the unit vector e_n. Now, observe that $0 \in \partial\{u_0 > 0\}$, $B_1(-e_n) \subset \{u_0 > 0\}$, and

$$|\nabla u_0| \le \ell \text{ in } \{u_0 > 0\}, \quad |\nabla u_0(-e_n)| = \ell.$$

This implies $\ell > 0$. Since u_0 is also p-harmonic in $\{u_0 > 0\}$, it is locally $C^{1,\alpha}$ there. In particular, there exists a small $\delta > 0$ such that $|\nabla u_0| > \ell/2$ in $B_\delta(-e_n)$.

If e denotes a unit vector such that $\nabla u_0(-e_n) = |\nabla u_0(-e_n)|e$, the directional derivative $v = D_e u_0$ will satisfy in $B_\delta(-e_n)$ an uniformly elliptic equation

$$D_i(a_{ij}D_j v) = 0, \quad a_{ij} = |\nabla u_0|^{p-2}\left(\delta_{ij} + (p-2)\frac{D_i u_0 D_j u_0}{|\nabla u_0|^2}\right).$$

By the strong maximum principle, we must have $D_e u_0 = \ell$ in $B_\delta(-e_n)$, implying that $\nabla u_0 = \ell e$. By continuation, we can prove that this is true in the whole $B_1(-e_n)$. Therefore

$$u_0(x) = \ell(x \cdot e) + C \quad \text{in } B_1(-e_n).$$

Since $u_0(0) = 0$ and $u_0 > 0$ in $B_1(-e_n)$, we obtain that $C = 0$ and $e = -e_n$. Thus,

$$u_0(x) = -\ell\, x_n \quad \text{in } B_1(-e_n).$$

Using the continuation method one more time, we see that

$$u_0(x) = -\ell\, x_n \quad \text{in } \{x_n < 0\}. \tag{5.6}$$

Next, we claim that

$$u_0 = 0 \quad \text{in } \{0 < x_n < \varepsilon_0\} \tag{5.7}$$

for some $\varepsilon_0 > 0$. Indeed, let

$$s = \limsup_{\substack{x_n \to 0+,\ x' \in \mathbf{R}^n \\ u(x', x_n) > 0}} D_n u_0(x', x_n).$$

Then $s < \infty$ since u_0 is uniformly Lipschitz. Assume that $s > 0$. Consider a sequence (z_k, h_k), $h_k \to 0+$, such that $D_n u_0(z_k, h_k) \to s$. Arguing as above one can show that the blow-up limit u_{00} of u_0 with respect to the balls $B_{h_k}(z_k, 0)$ satisfies

$$u_{00}(x) = s\, x_n \quad \text{in } \{x_n > 0\}.$$

On the other hand we have

$$u_{00}(x) = -\ell\, x_n \quad \text{in } \{x_n < 0\}.$$

We have reached a contradiction, since by Lemma 5.3 u_{00} is a minimizer and therefore by Theorem 4.4 the set $\{u_{00} = 0\}$ must have positive density. Thus $s = 0$ and consequently $u_0(x', x_n) = o(x_n)$. Hence, for every $\varepsilon > 0$

$$\frac{1}{r}\left(\fint_{B_r(x_0)} u_0^\gamma\right)^{1/\gamma} < \varepsilon$$

for any $x_0 = (z_0, h_0)$, $r = h_0$, if h_0 is small enough. But then the nondegeneracy lemma (Lemma 4.2) implies that $u_0 = 0$ in some strip $\{0 < x_n < \varepsilon_0\}$.

Having proved (5.6) and (5.7), we deduce from Theorem 2.1 that $\ell = 1$. $\square$

Theorem 5.5 *Let $x_0 \in \partial_{\mathrm{red}}\{u>0\}$ and suppose that the upper $\mathcal{H}^{n-1}$-density satisfies*

$$\Theta^{*n-1}\left(\mathcal{H}^{n-1}\llcorner\partial\{u>0\}, x_0\right) \le 1.$$

Then the topological tangent plane $\mathrm{Tan}(\partial\{u>0\}, x_0)$ *of* $\partial\{u>0\}$ *at* x_0 *is given by* $\{x : x\cdot\nu_u(x_0)=0\}$*. If, in addition,*

$$\int_{B_r(x_0)\cap\partial\{u>0\}} |q_u - q_u(x_0)|\,d\mathcal{H}^{n-1} = o(r^{n-1}), \quad \textit{as } r\to 0,$$

then $q_u(x_0)=1$ *and*

$$u(x_0+x) = (-x\cdot\nu_u(x_0))^+ + o(|x|), \quad \textit{as } x\to 0.$$

For definitions of $\Theta^{*m}(\mu,a)$ and $\mathrm{Tan}(S,a)$ see [7], 2.10.19 and 3.1.21.

Proof. Without loss of generality assume $\nu_u(x_0)=e_n$. Consider then the blow-up limit u_0 of u with respect to the sequence of balls $B_{\rho_k}(x_0)$, $\rho_k\to 0$. Using that $x_0\in\partial_{\mathrm{red}}\{u>0\}$ and that the upper $\mathcal{H}^{n-1}$-density of $\partial\{u>0\}$ at x_0 does not exceed 1, precisely as in the proofs of Theorem 4.8 in [2] and Theorem 3.5 in [3], one can show that $u_0>0$ in $\{x_n<0\}$ and $u_0=0$ on $\{x_n>0\}$. As a consequence, $\{x_n=0\}$ is the tangent space to $\partial\{u>0\}$ at x_0.

To prove the second part of the theorem, we again repeat the arguments from [3] and obtain that

$$|\nabla u_0|^{p-2}\nabla u_0\cdot e_n = q_u(x_0) \quad \text{on } \{x_n=0\}$$

in the sense that

$$-\int_{B_r\cap\{x_n<0\}} |\nabla u_0|^{p-2}\nabla u_0\cdot\nabla\zeta = q_u(x_0)\int_{B'_r}\zeta(x',0)\,d\mathcal{H}^{n-1}$$

for every $\zeta\in C^1_0(B_r)$. Since $\Delta_p u_0=0$ in $\{x_n<0\}$, from the boundary regularity it follows that the boundary condition above is satisfied in the classical sense. Hence from the Lemma 5.4 we obtain $q_u(x_0)=1$.

Finally, we need to show that $u_0=(-x_n)^+$. Define the function w_0 by

$$w_0(x) = \begin{cases} u_0(x) & \text{in } \{x_n<0\}\\ -u_0(x^*) & \text{in } \{x_n>0\},\end{cases}$$

where $x^*=(x',-x_n)$ for $x=(x',x_n)$. It is easy to see that w_0 is p-harmonic in the whole $\mathbf{R}^n$. Observe now, since $|\nabla u_0|\le 1$ in $\{x_n<0\}$ by Lemma 5.4,

$$|\nabla w_0|\le 1 \quad \text{in } \mathbf{R}^n.$$

On the other hand,

$$D_n w_0 = -1 \quad \text{on } \{x_n=0\}.$$

In a small ball $B_\delta(0)$, we have $|\nabla w_0| > 1/2$, hence $D_n w_0$ satisfies a uniformly elliptic equation in divergence form and from the strong comparison principle we infer that

$$D_n w_0 = -1 \quad \text{in } B_\delta(0).$$

By continuation, we can prove that, in fact,

$$D_n w_0 = -1 \quad \text{in } \mathbf{R}^n.$$

This implies that $w_0 = -x_n$ in $\mathbf{R}^n$, or equivalently, $u_0 = (-x_n)^+$ in $\mathbf{R}^n$. The proof is complete. □

Theorem 5.6 *For $\mathcal{H}^{n-1}$ a.e. x in $\partial_{\text{red}}\{u > 0\}$*

$$q_u(x) = 1.$$

Since also $\mathcal{H}^{n-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$ (from the positive density property)

$$\Delta_p u = \mathcal{H}^{n-1} \llcorner \partial_{\text{red}}\{u > 0\}$$

for any local minimizer u in Ω.

Proof. Just observe that the condition on q_u in Theorem 5.5 is satisfied for $\mathcal{H}^{n-1}$ a.e. $x_0 \in \partial_{\text{red}}\{u > 0\}$. This follows from [7], 4.5.6(2) and [7], 2.9.8, 2.9.9 applied to $\mathcal{H}^{n-1}$ on $\partial\{u > 0\}$ and the Vitali relation

$$\{(x, B_r(x)) : x \in \partial\{u > 0\} \text{ and } B_r(x) \subset\subset \Omega\}.$$

□

6 Flatness and nondegeneracy of the gradient

We define the relevant flatness classes as in [3], Definition 5.1.

Definition 6.1 Let $0 \le \sigma_+, \sigma_- \le 1$ and $\tau > 0$. We say that u belongs to the class $F(\sigma_+, \sigma_-; \tau)$ in $B_\rho = B_\rho(0)$ if u is a local minimizer in B_ρ with $0 \in \partial\{u > 0\}$, and

$$\begin{aligned} &u(x) = 0 && \text{for } x_n \ge \sigma_+\rho, \\ &u(x) \ge -(x_n + \sigma_-\rho) && \text{for } x_n \le -\sigma_-\rho, \\ &|\nabla u| \le 1 + \tau && \text{in } B_\rho. \end{aligned} \tag{6.1}$$

More generally, changing the direction e_n by ν and the origin by x_0 in the definition above, we obtain the flatness class $F(\sigma_+, \sigma_-; \tau)$ in $B_\rho(x_0)$ in direction ν.

Remark 6.2 If $x_0 \in \partial_{\text{red}}\{u > 0\} \cap \Omega$ then $u \in F(\sigma_\rho, 1; \infty)$ in $B_\rho(x_0)$ in direction $\nu_0 = \nu_u(x_0)$ with $\sigma_\rho \to 0$ as $\rho \to 0$. This follows from the fact that any blow-up u_0 at x_0 vanishes on $\{x : x \cdot \nu_0 \ge 0\}$.

The following two theorems play an important role in the iteration process of proving the $C^{1,\alpha}$ regularity of the free boundary.

Theorem 6.3 *There exists $\sigma_0 > 0$ and $C_0 > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad u \in F(2\sigma, C_0\sigma; \sigma) \text{ in } B_{1/2}$$

for $0 < \sigma < \sigma_0$.

Theorem 6.4 *For every $\delta > 0$ there exists $\sigma_\delta > 0$ and $C_\delta > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad |\nabla u| \geq 1 - \delta \text{ in } B_{1/2} \cap \{x_n \leq -C_\delta \sigma\}$$

for $0 < \sigma < \sigma_\delta$.

We first prove the following weak forms of the theorems.

Lemma 6.5 *For every $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad u \in F(2\sigma, \varepsilon; \sigma) \text{ in } B_{1/2}$$

for $0 < \sigma < \sigma_\varepsilon$.

Lemma 6.6 *For every $\varepsilon > 0$ and $\delta > 0$ there exists $\sigma_{\varepsilon,\delta} > 0$ such that*

$$u \in F(\sigma, 1; \sigma) \text{ in } B_1 \quad \text{implies} \quad |\nabla u| \geq 1 - \delta \text{ in } B_{1/2} \cap \{x_n \leq -\varepsilon\}$$

for $0 < \sigma < \sigma_{\varepsilon,\delta}$

Proof of Lemma 6.5. We use the following construction from [2, 3]. Let

$$\eta(y) = \exp\left(-\frac{9|y|^2}{1 - 9|y|^2}\right)$$

for $|y| < 1/3$ and $\eta(y) = 0$ for $|y| \geq 1/3$, and choose $s \geq 0$ maximal such that

$$B_1 \cap \{u > 0\} \subset D := \{x \in B_1 : x_n < \sigma - s\eta(x')\},$$

where $x = (x', x_n)$. Hence there exists a point

$$z \in B_{1/2} \cap \partial D \cap \partial\{u > 0\}.$$

Observe also that $s \leq \sigma$ since $0 \in \partial\{u > 0\}$.

Now, let $\xi \in \partial B_{3/4}$ and $\xi_n \leq -1/2$. We want to prove an estimate for $u(\xi)$ from below. Consider the solution $v = v_{\kappa,\rho}$ of

$$\begin{aligned} &\Delta_p v = 0 && \text{in } D \setminus \overline{B_\rho(\xi)}, \\ &v = 0 && \text{on } \partial D \cap B_1, \\ &v = (1 + \sigma)(\sigma - x_n) && \text{on } \partial D \setminus B_1, \\ &v = -(1 - \kappa\sigma)x_n && \text{on } \partial B_\rho(\xi), \end{aligned}$$

where $\kappa > 0$ is a large and $\rho > 0$ is a small constant to be chosen later. We claim that for large $\kappa = \kappa(\rho)$

$$u(x_\xi) \geq v(\xi) \quad \text{for some } x_\xi \in \partial B_\rho(\xi). \tag{6.2}$$

Indeed, otherwise $u \leq v$ on $\partial(D \setminus B_\rho(\xi))$ and, by the comparison principle, also $u \leq v$ on $D \setminus B_\rho(\xi)$. Then the contradiction follows from the following two statements, applied at the point z.

Claim 6.7 *If B is a ball in $\{u = 0\}$ touching $\partial\{u > 0\}$ at x_0, then*

$$\limsup_{x \to x_0} \frac{u(x)}{\operatorname{dist}(x, B)} = 1.$$

Claim 6.8 *The function $v = v_\kappa$ constructed above satisfies*

$$|\nabla v(z)| \leq 1 + C\sigma - c\kappa\sigma \quad \textit{for } z \in B_{1/2} \cap (\partial D \setminus \partial B_1)$$

for some positive constants $C = C(\rho)$ and $c = c(\rho)$ and if $0 < \sigma < \sigma(\kappa, \rho)$.

We postpone the proofs of these claims for a moment.

Now, choose $\kappa = \kappa(\rho) > C(\rho)/c(\rho)$ and $\sigma < \sigma(\kappa(\rho), \rho)$. Then (6.2) holds and

$$\begin{aligned} u(\xi) &\geq u(x_\xi) - \rho(1+\sigma) \geq v(x_\xi) - \rho(1+\sigma) \\ &= -(1-\kappa\sigma)(x_\xi)_n - \rho(1+\sigma) \geq -(x_\xi)_n - \kappa\sigma - 2\rho \geq -\xi_n - 4\rho \end{aligned}$$

for $\sigma < \sigma(\rho)$ sufficiently small. That is, we get

$$u(\xi) \geq -\xi_n - 4\rho \quad \text{on } \{\xi \in \partial B_{3/4}, \xi_n < -1/2\}. \tag{6.3}$$

Integrating along vertical lines and using that $|\nabla u| \leq 1 + \sigma$ we obtain

$$u(\xi + \alpha e_n) \geq u(\xi) - \alpha(1+\sigma) \geq -\xi_n - 4\rho - \alpha - \alpha\sigma \geq -(\xi_n + \alpha) - 5\rho$$

for $\sigma < \sigma(\rho)$. Choosing $\rho = \varepsilon/10$, we complete the proof of Lemma 6.5. □

Proof of Lemma 6.6. Assume the contrary. Then there exists a sequence $u_k \in F(1/k, 1; 1/k)$ such that

$$|\nabla u_k(x^k)| \leq 1 - \delta \quad \text{for some } x^k \in B_{1/2} \cap \{x_n \leq -\varepsilon\}.$$

Letting $k \to \infty$ we obtain from Lemma 6.5 that

$$u_k(x) \to u_0(x) = -x_n \quad \text{uniformly on } \overline{B_{3/4}}.$$

Moreover, on the positivity set $\{u_0 > 0\} = \{x_n < 0\}$ the convergence is locally $C^{1,\alpha}$. This implies that if a subsequence of x^k converges to $x^0 \in B_{1/2} \cap \{x_n \leq -\varepsilon\}$, then $|\nabla u_0(x^0)| \leq 1 - \delta$, which contradicts to the fact that $|\nabla u_0| = 1$. □

Proof of Theorem 6.3. We revisit the proof of Lemma 6.5. Choose $\rho = 1/10$ and $\kappa = \kappa(\rho)$ so that (6.2) holds. We can refine the estimate (6.3) as follows. Set

$$w(x) = (1+\sigma)(\sigma - x_n) - u(x).$$

Then $u \in F(\sigma, 1; \sigma)$ implies that $w(x) \geq 0$ in $B_{2\rho}(\xi)$ and

$$w(x_\xi) \leq -(x_\xi)_n - v(x_\xi) + C\sigma \leq C\sigma.$$

For σ sufficiently small we know from Lemma 6.6 that $|\nabla u| > 1/2$, so u will satisfy the linearized equation

$$L_u u = a_{ij} D_{ij} u = 0,$$

where

$$a_{ij} = \delta_{ij} + (p-2)\frac{D_i u D_j u}{|Du|^2}$$

Observe that the ellipticity constant of L_u is $\lambda = \lambda(p) = \max\{p-1, 1/(p-1)\}$. As a consequence w will satisfy the equation

$$L_u w = 0.$$

Applying the Harnack inequality we obtain that

$$w(\xi) \leq C w(x_\xi) \leq C\sigma$$

or

$$u(\xi) \geq -\xi_n - C\sigma \quad \text{on } \{\xi \in \partial B_{3/4}, \xi_n \leq -1/2\}.$$

Integrating along vertical lines and using that $|\nabla u| \leq 1 + \sigma$, we conclude that

$$u(\xi + \alpha e_n) \geq u(\xi) - (1+\sigma)\alpha \geq -(\xi_n + \alpha) - C\sigma,$$

which implies that $u \in F(2\sigma, C\sigma; \sigma)$ in $B_{1/2}$. □

Proof of Theorem 6.4. Assume the contrary. Then there exists a sequence $\sigma_k \to 0$ and $u_k \in F(\sigma_k, 1; \sigma_k)$ such that

$$|\nabla u_k(x^k)| \leq 1 - \delta \quad \text{for some } x^k \in B_{1/2} \cap \{x_n \leq -k\sigma_k\}.$$

Let $d_k = \operatorname{dist}(x^k, \partial\{u_k > 0\})$ and $y^k \in \partial\{u_k > 0\}$ be such that $|x^k - y^k| = d_k$. From Theorem 6.3 it follows that $d_k \geq (k - C_0)\sigma_k$. Define now

$$\widetilde{u}_k(x) = \frac{u_k(y^k + 2d_k x)}{2d_k}, \quad \widetilde{x}^k = \frac{x^k - y^k}{2d_k}.$$

Then one can easily verify that

$$\widetilde{u}_k \in F((C_0+1)/2(k - C_0), 1; \sigma_k) \quad \text{in } B_1,$$

$$|\widetilde{x}^k| = \frac{1}{2}, \quad (\widetilde{x}^k)_n \leq -\frac{1}{2}(1 - (C_0 + 1)/(k - C_0)),$$

and

$$|\nabla \widetilde{u}_k(\widetilde{x}^k)| \leq 1 - \delta.$$

This contradicts to Lemma 6.6. The proof is complete. □

We now prove the claims stated in the proof of Lemma 6.5.

Proof of Claim 6.7. Denote

$$\ell = \limsup_{x \to x^0} u(x)/\operatorname{dist}(x, B)$$

and let the sequence $x^k \to x^0$ be such that $u(x^k) > 0$ and

$$\frac{u(x^k)}{d_k} \to \ell,$$

where $d_k = \operatorname{dist}(x^k, B)$. Moreover, let $y^k \in \partial B$ be such that $|x^k - y^k| = d_k$. By nondegeneracy $\ell > 0$. Consider now a blow-up sequence

$$u_k(x) = \frac{u(y_k + d_k x)}{d_k}$$

and assume that for a subsequence

$$(x^k - y^k)/d_k \to e, \quad u_k \to u_0.$$

We claim that

$$u_0(x) = \ell(x \cdot e)^+.$$

Indeed, by construction $u_0(x) = 0$ when $x \cdot e \leq 0$, $u_0(x) \leq \ell(x \cdot e)$ when $x \cdot e > 0$, and $u_0(e) = \ell$. Both $u_0(x)$ and $\ell(x \cdot e)^+$ are p-harmonic in $\{u_0 > 0\}$ and from the strong maximum principle (applicable here since $\ell > 0$) it follows that they must coincide.

The only constant ℓ for which u_0 can have the form $\ell(x \cdot e)^+$ is $\ell = 1$ and the proof is complete. □

Proof of Claim 6.8. The idea of the proof is to construct an explicit p-superharmonic function w in $D \setminus B_\rho(\xi)$ to estimate v. Observe that if $|\nabla w| > 0$, then w will be p-superharmonic if we can verify that

$$a_{ij} D_{ij} w \leq 0$$

for any positive matrix a_{ij} of ellipticity $\lambda = \lambda(p) = \max\{1/(p-1), p-1\}$, i.e. if

$$\lambda^{-1}|\zeta|^2 \leq a_{ij}\zeta_i\zeta_j \leq \lambda|\zeta|^2 \quad \text{for every } \zeta \in \mathbf{R}^n.$$

Indeed, taking

$$a_{ij} = \delta_{ij} + (p-2)\frac{D_i w D_j w}{|Dw|^2},$$

we obtain that w satisfies $L_w(w) \le 0$ for the linearized p-Laplacian and hence w is p-superharmonic.

We construct w in the form $v_1 - \kappa\sigma v_2$ with v_1, v_2 defined as follows. First

$$v_1 = \frac{\gamma_1}{\mu_1}(1 - \exp(-\mu_1 d_1)) \quad \text{in } D,$$

where

$$d_1(x) = -x_n + \sigma - s\eta(x'),$$

and with positive constants μ_1, γ_1 depending on σ. Then

$$1 \le |\nabla d_1| \le 1 + C\sigma, \quad |D^2 d_1| \le C\sigma.$$

Hence, if a_{ij} is a positive matrix of ellipticity $\lambda(p)$

$$\begin{aligned} a_{ij}D_{ij}v_1 &= \gamma_1 \exp(-\mu_1 d_1) a_{ij}(D_{ij}d_1 - \mu_1 D_i d_1 D_j d_1) \\ &\le \gamma_1 \exp(-\mu_1 d_1)(C\sigma - c\mu_1) < 0 \end{aligned} \tag{6.4}$$

if

$$\mu_1 = C_1\sigma, \quad C_1 \text{ large enough.}$$

Next, if $\gamma_1 = 1 + C_2\sigma$, C_2 large enough and σ small, we obtain for $x \in D$

$$v_1(x) \ge \gamma_1 d_1(x)(1 - C\mu_1) \ge (1 + 2\sigma)d_1(x), \tag{6.5}$$

$$(1 + 2\sigma)d_1(x) \ge v(x) \quad \text{for } x \in \partial(D \setminus B_\rho(\xi)),$$

and

$$|\nabla v_1(x)| = \gamma_1 \exp(-\mu_1 d_1)|\nabla d_1| \ge \gamma_1(1 - 2\mu_1) \ge 1. \tag{6.6}$$

The inequality (6.4) implies that v_1 is p-superharmonic in D and therefore the maximum principle yields

$$v_1 \ge v \quad \text{in } D \setminus B_\rho(\xi).$$

At the point $z \in B_{1/2} \cap (\partial D \setminus \partial B_1)$ we compute

$$|\nabla v_1(z)| = \gamma_1 |\nabla d_1| \le 1 + C\sigma.$$

We define v_2 depending on $B_\rho(\xi)$ by

$$v_2 = \frac{\gamma_2}{\mu_2}(\exp(\mu_2 d_2) - 1) \quad \text{in } \widetilde{D} \setminus B_\rho(\xi),$$

with constants γ_2, μ_2 to be specified later. Here $\widetilde{D} \subset D$ is a domain with smooth boundary containing

$$D \setminus B_{1/10}(\partial B_1' \times \{0\}),$$

and d_2 is a function in $C^2(D \setminus B_\rho(\xi))$ satisfying

$$\begin{aligned} d_2 = 0 & \quad \text{on } \partial\widetilde{D}, \\ d_2 = 1 & \quad \text{on } \partial B_\rho(\xi), \\ C \ge |\nabla d_2| \ge c > 0 & \quad \text{in } \widetilde{D} \setminus \overline{B_\rho(\xi)}. \end{aligned}$$

Thus, for any matrix a_{ij} of ellipticity $\lambda(p)$

$$\begin{aligned} a_{ij}D_{ij}v_2 &= \gamma_2 \exp(\mu_2 d_2) a_{ij}(D_{ij}d_2 + \mu_2 D_i d_2 D_j d_2) \\ &\geq \gamma_2 \exp(\mu_2 d_2)(-C + c\mu_2) > 0 \end{aligned} \tag{6.7}$$

if μ_2 is large enough. Then choose γ_2 such that

$$v_2 = 1 \quad \text{on } \partial B_\rho(\xi),$$

or explicitly

$$\gamma_2 = \frac{\mu_2}{\exp(\mu_2) - 1}.$$

In $\widetilde{D} \setminus \overline{B_\rho(\xi)}$ we have

$$|\nabla v_2| = \gamma_2 \exp(\mu_2 d_2)|\nabla d_2| \leq C$$

and at the point z

$$|\nabla v_2(z)| = \gamma_2 |\nabla d_2(z)| \geq c > 0.$$

Thus the function

$$w = v_1 - \kappa\sigma v_2$$

satisfies $a_{ij}D_{ij}w \leq 0$ in $\widetilde{D} \setminus \overline{B_\rho(\xi)}$ with

$$w = v_1 \geq v \quad \text{on } \partial\widetilde{D},$$

and for $x \in \partial B_\rho(\xi)$

$$w(x) \geq d_1(x) - \kappa\sigma \geq -(1 - \kappa\sigma)x_n = v(x).$$

Also, the gradient of w is not degenerate since

$$|\nabla w| \geq |\nabla v_1| - \kappa\sigma|\nabla v_2| \geq 1 - C\kappa\sigma > 0$$

if $\sigma < \sigma(\kappa)$ is small enough. We conclude that w is p-superharmonic and the comparison principle yields $w \geq v$ in $\widetilde{D} \setminus \overline{B_\rho(\xi)}$. In particular,

$$|\nabla v(z)| \leq |\nabla w(z)| = |\nabla v_1(z)| - \kappa\sigma|\nabla v_2(z)| \leq 1 + C\sigma - c\kappa\sigma.$$

The claim is proved. □

7 Gradient estimates

Theorem 7.1 *Let u be a local minimizer. For any $D \subset\subset \Omega$ and a ball $B_r(x) \subset D$ such that $B_r(x) \cap \partial\{u > 0\}$ is nonempty,*

$$\sup_{B_r(x)} |\nabla u| \leq 1 + Cr^\alpha$$

with $C > 0$, $0 < \alpha < 1$ (depending only on D, Ω, n, p.)

Proof. For any $\varepsilon > 0$ consider the function

$$U_\varepsilon = (|\nabla u|^2 - 1 - \varepsilon)^+$$

Observe that it vanishes in a neighborhood of the free boundary. Since $U_\varepsilon > 0$ implies $|\nabla u| > 1 + \varepsilon$, the closure of $\{U_\varepsilon > 0\}$ is contained in $\{|\nabla u| > 1 + \varepsilon/2\}$. The function u satisfies the linearized p-Laplace equation

$$L_u(u) = a_{ij}(\nabla u) D_{ij} u \geq 0, \quad a_{ij}(\nabla u) = \delta_{ij} + (p-2)\frac{D_i u D_j u}{|\nabla u|^2},$$

which is $\lambda(p)$-uniformly elliptic in $\{|\nabla u| > 1+\varepsilon/2\}$. Hence, by [8], Section 13.3, the function $v = |\nabla u|^2$ satisfies

$$M(v) = D_i(e^{\gamma v} a_{ij}(\nabla u) D_j v) \geq 0 \quad \text{in } \{|\nabla u| > 1 + \varepsilon/2\},$$

where $\gamma = \gamma(p, \|\nabla u\|_{L^\infty(\Omega)})$ is some positive constant, and so U_ε satisfies

$$M(U_\varepsilon) \geq 0 \quad \text{in } \{|\nabla u| > 1 + \varepsilon/2\}.$$

Extending the operator M with the uniformly elliptic divergence-form operator

$$\widetilde{M}(w) = D_i(\widetilde{a}_{ij}(x) D_j w) \quad \text{in } \Omega$$

with measurable coefficients such that

$$\widetilde{a}_{ij}(x) = e^{\gamma v(x)} a_{ij}(\nabla u(x)) \quad \text{in } \{|\nabla u| > 1 + \varepsilon/2\},$$

we obtain

$$\widetilde{M}(U_\varepsilon) \geq 0 \quad \text{in } \Omega.$$

For any $r > 0$ set

$$h_\varepsilon(r) = \sup_{B_r} U_\varepsilon$$

where the origin is taken to be on the free boundary. Then $h_\varepsilon(r) - U_\varepsilon$ is $\widetilde{M}$-supersolution in B_r, and

$$\begin{aligned} h_\varepsilon(r) - U_\varepsilon &\geq 0 && \text{in } B_r \\ &= h_\varepsilon(r) && \text{in } B_r \cap \{u \equiv 0\}. \end{aligned}$$

By [8], Theorem 8.18, with $1 \leq q < n/(n-2)$,

$$\inf_{B_{r/2}} (h_\varepsilon(r) - U_\varepsilon) \geq c r^{-n/q} \|h_\varepsilon(r) - U_\varepsilon\|_{L^q(B_r)} \geq c h_\varepsilon(r),$$

since $|B_r \cap \{u \equiv 0\}| \geq c r^n$ by the positive density property. Taking $\varepsilon \to 0$ we get

$$\inf_{B_{r/2}} (h_0(r) - U_0) \geq c h_0(r) \quad (0 < c < 1)$$

or

$$\sup_{B_{r/2}} U_0 \leq (1-c) h_0(r).$$

In conclusion

$$h_0(r/2) \le (1-c)h(r)$$

and by a standard argument we deduce that $h_0(r) \le Cr^\alpha$ for some $C > 0$ and $0 < \alpha < 1$. This completes the proof of the theorem.

□

8 Nonhomogeneous blow-up

Lemma 8.1 *Let $u_k \in F(\sigma_k, \sigma_k; \tau_k)$ in B_{ρ_k} with $\sigma_k \to 0$, $\tau_k \sigma_k^{-2} \to 0$. For $y \in B_1'$, set*

$$\begin{aligned} f_k^+(y) &= \sup\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}, \\ f_k^-(y) &= \inf\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}. \end{aligned}$$

Then, for a subsequence,

$$f(y) := \limsup_{\substack{z \to y \\ k \to \infty}} f_k^+(z) = \liminf_{\substack{z \to y \\ k \to \infty}} f_k^-(z) \quad \text{for all } y \in B_1'.$$

Further, $f_k^+ \to f$, $f_k^- \to f$ uniformly, $f(0) = 0$, and f is continuous.

This is the analogue of Lemma 5.3 in [3]. The proof is based on Theorem 6.3 and is identical to the one of Lemma 7.3 in [2].

Lemma 8.2 *f is subharmonic.*

The proof is identical to the one of Lemma 5.4 in [3].

Lemma 8.3 *There exists a positive constant C such that, for any $y \in B_{r/2}'$,*

$$\int_0 \frac{1}{r^2} \left(\fint_{\partial B_r'(y)} f - f(y) \right) \le C$$

Proof. The proof below is a modification of that of Lemmas 5.6–5.7 in [3]. It should be pointed out that Theorem 6.4 is used in an essential way.

With no loss of generality we assume $\rho_k = 1$. Also, it suffices to prove the lemma for $y = 0$, since $u_k \in F(8\sigma_k, 8\sigma_k; \tau_k)$ in $B_{1/4}(y, \sigma_k f_k^+(y))$.

Set

$$w_k(y, h) = \frac{u_k(y, h) + h}{\sigma_k}.$$

Since the free boundary of u_k lies in the strip $|x_n| \le \sigma_k$, $|\nabla u_k| \le 1 + \tau_k$, $\tau_k \le \sigma_k$, and we have $w_k \le C$ in B_1^-. The flatness assumption also implies that $w_k \ge -C$ in B_1^-, and thus

$$|w_k| \le C \quad \text{in } B_1^-.$$

Claim 8.4 *For a subsequence,*

$$\lim_{k\to\infty} w_k =: w \quad \textit{exists everywhere in } B_1^-.$$

The convergence is uniform in compact subsets of B_1^-, *and* w *satisfies*

$$a_{ij}(e_n)D_{ij}w = \sum_{i=1}^{n-1} D_{ii}w + (p-1)D_{nn}w = 0 \quad \textit{in } B_1^-, \tag{8.1}$$

$$w(0,h) \le 0, \tag{8.2}$$

$$w(y,0) = f(y) \tag{8.3}$$

in the sense that $\lim_{h\to 0^-} w(y,h) = f(y)$,

$$|w| \le C. \tag{8.4}$$

Once we prove this claim, the lemma will follow by Lemma 5.5 in [3] after applying an affine transformation. □

Proof of Claim 8.4. By Theorem 6.4 we know that

$$|\nabla u_k| \ge 1/2 \quad \text{in } B_1 \cap \{h \le -C_0\sigma_k\} \tag{8.5}$$

for σ_k sufficiently small. Then u_k satisfies

$$a_{ij}(\nabla u_k)D_{ij}u_k = 0, \quad a_{ij}(\nabla u_k) = \delta_{ij} + (p-2)\frac{D_i u_k D_j u_k}{|\nabla u_k|^2}$$

for $h \le -C_0\sigma_k$. Therefore, we have

$$a_{ij}(\nabla u_k)D_{ij}w_k = 0 \quad \text{in } B_1 \cap \{h \le -C_0\sigma_k\}.$$

From the flatness assumption it is clear that, for a subsequence,

$$u_k(y,h) \to -h$$

in C^2 in compact subsets of B_1^-. Also, we may assume

$$w_k \to w$$

in C^2 in compact subsets of B_1^- and that w satisfies (8.1). Clearly, (8.4) is also valid. Also, since

$$-D_n w_k = -\frac{1}{\sigma_k}(D_n u_k + 1) \le \frac{|\nabla u_k| - 1}{\sigma_k} \le \frac{\tau_k}{\sigma_k} \tag{8.6}$$

and $w_k(0,0) = 0$, we obtain for $h \le 0$

$$w_k(0,h) \le |h|\frac{\tau_k}{\sigma_k} \to 0.$$

Thus $w(0,h)\le 0$ and (8.2) follows.

It remains to prove (8.3). First we show that for any small $\delta>0$ and a large constant K

$$w_k(y,h\sigma_k)\to f(y)\quad\text{uniformly for } y\in B'_{1-\delta},\ -K\le h\le -1. \tag{8.7}$$

By Lemma 8.1 it suffices to prove that

$$w_k(y,h\sigma_k)-f_k^+(y)\to 0. \tag{8.8}$$

From (8.6) we obtain

$$\begin{aligned} w_k(y,h\sigma_k)-f_k^+(y) &\le w_k(y,\sigma_k f_k^+(y))-f_k^+(y)+(f_k^+(y)-h)\frac{\tau_k}{\sigma_k}\\ &= (f_k^+(y)-h)\frac{\tau_k}{\sigma_k}\le (1+K)\frac{\tau_k}{\sigma_k}\to 0. \end{aligned}$$

To show (8.8) from below, take a sequence $y_k\in B'_{1-\delta}$, $-K\le h_k\le -C_0$ and consider u_k in $B_{R\sigma_k}(x_k)$, where x_k is the free boundary point

$$x_k=(y_k,\sigma_k f_k^+(y_k))$$

and R is a large constant. We know that

$$u_k\in F(\widetilde{\delta}_k,1;\tau_k)\quad\text{in } B_{R\sigma_k}(x_k),$$

if

$$\widetilde{\delta}_k=\frac{1}{R}\sup_{B'_{R\sigma_k}(y_k)}(f_k^+-f_k^+(y_k)).$$

Notice that $\widetilde{\delta}_k\to 0$ by Lemma 8.1. From Theorem 6.3 we have

$$u_k\in F(2\delta_k,C\delta_k;\tau_k)\quad\text{in } B_{(R/2)\sigma_k}(x_k)$$

for

$$\delta_k=\max(\widetilde{\delta}_k,\tau_k).$$

Hence for any h with $|h|\le R/2$

$$u_k(x_k+h\sigma_k e_n)\ge -(h\sigma_k+C\delta_k(R/2)\sigma_k)$$

for $h\le -C(R/2)\delta_k$. In other words,

$$w_k(x_k+h\sigma_k e_n)-f_k^+(y_k)=\frac{u_k(x_k+h\sigma_k e_n)+h\sigma_k}{\sigma_k}\ge -C(R/2)\delta_k\to 0.$$

This proves (8.8). Next, for $\varepsilon>0$ choose a C^3 function g_ε such that

$$f-2\varepsilon\le g_\varepsilon\le f-\varepsilon\quad\text{on } B'_1, \tag{8.9}$$

and let u_ε solve

$$\begin{aligned} a_{ij}(e_n)D_{ij}u_\varepsilon &= 1 && \text{in } B_1^-, \\ u_\varepsilon &= g_\varepsilon && \text{on } \partial B_{1-\delta}^- \cap \{h=0\}, \\ u_\varepsilon &= \inf_{B_1^-} w && \text{on } \partial B_{1-\delta}^- \cap \{h<0\}, \end{aligned}$$

with δ as in (8.7). By (8.7) and (8.9),

$$w_k > u_\varepsilon \quad \text{on } \partial(B_{1-\delta}^- \cap \{h < -K\sigma_k\}) \tag{8.10}$$

for any large K (independent of ε and δ) and $k \geq k(\varepsilon, \delta, K)$ sufficiently large. Assume also $K > C_0$, where C_0 is as in (8.5). The function w_k is bounded and satisfies a $\lambda(p)$-uniformly elliptic equation in $B_1^- \cap \{h \leq -C_0\sigma_k\}$. By elliptic estimates we deduce that

$$|\nabla w_k| \leq \frac{C}{(K-C_0)\sigma_k} \quad \text{in } B_{1-\delta}^- \cap \{h \leq -K\sigma_k\},$$

where C is independent of k, K. Hence

$$\begin{aligned} a_{ij}(\nabla u_k)D_{ij}u_\varepsilon &= (a_{ij}(\nabla u_k) - a_{ij}(-e_n))D_{ij}u_\varepsilon + 1 \\ &\geq 1 - \frac{C}{K}\|u_\varepsilon\|_{C^{1,1}(B_{1-\delta}^-)} > 0 \quad \text{in } B_{1-\delta}^- \cap \{h < -K\sigma_k\}, \end{aligned}$$

if $K = K(\delta, \varepsilon)$ is sufficiently large. Thus

$$a_{ij}(\nabla u_k)D_{ij}u_\varepsilon > a_{ij}(\nabla u_k)D_{ij}w_k \quad \text{in } B_{1-\delta}^- \cap \{h < -K\sigma_k\},$$

and recalling (8.10) we obtain that $u_\varepsilon \leq w_k$ in $B_{1-\delta}^- \cap \{h < -K\sigma_k\}$ if k is large enough. It follows now that $w(y,h) \geq u_\varepsilon(y,h)$ in $B_{1-\delta}^-$ and consequently,

$$\liminf_{h\to 0-} w(y,h) \geq f(y) - 2\varepsilon.$$

Similarly, working with the solution of

$$\begin{aligned} a_{ij}(e_n)D_{ij}\widetilde{u}_\varepsilon &= 1 && \text{in } B_1^-, \\ \widetilde{u}_\varepsilon &= \widetilde{g}_\varepsilon && \text{on } \partial B_{1-\delta}^- \cap \{h=0\}, \\ \widetilde{u}_\varepsilon &= \sup_{B_1^-} w && \text{on } \partial B_{1-\delta}^- \cap \{h<0\} \end{aligned}$$

for $\widetilde{g}_\varepsilon$ satisfying $f + \varepsilon < \widetilde{g}_\varepsilon < f + 2\varepsilon$, we obtain

$$\limsup_{h\to 0-} w(y,h) \leq f(y) + 2\varepsilon.$$

Since ε is arbitrary, (8.3) follows. □

9 Regularity of the free boundary

Everything is now ready to prove the $C^{1,\alpha}$ regularity of the free boundary near flat points.

Theorem 9.1 *Suppose u is a local minimizer and $D \subset\subset \Omega$. Then there exist positive constants α, β, σ_0, τ_0, C such that if*

$$u \in F(\sigma, 1; \infty) \quad \text{in } B_\rho(x_0) \subset D \text{ in direction } \nu$$

with $\sigma \leq \sigma_0$, $\rho \leq \tau_0\, \sigma^{2/\beta}$, then

$$B_{\rho/4}(x_0) \cap \partial\{u > 0\} \quad \text{is a } C^{1,\alpha} \text{ surface.}$$

More precisely, $B_{\rho/4}(x_0) \cap \partial\{u > 0\}$ is a graph in direction ν of a $C^{1,\alpha}$ function, and for any x_1, x_2 on this surface,

$$|\nu(x_1) - \nu(x_2)| \leq C\sigma \left| \frac{x_1 - x_2}{\rho} \right|^\alpha.$$

Proof. The proof follows the same scheme as the proof of Theorem 8.1 in [2]. Assume for a moment that $B_\rho(x_0) = B_1$ and $\nu = e_n$. Then Lemma 8.3 implies $C(n,p)$-Lipschitz regularity and then "better than" Lipschitz regularity of f, precisely as in [2], Lemmas 7.7 and 7.8. Namely, we have that there exists a large constant $C = C(n,p) < \infty$ and for any $\theta > 0$ a small constant $c_\theta = c_\theta(n,p) > 0$ such that we can find a ball B'_r and a vector $l \in \mathbf{R}^{n-1}$ with

$$c_\theta \leq r \leq \theta,\ |l| \leq C \quad \text{and} \quad f(y) \leq l \cdot y + (\theta/2)r,\ \text{for } y \in B'_r.$$

This, in conjunction with Theorem 6.3 and the proof of Theorem 7.1, implies the flatness improvement in a smaller ball: if

$$u \in F(\sigma, 1; \tau) \text{ in } B_\rho(x_0) \text{ in direction } \nu$$

with $\sigma < \sigma_0$ and $\tau \leq \sigma_0\, \sigma^2$ for sufficiently small σ_0, then

$$u \in F(\theta\sigma, \theta\sigma; \theta^2\tau) \text{ in } B_{\bar\rho}(x_0) \text{ in direction } \bar\nu$$

for some $\bar\rho$, $\bar\nu$ with $c_\theta \rho \leq \bar\rho \leq \rho/4$ and $|\bar\nu - \nu| < C\theta$, see Lemmas 7.9 and 7.10 in [2]. Finally, using Theorem 7.1 and iteratively applying the flatness improvement, we conclude the proof as in Theorem 8.1 in [2]. □

Corollary 9.2 *Let u be a local minimizer. Then $\partial_{\text{red}}\{u > 0\}$ is an analytic surface locally in Ω and the remainder of $\partial\{u > 0\}$ has $\mathcal{H}^{n-1}$ measure 0.*

Proof. The $C^{1,\alpha}$ regularity of $\partial_{\text{red}}\{u > 0\}$ follows from Theorem 9.1 and Remark 6.2. Once we have $C^{1,\alpha}$ regularity, it follows that $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is uniformly elliptic locally near $\partial_{\text{red}}\{u > 0\}$ and we obtain the analyticity by [12]. □

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