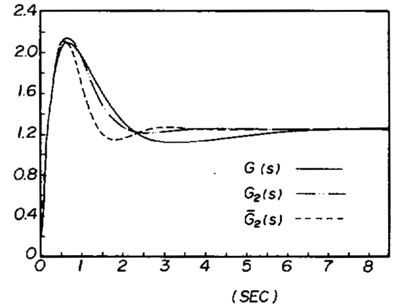

Fig. 2. Comparison of unit step responses.

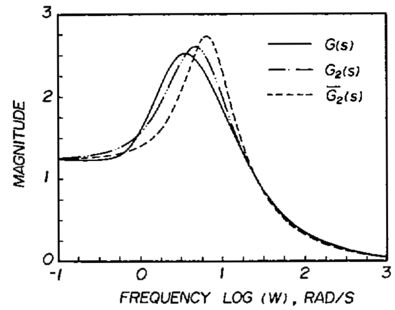

Fig. 3. Comparison of magnitude frequency responses.

TABLE I

Model	Impulse Response Energy	Integral of Squared Errors: Impulse Response	Unit Step Response
$G(s)$	11.400556	—	—
$\simeq g_2(s)$	11.551725	0.082905	0.035273
$\bar{G}_2(s)$	12.254919	0.578921	0.159451

Here, it should be noted that in this example an unstable second-order reduced model,

$$G_2^*(s)=\frac{-0.46358-2.92242s}{-0.37086-2.20814s+s^2}$$

will be obtained if we use the method of Chen and Shieh [1] which uses the Cauer-type continued-fraction expansion about $s = 0$. It is an important advantage of the present method that the instability problem of yielding unstable reduced models for a full system can be partially overcome by choosing suitable expansion points a_i.

V. Concluding Remarks

A multipoint continued-fraction expansion about arbitrary points in the real axis has been proposed for reduced-order modeling of linear time-invariant systems. Both the frequency-domain and time-domain MCFE modeling procedures have also been presented. As compared to the MCFE of [9], the present MCFE has the following advantages: 1) it involves only the operation of real values; 2) it also provides a time-domain modeling procedure; 3) it is equally applicable to discrete-time systems.

Finally, it should be mentioned that the problem of yielding unstable reduced models for a stable full model can be partially overcome by an appropriate choice of the values of a_i. However, it is difficult to derive a relation between the points a_i and stability preservation of the full models.

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A Miscellany of Results on an Equation of Count J. F. Riccati

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Abstract—**A collection of results on the Riccati equation is presented. The questions addressed are the existence of strong solutions of the algebraic Riccati equation and the convergence of solutions of the Riccati difference equation to those of the algebraic equation. The results derived utilize detestability conditions only.**

I. Introduction

In this note we present some new properties of the following equation:

$$\dot{P}(t)=P(t)F^T+FP(t)-P(t)H^THP(t)+Q,\ P(0)=P_0 \tag{1.1}$$

and its discrete-time counterpart

$$P(t+1)=FP(t)F^T-FP(t)H^T[HP(t)H^T+I]^{-1}HP(t)F^T+Q,\ P(0)=P_0 \tag{1.2}$$

where F, H, Q are constant real matrices with Q symmetric and nonnegative definite, and where the dimensions of the matrices $P(t)$, F, H, Q are, respectively, $n \times n$, $n \times n$, $p \times n$, and $n \times n$. The particular variants of these equations which we consider are familiar from an optimal filtering context. The same equations (modulo the classical duality substitutions) arise in optimal control problems.

Although Count Jacobo Francesco Riccati (1676–1754) did not originate these precise equations, they have been widely attributed to him

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ever since d'Alembert did so after reading [1]. Therefore, following popular folklore, we shall call these equations the Riccati differential and Riccati difference equations (RDE). Associated with these two RDE's are a continuous-time and a discrete-time algebraic Riccati equation (ARE):

$$PF^T+FP-PH^THP+Q=0 \tag{1.3}$$

$$P=FPF^T-FPH^T[HPH^T+I]^{-1}HPF^T+Q. \tag{1.4}$$

The RDE's and their corresponding ARE's are among the most widely studied equations in linear systems theory. As is well known, the solution of the linear quadratic (LQ) optimal control problem and the dual optimal filtering problem (i.e., Kalman filtering problem) require the solution of these RDE's.

The solution of the ARE is of interest because if the RDE has a convergence point, then the limiting solution of the RDE must obey the corresponding ARE.

Ever since Kalman's pioneering work of the early 1960's, there has been abundant literature (both papers and books) devoted to these equations, and we shall not attempt to survey this literature in this brief note. The most central question in all this work has been concerned with deriving properties of H, F, Q that would guarantee asymptotic stability of the time-varying closed-loop system

$$\dot{x}(t)=[F-P(t)H^TH]x(t) \tag{1.5}$$

in continuous time, or

$$x(t+1)=\{F-FP(t)H^T[HP(t)H^T+I]^{-1}\}x(t) \tag{1.6}$$

in discrete time. Until 1984, the most classical result was that the closed-loop system is exponentially asymptotically stable if the pair $[H, F]$ is completely detectable and the pair $[F, L]$ is completely stabilizable (where L is a square root of Q: $Q = LL^T$)(see, e.g., [2]). It was widely believed that this was a necessary and sufficient condition. In [3] Chan, Goodwin, and Sin presented a series of new results for the discrete-time ARE and RDE. Assuming only the detectability of $[H, F]$, they studied the properties of the solution of the ARE and also the convergence properties of the RDE under a variety of assumptions on the pair $[F, L]$. They introduced the notion of *strong solution* defined as follows.

Definition 1: A real symmetric nonnegative definite solution $\bar{P}$ of the discrete-time (respectively, continuous-time) ARE is called *strong* if the corresponding closed-loop state-transition matrix $\bar{F} = F - F\bar{P}H^T(H\bar{P}H^T + I)^{-1}H$ (respectively, $\bar{F} = F - \bar{P}H^TH$) has all its eigenvalues inside or on the unit circle (respectively, in the closed left half plane).

The central ARE result established in [3] (and the hardest one to prove) was the existence and uniqueness of a strong solution of (1.4) under an assumption that involved only the detectability of $[H, F]$. In [4] the ARE results of [3] were strengthened to necessary and sufficient conditions, and the following discrete-time RDE result was established.

Proposition 1 [4, Theorem 4.2]: Subject to $P_o \geq \bar{P}$, then $\lim_{t\to\infty} P(t) = P$ if and only if $[H, F]$ is detectable, where $P(t)$ is the solution of the RDE (1.2) and $\bar{P}$ is the unique strong solution of the ARE (1.4).

In [5] and [6] a new closed-loop stability problem was considered, different from that of (1.5), (1.6). Sufficient conditions on H, F, Q, and P_o were derived that guarantee the asymptotic stability of the time-invariant closed-loop system

$$\dot{x}(t)=[F-P(s)H^TH]x(t) \tag{1.7}$$

in continuous time (see [6]), or

$$x(t+1)=\{F-FP(s)H^T[HP(s)H^T+I]^{-1}H\}x(t) \tag{1.8}$$

in discrete time [5], for *arbitrary but fixed s*.

The results of [3] and [4] were instrumental in establishing the discrete-time results of [5]. In the process of proving the continuous-time results of [6], it was found that the continuous-time counterparts of the most crucial discrete-time results of [3] and [4] were not only not available in the literature, but were apparently unknown to authoritative experts in the field. For example, all the theorems of [7] on the continuous-time ARE (1.3) assume that $[H, F]$ is detectable and start with "If a strong solution P exists, it has the following properties." Our first theorem will prove the existence of a strong solution when $[H, F]$ is detectable. The results of this note therefore fill gaps in the existing literature; they are not surprising because they have either discrete-time counterparts (in [3]–[5]) or continuous-time antecedents (in [7]). However, we believe that it serves a useful purpose to make these results available, and they were actually needed to establish the results of [5] and [6]; as a matter of fact, the proof of the main theorem of [5] (Theorem 2) contains an abysmal gap, which the present note fills. Finally, it is worth noting that the proof techniques of all but one of our theorems are completely different, and for the most part much simpler, than those of [3]–[7].

II. A Continuous-Time ARE Result

Theorem 1: If the pair $[H, F]$ is detectable, then the continuous-time ARE (1.3) has a unique strong solution $\bar{P}$.

Proof: We construct the strong solution of (1.3) by considering a sequence of algebraic Riccati equations and then using a limiting process. Indeed, for each integer k, we consider the ARE

$$PF^T+FP-PH^THP+Q+\frac{1}{k}I=0. \tag{2.1}$$

Given that the matrix $Q + (1/k)I$ is positive-definite and the pair $[H, F]$ is detectable, it follows that there exists a unique positive-semidefinite symmetric solution P_k to (2.1) such that

$$F-P_kH^TH$$

is a stability matrix; see [8, Theorem 4.11]. Now using [7, Theorem 1] it follows that $P_1 \geqslant P_2 \cdots \geqslant P_k \cdots$. Thus, each matrix P_k is contained in the set $S \triangleq \{P \in \mathbb{R}^{n\times n}: 0 \leqslant P \leqslant P_1$ and P is symmetric$\}$. This set is bounded, since if we consider the matrix norm $\|P\| = \lambda_{\max}(P^TP)$ then for any $P \in S$, $\|P\| \leqslant \|P_1\|$. Furthermore, S is closed, and hence S is a compact set. Therefore, the sequence $P_1, P_2, \cdots$ must have a convergent subsequence $P_{\bar{k}} \to \bar{P} \in S$ (see [9, Ch. 9]). This is the crucial existence step. We will show that $\bar{P}$ is the required strong solution of (1.3). We have

$$P_{\bar{k}}F^T+FP_{\bar{k}}-P_{\bar{k}}H^THP_{\bar{k}}+Q+\frac{1}{\bar{k}}I=0.$$

Taking the limit as $\bar{k} \to \infty$, it follows that

$$\bar{P}F^T+F^T\bar{P}-\bar{P}H^TH\bar{P}+Q=0,$$

i.e., $\bar{P}$ satisfies (1.3). Furthermore, for each $\bar{k}$ the eigenvalues of $F - P_{\bar{k}}H^TH$ lie in the open left half plane. We note that the eigenvalues of $F - PH^TH$ are continuous functions of P. Thus, taking the limit as $\bar{k} \to \infty$, it follows that the eigenvalues of $F - \bar{P}H^TH$ lie in the closed left half plane. This proves that $\bar{P}$ is a strong solution to (1.3). Given that there exists a strong solution to (1.3), it follows from [7, Theorem 3] that this is the unique strong solution. □

In [14] Molinari establishes the existence of a unique strong solution when either $[H, F]$ is observable [14, Theorem 5a] or $[H, F]$ is detectable and the associated Hamiltonian matrix has no imaginary eigenvalues [14, Corollary p. 354]. Our result fills the gap between these two.

III. Two Continuous-Time RDE Results

We first prove an inequality, whose discrete-time counterpart was crucial in establishing monotonicity of the solutions of the RDE (1.2) under certain initial conditions in [5]. The proof of the discrete-time counterpart (see [5] or [6, Lemma 1]) relies on an inequality obtained by Nishimura [10]. Our continuous-time proof uses a simple optimal control argument.

Theorem 2: Consider the continuous-time RDE (1.1) with initial condition $P_o \geqslant 0$ and the following RDE:

$$\dot{S}(t)=S(t)F^T+FS(t)-S(t)H^THS(t)+M,\ S(0)=S_o \geqslant 0. \tag{3.1}$$

Suppose the solutions to both RDE's exist. If $P_o \geqslant S_o$ and $Q \geqslant M$, then $P(t) \geqslant S(t)$ for all $t \geqslant 0$.

Proof: If the solutions exist, then given arbitrary but fixed x_1, and $t_1 \geqslant 0$, we have (see [11, Ch. 21]):

$$x_1^T P(t_1) x_1 = \min_{u(\cdot)} \left\{ \int_0^{t_1} [u^T(\sigma)u(\sigma) + x^T(\sigma)Qx(\sigma)]\, d\sigma + x^T(0)P_o x(0) \right\} \tag{3.2}$$

and

$$x_1^T S(t_1) x_1 = \min_{u(\cdot)} \left\{ \int_0^{t_1} [u^T(\sigma)u(\sigma) + x^T(\sigma)Mx(\sigma)]\, d\sigma + x^T(0)S_o x(0) \right\} \tag{3.3}$$

subject to

$$\dot{x} = -F^T x + H^T u, \quad x(t_1) = x_1. \tag{3.4}$$

Let $\tilde{u}(\cdot)$ and $\tilde{x}(\cdot)$ be the optimal solution to (3.2) and (3.4), and evaluate the right-hand side of (3.3) for this control $\tilde{u}(\cdot)$. Then the corresponding cost is

$$\eta = \int_0^{t_1} [\tilde{u}^T(\sigma)\tilde{u}(\sigma) + \tilde{x}^T(\sigma)M\tilde{x}(\sigma)]\, d\sigma + \tilde{x}^T(0)S_o\tilde{x}(0)$$

$$= x_1^T P(t_1)x_1 + \int_0^{t_1} \tilde{x}^T(\sigma)[M-Q]\tilde{x}(\sigma)\, d\sigma + \tilde{x}^T(0)[S_o - P_o]\tilde{x}(0)$$

$$\leqslant x_1^T P(t_1)x_1 \quad \text{by our assumptions.}$$

Since the optimal cost for problems (3.3), (3.4) is $x_1^T S(t_1)x_1$, we have

$$x_1^T S(t_1)x_1 \leqslant \eta \leqslant x_1^T P(t_1)x_1.$$

Since this is true for any x_1 and any $t_1 \geqslant 0$, we have $S(t) \leqslant P(t)\ \forall t \geqslant 0$. □

The next theorem is a continuous-time version of [4, Theorem 4.2].

Theorem 3: Consider the RDE (1.1) with $[H, F]$ detectable. Let $\bar{P}$ be the unique strong solution of the corresponding ARE (1.3) and let $P(0) \geqslant \bar{P}$. Then $\lim_{t\to\infty} P(t) = \bar{P}$.

Proof: Note first that $\bar{P}$ exists and is unique by Theorem 1. Furthermore, with the given initial condition, the existence of a solution to (1.1) follows from remark M in [12]. Now define

$$V(t) \triangleq P(t) - \bar{P}, \quad \bar{F} \triangleq F - \bar{P}H^T H.$$

Then $V(t)$ obeys the RDE

$$\dot{V}(t) = V(t)\bar{F}^T + \bar{F}V(t) - V(t)H^T H V(t) \tag{3.6a}$$

and by assumption

$$V(0) = P(0) - \bar{P} \geqslant 0. \tag{3.6b}$$

Notice that $\mathrm{Re}\,\lambda_i(\bar{F}) \leqslant 0$, $i = 1, \cdots, n$ since $\bar{P}$ is strong.

Now, for a given $V(0)$ satisfying (3.6b), consider the following linear matrix equation:

$$\dot{S}(t) = -\bar{F}^T S(t) - S(t)\bar{F} + H^T H \tag{3.7a}$$

with

$$S(0) \triangleq S_o \triangleq \hat{V}^{-1}(0) > 0 \text{ and } \hat{V}(0) \geqslant V(0). \tag{3.7b}$$

It can be verified by substitution that the solution of (3.7) is

$$S(t) = e^{-\bar{F}^T t} S_o e^{-\bar{F}t} + W(t) \tag{3.8}$$

where

$$W(t) \triangleq \int_0^t e^{-\bar{F}^T(t-\tau)} H^T H e^{-\bar{F}(t-\tau)}\, d\tau. \tag{3.9}$$

Since $S_o > 0$, it follows that $S(t) > 0$ for all $t \geqslant 0$. Therefore, we can define $\hat{V}(t) \triangleq S^{-1}(t)$. Then $\hat{V}(t)$ satisfies the RDE (3.6a) with initial condition $\hat{V}(0)$ satisfying (3.7b). The remainder of the proof will be to show that $\lim_{t\to\infty} \lambda_{\min}(S(t)) = \infty$. This will imply $\lim_{t\to\infty} \hat{V}(t) = 0$, from which $\lim_{t\to\infty} V(t) = 0$ will follow by Theorem 2, using (3.7b).

Since $[H, \bar{F}]$ is detectable, we shall assume without loss of generality that $[H, \bar{F}]$ has been transformed into

$$H = [0 \ \vdots\ H_2], \quad \bar{F} = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}$$

where H_2 and F_{22} have compatible dimensions, with $\mathrm{Re}\,\lambda_i(F_{11}) < 0$ for all i, and $[H_2, F_{22}]$ completely observable. Consider now an arbitrary constant nonzero vector $x \in \mathbb{R}^n$ and partition it as $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ with $\dim x_1 = \dim F_{11}$. We show that $\lim_{t\to\infty} x^T S(t)x = \infty$, by considering two cases separately.

Case 1: Suppose first that $x_2 = 0$, so that $x_1 \neq 0$. Then

$$\begin{aligned} x^T S(t) x &= x_1^T e^{-F_{11}^T t} S_{11} e^{-F_{11}t} x_1 \\ &\geqslant [\lambda_{\min}(S_{11})] x_1^T e^{-F_{11}^T t} e^{-F_{11}t} x_1 \\ &= \|y(t)\|^2 \lambda_{\min}(S_{11}) \end{aligned} \tag{3.10}$$

where

$$S_o = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

with $\dim S_{11} = \dim F_{11}$, and where $y(t)$ is the solution of $\dot{y}(t) = -F_{11}y(t)$ with $y(0) = x_1 \neq 0$. Since $\mathrm{Re}\,\lambda_i(-F_{11}) > 0$, it follows that $\lim_{t\to\infty} \|y(t)\|^2 = \infty$ and since $S_o > 0$ it follows that $\lim_{t\to\infty} x^T S(t)x = \infty$ for all such x.

Case 2: Suppose now that $x_2 \neq 0$. Then

$$x^T W(t, 0) x = x_2^T \left\{ \int_0^t e^{-F_{22}^T(t-\tau)} H_2^T H_2 e^{-F_{22}(t-\tau)}\, d\tau \right\} x_2.$$

Call $R(t)$ the matrix within the braces. Then $R(t)$ is the solution of the equation

$$\dot{R}(t) = -F_{22}^T R(t) - R(t)F_{22} + H_2^T H_2, \qquad R(0) = 0 \tag{3.11}$$

and $R(t) \geqslant 0$ by definition. We prove that $\lim_{t\to\infty} \lambda_{\min} R(t) = \infty$. Since the pair $[H_2, -F_{22}]$ is completely observable (and $R(t)$ is the observability Grammian of that pair), it follows that $R(\delta) > 0$ for some $\delta > 0$. Also $R(t)$ is monotonically nondecreasing, i.e., $R(t_1) \geqslant R(t_2)$ for $t_1 \geqslant t_2$ since the integrand is nonnegative definite.

Therefore, $R(t) > 0$ for all $t \geqslant \delta$. Define $M(t) \triangleq R^{-1}(t)$ for all $t \geqslant \delta$. Then for $t \geqslant \delta$, $M(t)$ satisfies the RDE

$$\dot{M}(t) = M(t)F_{22}^T + F_{22}M(t) - M(t)H_2^T H_2 M(t), \quad M(\delta) = R^{-1}(\delta). \tag{3.12}$$

The fact that $R(t)$ is monotonically nondecreasing implies that $M(t)$ is monotonically nonincreasing and since $R(t) > 0$ and finite for all finite $t \geqslant \delta$, it follows that $M(t) \geqslant 0$ for all $t \geqslant \delta$. Therefore, applying the "remarkable monotone sequence theorem" (see [13, Theorem 1, p. 169]) it follows that $M(t)$ converges. Call $\bar{M}$ its convergence point; then M is a nonnegative definite solution of the ARE

$$MF_{22}^T + F_{22}M - MH_2^T H_2 M = 0. \tag{3.13}$$

However, $\lambda_i(F_{22}) \leqslant 0$ for all i and, hence, $M = 0$ is the unique strong solution of (3.13) by Theorem 1. Moreover, by [7, Theorem 2] the strong solution of (3.13) is its maximal solution; therefore $\bar{M}$ must be the strong solution; i.e., $\bar{M} = 0$. We have thus shown that $\lim_{t\to\infty} M(t) = 0$; hence $\lim_{t\to\infty} \lambda_{\min} R(t) = \infty$ and therefore $\lim_{t\to\infty} x^T S(t)x = \infty$ for all x with $x_2 \neq 0$, and (by Cases 1 and 2) for all $x \neq 0$.

It follows that $\lim_{t\to\infty} \hat{V}(t) = 0$. Since $V(0) \leqslant \hat{V}(0)$, and $V(t)$ and $\hat{V}(t)$ are both solutions of (3.6a), it follows by Theorem 2 that $\lim_{t\to\infty} V(t) = 0$. Hence, $\lim_{t\to\infty} P(t) = \bar{P}$.

Under the additional assumption that the associated Hamiltonian matrix has no imaginary eigenvalues and has a simplified eigenstructure, Theorem 3 can also be obtained from [12, Theorem 4]. One motivation for our deriving these new Riccati results was a signal processing application where the Hamiltonian naturally has multiple imaginary eigenvalues [15].

IV. A Discrete-Time ARE Result

We now prove a result for the discrete-time ARE (1.4): it is a special case of a continuous-time result proved by Wimmer [7].

Theorem 4: Consider the ARE (1.4) with $[H, F]$ detectable. Let P_1 be the unique strong solution for $Q = Q_1 \geqslant 0$, and let P_2 be the unique strong solution for $Q = Q_2 \geqslant 0$. Then $Q_1 \geqslant Q_2$ implies $P_1 \geqslant P_2$.

Proof: The existence and uniqueness of the strong solutions follows from [3] and [4]. We prove the inequality using a dual optimal control problem. Define

$$J_i(x_o, N, U) = x^T(N)P_f x(N) + \sum_{t=0}^{N-1} [x^T(t)Q_i x(t) + u^T(t)u(t)], \quad i=1, 2 \tag{4.1}$$

where P_f is a symmetric nonnegative definite matrix such that $P_f \geqslant P_1$ and $P_f \geqslant P_2$, x_o is an arbitrary nonzero vector and U denotes $\{u(0), \cdots, u(N-1)\}$. If we minimize J_1 and J_2 w.r.t. U subject to

$$x(t+1) = F^T x(t) + H^T u(t), \; x(0) = x_o \tag{4.2}$$

then the optimal costs are given, respectively, by

$$J_1^*(x_o, N) = x_o^T P_1(N) x_o \text{ and } J_2^*(x_o, N) = x_o^T P_2(N) x_o \tag{4.3}$$

where $P_i(N)$, $i = 1, 2$ are the solutions of the RDE

$$\begin{cases} P_i(t+1) = FP_i(t)F^T - FP_i(t)H^T[HP_i(t)H^T + I]^{-1}HP_i(t)F^T + Q_i & (4.4a) \\ P_i(0) = P_f \qquad i=1, 2. & (4.4b) \end{cases}$$

We now show that $P_1(N) \geqslant P_2(N)$ by contradiction. Suppose there exists an x_o such that $x_o^T P_1(N) x_o < x_o^T P_2(N) x_o$ and let U^* be the corresponding optimal control sequence for the problem $\min_U J_1(x_o, N, U)$. Applying this sequence U^* to $J_2(x_o, N, U)$, we then have

$$J_2[x_o, N, U^*] = J_1^*(x_o, N) + \sum_{t=0}^{N-1} [x^T(t)[Q_2 - Q_1]x(t)]$$

$$\leqslant J_1^*(x_o, N) = x_o^T P_1(N) x_o < x_o^T P_2(N) x_o = J_2^*(x_o, N).$$

This is a contradiction, since $J_2^*(x_o, N)$ is the optimal cost. Hence, $P_1(N) \geqslant P_2(N)$ for all $N \geqslant 0$. Now, since $P_f \geqslant P_i$, $i = 1, 2$, it follows by [4, Theorem 4.3] that $\lim_{N\to\infty} P_i(N) = P_i$, $i = 1, 2$. Therefore, $P_1 \geqslant P_2$. □

V. Comments on a Result of [5]

The results of the previous sections allow us to correct an erroneous argument in the proof of Theorem 2 of [5]; at the time of the writing of [5] the present results were not available. Theorem 2 of [5] gives a set of sufficient conditions under which the system (1.8) is exponentially asymptotically stable, where $P(s)$ is the solution of the RDE (1.2) frozen at an arbitrary but fixed iteration s.

In the proof of Theorem 2 of [5], Proposition 2 of the same paper was used to establish that the solution $P(t)$ of (1.2) converges to the strong solution $\bar{P}$ of (1.4). However, to use Proposition 2 requires $[F, Q^{1/2}]$ stabilizable in (1.2), which is not assumed in the theorem. The argument should go as follows. Condition (3) of Theorem 2 of [5] implies, by Theorem 4 of the present note, that $P_o \geqslant \bar{P}$, where $\bar{P}$ is the unique strong solution of (1.4), which exists because $[H, F]$ is detectable. It then follows from [4, Theorem 4.3] that $\lim_{t\to\infty} P(t) = \bar{P}$, where $P(t)$ is the solution of (1.2). The other parts of the proof remain unchanged.

VI. Conclusions

We have filled some existing gaps in the literature on the Riccati equation, both the RDE and the ARE, in both discrete-time and continuous-time. Although none of our results are surprising, these are all fundamental properties. Three of the authors of this note needed them in proving results of [5] and [6]. We hope that their publication will prove useful to other researchers in deriving their own results. To paraphrase Frank Zappa: "All our results are Australian made; they are a little bit cheesy but nicely displayed."

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Trace Bounds on the Solution of the Algebraic Matrix Riccati and Lyapunov Equation

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Abstract—**Lower and upper bounds on the trace of the positive semidefinite solution of the algebraic matrix Riccati and Lyapunov**

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