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A MIXED–FEM AND BEM COUPLING FOR A THREE-DIMENSIONAL EDDY CURRENT PROBLEM *, **

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Abstract. We study in this paper the electromagnetic field generated in a conductor by an alternating current density. The resulting interface problem (see Bossavit (1993)) between the metal and the dielectric medium is treated by a mixed–FEM and BEM coupling method. We prove that our BEM-FEM formulation is well posed and that it leads to a convergent Galerkin method.

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1. Introduction

The eddy current model is commonly used as an approximation of Maxwell equations in problems related to machines working at power frequencies. Formally speaking, this sub-model is obtained by neglecting the displacement currents in Ampère's law. The paper of Ammari *et al.* [2] gives conditions under which the approximation of Maxwell equations by the eddy current model is reliable.

The purpose of this paper is to provide a new method, based on a combination of finite elements (FEM) and boundary elements (BEM), to compute the eddy currents generated in a passive conductor $\Omega \subset \mathbb{R}^3$ by a divergence free source current $\widehat{\mathbf{j}}$ featuring sinusoidal dependence in time. We point out that, generally, the eddy current problem is reformulated by expressing the magnetic field in terms of the electric field and *vice versa*. The two formulations resulting from these two *dual* proceedings are equivalent at the continuous level but they lead to different numerical schemes; in spite of the fact that in both cases the discretization process relays upon the edge finite element method, which is recognized now to be well suited for the approximation of electromagnetic vector-fields, see [4, 7, 9].

The idea of using FEM-BEM methods for solving eddy current problems in dimension three has been exploited in several works of Bossavit [5,6,8,9]. The main idea of such an approach consists in providing non-local boundary conditions for a finite element treatment of the problem in the conductor Ω where the eddy currents are to be computed. These non-local boundary conditions are deduced from an integral representation of the solution in the domain occupied by the dielectric medium. The formulations of Bossavit relay on the so-called *one boundary integral approach* introduced by Johnson and Nédélec for the Laplace equation [21]. However, it is not written

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in the standard form of the Johnson–Nédélec formulation since the normal derivative on the interface boundary is eliminated *via* Steklov–Poincaré operator. This method has been implemented in a code named TRIFOU (*cf.* [5]) but, at the authors knowledge, an analysis of this coupling scheme is not available. In any case, the analysis of the Johnson–Nédélec method requires an hypothesis that constitutes a severe limitation in practice: the interface boundary $\Gamma := \partial\Omega$ must be regular since the term associated to the double layer potential must be considered as a compact perturbation, *cf.* [21, 23, 25].

In recent years, a symmetric method for the coupling of finite element and boundary element methods has been developed and applied to various interface problems, see [16, 18]. The most important feature of this FEM-BEM approach is that it preserves the coercivity of the original problem and, by the way, relaxes the regularity requirements on the coupling boundary. In this paper, we show that this coupling method can be successfully applied to the eddy current interface problem studied by Bossavit. We prove convergence and error estimates for this new coupling method. Moreover, thanks to the important tools given recently in [11, 12, 14], our analysis can be carried out in the case of a Lipschitz domain Ω with no restrictions on its topology.

A similar program has been realized by Hiptmair in [20]. Hiptmair chooses in his formulation the electric field as primary unknown. Furthermore, the non-local boundary condition on the FEM-BEM coupling interface are deduced directly from a Stratton–Chu integral representation of the electric field. Our method, which has been first presented in the two dimensional case in [24], is inspired from the TRIFOU method. It uses the magnetic field and its scalar potential as principle unknowns.

Both approaches lead to systems of linear equations that have the same structure and the same size. However, when Ω is non-simply connected, our method is more expensive than that of Hiptmair since the construction of the discrete problem requires in our case the resolution of some auxiliary problems that depend on the geometry and the topology of the conductor, see Section 6.

We finally point out that Hiptmair assumes in his paper [20] that the exiting current $\widehat{\mathbf{j}}$ is supported inside the conductor. This case can also be covered by our method without difficulty. Nevertheless, here we make the opposite assumption: $\mathrm{support}\left(\widehat{\mathbf{j}}\right) \subset \mathbb{R}^3 \setminus \overline{\Omega}$. In the latter case, it is necessary to remove the non-homogeneity of the equation in order to obtain an adequate representation formula of the solution in the unbounded domain $\mathbb{R}^3 \setminus \overline{\Omega}$. Following [9], we show that it is easy to handle this additional difficulty when considering the magnetic field as state variable.

The paper is organized as follows. In Section 2 we summarize some results from [11, 12, 14] concerning function spaces of tangential traces and tangential differential operators defined on a Lipschitz boundary. In Section 3, we introduce the model problem and derive its FEM-BEM formulation staring from an integral representation formula for the scalar magnetic potential. We prove that the resulting weak formulation is uniquely solvable. We prepare the discrete scheme of our variational problem by introducing in Section 4 the edge finite element and by recalling some fundamental properties of the corresponding interpolation operator. In Section 5, we treat separately the discretization of our problem in the simply connected case. The derivation of the Galerkin scheme is straightforward and its convergence analysis relays on Céa's lemma and the interpolation error estimates given in the previous section. An optimal order of convergence in obtained. We also illustrate in this section the applicability of our discrete scheme by describing its matrix representation. The derivation of the discrete problem in the case of a non-simply connected conductor is less obvious, it requires several types of approximations of the harmonic Neumann vector fields. These approximations are obtained by solving the discrete auxiliary problems described in Section 6. The convergence analysis of the resulting discrete scheme (in the non-simply connected case) is reported in Section 7.

2. Preliminaries

In the sequel we deal with complex valued functions and the symbol $\imath$ is used for $\sqrt{-1}$. Boldface letters will denote vectors or vector–valued functions (in $\mathbb{C}^3$). We denote by $\overline{\alpha}$ the conjugate of a complex number $\alpha \in \mathbb{C}$. We also denote by $|\alpha|$ its modulus and by $\Re\alpha$ its real part. The symbol $|\cdot|$ will represent as well the 2-norm for vectors:

$$|\mathbf{q}|^2 = \mathbf{q} \cdot \overline{\mathbf{q}} := \sum_{i=1}^{3} q_i \overline{q}_i.$$

Throughout this paper C, with or without subscripts, will denote positive constants, not necessarily the same at different occurrences, which are independent of the parameter h and functions involved.

In all the paper Ω is a bounded Lipschitz domain in $\mathbb{R}^3$ such that the complement $\Omega_c := \mathbb{R}^3 \setminus \overline{\Omega}$ is connected. The domain Ω is the union of m connected components Ω_j, $j = 1, \cdots, m$ whose boundaries Γ_j are closed and disjoint surfaces. Setting $\Gamma = \cup_{j=1}^m \Gamma_j$ we get $\Gamma = \partial\Omega = \partial\Omega_c$. Since the analysis given in the following can be extended straightforwardly to the multi-connected case, for the sake of simplicity in exposition, we will assume that Ω is also connected.

We remark that, under the above conditions, Ω and Ω_c have the same first Betti number B. There exist B disjoint connected open surfaces $\Sigma_k^{\text{ext}} \subset \Omega_c$ (respectively $\Sigma_k^{\text{int}} \subset \Omega$), $k = 1, \cdots, B$, such that $\Omega_c^0 := \Omega_c \setminus \cup_{k=1}^B \Sigma_k^{\text{ext}}$ (respectively $\Omega^0 := \Omega \setminus \cup_{k=1}^B \Sigma_k^{\text{int}}$) is simply connected. The boundary curves $\gamma_k^{\text{ext}} := \partial\Sigma_k^{\text{ext}}$ and $\gamma_k^{\text{int}} := \partial\Sigma_k^{\text{int}}$ lie on Γ.

We denote by

$$(f, g)_{0,\Omega} := \int_\Omega f\overline{g}\, \mathrm{d}x$$

the inner product in $\mathrm{L}^2(\Omega)$ and $\|\cdot\|_{0,\Omega}$ the corresponding norm. As usual, $\|\cdot\|_{s,\Omega}$ stands for the norm of the Hilbertian Sobolev space $\mathrm{H}^s(\Omega)$ $\forall s \in \mathbb{R}$. We also recall that, for any $t \in [-1, 1]$, the spaces $\mathrm{H}^t(\Gamma)$ have an intrinsic definition (by localization) on the Lipschitz surface Γ due to their invariance under Lipschitz coordinate transformations. We denote by $\|\cdot\|_{t,\Gamma}$ the norm in $\mathrm{H}^t(\Gamma)$ and by $\langle \cdot, \cdot \rangle_{t,\Gamma}$ the duality pairing between $\mathrm{H}^{-t}(\Gamma)$ and $\mathrm{H}^t(\Gamma)$. Here $\mathrm{L}^2(\Gamma)$ is taken as pivot space. Hence, if $0 < t \leq 1$ and $\lambda \in \mathrm{L}^2(\Gamma)$ we have

$$\langle \lambda, \eta \rangle_{t,\Gamma} = \int_\Gamma \lambda \overline{\eta}\, \mathrm{d}\xi \quad \forall \eta \in \mathrm{H}^t(\Gamma).$$

In this paper, the spaces that are product forms of the previous function spaces are endowed with the natural product norm and duality pairing without changing the notations since it will be clear from the context when scalar or vector functions are used.

We set:

$$\mathbf{H}^s(\Omega) := (\mathrm{H}^s(\Omega))^3, \quad \mathbf{V} := \left(\mathrm{H}^{1/2}(\Gamma)\right)^3, \quad \mathbf{V}' := \left(\mathrm{H}^{-1/2}(\Gamma)\right)^3,$$
$$\mathbf{L}_t^2(\Gamma) := \left\{\mathbf{q} \in \mathbf{L}^2(\Gamma); \quad \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \Gamma\right\},$$

where $\mathbf{n}$ is the unit normal to Γ that points from Ω into Ω_c. We also introduce the functional space

$$\mathbf{H}(\mathbf{curl}, \Omega) := \left\{\mathbf{q} \in \mathbf{L}^2(\Omega);\ \mathbf{curl}\, \mathbf{q} \in \mathbf{L}^2(\Omega)\right\};$$

endowed with the natural norm: $\|\mathbf{q}\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 := \|\mathbf{q}\|_{0,\Omega}^2 + \|\mathbf{curl}\, \mathbf{q}\|_{0,\Omega}^2$.

We shall make use of some recent results on the characterization of traces of functions in $\mathbf{H}(\mathbf{curl}, \Omega)$ and on the properties of tangential differential operators defined on the Lipschitz manifold Γ. We present here a brief description of these results and refer to [11–15] for details and proofs.

Let $\mathcal{D}(\overline{\Omega})$ be the space of indefinitely differentiable functions on the closure of Ω. We define the tangential components trace mapping $\pi_\tau : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{L}_t^2(\Gamma)$ and the tangential trace mapping $\gamma_\tau : \mathcal{D}(\overline{\Omega})^3 \to \mathbf{L}_t^2(\Gamma)$ as $\mathbf{q} \mapsto \mathbf{n} \wedge (\mathbf{q}_{|\Gamma} \wedge \mathbf{n})$ and $\mathbf{q} \mapsto \mathbf{q}_{|\Gamma} \wedge \mathbf{n}$ respectively.

We denote by γ the standard trace operator acting on vectors: $\gamma : \mathbf{H}^1(\Omega) \to \mathbf{V}$, $\gamma(\mathbf{q}) = \mathbf{q}_{|\Gamma}$. Let γ^{-1} be one of its continuous right inverses. We will also use the notation π_τ (resp. γ_τ) for the composite operator $\pi_\tau \circ \gamma^{-1}$

(resp. $\gamma_\tau \circ \gamma^{-1}$) which acts only on traces. By density of $\mathcal{D}(\overline{\Omega})^3_{|\Gamma}$ into $\mathbf{L}^2(\Gamma)$, the operators π_τ and γ_τ can be extended to linear continuous operators in $\mathbf{L}^2(\Gamma)$.

The spaces $\mathbf{V}_\gamma := \gamma_\tau(\mathbf{V})$ and $\mathbf{V}_\pi := \pi_\tau(\mathbf{V})$ endowed with the norms

$$\|\boldsymbol{\lambda}\|_{\mathbf{V}_\gamma} := \inf_{\mathbf{q}\in\mathbf{V}} \{\|\mathbf{q}\|_{\mathbf{V}}\,;\ \gamma_\tau(\mathbf{q}) = \boldsymbol{\lambda}\} \quad \text{and} \quad \|\boldsymbol{\lambda}\|_{\mathbf{V}_\pi} := \inf_{\mathbf{q}\in\mathbf{V}} \{\|\mathbf{q}\|_{\mathbf{V}}\,;\ \pi_\tau(\mathbf{q}) = \boldsymbol{\lambda}\}$$

are Hilbert spaces. We denote by $\mathbf{V}'_\gamma$ and $\mathbf{V}'_\pi$ their dual spaces respectively with $\mathbf{L}^2_t(\Gamma)$ as pivot. We remark that if Γ is a sufficiently smooth variety then $\mathbf{V}_\gamma = \mathbf{V}_\pi = \mathbf{TH}^{1/2}(\Gamma)$, where $\mathbf{TH}^{1/2}(\Gamma)$ denotes the space of vectors tangent to Γ that have components in $\mathrm{H}^{1/2}(\Gamma)$. However, these two spaces are in general different; see for example the characterizations given in [13] for $\mathbf{V}_\gamma$ and $\mathbf{V}_\pi$ in the case of a polyhedral domain Ω.

We introduce the tangential differential operators

$$\nabla_\Gamma \varphi := \pi_\tau(\nabla\varphi) \quad \text{and} \quad \mathbf{curl}_\Gamma\, \varphi := \gamma_\tau(\nabla\varphi) \qquad \forall \varphi \in \mathrm{H}^2(\Omega).$$

It is clear that $\nabla_\Gamma : \mathrm{H}^{3/2}(\Gamma) \to \mathbf{V}_\pi$ and $\mathbf{curl}_\Gamma : \mathrm{H}^{3/2}(\Gamma) \to \mathbf{V}_\gamma$ are continuous, where the Hilbert space $\mathrm{H}^{3/2}(\Gamma) := \left\{\varphi_{|\Gamma},\ \varphi \in \mathrm{H}^2(\Omega)\right\}$ is endowed with the norm

$$\|\lambda\|_{3/2,\Gamma} := \inf_{\varphi\in \mathrm{H}^2(\Omega)} \left\{\|\varphi\|_{2,\Omega}\,;\ \gamma(\varphi) = \lambda\right\}.$$

Let $\mathrm{H}^{-3/2}(\Gamma)$ be the dual space of $\mathrm{H}^{3/2}(\Gamma)$ with $\mathrm{L}^2(\Gamma)$ as pivot space. We define $\mathrm{div}_\Gamma : \mathbf{V}'_\pi \to \mathrm{H}^{-3/2}(\Gamma)$ by the duality

$$\langle \mathrm{div}_\Gamma\, \boldsymbol{\lambda},\ \varphi\rangle_{3/2,\Gamma} := -\,\langle \boldsymbol{\lambda},\ \nabla_\Gamma \varphi\rangle_{\mathbf{V}'_\pi\times\mathbf{V}_\pi} \quad \forall \varphi \in \mathrm{H}^2(\Omega),$$

where $\langle\cdot,\ \cdot\rangle_{3/2,\Gamma}$ denotes the duality pairing between $\mathrm{H}^{-3/2}(\Gamma)$ and $\mathrm{H}^{3/2}(\Gamma)$ while $\langle\cdot,\ \cdot\rangle_{\mathbf{V}'_\pi\times\mathbf{V}_\pi}$ denotes the duality pairing between $\mathbf{V}'_\pi$ and $\mathbf{V}_\pi$. Similarly, $\mathrm{curl}_\Gamma : \mathbf{V}'_\gamma \to \mathrm{H}^{-3/2}(\Gamma)$ is defined by

$$\langle \mathrm{curl}_\Gamma \boldsymbol{\lambda},\ \varphi\rangle_{3/2,\Gamma} := \langle \boldsymbol{\lambda},\ \mathbf{curl}_\Gamma\, \varphi\rangle_{\mathbf{V}'_\gamma\times\mathbf{V}_\gamma} \quad \forall \varphi \in \mathrm{H}^2(\Omega).$$

Now, the Green's formula

$$(\mathbf{u},\ \mathbf{curl}\,\mathbf{v})_{0,\Omega} - (\mathbf{curl}\,\mathbf{u},\ \mathbf{v})_{0,\Omega} = \langle \gamma_\tau \mathbf{u},\ \pi_\tau \mathbf{v}\rangle_{\mathbf{V}'_\pi\times\mathbf{V}_\pi} \quad \forall \mathbf{u} \in \mathcal{D}(\overline{\Omega})^3,\ \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{1}$$

and the density of $\mathcal{D}(\overline{\Omega})^3$ in $\mathbf{H}(\mathbf{curl},\Omega)$ permit one to prove that $\gamma_\tau : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{V}'_\pi$ is linear continuous since by construction $\pi_\tau(\mathbf{H}^1(\Omega)) = \mathbf{V}_\pi$. Actually, a more precise result has been proved in [15]. Let us introduce the space

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \left\{\boldsymbol{\lambda} \in \mathbf{V}'_\pi;\quad \mathrm{div}_\Gamma\, \boldsymbol{\lambda} \in \mathrm{H}^{-1/2}(\Gamma)\right\}$$

endowed with the graph norm $\|\boldsymbol{\lambda}\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} := \|\boldsymbol{\lambda}\|^2_{\mathbf{V}'_\pi} + \|\mathrm{div}_\Gamma\, \boldsymbol{\lambda}\|^2_{-1/2,\Gamma}$.

Theorem 2.1. *The tangential trace mapping* $\gamma_\tau : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *is continuous, surjective and possesses continuous right inverses.*

We associate to any $\mathbf{g} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ the unique solution $\mathcal{R}\mathbf{g}$ of

$$\begin{aligned} &\text{find } \mathcal{R}\mathbf{g} \in \mathbf{H}(\mathbf{curl},\Omega) \text{ such that } \gamma_\tau\mathcal{R}\mathbf{g} = \mathbf{g} \text{ and}\\ &(\mathcal{R}\mathbf{g},\ \mathbf{q})_{\mathbf{H}(\mathbf{curl},\Omega)} = 0 \quad \forall \mathbf{q} \in \mathbf{H}(\mathbf{curl},\Omega) \cap \ker \gamma_\tau, \end{aligned} \tag{2}$$

where $(\cdot,\cdot)_{\mathbf{H}(\mathbf{curl},\Omega)}$ stands for the inner product of $\mathbf{H}(\mathbf{curl},\Omega)$. We remark that we have just defined a continuous right inverse $\mathcal{R}:\ \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\to\mathbf{H}(\mathbf{curl},\Omega)$ of γ_τ since, by virtue of the closed graph theorem,

$$\|\mathcal{R}\mathbf{g}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\le C\,\|\mathbf{g}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\quad\forall\mathbf{g}\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma).\tag{3}$$

We point out that taking $\mathbf{u}=\nabla\varphi$ in (1), with $\varphi\in\mathrm{H}^1(\Omega)$, one may also deduce that

$$\mathbf{curl}_\Gamma:\ \mathrm{H}^{1/2}(\Gamma)\to\mathbf{V}'_\pi$$

is linear and bounded.

We introduce the harmonic Neumann vector-fields associated with Ω_c and Ω by

$$\mathbb{H}(\Omega_c):=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega_c);\quad\mathbf{curl}\,\mathbf{v}=0,\quad\mathrm{div}\,\mathbf{v}=0,\quad\mathbf{v}\cdot\mathbf{n}_{|\Gamma}=0\right\}$$

and

$$\mathbb{H}(\Omega):=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega);\quad\mathbf{curl}\,\mathbf{v}=0,\quad\mathrm{div}\,\mathbf{v}=0,\quad\mathbf{v}\cdot\mathbf{n}_{|\Gamma}=0\right\}$$

respectively and let

$$\mathbb{H}:=\left\{\boldsymbol{\lambda}\in\mathbf{L}^2_t(\Gamma);\quad\mathrm{curl}_\Gamma\boldsymbol{\lambda}=0,\quad\mathrm{div}_\Gamma\,\boldsymbol{\lambda}=0\right\}\cdot$$

In fact, $\mathbb{H}$ is none other than the direct sum of the tangential traces of the Neumann fields associated with Ω and Ω_c: $\mathbb{H}=\gamma_\tau(\mathbb{H}(\Omega_c))\oplus\gamma_\tau(\mathbb{H}(\Omega))$, see [11]. We point out that we are denoting here by γ_τ the tangential traces from both sides of Γ.

Let $\mathbf{X}$ be the subspace of $\mathbf{H}(\mathbf{curl},\Omega)$ defined by

$$\mathbf{X}:=\left\{\mathbf{q}\in\mathbf{H}(\mathbf{curl},\Omega);\quad\mathbf{curl}\,\mathbf{q}\cdot\mathbf{n}=0\ \text{ in }\mathrm{H}^{-1/2}(\Gamma)\right\}\cdot$$

Proposition 2.2. *The space* $\mathbf{X}$ *is closed in* $\mathbf{H}(\mathbf{curl},\Omega)$*. Moreover, we have the direct sum*

$$\gamma_\tau(\mathbf{X})=\mathbf{curl}_\Gamma\left(\mathrm{H}^{1/2}(\Gamma)\right)\oplus\mathbb{H}.$$

Proof. If $\mathbf{q}\in\mathbf{H}(\mathbf{curl},\Omega)$ then it is clear that $\mathbf{curl}\,\mathbf{q}\in\mathbf{H}(\mathrm{div},\Omega)$ where

$$\mathbf{H}(\mathrm{div},\Omega):=\left\{\mathbf{u}\in\mathbf{L}^2(\Omega);\ \mathrm{div}\,\mathbf{u}\in\mathrm{L}^2(\Omega)\right\}\cdot$$

This space is endowed with the norm $\|\mathbf{u}\|^2_{\mathbf{H}(\mathrm{div},\Omega)}:=\|\mathbf{u}\|^2_{0,\Omega}+\|\mathrm{div}\,\mathbf{u}\|^2_{0,\Omega}$. It is well known that the normal trace operator $\mathbf{v}\mapsto\mathbf{v}\cdot\mathbf{n}$ is linear continuous from $\mathbf{H}(\mathrm{div},\Omega)$ onto $\mathrm{H}^{-1/2}(\Gamma)$. Thus $\mathbf{X}$ is the kernel of a continuous linear operator on $\mathbf{H}(\mathbf{curl},\Omega)$ and the first assertion of the proposition follows.

Taking $\mathbf{v}=\nabla\varphi$ in (1) with $\varphi\in\mathrm{H}^2(\Omega)$ we obtain:

$$-(\mathbf{u}\wedge\mathbf{n},\ \nabla_\Gamma\varphi)_{0,\Gamma}=(\mathbf{curl}\,\mathbf{u},\ \nabla\varphi)_{0,\Omega}=(\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n},\ \varphi)_{0,\Gamma},\quad\forall\mathbf{u}\in\mathcal{D}(\overline{\Omega})^3.$$

Now, by definition of div_Γ

$$\langle\mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n}),\ \varphi\rangle_{1/2,\Gamma}=\langle\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n},\ \varphi\rangle_{1/2,\Gamma}\quad\forall\mathbf{u}\in\mathcal{D}(\overline{\Omega})^3,\ \forall\varphi\in\mathrm{H}^{1/2}(\Gamma),$$

since $\mathrm{H}^2(\Omega)_{|\Gamma}$ is dense in $\mathrm{H}^{1/2}(\Gamma)$. Therefore, we deduce the identity

$$\mathrm{div}_\Gamma\,\gamma_\tau\mathbf{u}=\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}\qquad\forall\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega),\tag{4}$$

by virtue of the density of $\mathrm{D}(\overline{\Omega})^3$ in $\mathbf{H}(\mathbf{curl},\Omega)$.

The result follows now from (4) and the following characterization of the kernel of div_Γ (*cf.* Th. 3.2 in [11])

$$\ker(\mathrm{div}_\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma(\mathrm{H}^{1/2}(\Gamma)) \oplus \mathbb{H}. \tag{5}$$

□

Proposition 2.3. *The following identity holds true*

$$(\mathbf{u},\ \mathbf{curl}\,\mathbf{v})_{0,\Omega} - (\mathbf{curl}\,\mathbf{u},\ \mathbf{v})_{0,\Omega} = (\mathbf{g}_u,\ \mathbf{n}\wedge\mathbf{g}_v)_{0,\Gamma} \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{X}, \tag{6}$$

where $\mathbf{g}_u$ *(respectively* $\mathbf{g}_v$*) is the component of* $\gamma_\tau\mathbf{u}$ *(respectively* $\gamma_\tau\mathbf{v}$*) that belongs to* $\mathbb{H}$*.*

Proof. It turns out that Green's formula (1) is still valid for functions $\mathbf{u},\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega)$ if the right hand side of the identity is adequately interpreted by means of the Hodge decomposition, see Proposition 3.7 in [11]. Identity (6) is then a direct consequence of the resulting integration by part formula. □

Theorem 2.4. *Let* $\widehat{\mathbf{X}}$ *be the closed subspace of* $\mathbf{X}$ *defined by*

$$\widehat{\mathbf{X}} := \{\mathbf{q}\in\mathbf{X};\quad (\mathbf{q},\ \mathbf{curl}\,\mathcal{R}\mathbf{g})_{0,\Omega} = (\mathbf{curl}\,\mathbf{q},\ \mathcal{R}\mathbf{g})_{0,\Omega}\quad \forall\mathbf{g}\in\mathbb{H}\}\cdot$$

Then, $\mathbf{X}$ *may be written as a direct sum of* $\widehat{\mathbf{X}}$ *and* $\mathcal{R}(\mathbb{H})$*:*

$$\mathbf{X} = \widehat{\mathbf{X}} \oplus \mathcal{R}(\mathbb{H})$$

and there exists a constant $\alpha_0 > 0$ *such that*

$$\alpha_0(\|\widehat{\mathbf{q}}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\mathcal{R}\mathbf{g}_q\|_{\mathbf{H}(\mathbf{curl},\Omega)}) \le \|\mathbf{q}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \qquad \forall \mathbf{q} = \widehat{\mathbf{q}} + \mathcal{R}\mathbf{g}_q \in \mathbf{X} \tag{7}$$

with $\widehat{\mathbf{q}}\in\widehat{X}$ *and* $\mathbf{g}_q\in\mathbb{H}$*.*

Proof. First of all, it is clear that $\widehat{\mathbf{X}}$ is a closed subspace of $\mathbf{X}$. Now, notice that $\mathbf{g}\in\mathbb{H}$ if and only if $\mathbf{n}\wedge\mathbf{g}$ belongs to $\mathbb{H}$ since $\mathrm{div}_\Gamma(\mathbf{n}\wedge\mathbf{g}) = -\mathrm{curl}_\Gamma\mathbf{g}$ and $\mathrm{curl}_\Gamma(\mathbf{n}\wedge\mathbf{g}) = \mathrm{div}_\Gamma\,\mathbf{g}$, *cf.* [14]. Given $\mathbf{q}\in\mathbf{X}$, Proposition 2.2 implies that $\gamma_\tau\mathbf{q}\in\mathbf{curl}_\Gamma(\mathrm{H}^{1/2}(\Gamma))\oplus\mathbb{H}$. Let $\mathbf{g}_q$ be the component of $\gamma_\tau\mathbf{q}$ that belongs to $\mathbb{H}$. Taking $\mathbf{u}=\mathbf{q}$ and $\mathbf{v}=\mathcal{R}(\mathbf{n}\wedge\mathbf{g}_q)$ in (6) we deduce that

$$\|\mathbf{g}_q\|_{0,\Gamma}^2 = (\mathbf{curl}\,\mathbf{q},\ \mathcal{R}(\mathbf{n}\wedge\mathbf{g}_q))_{0,\Omega} - (\mathbf{q},\ \mathbf{curl}\,\mathcal{R}(\mathbf{n}\wedge\mathbf{g}_q))_{0,\Omega}. \tag{8}$$

It follows that a function $\mathbf{q}\in\mathbf{X}$ belongs to $\widehat{\mathbf{X}}$ if and only if $\gamma_\tau\mathbf{q}\in\mathbf{curl}_\Gamma(\mathrm{H}^{1/2}(\Gamma))$. Furthermore, the decomposition $\mathbf{q} = (\mathbf{q}-\mathcal{R}\mathbf{g}_q) + \mathcal{R}\mathbf{g}_q$ shows that the direct sum asserted in the Theorem holds true since $\mathbf{q}-\mathcal{R}\mathbf{g}_q\in\widehat{\mathbf{X}}$.

In order to prove (7) we first observe that the continuity of the lifting $\mathcal{R}$ and the imbedding $\mathbf{L}_t^2(\Gamma)\hookrightarrow\mathbf{V}'_\pi$ give

$$\|\mathcal{R}\mathbf{g}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C_0\,\|\mathbf{g}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} = C_0\,\|\mathbf{g}\|_{\mathbf{V}'_\pi} \le C_0\,\|\mathbf{g}\|_{0,\Gamma} \quad \forall\mathbf{g}\in\mathbb{H}.$$

We use this estimate to deduce from (8) that

$$\|\mathbf{g}_q\|_{0,\Gamma}^2 \le C_0\ \|\mathbf{q}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,\|\mathbf{n}\wedge\mathbf{g}_q\|_{0,\Gamma}\,.$$

Combining the last two inequalities we obtain that

$$\|\mathcal{R}\mathbf{g}_q\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C_0^2\ \|\mathbf{q}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \quad \forall\mathbf{q}\in\mathbf{X}$$

and (7) follows. □

Proposition 2.5. *The linear operator* $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma)/\mathbb{C} \to \gamma_\tau(\widehat{\mathbf{X}})$ *is an isomorphism.*

Proof. As $\mathrm{div}_\Gamma : \mathbf{V}'_\pi \to \mathrm{H}^{-3/2}(\Gamma)$ is continuous, $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is a closed subspace of $\mathbf{V}'_\pi$ and $\mathbb{H}$ has a finite dimension, we deduce from (5) that $\mathbf{curl}_\Gamma(\mathrm{H}^{1/2}(\Gamma))$ is closed in $\mathbf{V}'_\pi$. Taking into account that $\ker(\mathbf{curl}_\Gamma) \cap \mathrm{H}^{1/2}(\Gamma) = \mathbb{C}$ (*cf.* Cor. 3.7 in [14]) we deduce from the closed graph theorem that $\mathbf{curl}_\Gamma : \mathrm{H}^{1/2}(\Gamma)/\mathbb{C} \to \mathbf{curl}_\Gamma(\mathrm{H}^{1/2}(\Gamma)) = \gamma_\tau(\widehat{\mathbf{X}})$ is an isomorphism. □

3. Statement of the problem

Our purpose is to compute eddy currents induced in the conductor Ω by an inductive coil which carries low-frequency alternative current. We assume that there is no fields at the initial time and that $\widehat{\mathbf{j}}^s$ is a given current density. We suppose $\widehat{\mathbf{j}}^s$ of the form

$$\widehat{\mathbf{j}}^s(t, \mathbf{x}) := \Re[\mathrm{e}^{\imath\omega t}\mathbf{j}^s(\mathbf{x})],$$

where $\mathbf{j}^s(\mathbf{x})$ has a bounded support included in Ω_c and satisfies the compatibility condition $\mathrm{div}\,\mathbf{j}^s(\mathbf{x}) = 0$.

We may assume that all quantities occurring in Maxwell's equations feature sinusoidal dependence on time with a small angular frequency $\omega > 0$. We have the representations

$$\widehat{\mathbf{E}}(t, \mathbf{x}) := \Re[\mathrm{e}^{\imath\omega t}\mathbf{E}(\mathbf{x})] \quad \text{and} \quad \widehat{\mathbf{H}}(t, \mathbf{x}) := \Re[\mathrm{e}^{\imath\omega t}\mathbf{H}(\mathbf{x})]$$

for the electric and magnetic fields respectively. The complex amplitudes $\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ are then found to be solutions of the following eddy currents model which is derived from Maxwell's equations in time harmonic regime after neglecting displacement currents (see [2] for a justification of this approximation):

$$\imath\omega\mu\mathbf{H} + \mathbf{curl}\,\mathbf{E} = 0 \qquad \text{in } \mathbb{R}^3; \tag{9}$$

$$\mathbf{curl}\,\mathbf{H} = \mathbf{j}^s + \sigma\mathbf{E} \qquad \text{in } \mathbb{R}^3; \tag{10}$$

$$\mathrm{div}(\varepsilon\mathbf{E}) = 0 \qquad \text{in } \Omega_c; \tag{11}$$

$$\mathbf{H}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{as } |\mathbf{x}| \to \infty; \tag{12}$$

$$\mathbf{E}(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \text{as } |\mathbf{x}| \to \infty. \tag{13}$$

Here, the conductivity σ, the permeability μ and the electric permittivity ε are real valued and bounded functions that satisfy the conditions:

$$\mathrm{support}(\sigma) = \overline{\Omega}, \quad \text{and} \quad \sigma(\mathbf{x}) \geq \sigma_0 > 0 \quad \forall \mathbf{x} \in \Omega;$$

$$\mu(\mathbf{x}) \geq \mu_0 > 0 \quad \forall \mathbf{x} \in \Omega \quad \text{with} \quad \mu(\mathbf{x}) = \mu_0 \quad \text{in } \Omega_c;$$

$$\varepsilon(\mathbf{x}) \geq \varepsilon_0 > 0 \quad \forall \mathbf{x} \in \Omega \cup \mathrm{support}(\mathbf{j}^s) \quad \text{and } \varepsilon(\mathbf{x}) = \varepsilon_0 \text{ elsewhere.}$$

Proposition 3.1 in [2] shows that if $\mathbf{H}(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ are solutions of (9)–(13), then, they behave as $O\left(\frac{1}{|\mathbf{x}|^2}\right)$ when $|\mathbf{x}|$ goes uniformly to infinity. Consequently, both vector fields are square integrable on all $\mathbb{R}^3$. This is the starting point in [2] to derive a variational formulation of the eddy currents model in terms of the unknown $\mathbf{E}$ and study its well-posedness.

In this paper, we proceed as in [9] to deduce a weak formulation that only conserves the unknown $\mathbf{H}$. We begin by testing equation (9) by a smooth function $\mathbf{q}$ in $\mathbb{R}^3$ which is divergence and rotational free in Ω_c. Then, we use a Green formula to obtain

$$\imath\omega(\mu\mathbf{H},\mathbf{q})_{0,\mathbb{R}^3} + (\mathbf{E},\mathbf{curl}\,\mathbf{q})_{0,\Omega} = 0.$$

Furthermore, (10) implies that $\mathbf{E} = \sigma^{-1}\,\mathbf{curl}\,\mathbf{H}$ in Ω. This permits us to eliminate the electric field $\mathbf{E}$ from the last equation:

$$\imath\omega(\mu\mathbf{H},\mathbf{q})_{0,\mathbb{R}^3} + (\sigma^{-1}\,\mathbf{curl}\,\mathbf{H},\mathbf{curl}\,\mathbf{q})_{0,\Omega} = 0, \tag{14}$$

for all smooth functions $\mathbf{q}$ that are divergence and rotational free in Ω_c.

Equation (9) shows that $\mathbf{H}$ is divergence free in Ω_c and (10) reduces to $\mathbf{curl}\,\mathbf{H} = \mathbf{j}^s$ in Ω_c. It is convenient to change the unknown $\mathbf{H}$ in (14) for a function which is also rotational free in Ω_c. To this end, we introduce the vector-field

$$\mathbf{a}^s(\mathbf{x}) := \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\mathbf{j}^s(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}, \tag{15}$$

and consider $\mathbf{h} := (\mathbf{H} - \mathbf{curl}\,\mathbf{a}^s)$. It follows from the Biot and Savart formula that $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$ in Ω_c and $\mathbf{h}$ is evidently still divergence free in Ω_c. Let us now substitute $\mathbf{H}$ by $\mathbf{h}$ in (14):

$$\imath\omega(\mu\mathbf{h},\mathbf{q})_{0,\mathbb{R}^3} + (\sigma^{-1}\,\mathbf{curl}\,\mathbf{h},\mathbf{curl}\,\mathbf{q})_{0,\Omega} = -\imath\omega(\mu\,\mathbf{curl}\,\mathbf{a}^s,\mathbf{q})_{0,\mathbb{R}^3} - (\sigma^{-1}\mathbf{j}^s,\mathbf{curl}\,\mathbf{q})_{0,\Omega}. \tag{16}$$

Here, the test function $\mathbf{q}$ satisfies the same conditions as above.

3.1. The scalar magnetic potential

We introduce the Beppo–Levi space:

$$W^1(\Omega_c) := \left\{\text{distributions } p \text{ in } \Omega_c;\ \frac{p}{\sqrt{1+|\mathbf{x}|^2}} \in L^2(\Omega_c),\ \nabla p \in \mathbf{L}^2(\Omega_c)\right\},$$

and recall that the semi-norm $\|\nabla(\cdot)\|_{0,\Omega_c}$ is a norm in $W^1(\Omega_c)$ equivalent to the natural norm; *i.e.*, there exists a constant $C > 0$ such that (see [28]):

$$\left\|\frac{p}{\sqrt{1+|\mathbf{x}|^2}}\right\|_{0,\Omega_c}^2 + \|\nabla p\|_{0,\Omega_c}^2 \le C\ \|\nabla p\|_{0,\Omega_c}^2 \quad \forall p \in W^1(\Omega_c). \tag{17}$$

We will need a basis of the finite dimensional space $\mathbb{H}(\Omega_c)$ of harmonic Neumann vector-fields associated with Ω_c. To this end, we follow [17] and consider the set $\{\Sigma_1^{\mathrm{ext}},\cdots,\Sigma_B^{\mathrm{ext}}\}$ of orientable cutting surfaces in Ω_c introduced in Section 2. We fix a unit normal $\mathbf{n}_k$ on each Σ_k^{ext} pointing from the face Σ_k^+ of Σ_k^{ext} into the face Σ_k^-. Recall that we denoted $\Omega_c^0 := \Omega_c \setminus \cup_{k=1}^B \Sigma_k^{\mathrm{ext}}$. For any function $z \in W^1(\Omega_c^0)$ we use the notation $[z]_k := z|_{\Sigma_k^+} - z|_{\Sigma_k^-}$. Moreover, we denote by ∇z the gradient of z in the sense of distributions in Ω_c^0. It belongs to $\mathbf{L}^2(\Omega_c^0)$ and therefore it can be extended to $\mathbf{L}^2(\Omega_c)$. We denote $\widetilde{\nabla} z$ the resulting extended vector-field.

Theorem 3.1. *For any $k = 1, \cdots, B$, the problems: Find $z_k \in W^1(\Omega_e^0)$ such that*

$$\begin{aligned} \Delta z_k &= 0 && \text{in } \Omega_c^0; \\ \frac{\partial z_k}{\partial \mathbf{n}} &= 0 && \text{on } \Gamma; \\ \left[\frac{\partial z_k}{\partial \mathbf{n}}\right]_\ell &= 0 && \text{on } \Sigma_\ell^{\text{ext}}, \quad \ell = 1, \cdots, B; \\ [z_k]_\ell &= \delta_{k\ell} && \text{on } \Sigma_\ell^{\text{ext}}, \quad \ell = 1, \cdots, B; \end{aligned} \tag{18}$$

admit unique solutions. Moreover, the set $\left\{\widetilde{\nabla} z_k,\ k = 1, \cdots, B\right\}$ is a basis of $\mathbb{H}(\Omega_c)$.

We have the following representation of rotational free vector-fields in Ω_c, see Remark 7 in [17].

Lemma 3.2. *It holds that*

$$\left\{\mathbf{u} \in \mathbf{L}^2(\Omega_c);\ \mathbf{curl}\,\mathbf{u} = 0 \text{ in } \Omega_c\right\} = \nabla(W^1(\Omega_c)) \oplus \mathbb{H}(\Omega_c).$$

Moreover the sum is $L^2(\Omega_c)$-orthogonal.

It follows that there exist functions ψ and φ in $\mathrm{W}^1(\Omega_c)$ and unique sets of coefficients $\{\beta_k,\ k = 1, \cdots, B\}$ and $\{\zeta_k,\ k = 1, \cdots, B\}$ such that $\mathbf{h}|_{\Omega_c} = \nabla\psi + \sum_{k=1}^B \beta_k \widetilde{\nabla} z_k$ and $\mathbf{q}|_{\Omega_c} := \nabla\varphi + \sum_{k=1}^B \zeta_k \widetilde{\nabla} z_k$. The $\mathrm{L}^2(\Omega_c)$-orthogonality of $\nabla(\mathrm{W}^1(\Omega_c))$ and $\mathbb{H}(\Omega_c)$ gives

$$(\mathbf{h},\ \mathbf{q})_{0,\Omega_c} = (\nabla\psi,\ \nabla\varphi)_{0,\Omega_c} + \sum_{k,\ell=1}^{B} \beta_k \overline{\zeta}_\ell \left(\widetilde{\nabla} z_k,\ \widetilde{\nabla} z_\ell\right)_{0,\Omega_c}$$

and, recalling that $\Delta\psi = \operatorname{div}\mathbf{h} = 0$, Green's formula yields (Lem. 3.10 in [3]):

$$(\mathbf{h},\ \mathbf{q})_{0,\Omega_c} = \left\langle \frac{\partial\psi}{\partial\mathbf{n}}, \varphi \right\rangle_{1/2,\Gamma} + \sum_{k,\ell=1}^{B} \beta_k \overline{\zeta}_\ell \left\langle \frac{\partial z_k}{\partial \mathbf{n}}, 1 \right\rangle_{1/2,\Sigma_\ell^{\text{ext}}}.$$

We introduce the matrix $\mathbf{N} := \left(\langle \frac{\partial z_k}{\partial \mathbf{n}}, 1\rangle_{1/2,\Sigma_\ell^{\text{ext}}}\right)_{k,\ell=1,\cdots,B}$. It is clear that $\mathbf{N}_{k,\ell} = \overline{\mathbf{N}}_{\ell,k}$ and, due to (17), a vector $\boldsymbol{\zeta} \in \mathbb{C}^B$ satisfies

$$\mathbf{N}\boldsymbol{\zeta} \cdot \overline{\boldsymbol{\zeta}} = \left(\nabla\left(\sum_{k=1}^{B} \zeta_k z_k\right),\ \nabla\left(\sum_{k=1}^{B} \zeta_\ell z_\ell\right)\right)_{\Omega_c^0} = 0$$

if and only if $\boldsymbol{\zeta} = \mathbf{0}$. This means that $\mathbf{N}$ is Hermitian and positive definite.

After multiplying equation (16) by $\frac{1-\imath}{\omega\mu_0}$ and substituting the term $(\mathbf{h}, \mathbf{q})_{0,\Omega_c}$ by the expression obtained above we arrive at the following identity:

$$\begin{aligned} a(\mathbf{h},\ \mathbf{q}) - (1+\imath)\left\langle \frac{\partial\psi}{\partial\mathbf{n}}, \varphi\right\rangle_{1/2,\Gamma} + (1+\imath)\mathbf{N}\boldsymbol{\beta}\cdot\overline{\boldsymbol{\zeta}} = \\ -(1+\imath)(\mu/\mu_0\,\mathbf{curl}\,\mathbf{a}^s, \mathbf{q})_{0,\Omega} + (1+\imath)\left\langle \mathbf{curl}\,\mathbf{a}^s\cdot\mathbf{n}, \varphi\right\rangle_{1/2,\Gamma}, \end{aligned} \tag{19}$$

where $\boldsymbol{\beta} := (\beta_k)_{k=1,\cdots,B}$, $\boldsymbol{\zeta} := (\zeta_k)_{k=1,\cdots,B}$ and

$$a(\mathbf{h},\ \mathbf{q}) = (1+\imath)(\mu/\mu_0\mathbf{h}, \mathbf{q})_{0,\Omega} + (1-\imath)/(\omega\mu_0)\left(\sigma^{-1}\,\mathbf{curl}\,\mathbf{h}, \mathbf{curl}\,\mathbf{q}\right)_{0,\Omega}.$$

3.2. **Boundary integral equations**

Let $E(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$ be the fundamental solution of the Laplace equation in $\mathbb{R}^3$. As ψ is harmonic in Ω_c, it has the following integral representation:

$$\psi(\mathbf{x}) = \int_\Gamma \frac{\partial E(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\psi(\mathbf{y})\,\mathrm{d}\xi_y - \int_\Gamma E(\mathbf{x},\mathbf{y})\frac{\partial\psi}{\partial\mathbf{n}}\,\mathrm{d}\xi_y \qquad \forall \mathbf{x}\in\Omega_c,$$

which, in its turn, provides boundary integral identities relating on Γ the trace of ψ and its normal derivative $\frac{\partial\psi}{\partial\mathbf{n}} = \mu/\mu_0\mathbf{h}\cdot\mathbf{n}$, denoted from now on λ (*cf.* [16]):

$$\psi = \left(\frac{1}{2}\mathcal{I}+\mathcal{K}\right)\psi - \mathcal{V}\lambda; \tag{20}$$

$$\lambda = -\mathcal{H}\psi + \left(\frac{1}{2}\mathcal{I}-\mathcal{K}'\right)\lambda. \tag{21}$$

The operators involved in (20) are the single layer potential $\mathcal{V}$ and the double layer potential $\mathcal{K}$. They are formally defined by

$$\mathcal{V}\lambda(\mathbf{x}) := \int_\Gamma E(\mathbf{x},\mathbf{y})\lambda(\mathbf{x})\,\mathrm{d}\xi_y, \quad \text{and} \quad \mathcal{K}\varphi(\mathbf{x}) := \int_\Gamma \frac{\partial E(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})}\varphi(\mathbf{y})\,\mathrm{d}\xi_y \quad (\mathbf{x}\in\Gamma),$$

respectively. We recall below some well-known properties of $\mathcal{V}$ and $\mathcal{K}$ (see [22]).

Lemma 3.3. *The operators* $\mathcal{K} : \mathrm{H}^{1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)$ *and* $\mathcal{V} : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)$ *are bounded. Moreover there exists* $\alpha_1 > 0$ *such that*

$$\langle \eta, \mathcal{V}\eta\rangle_{1/2,\Gamma} \geq \alpha_1 \|\eta\|^2_{-1/2}, \qquad \forall \eta\in \mathrm{H}^{-1/2}(\Gamma).$$

We introduce the adjoint operator $i_\pi : \mathbf{V}'_\pi \to \mathbf{V}'$ of π_τ and define $\widetilde{\mathcal{V}} := \mathcal{V}\circ i_\pi$ where $\mathcal{V}$ is assumed here to act component wise on vector-fields $i_\pi(\boldsymbol{\lambda}) \in \mathbf{V}'$ for any $\boldsymbol{\lambda}\in\mathbf{V}'_\pi$. It is evident that $\widetilde{\mathcal{V}} : \mathbf{V}'_\pi \to \mathbf{V}$ is linear and continuous. The boundary integral operators involved in equation (21) are $\mathcal{K}'$, the adjoint of $\mathcal{K}$, and the hypersingular operator

$$\mathcal{H}\varphi(\mathbf{x}) := -\frac{\partial}{\partial\mathbf{n}(\mathbf{x})}\int_\Gamma \frac{\partial E(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})}\varphi(\mathbf{y})\,\mathrm{d}\xi_y \quad (\mathbf{x}\in\Gamma),$$

which is related, as we will see in the following result, to the single layer operator $\widetilde{\mathcal{V}}$ *via* tangential derivatives.

Lemma 3.4. *The operator* $\mathcal{H} : \mathrm{H}^{1/2}(\Gamma) \to \mathrm{H}^{-1/2}(\Gamma)$ *is bounded and it is related to* $\widetilde{\mathcal{V}}$ *by means of the following identity:*

$$\langle \mathcal{H}\psi,\varphi\rangle_{1/2,\Gamma} = \left\langle \mathbf{curl}_\Gamma\,\overline{\varphi},\ \pi_\tau\widetilde{\mathcal{V}}\left(\mathbf{curl}_\Gamma\,\overline{\psi}\right)\right\rangle_{\mathbf{V}'_\pi\times\mathbf{V}_\pi} \qquad \forall\psi,\varphi\in \mathrm{H}^{1/2}(\Gamma). \tag{22}$$

Moreover, there exists a constant $\alpha_2 > 0$ *such that*

$$\langle \mathcal{H}\varphi,\varphi\rangle_{1/2,\Gamma} \geq \alpha_2\ \|\varphi\|^2_{\mathrm{H}^{1/2}(\Gamma)/\mathbb{C}} \quad \forall\varphi\in \mathrm{H}^{1/2}(\Gamma)/\mathbb{C}.$$

Proof. The proof given in Theorem 3.3.2 in [28] for the continuity of $\mathcal{H} : \mathrm{H}^{1/2}(\Gamma) \to \mathrm{H}^{-1/2}(\Gamma)$ is valid verbatim for Lipschitz boundaries. See also Theorem 7.8 in [22].

We will also show that the reasoning given in Theorem 3.3.2 in [28] to deduce the relationship between $\widetilde{\mathcal{V}}$ and $\mathcal{H}$ in the case of a regular boundary Γ can be adapted to our case.

Let us introduce the Hilbert space

$$Z := \left\{ v\in \mathrm{H}^1(\Omega)/\mathbb{C}\times \mathrm{W}^1(\Omega_c);\quad \Delta v = 0 \quad \text{in } \Omega \text{ and } \Omega_c, \quad \left[\frac{\partial v}{\partial\mathbf{n}}\right]_\Gamma = 0\right\},$$

where the brackets $[\cdot]_\Gamma$ represent the jump across Γ. Thus,

$$\left[\frac{\partial v}{\partial \mathbf{n}}\right]_\Gamma := \frac{\partial (v|_\Omega)}{\partial \mathbf{n}} - \frac{\partial (v|_{\Omega_c})}{\partial \mathbf{n}}\cdot$$

We associate to any function $\varphi \in \mathrm{H}^{1/2}(\Gamma)$ the unique solution $u_\varphi \in Z$ of the variational formulation

$$(\nabla u_\varphi,\, \nabla v)_{0,\Omega} + (\nabla u_\varphi,\, \nabla v)_{0,\Omega_c} = \left\langle \varphi,\, \frac{\partial v}{\partial \mathbf{n}} \right\rangle_{1/2,\Gamma} \quad \forall v \in Z. \tag{23}$$

The gradient of u_φ in the sense of $\mathcal{D}'(\mathbb{R}^3 \setminus \Gamma)$ belongs to $\mathbf{L}^2(\mathbb{R}^3 \setminus \Gamma)$. We extend it to $\mathbf{L}^2(\mathbb{R}^3)$ and denote, as before, the resulting vector-field $\widetilde{\nabla} u_\varphi$. A proof similar to the one given in [15] for Lemma 3.1 permits one to obtain the following jump relation

$$\nabla u_\varphi = \widetilde{\nabla} u_\varphi - \varphi \mathbf{n} \delta_\Gamma \quad \text{in } \mathcal{D}'\left(\mathbb{R}^3\right)^3,$$

where $\langle \varphi \mathbf{n} \delta_\Gamma,\, \mathbf{z} \rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} := \langle \varphi,\, \mathbf{z}|_\Gamma \cdot \mathbf{n} \rangle_{0,\Gamma}$ for all $\mathbf{z} \in \mathcal{D}(\mathbb{R}^3)^3$. Besides, we deduce from the fact that $\operatorname{div}\left(\widetilde{\nabla} u_\varphi\right) = 0$ the equation

$$-\Delta\left(\widetilde{\nabla} u_\varphi\right) = \mathbf{curl}\,\mathbf{curl}(\varphi \mathbf{n} \delta_\Gamma)$$

and thus

$$\widetilde{\nabla} u_\varphi = E * \mathbf{curl}\,\mathbf{curl}(\varphi \mathbf{n} \delta_\Gamma).$$

Now, notice that the identities

$$\langle \mathbf{curl}(\varphi \mathbf{n} \delta_\Gamma),\, \mathbf{z} \rangle_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} = (\varphi,\, \mathbf{curl}\,\mathbf{z} \cdot \mathbf{n})_{0,\Gamma} = (\varphi,\, \mathrm{curl}_\Gamma(\pi_\tau \mathbf{z}))_{0,\Gamma} \quad \forall \mathbf{z} \in \mathcal{D}\left(\mathbb{R}^3\right)^3$$

lead to the expression $\mathbf{curl}(\varphi \mathbf{n} \delta_\Gamma) = i_\pi(\mathbf{curl}_\Gamma\, \varphi)\delta_\Gamma$. Consequently, we may write

$$\widetilde{\nabla} u_\varphi = \mathbf{curl}\,\mathcal{S}(i_\pi\, \mathbf{curl}_\Gamma\, \varphi)$$

where $\mathcal{S}$ is the vector layer potential:

$$\mathcal{S}\mathbf{z}(\mathbf{x}) := \int_\Gamma E(\mathbf{x},\mathbf{y})\mathbf{z}(\mathbf{y})\,\mathrm{d}\xi_\mathbf{y}, \quad \forall \mathbf{z} \in \mathcal{C}^0(\Gamma)^3 \quad \left(\mathbf{x} \in \mathbb{R}^3 \setminus \Gamma\right).$$

We recall here that $\mathcal{S}$ can be extended to a continuous mapping $\mathcal{S} : \mathbf{V}' \to \mathrm{W}^1\left(\mathbb{R}^3\right)^3$, see Proposition 4.1 in [15].

We deduce from (23) and the fact that $\mathcal{H}\psi = \frac{\partial u_\psi}{\partial \mathbf{n}}$ the identity (see the proof of Th. 3.3.2 in [28])

$$\langle \mathcal{H}\psi,\, \varphi \rangle_{1/2,\Gamma} = \left(\widetilde{\nabla} u_\psi,\, \widetilde{\nabla} u_\varphi\right)_{0,\mathbb{R}^3} \quad \forall \psi, \varphi \in \mathrm{H}^{1/2}(\Gamma).$$

Using the expression obtained above for $\widetilde{\nabla} u_\psi$ gives

$$\langle \mathcal{H}\psi,\, \varphi \rangle_{1/2,\Gamma} = \left(\mathbf{curl}\,\mathcal{S}(i_\pi\, \mathbf{curl}_\Gamma\, \psi),\, \widetilde{\nabla} u_\varphi\right)_{0,\Omega} + \left(\mathbf{curl}\,\mathcal{S}(i_\pi\, \mathbf{curl}_\Gamma\, \psi),\, \widetilde{\nabla} u_\varphi\right)_{0,\Omega_c}.$$

Notice now that Green's formula (1) is valid in Ω_c for functions $\mathbf{v} \in \mathrm{W}^1(\Omega_c)^3$ and $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega_c)$ by virtue of the density of $\mathcal{D}(\overline{\Omega}_c)^3$ in both $\mathrm{W}^1(\Omega_c)^3$ and $\mathbf{H}(\mathbf{curl}, \Omega_c)$, *cf.* [19, 28]. Then, we may apply (1) separately in Ω and Ω_c and add up to arrive at

$$\left(\mathbf{curl}\,\mathcal{S}(i_\pi\, \mathbf{curl}_\Gamma\, \psi),\, \widetilde{\nabla} u_\varphi\right)_{0,\Omega} + \left(\mathbf{curl}\,\mathcal{S}(i_\pi\, \mathbf{curl}_\Gamma\, \psi),\, \widetilde{\nabla} u_\varphi\right)_{0,\Omega_c} = \left\langle \left[\gamma_\tau\left(\widetilde{\nabla} \overline{u}_\varphi\right)\right]_\Gamma,\, \pi_\tau \mathcal{S}\left(i_\pi\, \mathbf{curl}_\Gamma\, \overline{\psi}\right) \right\rangle_{\mathbf{V}'_\pi \times \mathbf{V}_\pi}.$$

Taking into account that $\left[\gamma_\tau\left(\widetilde{\nabla} u_\varphi\right)\right]_\Gamma = \mathbf{curl}_\Gamma[u_\varphi] = \mathbf{curl}_\Gamma\,\varphi$ we obtain

$$\langle \mathcal{H}\psi,\, \varphi\rangle_{1/2,\Gamma} = \left\langle \mathbf{curl}_\Gamma\,\overline{\varphi},\, \pi_\tau \mathcal{S}\left(i_\pi\,\mathbf{curl}_\Gamma\,\overline{\psi}\right)\right\rangle_{\mathbf{V}'_\pi\times\mathbf{V}_\pi}.$$

and identity (22) follows since $\pi_\tau\circ\mathcal{S}\circ i_\pi = \pi_\tau\circ\widetilde{\mathcal{V}}$.

Finally, the $\mathrm{H}^{1/2}(\Gamma)/\mathbb{C}$-coercivity of $\mathcal{H}$ is a direct consequence of Proposition 2.5 and Theorem 4.2 in [15], which states that there exists a constant $\alpha_3>0$ such that

$$\left\langle \mathbf{u}, \pi_\tau\left(\widetilde{\mathcal{V}}\mathbf{u}\right)\right\rangle_{\mathbf{V}'_\pi\times\mathbf{V}_\pi} \geq \alpha_3\,\|\mathbf{u}\|^2_{\mathbf{V}'_\pi} \qquad \forall \mathbf{u}\in\mathbf{V}'_\pi. \tag{24}$$

□

3.3. The variational formulation

Our purpose now is to perform the coupling of the boundary integral equations (20) and (21) with (19). First of all, the continuity of the tangential trace of the magnetic field on Γ implies that $\gamma_\tau\mathbf{h} = \mathbf{curl}_\Gamma\,\psi + \sum_{k=1}^B \beta_k\mathbf{g}_k$, where $\mathbf{g}_k := \gamma_\tau(\widetilde{\nabla} z_k)\in\mathbb{H}$. Thus, it is clear that we have to ask $\mathbf{h}$ to be in the Hilbert space $\mathbf{X}$. Furthermore, Theorem 2.4 ensures the existence of a unique vector-field $\widehat{\mathbf{h}}\in\widehat{\mathbf{X}}$ such that $\mathbf{h}|_\Omega = \widehat{\mathbf{h}} + \sum_{k=1}^B\beta_k\mathcal{R}\mathbf{g}_k$ with $\mathbf{curl}_\Gamma\,\psi = \gamma_\tau\widehat{\mathbf{h}}$. For any test function $\mathbf{q}\in\widehat{\mathbf{X}}$, we will also write $\mathbf{q}|_\Omega = \widehat{\mathbf{q}} + \sum_{k=1}^B\zeta_k\mathcal{R}\mathbf{g}_k$ with self evident notations. Combining equations (20) and (21) with (19) we deduce that our problem may be formulated as follows: find $\widehat{\mathbf{h}}\in\widehat{\mathbf{X}}$, $\boldsymbol{\beta}\in\mathbb{C}^B$ and $\lambda\in\mathrm{H}_0^{-1/2}(\Gamma)$ such that

$$\begin{aligned} A\left(\left(\widehat{\mathbf{h}},\boldsymbol{\beta}\right),(\widehat{\mathbf{q}},\boldsymbol{\zeta})\right) - (1+\imath)\left\langle\lambda,\left(\frac{1}{2}\mathcal{I}-\mathcal{K}\right)\varphi\right\rangle_{1/2,\Gamma} &= -(1+\imath)(\mu/\mu_0\,\mathbf{curl}\,\mathbf{a}^s,\mathbf{q})_{0,\Omega} + (1+\imath)\left\langle\mathbf{curl}\,\mathbf{a}^s\cdot\mathbf{n},\varphi\right\rangle_{1/2,\Gamma}\\ \left(\frac{1}{2}\mathcal{I}-\mathcal{K}\right)\psi + \mathcal{V}\lambda &= 0, \end{aligned} \tag{25}$$

for all $\widehat{\mathbf{q}}\in\widehat{\mathbf{X}}$ and for all $\boldsymbol{\zeta}\in\mathbb{C}^B$ where $\widehat{\mathbf{q}}$ and $\varphi\in\mathrm{H}^{1/2}(\Gamma)/\mathbb{C}$ are related on Γ by $\mathbf{curl}_\Gamma\,\varphi = \gamma_\tau\mathbf{q}$. The sesquilineal form $A(\cdot,\cdot)$ is defined by

$$A\left(\left(\widehat{\mathbf{h}},\boldsymbol{\beta}\right),\,(\widehat{\mathbf{q}},\boldsymbol{\zeta})\right) := a\left(\widehat{\mathbf{h}} + \sum_{k=1}^B\beta_k\mathcal{R}\mathbf{g}_k,\ \widehat{\mathbf{q}} + \sum_{\ell=1}^B\zeta_\ell\mathcal{R}\mathbf{g}_\ell\right) + (1+\imath)\mathbf{N}\boldsymbol{\beta}\cdot\overline{\boldsymbol{\zeta}} + d\left(\gamma_\tau\widehat{\mathbf{h}},\gamma_\tau\widehat{\mathbf{q}}\right)$$

where

$$d\left(\gamma_\tau\widehat{\mathbf{h}},\gamma_\tau\widehat{\mathbf{q}}\right) := (1+\imath)\left\langle\gamma_\tau\overline{\widehat{\mathbf{q}}},\,\pi_\tau\left(\widetilde{\mathcal{V}}\gamma_\tau\overline{\widehat{\mathbf{h}}}\right)\right\rangle_{\mathbf{V}'_\pi\times\mathbf{V}_\pi}.$$

By virtue of Proposition 2.5 we may write $\psi = \mathbf{curl}_\Gamma^{-1}\gamma_\tau\widehat{\mathbf{h}}$ and $\varphi = \mathbf{curl}_\Gamma^{-1}\gamma_\tau\widehat{\mathbf{q}}$. Furthermore, Lemma 3.3 and the fact that $\mathcal{K}1 = -1/2$ prove that $\frac{1}{2}\mathcal{I}-\mathcal{K}$ is bounded from $\mathrm{H}^{1/2}(\Gamma)/\mathbb{C}$ into itself. Consequently, $(\frac{1}{2}\mathcal{I}-\mathcal{K})\,\mathbf{curl}_\Gamma^{-1}:\ \gamma_\tau(\mathbf{X})\to\mathrm{H}^{1/2}(\Gamma)/\mathbb{C}$ is continuous. We may then eliminate the scalar potential ψ from our weak formulation. Indeed, testing the complex conjugate of the second equation of (25) with $(1-\imath)\overline{\eta}$ we obtain:

$$\begin{aligned} &\text{find } \left(\widehat{\mathbf{h}},\,\boldsymbol{\beta}\right)\in\widehat{\mathbf{X}}\times\mathbb{C}^B \text{ and } \lambda\in H_0^{-1/2}(\Gamma) \text{ such that;}\\ &A\left(\left(\widehat{\mathbf{h}},\boldsymbol{\beta}\right),\,(\widehat{\mathbf{q}},\boldsymbol{\zeta})\right) - b\,(\gamma_\tau\widehat{\mathbf{q}},\,\lambda) = L((\widehat{\mathbf{q}},\boldsymbol{\zeta})) \qquad \forall\widehat{\mathbf{q}}\in\widehat{\mathbf{X}},\ \forall\zeta\in\mathbb{C}^B\\ &\qquad b^*(\gamma_\tau\widehat{\mathbf{h}},\,\eta) + c(\lambda,\,\eta) = 0 \qquad\qquad \forall\eta\in\mathrm{H}_0^{-1/2}(\Gamma), \end{aligned} \tag{26}$$

where

$$b(\gamma_\tau \mathbf{q}, \eta) := (1+\imath) \left\langle \eta, \left(\frac{1}{2}\mathcal{I} - \mathcal{K}\right) \mathbf{curl}_\Gamma^{-1}(\gamma_\tau \mathbf{q}) \right\rangle_{1/2,\Gamma} \qquad b^*(\gamma_\tau \mathbf{q}, \eta) := \overline{b(\gamma_\tau \mathbf{q}, \eta)}$$

and

$$c(\lambda, \eta) := (1-\imath) \left\langle \overline{\eta}, \mathcal{V}\overline{\lambda} \right\rangle_{1/2,\Gamma}.$$

The right-hand side is given by

$$L((\widehat{\mathbf{q}}, \boldsymbol{\zeta})) := -(1+\imath) \left(\mu/\mu_0 \, \mathbf{curl}\, \mathbf{a}^s, \widehat{\mathbf{q}} + \sum_{k=1}^{B} \zeta_k \mathcal{R}\mathbf{g}_k \right)_{0,\Omega} + (1+\imath) \overline{\langle \gamma_\tau \widehat{\mathbf{q}}, \pi_\tau \mathbf{a}^s \rangle}_{\mathbf{V}_\pi' \times \mathbf{V}_\pi}.$$

Let $\mathbf{W} := \left(\widehat{\mathbf{X}} \times \mathbb{C}^B\right) \times \mathrm{H}_0^{-1/2}(\Gamma)$. We simplify the notation and denote $\widetilde{\mathbf{h}} := \left(\left(\widehat{\mathbf{h}}, \boldsymbol{\beta}\right), \lambda\right)$ and $\widetilde{\mathbf{q}} := ((\widehat{\mathbf{q}}, \boldsymbol{\zeta}), \eta)$ the elements of $\mathbf{W}$. The space $\mathbf{W}$ is provided with its natural Hilbertian norm:

$$\|\widetilde{\mathbf{q}}\|_W^2 := \|\mathbf{q}\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + |\boldsymbol{\zeta}|^2 + \|\eta\|_{-1/2,\Gamma}^2.$$

We introduce the sesquilinear form

$$\mathcal{A}\left(\widetilde{\mathbf{h}}, \widetilde{\mathbf{q}}\right) := A\left(\left(\widehat{\mathbf{h}}, \boldsymbol{\beta}\right), \, (\widehat{\mathbf{q}}, \boldsymbol{\zeta})\right) - b(\gamma_\tau \widehat{\mathbf{q}}, \lambda) + b^*\left(\gamma_\tau \widehat{\mathbf{h}}, \eta\right) + c(\lambda, \eta)$$

and denote

$$\mathcal{F}(\widetilde{\mathbf{q}}) := L((\widehat{\mathbf{q}}, \boldsymbol{\zeta})).$$

Problem (26) may be written in terms of these notations:

$$\begin{aligned} &\text{find } \widetilde{\mathbf{h}} \in \mathbf{W}; \\ &\mathcal{A}\left(\widetilde{\mathbf{h}}, \widetilde{\mathbf{q}}\right) = \mathcal{F}(\widetilde{\mathbf{q}}) \quad \forall \widetilde{\mathbf{q}} \in \mathbf{W}. \end{aligned} \tag{27}$$

Proposition 3.5. *Problem* (27) *has a unique solution.*

Proof. We deduce from the results given in Section 2 and from Lemma 3.3 that $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{L}(\cdot)$ are bounded. Moreover, we have the identity

$$\Re[\mathcal{A}(\widetilde{\mathbf{q}}, \widetilde{\mathbf{q}})] = \Re[A((\widehat{\mathbf{q}}, \boldsymbol{\zeta}), \, (\widehat{\mathbf{q}}, \boldsymbol{\zeta})) + c(\eta, \eta)].$$

Thus, Lemma 3.3 gives rise to the following inequality

$$\Re[\mathcal{A}(\widetilde{\mathbf{q}}, \widetilde{\mathbf{q}})] \geq \min\left\{1, \frac{\min_{\mathbf{x}\in\overline{\Omega}} \sigma^{-1}(\mathbf{x})}{\omega\mu_0}\right\} \left\| \widehat{\mathbf{q}} + \sum_{k=1}^{B} \zeta_k \mathcal{R}\mathbf{g}_k \right\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + \rho_{min}(\mathbf{N}) \, |\boldsymbol{\zeta}|^2 + \alpha_1 \|\eta\|_{-1/2,\Gamma}^2 \qquad \forall \widetilde{\mathbf{q}} \in \mathbf{W},$$

where $\rho_{min}(\mathbf{N}) > 0$ denotes the smallest eigenvalue of $\mathbf{N}$. Applying (7) we deduce that there exists a constant $\alpha^* > 0$ such that

$$\Re[\mathcal{A}(\widetilde{\mathbf{q}}, \widetilde{\mathbf{q}})] \geq \alpha^* \|\widetilde{\mathbf{q}}\|_W^2 \qquad \forall \widetilde{\mathbf{q}} \in \mathbf{W},$$

where we used that the norms $\boldsymbol{\zeta} \to \left\|\sum_{k=1}^B \zeta_k \mathcal{R}\mathbf{g}_k\right\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ and $\boldsymbol{\zeta} \to |\boldsymbol{\zeta}|$ are equivalent in $\mathbb{C}^B$. The result is then a consequence of the Lax–Milgram lemma. □

Notice that in the particular case of a simply connected domain Ω, the unknown $\boldsymbol{\beta}$ equals zero and then problem (27) reduces to:

$$\begin{aligned}&\text{find } \left(\widehat{\mathbf{h}},\lambda\right)\in\widehat{\mathbf{X}}\times \mathrm{H}_0^{-1/2}(\Gamma);\\ &\mathcal{A}_{\mathrm{sc}}\left(\left(\widehat{\mathbf{h}},\lambda\right),(\widehat{\mathbf{q}},\eta)\right)=L((\widehat{\mathbf{q}},\mathbf{0}))\quad\forall(\widehat{\mathbf{q}},\eta)\in\widehat{\mathbf{X}}\times\mathrm{H}_0^{-1/2}(\Gamma),\end{aligned}\tag{28}$$

where

$$\mathcal{A}_{\mathrm{sc}}\left(\left(\widehat{\mathbf{h}},\lambda\right),(\widehat{\mathbf{q}},\eta)\right):=A\left(\left(\widehat{\mathbf{h}},\mathbf{0}\right),\,(\widehat{\mathbf{q}},\mathbf{0})\right)-b(\gamma_\tau\widehat{\mathbf{q}},\lambda)+b^*\left(\gamma_\tau\widehat{\mathbf{h}},\eta\right)+c(\lambda,\eta).$$

4. The discrete spaces

In what follows we assume that Ω has a polyhedral boundary. Let $\mathcal{T}_h$ be a regular family of tetrahedral meshes of Ω. As usual, h stands for the largest diameter of all tetrahedron T in τ_h. Notice that $\mathcal{T}_h$ induces on Γ a triangulation $\mathcal{T}_h(\Gamma)$ composed of triangles. We may assume, without loss of generality, that for all $k=1,\cdots,B$, the boundaries $\gamma_k^{\mathrm{ext}}:=\partial\Sigma_k^{\mathrm{ext}}$ and $\gamma_k^{\mathrm{int}}:=\partial\Sigma_k^{\mathrm{int}}$ are closed paths of Γ formed by edges of $\mathcal{T}_h(\Gamma)$ and that they intersect only in one point.

Let us introduce Nédélec's edge finite elements, [27]. The local representation on T of the lowest order finite element of Nédélec is given by

$$\mathcal{ND}(T):=\left\{\mathbf{a}\wedge\mathbf{x}+\mathbf{b};\quad \mathbf{a},\mathbf{b}\in\mathbb{C}^3\right\}\cdot$$

The corresponding global space is made of functions that are locally in $\mathcal{ND}(T)$ and that have continuous tangential components across the faces of the triangulation $\mathcal{T}_h$:

$$\mathcal{ND}_h(\Omega):=\left\{\mathbf{q}\in\mathbf{H}(\mathbf{curl},\Omega);\quad \mathbf{q}_{|T}\in\mathcal{ND}(T),\ \forall T\in\mathcal{T}_h\right\}\cdot$$

Let us introduce the space $\mathbf{X}_h:=\mathcal{ND}_h(\Omega)\cap\mathbf{X}$. We will seek an approximation of the first unknown $\widehat{\mathbf{h}}$ of problem (27) in

$$\widehat{\mathbf{X}}_h:=\left\{\mathbf{q}\in\mathbf{X}_h;\quad\int_\gamma\mathbf{q}\cdot\mathbf{t}\,\mathrm{d}s=0,\quad\text{for all closed paths }\gamma\text{ of edges in }\mathcal{T}_h(\Gamma)\right\},$$

where $\mathbf{t}$ is a unit tangential vector along γ. The finite dimensional space M_h corresponding to the unknown λ is given by piecewise constant functions relative to the triangulation of Γ:

$$M_h:=\left\{\eta\in\mathrm{L}^2(\Gamma);\quad\int_\Gamma\eta\,\mathrm{d}\xi=0,\quad\eta_{|F}\in\mathbb{C}\text{ for all face }F\text{ of }\mathcal{T}_h(\Gamma)\right\}\cdot$$

Let us show that we are constructing a conforming approximation scheme. We first introduce some notations: For any oriented closed path $\gamma\subset\Gamma$ and for any $p\in\mathrm{H}^1(\Gamma\setminus\gamma)$ we denote by $[p]_\gamma$ the jump of p across γ. We will also denote by $\widetilde{\nabla}_\Gamma p$ the extension to $\mathbf{L}_t^2(\Gamma)$ of the tangential gradient $\nabla_\Gamma p$ of p in the sense of $\mathcal{D}'(\Gamma\setminus\gamma)$. Similarly, $\widetilde{\mathbf{curl}}_\Gamma p$ will represent the extension of the tangential rotational of p to $\mathbf{L}_t^2(\Gamma)$.

Proposition 4.1. *The finite dimensional space $\widehat{\mathbf{X}}_h$ is a subspace of $\widehat{\mathbf{X}}$.*

Proof. Let us first report a result due to Buffa [11]: The space $\mathbb{H}$ is spanned by $\left\{\widetilde{\nabla}_\Gamma p_k^{\mathrm{ext}},\ k=1,\cdots,B\right\}\cup\left\{\widetilde{\nabla}_\Gamma p_k^{\mathrm{int}},\ k=1,\cdots,B\right\}$ where, for each k, $p_k^{\mathrm{ext}}\in\mathrm{H}^1(\Gamma\setminus\gamma_k^{\mathrm{ext}})$ and $p_k^{\mathrm{int}}\in\mathrm{H}^1(\Gamma\setminus\gamma_k^{\mathrm{int}})$ are the unique (up to a constant) solutions of

$$\begin{aligned}&-\Delta_\Gamma p_k^{\mathrm{ext}}=0\quad\text{in }\Gamma\setminus\gamma_k^{\mathrm{ext}}\\ &[p_k^{\mathrm{ext}}]_{\gamma_k^{\mathrm{ext}}}=1,\quad[\nabla_\Gamma p_k^{\mathrm{ext}}\cdot\boldsymbol{\nu}_k^{\mathrm{ext}}]_{\gamma_k^{\mathrm{ext}}}=0,\end{aligned}\tag{29}$$

and

$$-\Delta_\Gamma p_k^{\mathrm{int}} = 0 \quad \text{in } \Gamma \setminus \gamma_k^{\mathrm{int}}$$
$$[p_k]_{\gamma_k^{\mathrm{int}}} = 1, \quad \left[\nabla_\Gamma p_k^{\mathrm{int}} \cdot \boldsymbol{\nu}_k^{\mathrm{int}}\right]_{\gamma_k^{\mathrm{int}}} = 0, \tag{30}$$

respectively. We denoted here by $\boldsymbol{\nu}_k^{\mathrm{ext}}$ (respectively $\boldsymbol{\nu}_k^{\mathrm{int}}$) a normal unit vector to γ_k^{ext} (respectively γ_k^{int}) that lies in the tangent plane of Γ.

Let $\mathbf{q}$ be an arbitrary element of $\widehat{\mathbf{X}}_h$ and $\mathbf{g}_k := \widetilde{\nabla}_\Gamma p_k^{\mathrm{ext}} \wedge \mathbf{n} = \widetilde{\mathbf{curl}}_\Gamma p_k^{\mathrm{ext}} \in \mathbb{H}$. We apply (1) to obtain

$$(\mathbf{q},\, \mathbf{curl}\, \mathcal{R}\mathbf{g}_k)_{0,\Omega} - (\mathbf{curl}\, \mathbf{q},\, \mathcal{R}\mathbf{g}_k)_{0,\Omega} = \int_\Gamma \gamma_\tau \mathbf{q} \cdot (\mathbf{n} \wedge \overline{\mathbf{g}}_k)\, \mathrm{d}\xi$$

and, recalling that $\mathrm{div}_\Gamma(\gamma_\tau \mathbf{q}) = 0$, an integration by part formula yields:

$$\int_\Gamma \gamma_\tau \mathbf{q} \cdot (\mathbf{n} \wedge \overline{\mathbf{g}}_k)\, \mathrm{d}\xi = \int_{\Gamma \setminus \gamma_k^{\mathrm{ext}}} \gamma_\tau \mathbf{q} \cdot \widetilde{\nabla}_\Gamma \overline{p}_k^{\mathrm{ext}}\, \mathrm{d}\xi = \int_{\gamma_k^{\mathrm{ext}}} \gamma_\tau \mathbf{q} \cdot \boldsymbol{\nu}_k^{\mathrm{ext}} \overline{[p_k^{\mathrm{ext}}]}_{\gamma_k^{\mathrm{ext}}}\, \mathrm{d}s = \int_{\gamma_k^{\mathrm{ext}}} (\mathbf{q} \wedge \mathbf{n}) \cdot \boldsymbol{\nu}_k^{\mathrm{ext}}\, \mathrm{d}s.$$

The last integral vanishes since $(\mathbf{q} \wedge \mathbf{n}) \cdot \boldsymbol{\nu}_k^{\mathrm{ext}} = \mathbf{q} \cdot (\mathbf{n} \wedge \boldsymbol{\nu}_k^{\mathrm{ext}})$ and $\mathbf{n} \wedge \boldsymbol{\nu}_k^{\mathrm{ext}}$ is a unit tangential vector along γ_k^{ext}. Repeating the same steps with $\mathbf{g}_k := \widetilde{\nabla}_\Gamma p_k^{\mathrm{int}} \wedge \mathbf{n}$, $(k = 1, \cdots, B)$, we arrive at the conclusion that

$$(\mathbf{q},\, \mathbf{curl}\, \mathcal{R}\mathbf{g})_{0,\Omega} - (\mathbf{curl}\, \mathbf{q},\, \mathcal{R}\mathbf{g})_{0,\Omega} = 0, \quad \forall \mathbf{g} \in \mathbb{H},$$

and the result follows. □

4.1. An explicit basis of $\widehat{\mathbf{X}}_h$

We introduce

$$\mathcal{E}_h^0 := \{E \text{ edge of } \mathcal{T}_h;\ E \not\subseteq \Gamma\}, \qquad \mathcal{V}_h^\Gamma := \{v \text{ vertex of } \mathcal{T}_h(\Gamma)\} \quad \text{and} \quad \mathcal{F}_h^\Gamma := \{F \text{ face of } \mathcal{T}_h(\Gamma)\} \cdot$$

We assume that the sets $\mathcal{E}_h^0$, $\mathcal{V}_h^\Gamma$ and $\mathcal{F}_h^\Gamma$ are numbered and, in the sequel, v_0 stands for the last vertex of $\mathcal{V}_h^\Gamma$ and F_0 stands for the last face of $\mathcal{F}_h^\Gamma$.

For any vertex v of $\mathcal{T}_h$ we denote by φ_v the piecewise linear and continuous function characterized by $\varphi_v(v') = \delta_{vv'}$ for all vertex v' of $\mathcal{T}_h$. Now, given any edge E of $\mathcal{T}_h$ we denote by $\mathbf{q}_E$ the function uniquely determined in $\mathcal{ND}_h(\Omega)$ by the conditions $\int_E \mathbf{q}_E \cdot \mathbf{t}_{E'}\, \mathrm{d}s = \delta_{EE'}$ for all edge E' of $\mathcal{T}_h$. The following result provides an explicit basis of the space $\widehat{\mathbf{X}}_h$.

Proposition 4.2. *The set*

$$\mathbf{B}_h := \left\{\mathbf{q}_E;\ E \in \mathcal{E}_h^0\right\} \cup \left\{\nabla\varphi_v;\ v \in \mathcal{V}_h^\Gamma \setminus \{v_0\}\right\}$$

forms a basis of $\widehat{\mathbf{X}}_h$.

Proof. Let us first prove that $\mathbf{B}_h$ is a free set of vector fields of $\widehat{\mathbf{X}}_h$. We consider coefficients $\left\{\alpha_E,\ E \in \mathcal{E}_h^0\right\}$ and $\left\{\alpha_v;\ v \in \mathcal{V}_h^\Gamma \setminus \{v_0\}\right\}$ such that

$$\sum_{E \in \mathcal{E}_h^0} \alpha_E \mathbf{q}_E + \sum_{v \neq v_0} \alpha_v \nabla\varphi_v = 0.$$

Then

$$\gamma_\tau \left(\sum_{E \in \mathcal{E}_h^0} \alpha_E \mathbf{q}_E + \sum_{v \neq v_0} \alpha_v \nabla\varphi_v \right) = \mathbf{curl}_\Gamma \left(\sum_{v \neq v_0} \alpha_v \varphi_v \right) = 0.$$

Consequently, $\sum_{v \neq v_0} \alpha_v \varphi_v$ must be constant on Γ. However, we already know that $\sum_{v \neq v_0} \alpha_v \varphi_v(v_0) = 0$. Thus, the function $\sum_{v \neq v_0} \alpha_v \varphi_v$ vanishes identically on Γ and therefore $\alpha_v = 0$ for all $v \in \mathcal{V}_h^\Gamma \setminus \{v_0\}$. Now, it is also clear that $\alpha_E = 0$ for all $E \in \mathcal{E}_h^0$.

It only remains to show the inclusion

$$\widehat{\mathbf{X}}_h \subseteq \operatorname{span}\{\mathbf{B}_h\}$$

where $\operatorname{span}\{\mathbf{B}_h\}$ is the vectorial space spanned by $\mathbf{B}_h$. We know that the tangential trace $\gamma_\tau \mathbf{q}$ of a function $\mathbf{q} \in \widehat{\mathbf{X}}_h$ is a divergence free piecewise constant tangential vector-field on $\mathcal{T}_h(\Gamma)$. Furthermore, as it has a vanishing circulation along any closed path γ: $\int_\gamma \gamma_\tau \mathbf{q} \cdot \mathbf{t}\, \mathrm{d}s = 0$. Then, there exists a (unique up to an additive constant) piecewise linear and continuous function ψ such that $\gamma_\tau \mathbf{q} = \mathbf{curl}_\Gamma\, \psi$. The function ψ can be uniquely determined by imposing the condition $\psi(v_0) = 0$. Let us consider now the extension $\widetilde{\psi}$ of ψ to $\overline{\Omega}$ given by the continuous and piecewise linear function that vanishes at all interior vertices. It follows that $\gamma_\tau \mathbf{q} = \gamma_\tau \nabla \widetilde{\psi}$ and hence

$$\mathbf{q} - \nabla \widetilde{\psi} \in \ker\, \gamma_\tau \cap \mathcal{ND}_h(\Omega) = \operatorname{span}\left\{\mathbf{q}_E;\ E \in \mathcal{E}_h^0\right\}\cdot$$

We conclude by noting that $\nabla \widetilde{\psi} \in \operatorname{span}\left\{\nabla \varphi_v;\ v \in \mathcal{V}_h^\Gamma \setminus \{v_0\}\right\}$. □

4.2. The interpolation operator

Now, we need to introduce some useful properties of the interpolation operator $\mathcal{I}_{h,\Omega}$ relative to the Nédélec finite element space $\mathcal{ND}_h(\Omega)$. For any $r \geq 0$, we introduce the Sobolev space

$$\mathbf{H}^r(\mathbf{curl}, \Omega) := \{\mathbf{q} \in \mathbf{H}^r(\Omega),\ \ \mathbf{curl}\, \mathbf{q} \in \mathbf{H}^r(\Omega)\}$$

and endow it with its Hilbertian norm $\|\mathbf{q}\|^2_{\mathbf{H}^r(\mathbf{curl},\Omega)} = \|\mathbf{q}\|^2_{r,\Omega} + \|\mathbf{curl}\, \mathbf{q}\|^2_{r,\Omega}$. For any edge E of $\mathcal{T}_h$, we denote by $\mathbf{t}_E$ a unit tangential vector along E and introduce the moments $\mathbf{m}_E(\mathbf{q})$ defined for a regular function $\mathbf{q}$ by

$$\mathbf{m}_E(\mathbf{q}) := \int_E \mathbf{q} \cdot \mathbf{t}_E \, \mathrm{d}s.$$

If γ is a path formed by edges E of $\mathcal{T}_h$ we will also denote $\mathbf{m}_\gamma(\mathbf{q}) := \sum_{E \in \gamma} \int_E \mathbf{q} \cdot \mathbf{t}_E \, \mathrm{d}s$.

It turns out that if $r > 1/2$ then, one may use a duality argument to extend the definition of $\mathbf{m}_E(\mathbf{q})$ to any function $\mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega)$, see Lemma 5.1 in [1] or Lemma 4.7 in [3]. It follows that the interpolation operator $\mathcal{I}_{h,\Omega} :\ \mathbf{H}^r(\mathbf{curl}, \Omega) \to \mathcal{ND}_h(\Omega)$ associated to the edge finite element, which is characterized by

$$\int_E \mathcal{I}_{h,\Omega} \mathbf{q} \cdot \mathbf{t}_E \, \mathrm{d}s = \mathbf{m}_E(\mathbf{q}) \qquad \text{for all edge } E \text{ of } \mathcal{T}_h, \tag{31}$$

is uniformly bounded and we have the following interpolation error estimate (Prop. 5.6 in [1]):

$$\|\mathbf{q} - \mathcal{I}_{h,\Omega}\mathbf{q}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C\, h^r\, \|\mathbf{q}\|_{\mathbf{H}^r(\mathbf{curl},\Omega)} \quad \forall \mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega), \quad (1/2 < r \leq 1). \tag{32}$$

Proposition 4.3. *Let r be a real number $> 1/2$. If $\mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega) \cap \widehat{\mathbf{X}}$ then $\mathcal{I}_{h,\Omega}\mathbf{q}$ belongs to $\widehat{\mathbf{X}}_h$.*

Proof. We will first prove that

$$\mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega); \quad \mathbf{curl}\, \mathbf{q} \cdot \mathbf{n} = 0 \quad \Rightarrow \quad \mathbf{curl}(\mathcal{I}_{h,\Omega}\mathbf{q}) \cdot \mathbf{n} = 0. \tag{33}$$

Let F be an arbitrary face of $\mathcal{T}_h(\Gamma)$. Applying Stokes' theorem yields

$$\int_F \mathbf{curl}(\mathcal{I}_{h,\Omega}\mathbf{q}) \cdot \mathbf{n}\, \mathrm{d}\xi = \int_{\partial F} (\gamma_\tau \mathcal{I}_{h,\Omega}\mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}s = \int_{\partial F} \mathcal{I}_{h,\Omega}\mathbf{q} \cdot \mathbf{t}_F \, \mathrm{d}s,$$

where $\mathbf{n}_F$ is the unit normal vector along ∂F and $\mathbf{t}_F := \mathbf{n} \wedge \mathbf{n}_F$ is a unit tangential vector along ∂F. Hence, by virtue of (31)

$$\int_F \mathbf{curl}(\mathcal{I}_{h,\Omega}\mathbf{q}) \cdot \mathbf{n}\, \mathrm{d}\xi = \int_{\partial F} \mathcal{I}_{h,\Omega}\mathbf{q} \cdot \mathbf{t}_F \, \mathrm{d}s = \mathbf{m}_{\partial F}(\mathbf{q}) = \int_F \mathbf{curl}\, \mathbf{q} \cdot \mathbf{n}\, \mathrm{d}\xi = 0.$$

It follows that (33) holds true since $\mathbf{curl}(\mathcal{I}_{h,\Omega}\mathbf{q}) \cdot \mathbf{n} \in M_h$ for all $\mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega) \cap \mathbf{X}$.

Now, using Green's formula (1) and following the steps given in the proof of Proposition 4.1 we obtain that

$$\mathbf{H}^r(\mathbf{curl}, \Omega) \cap \widehat{\mathbf{X}} = \left\{ \mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega) \cap \mathbf{X}; \quad \mathbf{m}_{\gamma_k^{\text{int}}}(\mathbf{q}) = 0 \quad \text{and} \quad \mathbf{m}_{\gamma_k^{\text{ext}}}(\mathbf{q}) = 0 \quad \forall k = 1, \cdots, B \right\} \cdot$$

Let γ be an arbitrary closed path of edges in $\mathcal{T}_h(\Gamma)$ and let Σ be a connected open surface contained in $\overline{\Omega}$ and such that $\partial\Sigma = \gamma$. It is clear that we can choose Σ in such a way that $\Sigma_k^{\text{ext}} \cap \Sigma = \emptyset$ or $\Sigma_k^{\text{int}} \cap \Sigma = \emptyset$ for some index $k \in \{1, \cdots, B\}$. To fix the ideas, let us assume that $k_0 \in \{1, \cdots, B\}$ is such that $\Sigma_{k_0}^{\text{ext}} \cap \Sigma = \emptyset$. We consider a Lipschitz open set V that satisfies $\overline{V} \subset \overline{\Omega}$, $\Sigma \cup \Sigma_{k_0}^{\text{ext}} \subset \partial V$ and $\partial V \setminus (\Sigma \cup \Sigma_{k_0}^{\text{ext}}) \subset \Gamma$. Then, by Gauss' formula,

$$0 = \int_{\partial V} \mathbf{curl}\, \mathbf{q} \cdot \mathbf{n}\, \mathrm{d}\xi = \int_{\Sigma} \mathbf{curl}\, \mathbf{q} \cdot \mathbf{n}\, \mathrm{d}\xi + \int_{\Sigma_{k_0}^{\text{ext}}} \mathbf{curl}\, \mathbf{q} \cdot \mathbf{n}\, \mathrm{d}\xi \quad \forall \mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega) \cap \widehat{\mathbf{X}}$$

and by Stokes' formula

$$\mathbf{m}_{\gamma}(\mathbf{q}) = \int_{\Sigma} \mathbf{curl}\, \mathbf{q} \cdot \mathbf{n}\, \mathrm{d}\xi = -\int_{\Sigma_{k_0}^{\text{ext}}} \mathbf{curl}\, \mathbf{q} \cdot \mathbf{n}\, \mathrm{d}\xi = -\mathbf{m}_{\gamma_{k_0}^{\text{ext}}}(\mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega) \cap \widehat{\mathbf{X}}.$$

Thus, we have just shown that

$$\mathbf{H}^r(\mathbf{curl}, \Omega) \cap \widehat{\mathbf{X}} = \{ \mathbf{q} \in \mathbf{H}^r(\mathbf{curl}, \Omega) \cap \mathbf{X}; \quad \mathbf{m}_{\gamma}(\mathbf{q}) = 0, \quad \text{for all closed paths } \gamma \text{ of edges in } \mathcal{T}_h(\Gamma) \} \cdot$$

After this new characterization of the space $\mathbf{H}^r(\mathbf{curl}, \Omega) \cap \widehat{\mathbf{X}}$, the result is a direct consequence of (33) and the definition (31) of the interpolation operator $\mathcal{I}_{h,\Omega}$. □

5. The Galerkin method in the simply connected case

In this section we will study separately the discrete scheme of problem (27) when Ω is simply connected. By virtue of the finite element spaces introduced in the previous section, the Galerkin scheme reads

$$\begin{aligned} &\text{find } \left(\widehat{\mathbf{h}}_h, \lambda_h\right) \in \widehat{\mathbf{X}}_h \times M_h; \\ &\mathcal{A}_{\text{sc}}\left(\left(\widehat{\mathbf{h}}_h, \lambda_h\right), (\widehat{\mathbf{q}}, \eta)\right) = L((\widehat{\mathbf{q}}, \mathbf{0})) \quad \forall (\widehat{\mathbf{q}}, \eta) \in \widehat{\mathbf{X}}_h \times M_h. \end{aligned} \tag{34}$$

Theorem 5.1. *Assume that the exact solution* $\left(\widehat{\mathbf{h}}, \lambda\right)$ *of* (28) *belongs to* $(\mathbf{H}^r(\mathbf{curl}, \Omega) \times \mathrm{H}^{r-1/2}(\Gamma)) \cap (\widehat{\mathbf{X}} \times \mathrm{H}_0^{-1/2}(\Gamma))$ *for some* $1/2 < r \leq 1$. *Then, there exists a constant* C *independent of* h *such that*

$$\left\| \widehat{\mathbf{h}} - \widehat{\mathbf{h}}_h \right\|_{\mathbf{H}(\mathbf{curl}, \Omega)} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \leq C\, h^r \left(\|\mathbf{h}\|_{r,\Omega} + \|\mathbf{curl}\, \mathbf{h}\|_{r,\Omega} + \|\lambda\|_{r-1/2,\Gamma} \right).$$

Proof. We deduce from Proposition 4.3 and Céa's estimate

$$\left(\left\| \widehat{\mathbf{h}} - \widehat{\mathbf{h}}_h \right\|^2_{\mathbf{H}(\mathbf{curl}, \Omega)} + \|\lambda - \lambda_h\|^2_{-1/2,\Gamma} \right)^{1/2} \leq C \inf_{(\widehat{\mathbf{q}}, \eta) \in \widehat{\mathbf{X}}_h \times M_h} \left(\left\| \widehat{\mathbf{h}} - \widehat{\mathbf{q}} \right\|^2_{\mathbf{H}(\mathbf{curl}, \Omega)} + \|\lambda - \eta\|^2_{-1/2,\Gamma} \right)^{1/2}$$

that

$$\left(\left\| \widehat{\mathbf{h}} - \widehat{\mathbf{h}}_h \right\|^2_{\mathbf{H}(\mathbf{curl}, \Omega)} + \|\lambda - \lambda_h\|^2_{-1/2,\Gamma} \right)^{1/2} \leq C(\|\mathbf{h} - \mathcal{I}_{h,\Omega}\mathbf{h}\|^2_{\mathbf{H}(\mathbf{curl}, \Omega)} + \|\lambda - \rho_h \lambda\|^2_{-1/2,\Gamma})^{1/2},$$

where ρ_h is $L^2(\Gamma)$-orthogonal projection operator onto the subspace of piecewise constant functions with respect to $\mathcal{T}_h(\Gamma)$. Then, the theorem follows directly from (32) and the fact that

$$\|\lambda - \rho_h\lambda\|_{-1/2,\Gamma} \le C\, h^r\, \|\lambda\|_{r-1/2,\Gamma}\,, \qquad \forall\lambda \in \mathrm{H}^{r-1/2}(\Gamma). \tag{35}$$

□

5.1. Matrix form of the discrete problem

We denote by $1_F(\mathbf{x})$ the indicator function of a face $F \in \mathcal{F}_h^\Gamma$ and by $|F|$ its area. Then, it is straightforward that the set

$$\left\{\rho_F(\mathbf{x}) := \frac{1}{|F|}1_F(\mathbf{x}) - \frac{1}{|F_0|}1_{F_0}(\mathbf{x}); \quad F \in \mathcal{F}_h^\Gamma \setminus \{F_0\}\right\}$$

is a basis of M_h.

We have at our disposal explicit basis of both $\widehat{\mathbf{X}}_h$ and M_h. Hence, we are in a position to give the matricial formulation of the discrete problem (34). If we set

$$\widehat{\mathbf{h}}_h(\mathbf{x}) := \sum_{E\in\mathcal{E}_h^0} h_E \mathbf{q}_E(\mathbf{x}) + \sum_{v\in\mathcal{V}_h^\Gamma\setminus\{v_0\}} h_v \nabla\varphi_v(\mathbf{x})$$

and

$$\lambda_h(\mathbf{x}) := \sum_{F\in\mathcal{F}_h^\Gamma\setminus\{F_0\}} \lambda_F \rho_F(\mathbf{x})$$

the linear system associated to (34) takes the following form

$$\begin{pmatrix} A^\Omega & (1+\imath)(A^{\Omega\Gamma})^\top & 0 \\ (1+\imath)A^{\Omega\Gamma} & A^\Gamma + D & (1+\imath)(B)^\top \\ 0 & (1-\imath)B & -V \end{pmatrix} \begin{pmatrix} \mathbf{h}^\Omega \\ \mathbf{h}^\Gamma \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} f^\Omega \\ f^\Gamma \\ 0 \end{pmatrix} \tag{36}$$

with the obvious definition for $\mathbf{h}^\Omega$, $\mathbf{h}^\Gamma$ and $\boldsymbol{\lambda}$ and where the superscript $(\cdot)^\top$ denotes transposition. The entries of the matrices that appears in (36) are defined by:

$$A^\Omega_{E,E'} := a(\mathbf{q}_E, \mathbf{q}_{E'}) \qquad A^{\Omega\Gamma}_{E,v} := (\mu/\mu_0 \mathbf{q}_E,\ \nabla\varphi_v)_{0,\Omega}$$

$$A^\Gamma_{v,v'} := (1+\imath)(\mu/\mu_0 \nabla\varphi_v,\ \nabla\varphi_{v'})_{0,\Omega} \qquad D_{v,v'} := (1+\imath)(\mathbf{curl}_\Gamma\,\varphi_{v'},\ \pi_\tau\mathcal{V}\,\mathbf{curl}_\Gamma\,\varphi_v,)_{0,\Gamma}$$

$$B_{v,F} := -(\rho_F, (1/2 - \mathcal{K})\varphi_v)_{0,\Gamma} \qquad V_{F,F'} := (1-\imath)(\rho_{F'},\ \mathcal{V}\rho_F)_{0,\Gamma}.$$

The right hand side is given by

$$f^\Omega_E := -(1+\imath)(\mu/\mu_0\,\mathbf{curl}\,\mathbf{a}^s,\ \mathbf{q}_E)_{0,\Omega} \qquad f^\Gamma_v := (1+\imath)(\mathbf{a}^s,\ \mathbf{curl}_\Gamma\,\varphi_v)_{0,\Gamma}.$$

Notice that although the inverse of the tangential operator $\mathbf{curl}_\Gamma$ is used in the definition of the form $b(\gamma_\tau\mathbf{q},\ \eta)$, it does not take place in the computation of the entries of the corresponding matrix B. Therefore the numerical scheme is implementable. Suitable choices of quadrature formulae for the computation of the singular boundary integrals and efficient strategies for the resolution of the linear system (whose matrix is not Hermitian) are the aim of forthcoming work.

6. The discrete problem in the non-simply connected case

When Ω is not simply connected, we need several types of approximations of the harmonic Neumann vector-fields $\mathbb{H}(\Omega_c)$ in order to derive the completely discrete problem. The discretization process requires the resolution of the following auxiliary problems.

6.1. **First auxiliary problem**

Our first goal consists in computing an approximation $\mathbf{N}_h$ of matrix $\mathbf{N}$. In other words, it is necessary to obtain approximations of the fluxes $\left(\langle \frac{\partial z_k}{\partial \mathbf{n}}, 1\rangle_{1/2,\Sigma_\ell^{\text{ext}}}\right)_{k,\ell=1,\cdots,B}$ of the harmonic Neumann vector-fields across the cutting surfaces Σ_ℓ^{ext}. To this end, for each k, we solve the exterior problem (18) by a scheme based on a coupling of the Raviart–Thomas finite element method with a boundary element method, [26].

Let us consider a connected and simply connected polyhedron $\mathcal{O}$ such that $\overline{\Omega} \cup \left(\cup_{k=1}^B \overline{\Sigma}_k^{\text{ext}}\right) \subset \mathcal{O}$. We set

$$Q^0 := \mathcal{O} \setminus \left\{\overline{\Omega} \cup \left(\cup_{k=1}^B \overline{\Sigma}_k^{\text{ext}}\right)\right\}, \quad Q := \mathcal{O} \setminus \overline{\Omega} \quad \text{and} \quad \Lambda := \partial\mathcal{O}.$$

We deduce immediately from (18) that $\mathbf{p}_k := \widetilde{\nabla} z_k|_Q$ belongs to the following closed subspace of $\mathbf{H}(\text{div}, Q)$

$$\mathbf{Y} := \left\{\mathbf{q} \in \mathbf{L}^2(Q); \quad \text{div}\, \mathbf{q} = 0 \text{ in } Q \text{ and } \mathbf{q}|_\Gamma \cdot \mathbf{n} = 0 \text{ in } \mathrm{H}^{-1/2}(\Gamma)\right\}$$

and satisfies the variational equation

$$(\mathbf{p}_k,\, \mathbf{q})_{0,Q} = \langle \overline{\mathbf{q}} \cdot \mathbf{n}_k,\, 1\rangle_{1/2,\Sigma_k^{\text{ext}}} + \langle \mathbf{q} \cdot \mathbf{n},\, \overline{z}_k\rangle_{1/2,\Lambda} \quad \forall \mathbf{q} \in \mathbf{Y},$$

where we are assuming that the normal vector $\mathbf{n}$ on Λ is pointing from Q into $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. Furthermore, as z_k is harmonic in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$ we deduce that the last equation may be coupled with boundary integral equations (similar to (20) and (21)) relating z_k and its normal derivative $\mathbf{p}_k \cdot \mathbf{n}$ on Λ. This leads to the following weak formulation (see [26] for more details):

find $\mathbf{p}_k \in \mathbf{Y}$ and $\phi_k \in \mathrm{H}^{1/2}(\Lambda)/\mathbb{C}$ such that;

$$\begin{aligned}(\mathbf{p}_k,\, \mathbf{q})_{0,Q} + \langle \overline{\mathbf{q}} \cdot \mathbf{n},\, \mathcal{V}\overline{\mathbf{p}}_k \cdot \mathbf{n}\rangle_{1/2,\Lambda} - \left\langle \overline{\mathbf{q}} \cdot \mathbf{n},\, \left(\frac{1}{2}\mathcal{I} + \mathcal{K}\right)\overline{\phi}_k\right\rangle_{1/2,\Lambda} &= \langle \overline{\mathbf{q}} \cdot \mathbf{n}_k,\, 1\rangle_{1/2,\Sigma_k^{\text{ext}}}, \\ -\left\langle \left(\frac{1}{2}\mathcal{I} + \mathcal{K}'\right)\mathbf{p}_k \cdot \mathbf{n},\, \chi\right\rangle_{1/2,\Lambda} - \langle \mathcal{H}\phi_k,\, \chi\rangle_{1/2,\Lambda} &= 0,\end{aligned} \tag{37}$$

for all functions $\mathbf{q} \in \mathbf{Y}$ and $\chi \in \mathrm{H}^{1/2}(\Lambda)/\mathbb{C}$. We point out here that the variable ϕ_k represents (up to an additive constant) the trace of z_k on Λ.

Let us now consider a regular family of triangulations $\{\mathcal{T}_h(Q)\}_h$ of Q by tetrahedra T of diameter no greater than $h > 0$. We assume that, for any h, the set $\mathcal{T}_h(\Omega) \cup \mathcal{T}_h(Q)$ is a triangulation of $\mathcal{O}$. This implies that the triangulation induced by $\mathcal{T}_h(Q)$ on Γ is identical to $\mathcal{T}_h(\Gamma)$. We may assume, without loss of generality, that the cutting surfaces Σ_ℓ^{ext} is union of faces of tetrahedra $T \in \mathcal{T}_h(Q)$ for each mesh $\mathcal{T}_h(Q)$. Finally, we denote by $\mathcal{T}_h(\Lambda)$ the triangulation induced by $\mathcal{T}_h(Q)$ on Λ.

We introduce a conforming discretization of $\mathbf{H}(\text{div}, \Omega)$ with the aid of lowest order Raviart–Thomas finite element space (*cf.* [10, 29])

$$\mathcal{RT}_h(Q) := \{\mathbf{q} \in \mathbf{H}(\text{div}, \Omega); \quad \mathbf{q}|_T \in \mathcal{RT}(T) \quad \forall T \in \mathcal{T}_h(Q)\}$$

where $\mathcal{RT}(T) := \left\{a\mathbf{x} + \mathbf{b},\ a \in \mathbb{C},\ \mathbf{b} \in \mathbb{C}^3\right\}$.

The discrete counterpart of problem (37) is formulated as follows:

$$\text{find } \mathbf{p}_{k,h} \in \mathbf{Y}_h \text{ and } \phi_{k,h} \in \Psi_h/\mathbb{C} \text{ such that;}$$

$$\begin{aligned}(\mathbf{p}_{k,h},\, \mathbf{q})_{0,Q} + \left\langle \overline{\mathbf{q}}\cdot\mathbf{n},\, \mathcal{V}\overline{\mathbf{p}}_{k,h}\cdot\mathbf{n}\right\rangle_{1/2,\Lambda} - \left\langle \overline{\mathbf{q}}\cdot\mathbf{n},\, \left(\frac{1}{2}\mathcal{I}+\mathcal{K}\right)\overline{\phi}_{k,h}\right\rangle_{1/2,\Lambda} &= \langle \overline{\mathbf{q}}\cdot\mathbf{n}_k,\, 1\rangle_{1/2,\Sigma_k^{\text{ext}}};\\ -\left\langle \left(\frac{1}{2}\mathcal{I}+\mathcal{K}'\right)\mathbf{p}_{k,h}\cdot\mathbf{n},\, \chi\right\rangle_{1/2,\Lambda} - \langle \mathcal{H}\phi_{k,h}, \chi\rangle_{1/2,\Lambda} &= 0;\end{aligned} \tag{38}$$

for all functions $\mathbf{q} \in \mathbf{Y}_h$ and $\chi \in \Psi_h/\mathbb{C}$, where $\mathbf{Y}_h := \mathcal{RT}_h(Q) \cap \mathbf{Y}$ and Ψ_h is the space of piecewise linear and continuous functions on Λ with respect to $\mathcal{T}_h(\Lambda)$.

We point out that, for the sake of brevity, we deliberately eliminated here the Lagrange multiplier associated to the constraint $\operatorname{div}\mathbf{p}_k = 0$. However, in practice it is a hard task to find a basis of $\mathbf{Y}_h$ because of the divergence free condition. Usually, it is more convenient to implement the discrete problem:

$$\text{find } \mathbf{p}_{k,h} \in \mathcal{RT}_h^0(Q),\ \phi_{k,h} \in \Psi_h/\mathbb{C} \text{ and } z_k \in M_h \text{ such that;}$$

$$\begin{aligned}(\mathbf{p}_{k,h},\, \mathbf{q})_{0,Q} + (\mathcal{V}\mathbf{p}_{k,h}\cdot\mathbf{n},\, \mathbf{q}\cdot\mathbf{n})_{0,\Lambda} - \left(\left(\frac{1}{2}\mathcal{I}+\mathcal{K}\right)\phi_{k,h},\, \mathbf{q}\cdot\mathbf{n}\right)_{0,\Lambda} &= (1,\, \mathbf{q}\cdot\mathbf{n}_k)_{0,\Sigma_k^{\text{ext}}};\\ -\left(\mathbf{p}_{k,h}\cdot\mathbf{n},\, \left(\tfrac{1}{2}\mathcal{I}+\mathcal{K}\right)\chi\right)_{0,\Lambda} - (\mathcal{V}(\mathbf{curl}_\Gamma\,\phi_{k,h}),\, \mathbf{curl}_\Gamma\,\chi)_{0,\Lambda} &= 0;\\ (\operatorname{div}\mathbf{p}_{k,h},\, v)_{0,Q} &= 0;\end{aligned} \tag{39}$$

for all functions $\mathbf{q} \in \mathcal{RT}_h^0(Q)$, $\chi \in \Psi_h/\mathbb{C}$ and $v \in M_h$, where $\mathcal{RT}_h^0(Q) := \{\mathbf{q} \in \mathcal{RT}_h(Q);\ \mathbf{q}|_\Gamma\cdot\mathbf{n} = 0\}$ and the multiplier $z_{k,h}$ lives in the space M_h of piecewise constant functions with respect to $\mathcal{T}_h(Q)$. We know from [26] that (39) is a well posed problem. Notice that, here again, we also take advantage of Lemma 3.4 and substitute $\mathcal{H}$ by $\mathcal{V}$ since the latter integral operator have a less severe singularity.

Once the function $\mathbf{p}_{k,h}$computed for $1 \le k \le B$, we may approximate $\mathbf{N}$ by the matrix

$$\mathbf{N}_h := \left(\int_{\Sigma_\ell^{\text{ext}}} \mathbf{p}_{k,h}\cdot\mathbf{n}\,\mathrm{d}\xi\right)_{1\le k,\ell\le B}.$$

6.2. Second auxiliary problem

We are now concerned with the problem of finding an adequate approximation of the lifting $\mathcal{R}\mathbf{g}_k$ to Ω of the tangential trace $\mathbf{g}_k := \gamma_\tau\mathbf{p}_k$. To this end, we must first provide an approximation $\mathbf{g}_{k,h}$ of $\mathbf{g}_k$. Notice that it is not suitable to obtain such an approximation from the solution of problem (38) since $\mathbf{p}_{k,h}$ furnishes only good approximations of the fluxes of $\mathbf{p}_k$. Thus, we are obliged to solve a new auxiliary problem. Fortunately, the problem can be posed in the bounded domain Q.

We deduce from Proposition 3.14 in [3] that the space of harmonic Neumann vector-fields $\mathbb{H}(\Omega_c)$ is also spanned by the functions $\left\{\widetilde{\nabla} w_k,\ k = 1,\cdots,B\right\}$, where for any $k = 1,\cdots,B$, $w_k \in W^1(\Omega_e^0)$ solves

$$\begin{aligned}\Delta w_k &= 0 \qquad \text{in } \Omega_c^0;\\ \frac{\partial w_k}{\partial\mathbf{n}} &= 0 \qquad \text{on } \Gamma;\\ [w_k]_\ell &= \text{const.} \quad \text{and} \quad \left[\frac{\partial w_k}{\partial\mathbf{n}}\right]_\ell = 0, \quad \ell = 1,\cdots,B;\\ \left\langle \frac{\partial w_k}{\partial\mathbf{n}},\, 1\right\rangle_{1/2,\Sigma_\ell^{\text{ext}}} &= \delta_{k\ell}, \quad \ell = 1,\cdots,B.\end{aligned} \tag{40}$$

It is straightforward that the two basis of $\mathbb{H}(\Omega_c)$ are related by

$$w_k = \sum_{\ell=1}^{B} [w_k]_\ell \, z_\ell \quad \text{and} \quad z_k = \sum_{\ell=1}^{B} \left\langle \frac{\partial z_k}{\partial \mathbf{n}},\, 1 \right\rangle_{1/2,\Sigma_\ell^{\text{ext}}} w_\ell.$$

The last identities may also be written

$$w_k = \sum_{\ell=1}^{B} \mathbf{N}_{k,\ell}^{-1} \, z_\ell \quad \text{and} \quad z_k = \sum_{\ell=1}^{B} \mathbf{N}_{k,\ell} \, w_\ell \tag{41}$$

since it is clear that $\sum_{j=1}^{B} \left\langle \frac{\partial z_k}{\partial \mathbf{n}},\, 1 \right\rangle_{1/2,\Sigma_j^{\text{ext}}} [w_j]_\ell = \delta_{k\ell}$, $1 \le k, \ell \le B$.

Let us introduce the closed subspace Θ of $\mathrm{H}^1(Q^0)$ defined by

$$\Theta := \left\{ \theta \in \mathrm{H}^1(Q^0);\ [\theta]_\ell = \text{const.},\ \ \ell = 1, \cdots, B \right\} \cdot$$

We notice that (41) yields $\frac{\partial w_k}{\partial \mathbf{n}}|_\Lambda = \sum_{\ell=1}^{B} \mathbf{N}_{k,\ell}^{-1} \frac{\partial z_\ell}{\partial \mathbf{n}}|_\Lambda$. Thus, following the steps given in the proof of Proposition 3.14 in [3], one can see that the solution of (40) satisfies the variational formulation

$$\begin{aligned} &\text{find } w_k \in \Theta/\mathbb{C} \text{ such that;} \\ &(\nabla w_k,\, \nabla \theta)_{0,Q^0} = [\theta]_k + \sum_{\ell=1}^{B} \mathbf{N}_{k,\ell}^{-1} \left\langle \mathbf{p}_\ell \cdot \mathbf{n},\, \theta \right\rangle_{1/2,\Lambda} \quad \forall \theta \in \Theta/\mathbb{C}; \end{aligned} \tag{42}$$

where we still denote here by $\mathbf{p}_k$ the solution of (37). This problem has a unique solution by virtue of the Lax–Milgram lemma.

We introduce

$$\Theta_h := \left\{ \theta \in \mathrm{H}^1(Q^0); \quad \theta|_T \in \mathbb{P}_1(T), \quad \forall T \in \mathcal{T}_h(Q) \text{ and } [\theta]_\ell = \text{const. } 1 \le \ell \le B \right\},$$

where $\mathbb{P}_1(T)$ the space of linear functions on T. Consider the following discrete version of problem (42):

$$\begin{aligned} &\text{find } w_{k,h} \in \Theta_h/\mathbb{C} \text{ such that;} \\ &(\nabla w_{k,h},\, \nabla \theta)_{0,Q^0} = [\theta]_k + \sum_{\ell=1}^{B} \left(\mathbf{N}_h^{-1}\right)_{k,\ell} \left\langle \mathbf{p}_{\ell,h} \cdot \mathbf{n},\, \theta \right\rangle_{1/2,\Lambda} \quad \forall \theta \in \Theta_h/\mathbb{C}; \end{aligned} \tag{43}$$

where $\mathbf{p}_{k,h}$ is the solution of (38) and $\left\{ \left(\mathbf{N}_h^{-1}\right)_{k,\ell},\ 1 \le k, \ell \le B \right\}$ are the entries of matrix $\mathbf{N}_h^{-1}$.

We can now take

$$\mathbf{g}_{k,h} := \gamma_\tau \left(\widetilde{\nabla} z_{k,h} \right) = \widetilde{\mathbf{curl}}_\Gamma z_{k,h}$$

where $z_{k,h} := \sum_{\ell=1}^{B} (\mathbf{N}_h)_{k,\ell} \, w_{\ell,h}$, for any $1 \le k \le B$.

6.3. Third auxiliary problem

Let us introduce a discrete version $\mathcal{R}_h : \gamma_\tau(\mathcal{ND}_h(\Omega)) \to \mathcal{ND}_h(\Omega)$ of the right inverse $\mathcal{R}$ of γ_τ. To this end, for any $\mathbf{g}_h \in \gamma_\tau(\mathcal{ND}_h(\Omega))$, we consider the unique solution $\mathcal{R}_h \mathbf{g}_h$ of

$$\begin{aligned} &\text{find } \mathcal{R}_h \mathbf{g}_h \in \mathcal{ND}_h(\Omega) \text{ such that } \gamma_\tau \mathcal{R}_h \mathbf{g}_h = \mathbf{g}_h \text{ and} \\ &(\mathcal{R}_h \mathbf{g}_h,\, \mathbf{q})_{\mathbf{H}(\mathbf{curl},\Omega)} = 0 \quad \forall \mathbf{q} \in \mathcal{ND}_h(\Omega) \cap \ker \gamma_\tau. \end{aligned} \tag{44}$$

Notice that the vector-fields $\widetilde{\nabla} z_{k,h},\ k = 1,\cdots,B$, belong to $\mathcal{ND}_h(Q)$. It follows that $\mathbf{g}_{k,h} := \gamma_\tau(\widetilde{\nabla} z_{k,h}) \in \gamma_\tau(\mathcal{ND}_h(Q)) = \gamma_\tau(\mathcal{ND}_h(\Omega))$. Hence, we are allowed to extend each tangential vector-field $\mathbf{g}_{k,h}$ to Ω by means of the discrete operator $\mathcal{R}_h$, *i.e.*, for each $k = 1,\cdots,B$, we can compute $\mathcal{R}_h\mathbf{g}_{k,h} \in \mathcal{ND}_h(\Omega)$.

With this last ingredient, we are now ready to define the discrete problem corresponding to (27).

6.4. Discrete version of problem (27)

Let us denote $\mathbf{W}_h := \left(\widehat{\mathbf{X}}_h \times \mathbb{C}^B\right) \times M_h$. We introduce the following discrete problem associated to (27):

$$\begin{aligned} &\text{find } \widetilde{\mathbf{h}}_h \in \mathbf{W}_h;\\ &\mathcal{A}_h\left(\widetilde{\mathbf{h}}_h, \widetilde{\mathbf{q}}\right) = \mathcal{F}_h(\widetilde{\mathbf{q}}) \quad \forall \widetilde{\mathbf{q}} \in \mathbf{W}_h, \end{aligned} \tag{45}$$

where

$$\mathcal{F}_h(\widetilde{\mathbf{q}}) := -(1+\imath)(\mu/\mu_0\, \mathbf{curl}\, \mathbf{a}^s, \widehat{\mathbf{q}} + \sum_{k=1}^{B} \zeta_k \mathcal{R}_h \mathbf{g}_{k,h})_{0,\Omega} + (1+\imath)\overline{\langle \gamma_\tau \widehat{\mathbf{q}}, \pi_\tau \mathbf{a}^s \rangle}_{\mathbf{V}'_\pi \times \mathbf{V}_\pi}$$

and

$$\mathcal{A}_h\left(\widetilde{\mathbf{h}}, \widetilde{\mathbf{q}}\right) := A_h\left(\left(\widehat{\mathbf{h}}, \boldsymbol{\beta}\right),\, (\widehat{\mathbf{q}}, \boldsymbol{\zeta})\right) - b(\gamma_\tau \widehat{\mathbf{q}}, \lambda) + b^*\left(\gamma_\tau \widehat{\mathbf{h}}, \eta\right) + c(\lambda, \eta)$$

with

$$A_h\left(\left(\widehat{\mathbf{h}}, \boldsymbol{\beta}\right),\, (\widehat{\mathbf{q}}, \boldsymbol{\zeta})\right) := a\left(\widehat{\mathbf{h}} + \sum_{k=1}^{B} \beta_k \mathcal{R}_h \mathbf{g}_{k,h},\ \widehat{\mathbf{q}} + \sum_{\ell=1}^{B} \zeta_\ell \mathcal{R}_h \mathbf{g}_{\ell,h}\right) + (1+\imath)\mathbf{N}_h \boldsymbol{\beta} \cdot \overline{\boldsymbol{\zeta}} + d\left(\gamma_\tau \widehat{\mathbf{h}}, \gamma_\tau \widehat{\mathbf{q}}\right).$$

7. Convergence analysis in the non-simply connected case

7.1. Analysis corresponding to the first auxiliary problem

In the sequel, we denote by $s_Q \in (1/2,\ 1)$ the exponent of maximal regularity in Q of the solution of Laplace operators with $\mathrm{L}^2(Q)$ right-hand side and homogeneous Neumann boundary datum (see Rem. 3.8 in [3]). Similarly, s_Ω is the exponent of maximal regularity in Ω of Laplace operators with $\mathrm{L}^2(\Omega)$ right-hand side and homogeneous Neumann boundary datum.

Theorem 7.1. *The principal unknown* $\mathbf{p}_k$ *of* (37) *belongs to* $\mathbf{H}^s(Q)$ *for all* $0 \le s < s_Q$.

Proof. We know from Theorem 4.3 in [1] that for each $\delta \in (0,\ 1/2)$ the space

$$\left\{\mathbf{q} \in \mathbf{H}(\mathbf{curl}, Q) \cap \mathbf{H}(\mathrm{div}, Q);\ \mathbf{q}\cdot\mathbf{n} \in H^\delta(\partial Q)\right\}$$

is continuously embedded in $\mathbf{H}^\epsilon(Q)$, where $0 \le \epsilon < \min(1/2 + \delta, s_Q)$.

In our case, $\mathbf{p}_k \cdot \mathbf{n} = 0$ on Γ and $\mathbf{p}_k$ is the gradient of a function that is harmonic in an open neighborhood of Λ. By virtue of the local character of the regularity of solutions of Laplace equation we deduce that the regularity of $\mathbf{p}_k$ depends only on s_Q. □

Theorem 7.2. *Problems* (37) *and* (38) *are well posed and there exists a constant* $C > 0$ *independent of* h *such that*

$$\|\mathbf{p}_k - \mathbf{p}_{k,h}\|_{0,Q} + \|\phi_k - \phi_{k,h}\|_{\mathrm{H}^{1/2}(\Lambda)/\mathbb{C}} \le C h^s \left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\} \quad \textit{for all } 1/2 < s < s_Q.$$

Proof. The well posedness of (37) and (38) can be obtained from [26] by reproducing verbatim the proof given there in the bidimensional case. However, as we are not considering the Lagrange multiplier corresponding to the divergence free condition, there is no need here of the so called inf-sup condition. The properties of the integral operators given in Lemma 3.3 and Lemma 3.4 shows that the Lax–Milgram theorem applies here directly to provide existence and uniqueness for problem (37) and Céa's lemma gives the following abstract error estimate

$$\|\mathbf{p}_k - \mathbf{p}_{k,h}\|_{0,Q} + \|\phi_k - \phi_{k,h}\|_{\mathrm{H}^{1/2}(\Lambda)/\mathbb{C}} \le C_0 \left\{ \inf_{(\mathbf{q},\chi)\in \mathbf{Y}_h\times\Theta_h/\mathbb{C}} \|(\mathbf{p}_k,\phi_k) - (\mathbf{q},\chi)\|_{\mathbf{L}^2(Q)\times \mathrm{H}^{1/2}(\Lambda)/\mathbb{C}} \right\}. \tag{46}$$

Let $\Pi_{h,Q}$ be the classical interpolation operator related to $\mathcal{RT}_h(Q)$, see [10, 29] for the definition and basic properties. We know that $\Pi_{h,Q} : \mathbf{Y}\cap\mathbf{H}^s(Q) \to \mathbf{Y}_h$ is uniformly bounded for all $s > 1/2$ and the interpolation error estimate

$$\|\mathbf{q} - \Pi_{h,Q}\mathbf{q}\|_{0,Q} \le C_1 h^s \|\mathbf{q}\|_{s,Q} \quad \forall \mathbf{q}\in \mathbf{Y}\cap\mathbf{H}^s(Q), \quad (1/2 < s \le 1)$$

holds true. In fact, in the classical literature [10, 29], the proof of the last estimate is only given for integer values of s. However, the proof for $1/2 < s < 1$ may also be obtained by the usual Bramble–Hilbert lemma and some scaling arguments close to those derived in Lemma 5.5 in [1].

Let us now consider the Lagrange interpolation operator $\mathcal{L}_{h,\Lambda}$ defined from the space of continuous functions on Λ onto Ψ_h. It is well known that the estimate

$$\|\chi - \mathcal{L}_{h,\Lambda}\chi\|_{1/2,\Lambda} \le C_2\, h^s \|\chi\|_{s+1/2,\Lambda} \quad \forall \chi \in \mathrm{H}^{s+1/2}(\Lambda) \quad (s>0)$$

is satisfied for some constant $C_2 > 0$ that only depends on the shape regularity of the mesh.

We conclude by using the last two interpolation error estimates and (46). □

Corollary 7.3. *There exists an $h_0 \in (0,\,1)$ such that $\mathbf{N}_h$ is invertible for all $0 < h \le h_0$. Moreover, we have the error estimate*

$$\|\mathbf{N} - \mathbf{N}_h\| + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| \le C\, h^s \max_k \left\{ \|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda} \right\}, \qquad (0 < h \le h_0),$$

for some constant C independent of h.

Proof. We deduce from the continuity of the normal trace on Σ_l in $\mathbf{Y}\subset\mathbf{H}(\mathrm{div}, Q)$ that

$$\left| \langle (\mathbf{p}_k - \mathbf{p}_{k,h})\cdot\mathbf{n},\, 1\rangle_{1/2,\Sigma_\ell} \right| \le C_1\ \|\mathbf{p}_k - \mathbf{p}_{k,h}\|_{0,Q} \quad (1 \le \ell \le B).$$

Hence, applying Theorem 7.2, we obtain the estimate

$$\|\mathbf{N} - \mathbf{N}_h\| \le C_2 \max_k \|\mathbf{p}_k - \mathbf{p}_{k,h}\|_{0,Q} \le C_2\, h^s \max_k \left\{ \|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda} \right\}. \tag{47}$$

Now, recall that $\mathbf{N}$ is Hermitian and positive definite and hence it is invertible. Considering the Neumann series, it is easy to prove that if $h_0 \in (0,\,1)$ is such that $\left\|\mathbf{N}^{-1}(\mathbf{N}-\mathbf{N}_h)\right\| \le 1/2$ for all $0 < h \le h_0$ then $\mathbf{N}_h^{-1}$ exists and

$$\left\|\mathbf{N}_h^{-1}\right\| \le \frac{\left\|\mathbf{N}^{-1}\right\|}{1 - \|\mathbf{N}^{-1}(\mathbf{N}_h - \mathbf{N})\|} \le 2\left\|\mathbf{N}^{-1}\right\| \quad (0 < h \le h_0).$$

Finally,

$$\left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| = \left\|\mathbf{N}^{-1}(\mathbf{N}_h - \mathbf{N})\mathbf{N}_h^{-1}\right\| \le 2\left\|\mathbf{N}^{-1}\right\|^2 \|\mathbf{N}_h - \mathbf{N}\|,$$

and the remaining estimate follows from (47). □

7.2. Analysis corresponding to the second auxiliary problem

Theorem 7.4. *There exists a constant C independent of h such that*

$$\|\nabla w_k - \nabla w_{k,h}\|_{0,Q} \leq C\, h^s \max_k \left\{ \|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda} \right\} \quad (1/2 < s < s_Q).$$

Proof. Straightforward manipulations yield to the following first Strang estimate

$$\|\nabla(w_k - w_{k,h})\|_{0,Q^0} \leq 2 \inf_{\theta\in\Theta_h} \|\nabla(w_k - \theta)\|_{0,Q^0} + \sup_{\theta\in\Theta_h/\mathbb{C}} \frac{\sum_{\ell=1}^{B} \left|\mathbf{N}^{-1}_{k,\ell} \langle \mathbf{p}_\ell \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda} - (\mathbf{N}_h^{-1})_{k,\ell} \langle \mathbf{p}_{\ell,h} \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda}\right|}{\|\nabla\theta\|_{0,Q^0}}.$$

The second term in the right-hand side of the last inequality is bounded as follows:

$$\left|\mathbf{N}^{-1}_{k,\ell} \langle \mathbf{p}_\ell \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda} - \left(\mathbf{N}_h^{-1}\right)_{k,\ell} \langle \mathbf{p}_{\ell,h} \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda}\right| \leq C \left\{ \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| \left|\langle \mathbf{p}_{\ell,h} \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda}\right| + \left\|\mathbf{N}^{-1}\right\| \left|\langle \mathbf{p}_\ell \cdot \mathbf{n} - \mathbf{p}_{\ell,h} \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda}\right| \right\}.$$

Trace theorems give

$$\left|\mathbf{N}^{-1}_{k,\ell} \langle \mathbf{p}_\ell \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda} - \left(\mathbf{N}_h^{-1}\right)_{k,\ell} \langle \mathbf{p}_{\ell,h} \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda}\right| \leq C \left\{ \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| \|\mathbf{p}_{\ell,h}\|_{0,Q} + \left\|\mathbf{N}^{-1}\right\| \|\mathbf{p}_\ell - \mathbf{p}_{\ell,h}\|_{0,Q} \right\} \|\theta\|_{1,Q^0}.$$

It is well known that the semi-norm $\|\nabla(\cdot)\|_{0,Q}$ is equivalent to $\|\cdot\|_{1,Q^0}$ on $\mathrm{H}^1(Q^0)/\mathbb{C}$. Thus, we arrive at the estimate

$$\frac{\left|\mathbf{N}^{-1}_{k,\ell} \langle \mathbf{p}_\ell \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda} - \left(\mathbf{N}_h^{-1}\right)_{k,\ell} \langle \mathbf{p}_{\ell,h} \cdot \mathbf{n},\, \theta\rangle_{1/2,\Lambda}\right|}{\|\nabla\theta\|_{0,Q^0}} \leq C \left\{ \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| \|\mathbf{p}_{\ell,h}\|_{0,Q} + \left\|\mathbf{N}^{-1}\right\| \|\mathbf{p}_\ell - \mathbf{p}_{\ell,h}\|_{0,Q} \right\}. \tag{48}$$

Now, let us introduce the Nédélec finite element space relative to the triangulation $\mathcal{T}_h(Q)$:

$$\mathcal{ND}_h(Q) := \left\{ \mathbf{q} \in \mathbf{H}(\mathbf{curl}, Q); \quad \mathbf{q}_{|T} \in \mathcal{ND}(T),\ \forall T \in \mathcal{T}_h(Q) \right\}.$$

We denote by $\mathcal{I}_{h,Q}$ the associated interpolation operator. Owing to Theorem 7.1, we have that $\widetilde{\nabla} w_k|_Q \in \mathbf{H}^s(Q)$ for all $1/2 < s < s_Q$. Thus, $\mathcal{I}_{h,Q}(\widetilde{\nabla} w_k)$ is well defined and Remark 5.6 in [19] implies that $\mathbf{curl}\, \mathcal{I}_{h,Q}(\widetilde{\nabla} w_k) = 0$ since $\mathbf{curl}(\widetilde{\nabla} w_k) = 0$. Applying Lemma 4.3 in [3] we deduce that $\mathcal{I}_{h,Q}(\widetilde{\nabla} w_k)$ is the gradient of a function in Θ_h. It follows that

$$\inf_{\theta\in\Theta_h} \|\nabla w_k - \nabla\theta\|_{0,Q^0} \leq \left\|\widetilde{\nabla} w_k - \mathcal{I}_{h,Q}\left(\widetilde{\nabla} w_k\right)\right\|_{0,Q}.$$

An interpolation error estimate similar to (32) gives now

$$\inf_{\theta\in\Theta_h} \|\nabla w_k - \nabla\theta\|_{0,Q^0} \leq C\, h^s \left\|\widetilde{\nabla} w_k\right\|_{s,Q} \quad (1/2 < s < s_Q),$$

and the result follows from the last inequality (48), Theorem 7.2 and Corollary 7.3. □

Corollary 7.5. *For any* $1 \le k \le B$*, let*

$$z_{k,h} := \sum_{\ell=1}^{B} (\mathbf{N}_h)_{k,\ell}\, w_{\ell,h}.$$

Then, there exists a constant C *independent of* h *such that*

$$\left\|\widetilde{\nabla} z_k - \widetilde{\nabla} z_{k,h}\right\|_{0,Q} \le C\, h^s \max_k \left\{ \|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda} \right\} \quad \text{for all } 1/2 < s < s_Q.$$

Proof. The triangle inequality yields

$$\left\|\widetilde{\nabla} z_k - \widetilde{\nabla} z_{k,h}\right\|_{0,Q} \le \|\mathbf{N}_h\| \max_k \left\|\widetilde{\nabla} w_k - \widetilde{\nabla} w_{k,h}\right\|_{0,Q} + \|\mathbf{N} - \mathbf{N}_h\| \max_k \left\|\widetilde{\nabla} w_k\right\|_{0,Q}$$

and the result is then a direct consequence of Theorem 7.4 and Corollary 7.3. □

7.3. Analysis corresponding to the third auxiliary problem

For any $t \in (0,\, 1)$ we denote $\mathbf{V}_\pi^t := \mathbf{n} \wedge (\mathbf{H}^t(\Gamma) \wedge \mathbf{n})$ and let

$$\mathbf{H}^t(\mathrm{div}_\Gamma, \Gamma) := \left\{ \boldsymbol{\lambda} \in \mathbf{V}_\pi^t;\ \mathrm{div}_\Gamma\, \boldsymbol{\lambda} \in \mathbf{V}_\pi^t \right\} \cdot$$

Endowed with the norm $\|\boldsymbol{\lambda}\|^2_{\mathbf{H}^t(\mathrm{div}_\Gamma,\Gamma)} := \|\boldsymbol{\lambda}\|^2_{t,\Gamma} + \|\mathrm{div}_\Gamma\, \boldsymbol{\lambda}\|^2_{t,\Gamma}$, $\mathbf{H}^t(\mathrm{div}_\Gamma, \Gamma)$ is a Hilbert space.

The following regularity result proved in Theorem A in [1] will be of utility in the sequel.

Theorem 7.6. *For any* $1/2 < s < s_\Omega$*, the extension operator* $\mathcal{R}$ *characterized by* (2) *is linear and bounded from* $\mathbf{H}^{s-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *onto* $\mathbf{H}^s(\mathbf{curl}, \Omega)$.

Moreover, we have the following stability result, see Proposition 3.3 in [1].

Proposition 7.7. *Assume that the family of triangulations* $\{\mathcal{T}_h(\Gamma)\}_h$ *induced by* $\{\mathcal{T}_h\}_h$ *on* Γ *is quasi-uniform. Then there exists a positive constant constant* C *independent of* h *such that*

$$\|\mathcal{R}_h \mathbf{g}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C\, \|\mathbf{g}_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \quad \forall \mathbf{g}_h \in \gamma_\tau(\mathcal{ND}_h(\Omega)).$$

Let us denote $\widetilde{\mathbf{g}}_{k,h} := \gamma_\tau(\mathcal{I}_{h,Q}\mathbf{p}_k) \in \gamma(\mathcal{ND}_h(\Omega))$, for all $k = 1, \cdots, B$. We have the following auxiliary result.

Proposition 7.8. *For any* $1/2 < s < \min(s_Q, s_\Omega)$*, there exists a constant* $C > 0$ *independent of* h *such that*

$$\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h \widetilde{\mathbf{g}}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C\, h^s\, \|\mathbf{p}_k\|_{\mathbf{H}^s(\Omega)}\, .$$

Proof. Let us first notice that, by virtue of Theorem 7.1, $\mathbf{g}_k \in \mathbf{H}^{s-1/2}(\mathrm{div}_\Gamma, \Gamma)$ for all $1/2 < s < s_Q$ (recall that $\mathrm{div}_\Gamma\, \mathbf{g}_k = 0$). Now, applying Theorem 7.6 we deduce that $\mathcal{R}\mathbf{g}_k \in \mathbf{H}^s(\mathbf{curl}, \Omega)$ for all $1/2 < s < \min(s_Q, s_\Omega)$. Thus $\mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k$ is well defined.

Now, as $\mathcal{R}_h \widetilde{\mathbf{g}}_{k,h} - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k \in \mathcal{ND}_h(\Omega) \cap \ker\, \gamma_\tau$, we have that

$$\begin{aligned}(\mathcal{R}_h \widetilde{\mathbf{g}}_{k,h} - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k,\ \mathcal{R}_h \widetilde{\mathbf{g}}_{k,h} - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k)_{\mathbf{H}(\mathbf{curl},\Omega)} &= (\mathcal{R}\mathbf{g}_k - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k,\ \mathcal{R}_h \widetilde{\mathbf{g}}_{k,h} - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k)_{\mathbf{H}(\mathbf{curl},\Omega)} \\ &\le \|\mathcal{R}\mathbf{g}_k - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k\|_{\mathbf{H}(\mathbf{curl},\Omega)}\, \|\mathcal{R}_h \widetilde{\mathbf{g}}_{k,h} - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k\|_{\mathbf{H}(\mathbf{curl},\Omega)}\, .\end{aligned}$$

Therefore, we obtain from the triangle inequality that

$$\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h \widetilde{\mathbf{g}}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le 2\, \|\mathcal{R}\mathbf{g}_k - \mathcal{I}_{h,\Omega}\mathcal{R}\mathbf{g}_k\|_{\mathbf{H}(\mathbf{curl},\Omega)}$$

and the the interpolation error estimate (32) gives

$$\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\widetilde{\mathbf{g}}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C_1\, h^s\, \|\mathcal{R}\mathbf{g}_k\|_{\mathbf{H}^s(\mathbf{curl},\Omega)} \quad (1/2 < s < \min(s_Q, s_\Omega)).$$

The result follows now from Theorem 7.6 and the trace theorem in $\mathbf{H}^s(Q)$. □

7.4. **Convergence analysis of problem** (45)

Let us begin with the following auxiliary lemma.

Lemma 7.9. *For any $1/2 < s < \min(s_Q, s_\Omega)$, there exist constants C_1 and C_2 independent of h such that*

$$|a(\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\mathbf{g}_{k,h},\, \widehat{\mathbf{q}})| \leq C_1\, h^s\, \|\widehat{\mathbf{q}}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \max_k \left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\} \quad \forall \widehat{\mathbf{q}} \in \widehat{\mathbf{X}}_h$$

and

$$|a(\mathcal{R}\mathbf{g}_k,\, \mathcal{R}\mathbf{g}_\ell) - a(\mathcal{R}_h\mathbf{g}_{k,h},\, \mathcal{R}_h\mathbf{g}_{\ell,h})| \leq C_2\, h^s \max_k \left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\}.$$

Proof. Let us first bound the term $\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\mathbf{g}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$. By the triangle inequality

$$\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\mathbf{g}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq \|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\widetilde{\mathbf{g}}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\mathcal{R}_h(\widetilde{\mathbf{g}}_{k,h} - \mathbf{g}_{k,h})\|_{\mathbf{H}(\mathbf{curl},\Omega)}.$$

Now, on the one hand, Proposition 7.8 yields

$$\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\widetilde{\mathbf{g}}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C_3 h^s\; \|\mathcal{R}\mathbf{g}_k\|_{\mathbf{H}^s(\mathbf{curl},\Omega)} \qquad (1/2 < s < \min(s_Q, s_\Omega))$$

and on the other hand, applying Proposition 7.7 we obtain

$$\|\mathcal{R}_h(\widetilde{\mathbf{g}}_{k,h} - \mathbf{g}_{k,h})\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C_4\; \|\widetilde{\mathbf{g}}_{k,h} - \mathbf{g}_{k,h}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}.$$

The triangle inequality, the continuity of the tangential trace in $\mathbf{H}(\mathbf{curl},\Omega)$ and the fact that the functions $\mathbf{p}_k$, $\mathcal{I}_{h,Q}\mathbf{p}_k$ and $\widetilde{\nabla} z_{k,h}$ are rotational free, permit one to arrive at the following estimate

$$\begin{aligned}\|\widetilde{\mathbf{g}}_{k,h} - \mathbf{g}_{k,h}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} &\leq \|\mathbf{g}_k - \widetilde{\mathbf{g}}_{k,h}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} + \|\mathbf{g}_k - \mathbf{g}_{k,h}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \\ &\leq C_5\, \left\{\|\mathbf{p}_k - \mathcal{I}_{h,Q}\mathbf{p}_k\|_{0,Q} + \left\|\mathbf{p}_k - \widetilde{\nabla} z_{k,h}\right\|_{0,Q}\right\}.\end{aligned}$$

Finally, Corollary 7.5 and the interpolation error estimate (32) give

$$\|\widetilde{\mathbf{g}}_{k,h} - \mathbf{g}_{k,h}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \leq C_6\, h^s \max_k \left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\} \quad \text{for all } 1/2 < s < s_Q.$$

Combining the last estimates we deduce that

$$\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\mathbf{g}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C_7\, h^s \max_k \left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\}, \tag{49}$$

for all $1/2 < s < \min(s_Q, s_\Omega)$.

The first estimate of the Theorem follows directly from (49) since the continuity of the sesquilinear form $a(\cdot,\, \cdot)$ on $\mathbf{H}(\mathbf{curl},\Omega)$ yields

$$|a(\mathcal{R}\mathbf{g}_k,\, \widehat{\mathbf{q}}) - a(\mathcal{R}_h\mathbf{g}_{k,h},\, \widehat{\mathbf{q}})| \leq C_8\, \|\widehat{\mathbf{q}}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\, \|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\mathbf{g}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)}.$$

On the other hand,

$$
\begin{aligned}
|a(\mathcal{R}\mathbf{g}_k,\, \mathcal{R}\mathbf{g}_\ell) - a(\mathcal{R}_h\mathbf{g}_{k,h},\, \mathcal{R}_h\mathbf{g}_{\ell,h})| &\le |a(\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\mathbf{g}_{k,h},\, \mathcal{R}\mathbf{g}_\ell)| + |a(\mathcal{R}_h\mathbf{g}_{k,h},\, \mathcal{R}\mathbf{g}_\ell - \mathcal{R}_h\mathbf{g}_{\ell,h})| \\
&\le C_8 \left\|\mathcal{R}\mathbf{g}_k - \mathcal{R}_h\mathbf{g}_{k,h}\right\|_{\mathbf{H}(\mathbf{curl},\Omega)} \left\|\mathcal{R}\mathbf{g}_\ell\right\|_{\mathbf{H}(\mathbf{curl},\Omega)} \\
&\quad + C_8 \left\|\mathcal{R}_h\mathbf{g}_{k,h}\right\|_{\mathbf{H}(\mathbf{curl},\Omega)} \left\|\mathcal{R}\mathbf{g}_\ell - \mathcal{R}_h\mathbf{g}_{\ell,h}\right\|_{\mathbf{H}(\mathbf{curl},\Omega)}
\end{aligned}
$$

and $\|\mathcal{R}_h\mathbf{g}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ is uniformly bounded in h. Indeed, Proposition 7.7 and the continuity of the tangential trace in $\mathbf{H}(\mathbf{curl},\Omega)$ may be used to obtain the estimate

$$
\begin{aligned}
\|\mathcal{R}_h\mathbf{g}_{k,h}\|_{\mathbf{H}(\mathbf{curl},\Omega)} &\le C_9\ \|\mathbf{g}_{k,h}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} := C_9\ \left\|\gamma_\tau \widetilde{\nabla} z_{k,h}\right\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \\
&\le C_{10}\ \left\|\widetilde{\nabla} z_{k,h}\right\|_{\mathbf{H}(\mathbf{curl},Q)} = C_{10}\ \left\|\widetilde{\nabla} z_{k,h}\right\|_{0,Q}
\end{aligned}
$$

and Corollary 7.5 gives

$$
\left\|\widetilde{\nabla} z_{k,h}\right\|_{0,Q} \le C_{11}\ \max_k \left\{\|\mathbf{p}_k\|_{s,Q^0} + \|\phi_k\|_{s+1/2,\Lambda}\right\}.
$$

Consequently, (49) permits also to obtain the second estimate of the Theorem. □

Theorem 7.10. *Assume that the family of triangulations $\{\mathcal{T}_h(\Gamma)\}_h$ induced by $\{\mathcal{T}_h\}_h$ on Γ is quasi-uniform. If the exact solution $\widetilde{\mathbf{h}} := \left(\left(\widehat{\mathbf{h}},\boldsymbol{\beta}\right),\lambda\right)$ of (27) belongs to $\left(\mathbf{H}^r(\mathbf{curl},\Omega)\cap\widehat{\mathbf{X}}\times\mathbb{C}^B\right)\times \mathrm{H}^{r-1/2}(\Gamma)\cap\mathrm{H}_0^{-1/2}(\Gamma)$ with $1/2 < r \le 1$, then, for h is sufficiently small, problem (45) admits a unique solution and there exists a constant C independent of h such that*

$$
\begin{aligned}
&\left\|\widehat{\mathbf{h}} - \widehat{\mathbf{h}}_h\right\|_{\mathbf{H}(\mathbf{curl},\Omega)} + |\boldsymbol{\beta} - \boldsymbol{\beta}_h| + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \\
&\qquad \le C\, h^s \left(\|\mathbf{h}\|_{s,\Omega} + \|\mathbf{curl}\,\mathbf{h}\|_{s,\Omega} + \|\lambda\|_{s-1/2,\Gamma} + \max_k\left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\}\right)
\end{aligned}
$$

for all $1/2 < s < \min(r, s_Q, s_\Omega)$.

Proof. It follows easily from (7.3) and (7.9) that

$$
\left|\mathcal{A}\left(\widetilde{\mathbf{h}},\widetilde{\mathbf{q}}\right) - \mathcal{A}_h\left(\widetilde{\mathbf{h}},\widetilde{\mathbf{q}}\right)\right| \le C_1\ \max_k\left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\} h^s \left\|\widetilde{\mathbf{h}}\right\|_{\mathbf{W}} \|\widetilde{\mathbf{q}}\|_{\mathbf{W}} \quad \forall\widetilde{\mathbf{h}},\widetilde{\mathbf{q}}\in\mathbf{W} \tag{50}
$$

and

$$
|\mathcal{F}(\widetilde{\mathbf{q}}) - \mathcal{F}_h(\widetilde{\mathbf{q}})| \le C_2\ \max_k\left\{\|\mathbf{p}_k\|_{s,Q} + \|\phi_k\|_{s+1/2,\Lambda}\right\} h^s\, \|\widetilde{\mathbf{q}}\|_{\mathbf{W}} \quad \forall\widetilde{\mathbf{q}}\in\mathbf{W} \tag{51}
$$

for all $1/2 < s < \min(s_Q, s_\Omega)$.

As a consequence of (50), problem (45) is well posed if h is sufficiently small and we deduce immediately the following Strang's type abstract error estimate:

$$
\left\|\widetilde{\mathbf{h}} - \widetilde{\mathbf{h}}_h\right\|_{\mathbf{W}} \le C \inf_{\widetilde{\mathbf{q}}\in\mathbf{W}_h}\left\{\left\|\widetilde{\mathbf{h}} - \widetilde{\mathbf{q}}\right\|_{\mathbf{W}} + \sup_{\widetilde{\mathbf{w}}\in\mathbf{W}_h}\frac{|\mathcal{A}(\widetilde{\mathbf{q}},\widetilde{\mathbf{w}}) - \mathcal{A}_h(\widetilde{\mathbf{q}},\widetilde{\mathbf{w}})|}{\|\widetilde{\mathbf{w}}\|_{\mathbf{W}}}\right\} + \sup_{\widetilde{\mathbf{q}}\in\mathbf{W}_h}\frac{|\mathcal{F}(\widetilde{\mathbf{q}}) - \mathcal{F}_h(\widetilde{\mathbf{q}})|}{\|\widetilde{\mathbf{q}}\|_{\mathbf{W}}}.
$$

The result is now a direct consequence of (50), (51), (35) and the interpolation error estimate (32). □

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