AD-A226 066

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE

3. REPORT TYPE AND DATES COVERED
Final Report, 1 Nov 88 to 31 Oct 90

TITLE AND SUBTITLE
A MIXED FINITE ELEMENT FORMULATION FOR THE BOUNDARY CONTROLLABILITY OF THE WAVE EQUATION

5. FUNDING NUMBERS
AFOSR-89-0025
61102F 2304/A3

AUTHOR(S)
R. Glowinski, W. Kinton, M. F. Wheeler

PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
University of Houston
Department of Mathematics
Houston, TX 77004

8. PERFORMING ORGANIZATION REPORT NUMBER
AEOSR-TR- 90 0879

SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AFOSR/NM
Bldg 410
Bolling AFB DC 20332-6448

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

This paper introduces the mixed finite element method as a viable numerical procedure for the boundary controllability of the linear wave equation. Another numerical implementation using Galerkin finite elements has been investigated by Glowinski, Li, and Lions in [4]. However, due to approximation problems of the normal derivative on the boundary, the method becomes unstable as the mesh is refined. To correct for the ill-posedness of the approximate problem, a Tychonoff regularization method was implemented in [4]. The aforementioned paper also presents other possible remedies; among them is the mixed finite element method. The mixed finite element approximation is a plausible procedure to overcome these difficulties since the derivative at certain nodal values arises naturally from the formulation.

14. SUBJECT TERMS

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

A Mixed Finite Element Formulation For the Boundary Controllability of the Wave Equation

R. Glowinski
Department of Mathematics
University of Houston
Houston, Texas

W. Kinton
Department of Mathematical Sciences
Rice University
Houston, Texas

M. F. Wheeler
Department of Mathematics
University of Houston
Houston, Texas

Dedicated to David Young for his 65th birhtday

This research was partially supported by the Department of Energy, the Air Force, and the NSF.

1 Introduction

This paper introduces the mixed finite element method as a viable numerical procedure for the boundary controllability of the linear wave equation. Another numerical implementation using Galerkin finite elements has been investigated by Glowinski, Li, and Lions in [4]. However, due to approximation problems of the normal derivative on the boundary, the method becomes unstable as the mesh is refined. To correct for the ill-posedness of the approximate problem, a Tychonoff regularization method was implemented in [4]. The aforementioned paper also presents other possible remedies; among them is the mixed finite element method. The mixed finite element approximation is a plausible procedure to overcome these difficulties since the derivative at certain nodal values arises naturally from the formulation.

This paper is numerical in nature; related theoretical results to this method will be presented at a later time. The first section gives a brief description of the control problem. For further details, we refer you to [4]. The second section of the paper describes the mixed finite element method along with the approximating spaces used in the procedure. The third section describes how the mixed method is applied to the controllability of the wave equation. The last section presents numerical results for a particular test problem constructed in such a fashion so that the exact solution is known. This test problem was taken from [4].

2 Formulation of the Control Problem

Let Ω be a bounded domain of R^n and let Γ be its boundary. Let T be a given positive number, where

$$Q = \Omega \times (0,T), \Sigma = \Gamma \times (0,T). \quad (1)$$

Let $p^0 \in L^2(\Omega)$, $p^1 \in H^{-1}(\Omega)$.

The linear wave equation, together with the initial conditions $p(x,0) = p^0(x)$ and $\frac{\partial p}{\partial t}(x,0) = p^1(x)$, is

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0. \quad (2)$$

The problem is the following: Is it possible to find $q \in L^2(\Sigma)$ such that adding the boundary condition $p = q$ on Σ, will imply $p(x,T) = 0$, $\frac{\partial p}{\partial t}(x,T) = 0$, a.e. ? The answer is *yes* if T is sufficiently large. The proof can be found [5] and [6].

The following subsections briefly describe a method, introduced in [5], [6], and [7], for constructing a boundary function, $q \in L^2(\Sigma)$, such that the conditions previously mentioned hold. This again has already been presented in [4].

2.1 Definition of the Operator Λ

Define E by

$$E = H_0^1(\Omega) \times L^2(\Omega); \tag{3}$$

then its dual E' is given by

$$E' = H^{-1}(\Omega) \times L^2(\Omega). \tag{4}$$

Now define $\Lambda \in L(E, E')$ as follows: with

$\underset{\sim}{e} = (e^0, e^1) \in E$, solve the linear wave problem,

$$\varphi_{tt} - \Delta\varphi = 0 \, in \, Q, \tag{5}$$

$$\varphi = 0 \, on \, \Sigma, \tag{6}$$

$$\varphi(x,0) = e^0(x), a.e.; \varphi_t(x,0) = e^1(x), a.e. \tag{7}$$

Then solve

$$\psi_{tt} - \Delta\psi = 0 \, in \, Q, \psi = \frac{\partial\phi}{\partial n} \, on \, \Sigma, \psi(x,T) = 0, a.e.; \psi_t(x,T) = 0, a.e. \tag{8}$$

Finally, define Λ by

$$\Lambda \underset{\sim}{e} = (\psi_t(0), -\psi(0)). \tag{9}$$

The fundamental result states the following:

If T is sufficiently large, then Λ is an isomorphism from E onto E'.

The proof can be found in [6] and [7].

2.2 Application to the Boundary Control of the Wave Equation

Let $\underset{\sim}{f} \in E'$ be defined to be

$$\underset{\sim}{f} = (p^1, -p^0). \tag{10}$$

Now consider the linear problem,

$$\Lambda \underset{\sim}{e} = \underset{\sim}{f}. \tag{11}$$

From the fundamental result it follows that (11) has a unique solution if T is sufficiently large. If one takes the solution $\underset{\sim}{e}$ as data to solve (5) – (7) and $q = \frac{\partial \psi}{\partial n}$ on Σ in (8), then from the construction of Λ, it follows that $p = \psi$ and $p(T) = p_t(T) = 0$.

It can be shown that for sufficiently large values of T, Λ is strongly elliptic from E onto E'. This follows from [6] and [7]. Λ is a self-adjoint operator, thereby allowing one to solve the problem using conjugate gradient methods. For further properties of Λ, see [4].

3 An Explicit Formulation of the mixed method for the linear wave equation

Let $H(\Omega; div)$ be the set of vector functions $\underset{\sim}{v} \in (L^2(\Omega))^n$ such that $\underset{\sim}{\nabla} \cdot \underset{\sim}{v} \in L^2(\Omega)$. Consider the linear wave equation:

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \, in \, Q \tag{12}$$

with $p(x,0) = p^o(x)$ and $\frac{\partial p}{\partial t}(x,0) = p^1(x)$. Set $\underset{\sim}{u} = -\underset{\sim}{\nabla} p$. Multiplying by $\underset{\sim}{v} \in H(\Omega; div)$ and integrating by parts yields

$$\int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{v} \, dx - \int_\Omega p \, \underset{\sim}{\nabla} \cdot \underset{\sim}{v} \, dx = -\int_\Gamma p \, \underset{\sim}{v} \cdot \underset{\sim}{\eta} \, d\gamma; \tag{13}$$

where $\underset{\sim}{\eta}$ is defined to be the outer normal to the boundary of Ω. Multiplying (12) by $w \in L^2(\Omega)$ and integrating gives

$$\int_\Omega \frac{\partial^2 p}{\partial t^2} w \, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \, w \, dx = 0. \tag{14}$$

The system is then approximated using finite elements. We define the finite dimensional subspaces , $\mathbf{V}$ and W, such that $\mathbf{V} \subset H(\Omega; div)$ and $W \subset L^2(\Omega)$.

For convergence, we further assume the property that $(\underset{\sim}{\nabla} \cdot \underset{\sim}{v} \mid \underset{\sim}{v} \in \mathbf{V}) \subset W$. For details for elliptic partial differential equations; see [1], [3], and [8]. For the linear wave equation, results for the continuous time case as well as convergence results and stability for the implicit and explicit time procedures can be found in Dupont, Kinton, and Wheeler [2]. In this paper we only treat the following explicit formulation.

Spaces satisfying the property that the $\mathbf{div}\,\mathbf{V} \subset W$ are the Raviart-Thomas spaces. An example of these spaces is:

$$W \equiv M^1_{-1}(\delta_x) \otimes M^1_{-1}(\delta_y) \tag{15}$$

$$\mathbf{V} \equiv M^2_0(\delta_x) \otimes M^1_{-1}(\delta_y) \times M^1_{-1}(\delta_x) \otimes M^2_0(\delta_y). \tag{16}$$

Here $M^1_{-1}(\delta_x) \otimes M^1_{-1}(\delta_y)$ is the tensor product of piecewise discontinuous linears. $M^2_0(\delta_x) \otimes M^1_{-1}(\delta_y)$ is the tensor product of piecewise continuous quadratics with piecewice discontinuous linears. $M^1_{-1}(\delta_x) \otimes M^2_0(\delta_y)$ is the tensor product of piecewise discontinuous linears with piecewise continuous quadratics.

Let $\Delta t > 0$ and $t^n = n\,\Delta t$. Let $\underset{\sim}{u}^n = \underset{\sim}{u}(\cdot, t^n)$ and $p^n = p(\cdot, t^n)$, for n a positive integer. Define $(\underset{\sim}{U}^n, P^n) \in \mathbf{V} \times W$ by

$$\int_\Omega \underset{\sim}{U}^n \cdot \underset{\sim}{v}\, dx - \int_\Omega P^n\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = -\int_\Gamma p^n\, \underset{\sim}{v} \cdot \underset{\sim}{\eta}\, d\gamma, \forall\, \underset{\sim}{v} \in \mathbf{V}, \tag{17}$$

$$\int_\Omega \frac{P^{n+1} - 2P^n + P^{n-1}}{\Delta t^2} w\, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{U}^n\, w = 0, \forall\, w \in W \tag{18}$$

$$P^0 = p(x, 0), \tag{19}$$

$$\frac{P^1 - P^{-1}}{2\Delta t} = \frac{\partial p}{\partial t}(x, 0), \tag{20}$$

$$\tag{21}$$

4 Discrete formulation of the Conjugate Gradient Method

Recalling from Section 2, e^0 and e^1 are initial data for solving the forward wave equation (5) – (7) so that the normal derivative of φ on Σ is the boundary function, q, such that $p(T) = p_t(T) = 0$. In this discrete formulation, we begin this iterative procedure with an initial guess for e^0 and e^1. The subscripts denote iteration count.

Assume

$$e_0^0 \in W,\ e_0^1 \in W \tag{22}$$

are given;

Now solve the discrete forward wave equation :

$$\Phi^n \approx \varphi^n, \tag{23}$$

$$\underset{\sim}{\Upsilon}^n \approx \underset{\sim}{\tau}^n, \text{ where } \underset{\sim}{\tau}^n = -\underset{\sim}{\nabla}\varphi^n, \tag{24}$$

$$\int_\Omega \underset{\sim}{\Upsilon}_0^n \cdot \underset{\sim}{v}\, dx - \int_\Omega \Phi_0^n\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = 0, \forall\, \underset{\sim}{v} \in \mathbf{V}, \tag{25}$$

$$\int_\Omega \frac{\Phi_0^{n+1} + \Phi_0^{n-1} - 2\Phi_0^n}{\Delta t^2} w\, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{\Upsilon}_0^n w\, dx = 0, \forall w \in W, \tag{26}$$

$$\Phi_0^{n+1} \in W, \tag{27}$$

$$\Phi_0^n \mid_\Gamma = 0, \tag{28}$$

$$n = 0, 1, \ldots. N, \tag{29}$$

where the forward equation is initialized by $\Phi_0^0 = e_0^0$, $\Phi_0^1 - \Phi_0^{-1} = 2\Delta t\, e_0^1$. Store Φ_0^N , Φ_0^{N+1} , $\underset{\sim}{\Upsilon}_0^N$.

Now for $n = N, N-1, \ldots., 0$, compute $\Phi_0^n \in W$, $\underset{\sim}{\Upsilon}_0^n \in \mathbf{V}$, $\Psi_0^{n-1} \in W$ by backward time integration.

If $n < N$, compute $\Phi_0^n \in W$ by solving

$$\int_\Omega \frac{\Phi_0^n + \Phi_0^{n+2} - 2\Phi_0^{n+1}}{\Delta t^2} w\, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{\Upsilon}_0^{n+1} w\, dx = 0, \forall w \in W, \tag{30}$$

$$\int_\Omega \underset{\sim}{\Upsilon}_0^n \cdot \underset{\sim}{v}\, dx - \int_\Omega \Phi_0^n\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = 0, \forall\, \underset{\sim}{v} \in \mathbf{V}, \tag{31}$$

$$\Phi_0^n \mid_\Gamma = 0. \tag{32}$$

If $n = N$, $\underset{\sim}{\Upsilon}_0^N \in \mathbf{V}$ is stored from forward time integration.

Then solve

$$\int_\Omega \underset{\sim}{Z}_0^n \cdot \underset{\sim}{v}\, dx - \int_\Omega \Psi_0^n \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = -\int_\Gamma Y_0^n \underset{\sim}{v} \cdot \underset{\sim}{\eta}\, d\gamma, \forall \underset{\sim}{v} \in \mathbf{V}, \tag{33}$$

$$\int_\Omega \frac{\Psi_0^{n-1} + \Psi_0^{n+1} - 2\Psi_0^n}{\Delta t^2} w\, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{Z}_0^n w\, dx = 0, \forall w \in W, \tag{34}$$

$$\Psi_0^{N+1} - \Psi_0^{N-1} = \Psi_0^N = 0, \tag{35}$$

$$\Psi_0^n \mid_\Gamma = Y_0^n, \tag{36}$$

$$\underset{\sim}{Z}_0^n \approx \underset{\sim}{z}_0^n, \text{ where } \underset{\sim}{z}_0^n = -\underset{\sim}{\nabla} \psi_0^n, \tag{37}$$

$$\underset{\sim}{\Upsilon}_0^n \approx \underset{\sim}{\tau}_0^n, \text{ where } \underset{\sim}{\tau}_0^n = -\underset{\sim}{\nabla} \varphi_0^n, \tag{38}$$

$$Y_0^n \approx -\underset{\sim}{\tau}_0^n \cdot \underset{\sim}{\eta}. \tag{39}$$

Now compute $\underset{\sim}{g}_0 = (g_0^0, g_0^1) \in W \times W$ by solving the discrete Dirichlet problem,

$$\int_\Omega \underset{\sim}{\Theta}_0 \cdot \underset{\sim}{v}\, dx - \int_\Omega g_0^0 \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = 0, \forall \underset{\sim}{v} \in \mathbf{V}, \tag{40}$$

$$\int_\Omega \underset{\sim}{\nabla} \cdot \underset{\sim}{\Theta}_0 w\, dx = \int_\Omega \frac{\Phi_0^1 - \Phi_0^{-1}}{2\Delta t} w\, dx - \int_\Omega p^1 w\, dx, \forall w \in W, \tag{41}$$

$$g_0^0 \mid_\Gamma = 0; \tag{42}$$

and then

$$\int_\Omega g_0^1 w\, dx = \int_\Omega p^0 w\, dx - \int_\Omega \Psi_0^0 w\, dx, \forall w \in W. \tag{43}$$

If $\underset{\sim}{g}_0 = 0$ or small then set $\underset{\sim}{e}_h = \underset{\sim}{e}_0$; else set $\underset{\sim}{w}_0 = \underset{\sim}{g}_0$.

Then for $k \geq 0$, compute

$$\underset{\sim}{e}_{k+1}, \underset{\sim}{g}_{k+1}, \underset{\sim}{w}_{k+1}, \Phi_{k+1}, \Psi_{k+1} \tag{44}$$

as follows:

Step 1: Descent: $(\overline{\Phi}_k^n, \underset{\sim}{\overline{\Upsilon}}_k^n) \in W \times \mathbf{V}$.

$$\int_\Omega \overline{\Upsilon}_k^n \cdot \underset{\sim}{v}\, dx - \int_\Omega \overline{\Phi}_k^n\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = 0, \forall\, \underset{\sim}{v} \in \mathbf{V}, \tag{45}$$

$$\int_\Omega \frac{\overline{\Phi}_k^{n+1} + \overline{\Phi}_k^{n-1} - 2\overline{\Phi}_k^n}{\Delta t^2} w\, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \overline{\Upsilon}_k^n\, w\, dx = 0, \forall\, w \in W, \tag{46}$$

$$\overline{\Phi}_k^n\, |_\Gamma = 0, \tag{47}$$

$$n = 0, 1, \ldots.N, \tag{48}$$

$$\tag{49}$$

where the forward equation is initialized by $\overline{\Phi}_k^0 = w_k^0$ and $\overline{\Phi}_k^1 - \overline{\Phi}_k^{-1} = 2\Delta t w_k^1$. Store $\overline{\Phi}_k^N$, $\overline{\Phi}_k^{N+1} \in W$, $\overline{\Upsilon}_k^N \in \mathbf{V}$.

Now for $n = N, N-1, \ldots.,0$, compute $\overline{\Phi}_k^n \in W$, $\overline{\Upsilon}_k^n \in \mathbf{V}$, $\overline{\Psi}_k^{n-1} \in W$ by backward time integration.

If $n < N$, compute $\overline{\Phi}_k^n \in W$ by solving

$$\int_\Omega \frac{\overline{\Phi}_k^n + \overline{\Phi}_k^{n+2} - 2\overline{\Phi}_k^{n+1}}{\Delta t^2} w\, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \overline{\Upsilon}_k^{n+1}\, w\, dx = 0, \forall\, w \in W, \tag{50}$$

$$\int_\Omega \overline{\Upsilon}_k^n \cdot \underset{\sim}{v}\, dx - \int_\Omega \overline{\Phi}_k^n\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = 0, \forall\, \underset{\sim}{v} \in \mathbf{V} \tag{51}$$

$$\overline{\Phi}_k^n\, |_\Gamma = 0. \tag{52}$$

If $n = N$, $\overline{\Upsilon}_k^N \in \mathbf{V}$ is stored from forward time integration.

Then solve

$$\int_\Omega \overline{\underset{\sim}{Z}}_k^n \cdot \underset{\sim}{v}\, dx - \int_\Omega \overline{\Psi}_k^n\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = -\int_\Gamma \overline{Y}_k^n\, \underset{\sim}{v} \cdot \underset{\sim}{n}\, d\gamma, \forall\, \underset{\sim}{v} \in \mathbf{V}, \tag{53}$$

$$\int_\Omega \frac{\overline{\Psi}_k^{n-1} + \overline{\Psi}_k^{n+1} - 2\overline{\Psi}_k^n}{\Delta t^2} w\, dx + \int_\Omega \underset{\sim}{\nabla} \cdot \overline{\underset{\sim}{Z}}_k^n w\, dx = 0, \forall\, w \in W, \tag{54}$$

$$\overline{\Psi}_k^{N+1} - \overline{\Psi}_k^{N-1} = \overline{\Psi}_k^N = 0, \tag{55}$$

$$\overline{\Psi}_k^n\, |_\Gamma = \overline{Y}_k^n, \tag{56}$$

$$\overline{\underset{\sim}{Z}}_k^n \approx \overline{\underset{\sim}{z}}_k^n, \text{where } \overline{\underset{\sim}{z}}_k^n = -\underset{\sim}{\nabla}\, \overline{\psi}_k^n, \tag{57}$$

$$\overline{\underset{\sim}{\Upsilon}}_k^n \approx \overline{\underset{\sim}{\tau}}_k^n, \text{where } \overline{\underset{\sim}{\tau}}_k^n = -\underset{\sim}{\nabla}\, \overline{\varphi}_k^n, \tag{58}$$

$$\overline{Y}_k^n \approx -\overline{\underset{\sim}{\tau}}_k^n \cdot \underset{\sim}{n}\,. \tag{59}$$

Now compute $\overline{\underset{\sim}{g}}_k = (\overline{g}_k^0, \overline{g}_k^1) \in W \times W$ by solving the discrete Dirichlet problem

$$\int_\Omega \overline{\underset{\sim}{\Theta}}_k \cdot \underset{\sim}{v}\, dx - \int_\Omega \overline{g}_k^0\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dx = 0, \forall\, \underset{\sim}{v} \in \mathbf{V}, \tag{60}$$

$$\int_\Omega \underset{\sim}{\nabla} \cdot \overline{\underset{\sim}{\Theta}}_k w\, dx = \int_\Omega \frac{\overline{\Phi}_k^1 - \overline{\Phi}_k^{-1}}{2\Delta t} w\, dx, \forall\, w \in W, \tag{61}$$

$$\overline{g}_k^0\, |_\Gamma = 0; \tag{62}$$

and then

$$\int_\Omega \overline{g}_k^1 w\, dx = -\int_\Omega \overline{\Psi}_k^0 w\, dx, \forall\, w \in W. \tag{63}$$

(64)

Then compute ρ_k by

$$\rho_k = \frac{\int_\Omega \underset{\sim}{\Theta}_k \cdot \underset{\sim}{\Theta}_k dx + \int_\Omega g_k^1 g_k^1 dx}{\int_\Omega \overline{\underset{\sim}{\Theta}}_k \cdot \underset{\sim}{\Upsilon}_k^0 dx + \int_\Omega \overline{g}_k^1 w_k^1 dx}. \tag{65}$$

Once ρ_k is known, compute

$$\underset{\sim}{e}_{k+1} = \underset{\sim}{e}_{k} - \rho_k \underset{\sim}{w}_{k}, \tag{66}$$
$$\Phi_{k+1} = \Phi_k - \rho_k \overline{\Phi}_k, \tag{67}$$
$$\Psi_{k+1} = \Psi_k - \rho_k \overline{\Psi}_k, \tag{68}$$
$$\underset{\sim}{g}_{k+1} = \underset{\sim}{g}_{k} - \rho_k \overline{\underset{\sim}{g}}_{k}. \tag{69}$$

If $\underset{\sim}{g}_{k+1} = 0$, or is small , then set $\underset{\sim}{e}_h = \underset{\sim}{e}_{k+1}$, $\Phi_h = \Phi_{k+1}$, $\Psi_h = \Psi_{k+1}$; else compute

$$\gamma_k = \frac{\int_\Omega \underset{\sim}{\Theta}_{k+1} \cdot \underset{\sim}{\Theta}_{k+1} dx + \int_\Omega g^1_{k+1} g^1_{k+1} dx}{\int_\Omega \underset{\sim}{\Theta}_{k} \cdot \underset{\sim}{\Theta}_{k} dx + \int_\Omega g^1_k g^1_k dx}. \tag{70}$$

Set

$$\underset{\sim}{w}_{k+1} = \underset{\sim}{g}_{k+1} + \gamma_k \underset{\sim}{w}_{k} \tag{71}$$

and $k = k + 1$ and go to <u>Step 1</u>.

Remarks:

- As pointed out in [4], substantial computer memory cost is reduced by solving the wave equation backward in time.
- This formulation is only valid for problems with smooth data. A variant of this conjugate gradient method is required to handle nonsmooth data. Procedure can be generalized to treat this case.

5 Numerical Results

This method was used for a test problem used in [4]. In [4], an exact solution is constructed for the problem $\Lambda \underset{\sim}{e} = \underset{\sim}{f}$ on the unit square. For details of this calculation, we refer you to [4]. Only the results will be presented here.

If

$$e^0(x) = \sin \pi x_1 \sin \pi x_2, \tag{72}$$
$$e^1(x) = \pi\sqrt{2} \sin \pi x_1 \sin \pi x_2, \tag{73}$$

then

$$\varphi(x,t)=\sqrt{2}\cos\pi\sqrt{2}(t-\frac{1}{4\sqrt{2}})\sin\pi x_1\sin\pi x_2. \tag{74}$$

Defining $\Gamma_i, i=1,2,3,4$ by

$$\begin{aligned}\Gamma_1&=(x\mid x\in\Gamma, x_1=0),\\ \Gamma_2&=(x\mid x\in\Gamma, x_1=1),\\ \Gamma_3&=(x\mid x\in\Gamma, x_2=0);\\ \Gamma_4&=(x\mid x\in\Gamma, x_2=1),\end{aligned}$$

we have

$$\frac{\partial\varphi}{\partial n}|_{\Gamma_1\cup\Gamma_2}=-\pi\sqrt{2}\cos\pi\sqrt{2}(t-\frac{1}{4\sqrt{2}})\sin\pi x_2, \tag{75}$$

$$\frac{\partial\varphi}{\partial n}|_{\Gamma_3\cup\Gamma_4}=-\pi\sqrt{2}\cos\pi\sqrt{2}(t-\frac{1}{4\sqrt{2}})\sin\pi x_1. \tag{76}$$

Using final time $T=\frac{1}{\sqrt{2}}(n+\frac{3}{4})$ (n is a nonnegative integer), $\psi=\psi_0+\psi_1$, where ψ_0 and ψ_1 are the following:

$$\psi_0 = -\pi\sqrt{2}\cos\pi\sqrt{2}(t-\frac{1}{4\sqrt{2}})(\sin\pi x_1\cos 2\pi x_2+\cos 2\pi x_1\sin\pi x_2), \tag{77}$$

$$\begin{aligned}\psi_1 = {} & 4\pi(T-t)\sin\pi\sqrt{2}(t-\frac{1}{4\sqrt{2}})+ \\ & (-1)^n\frac{28}{3\sqrt{2}}\sin\pi\sqrt{2}(t-T)\sin\pi x_1\sin\pi x_2 \\ & +4\sin\pi x_1\sum_{\substack{m\geq 3\\ m\ odd}}\frac{m}{m^2-1}\left[\frac{2(-1)^{n+1}}{\sqrt{1+m^2}}sin\pi\sqrt{1+m^2}(t-T)\right. \\ & \left.+\frac{3\sqrt{2}}{m^2-4}\cos\pi\sqrt{2}(t-\frac{1}{4\sqrt{2}})\right]\sin m\pi x_2 \\ & +4\sin\pi x_2\sum_{\substack{m\geq 3\\ m\ odd}}\frac{m}{m^2-1}\left[\frac{2(-1)^{n+1}}{\sqrt{1+m^2}}\sin\pi\sqrt{1+m^2}(t-T)\right. \\ & \left.+\frac{3\sqrt{2}}{m^2-4}\cos\pi\sqrt{2}(t-\frac{1}{4\sqrt{2}})\right]\sin m\pi x_1.\end{aligned} \tag{78}$$

Since $p=\psi$, we compute p^0 and p^1 from ψ_0 and ψ_1 so that

$$p^0(x) = \psi_0(x,0)+\psi_1(x,0), \tag{79}$$

$$p^1(x) = \frac{\partial\psi_0}{\partial t}(x,0)+\frac{\partial\psi_1}{\partial t}(x,0). \tag{80}$$

Since p^0 and p^1 involve infinite trigonometric series, Fast Fourier Transforms are used for these calculations (m is taken to be 255).

In the conjugate gradient algorithm, e^0 and e^1 are initialized to be zero and final T is $\frac{15}{4\sqrt{2}}$, $(n = 3)$. The following pages represent calculations for h = 1/16 , 1/32, and 1/64. The first six plots represent graphs of the calculated e_h^0 and e_h^1 along with the known e^0 and e^1. The last three plots represent variations of $\| q(t) \|_{L^2(\Gamma)}$ and $\| q_c(t) \|_{L^2(\Gamma)}$ with t. All approximate solutions are represented by dotted lines and known solutions are represented by solid lines.

The first table shows that the method is much better behaved as the mesh is refined However, the second table shows that the iteration count goes up as the mesh refined; roughly speaking like $h^{-\frac{1}{2}}$. This is substantially better than the Galerkin finite element procedure without the regularization discussed in [4].

h =	1/8	1/16	1/32	1/64
$\| e^0 - e_c^0 \|_{L^2(\Omega)}$	3.03×10^{-2}	1.00×10^{-2}	3.11×10^{-3}	1.25×10^{-3}
$\| e^1 - e_c^1 \|_{L^2(\Omega)}$	5.69×10^{-2}	1.79×10^{-2}	9.76×10^{-3}	4.22×10^{-3}
$\| e^0 - e_c^0 \|_{H_0^1(\Omega)}$	1.38×10^{-1}	4.95×10^{-2}	1.70×10^{-2}	7.39×10^{-3}
$\| q - q_c \|_{L^2(\Sigma)}$	2.85×10^{-2}	1.02×10^{-2}	3.31×10^{-3}	1.37×10^{-3}
$\| q_c \|_{L^2(\Sigma)}$	7.102	7.298	7.401	7.394

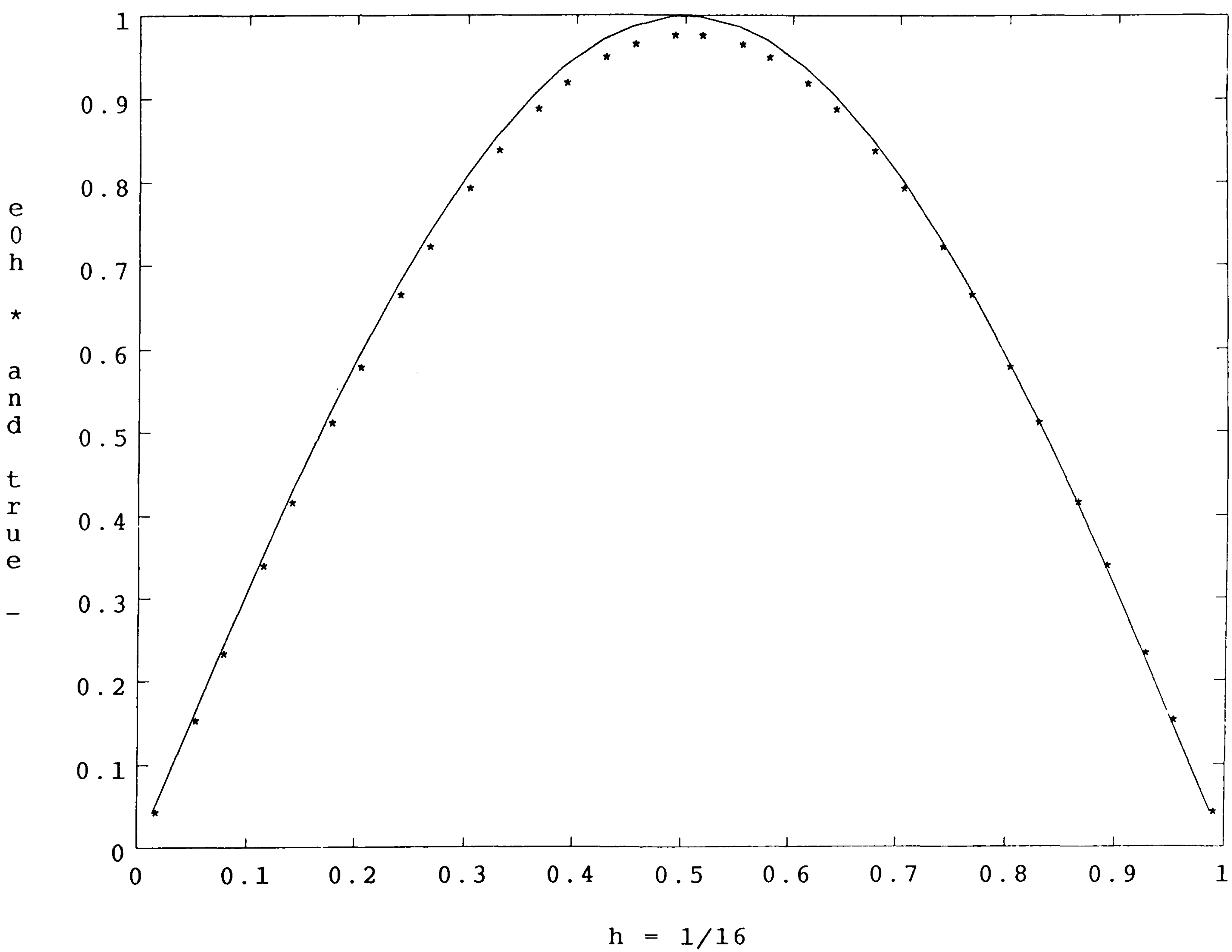
h	no. of iterations
$\frac{1}{4}$	19
$\frac{1}{8}$	30
$\frac{1}{16}$	48
$\frac{1}{32}$	72
$\frac{1}{64}$	119

6 Conclusion

From the numerical results, the ill-posedness of the approximate problem is allieviated considerably when using the mixed finite element procedure. Even though the iteration count goes up as the mesh is refined, there is no oscillatory behavior present as in [4] with no regularization.

Acknowledgement

The authors would like to thank Cray Research for providing the computer time required to obtain these calculations.


e0h * and true _
0
0.1
0.2
0.3
0.4
0.5
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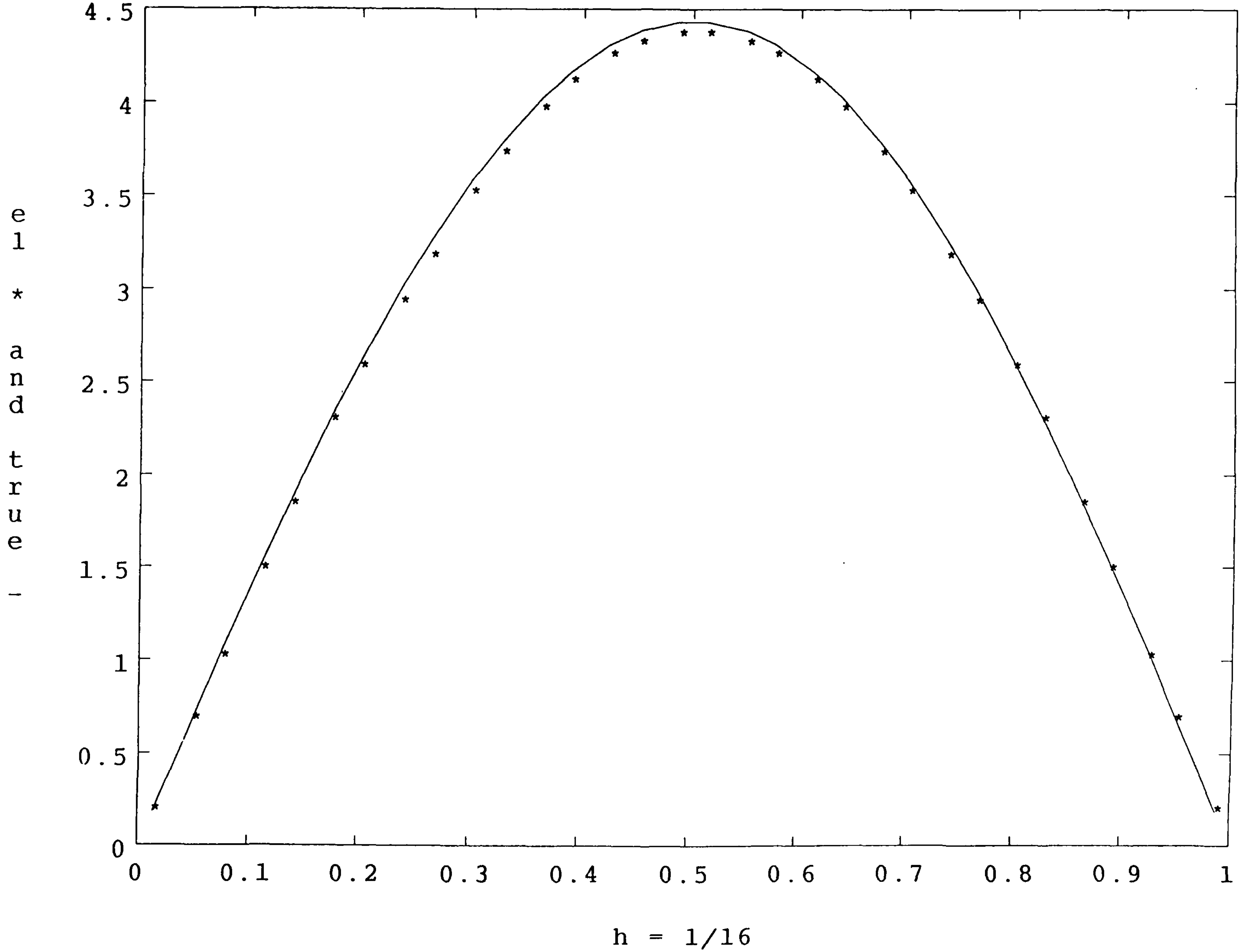
1
h = 1/16


e1 * and true -
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
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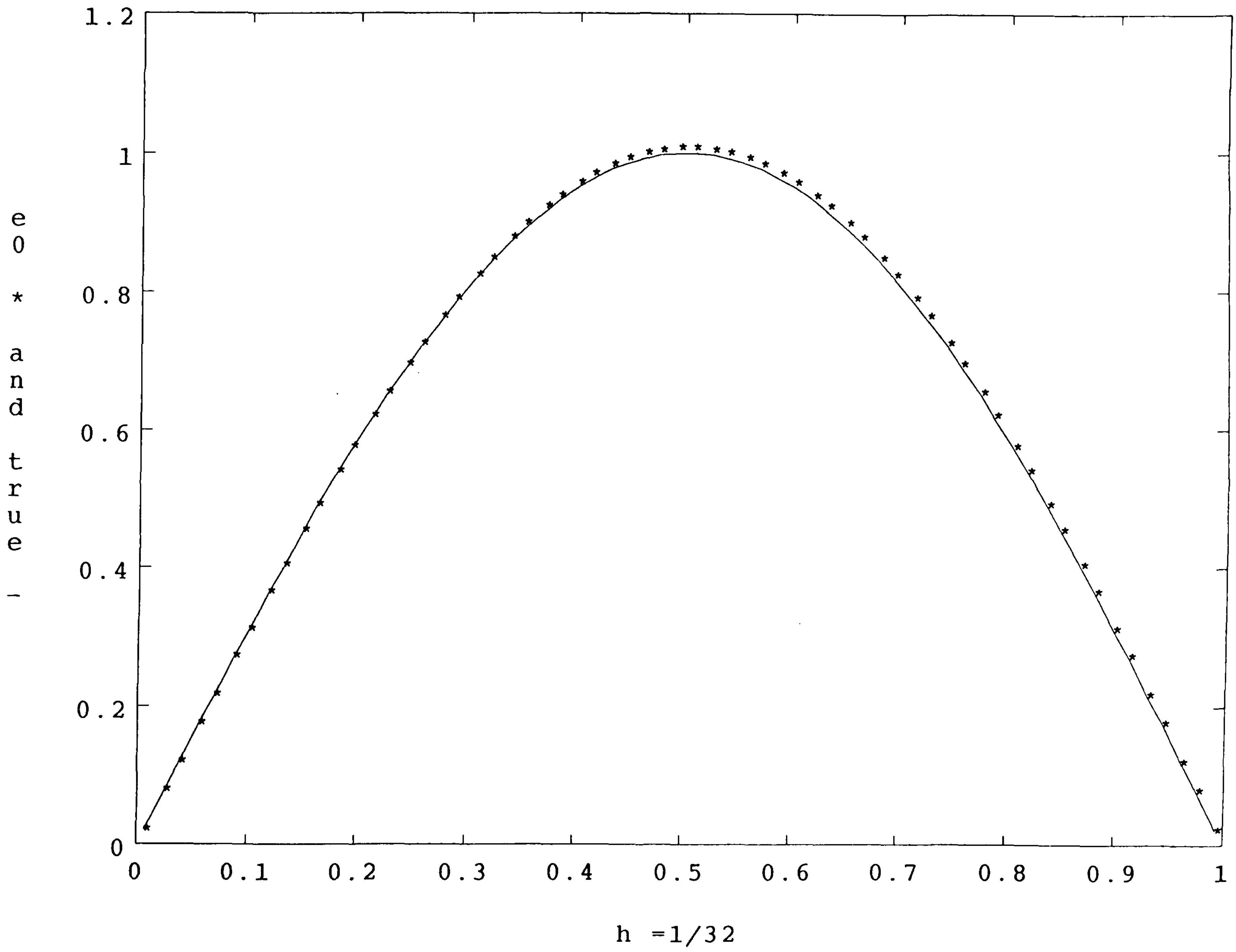
0.9
1
h = 1/16


1.2
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e0 * and true _
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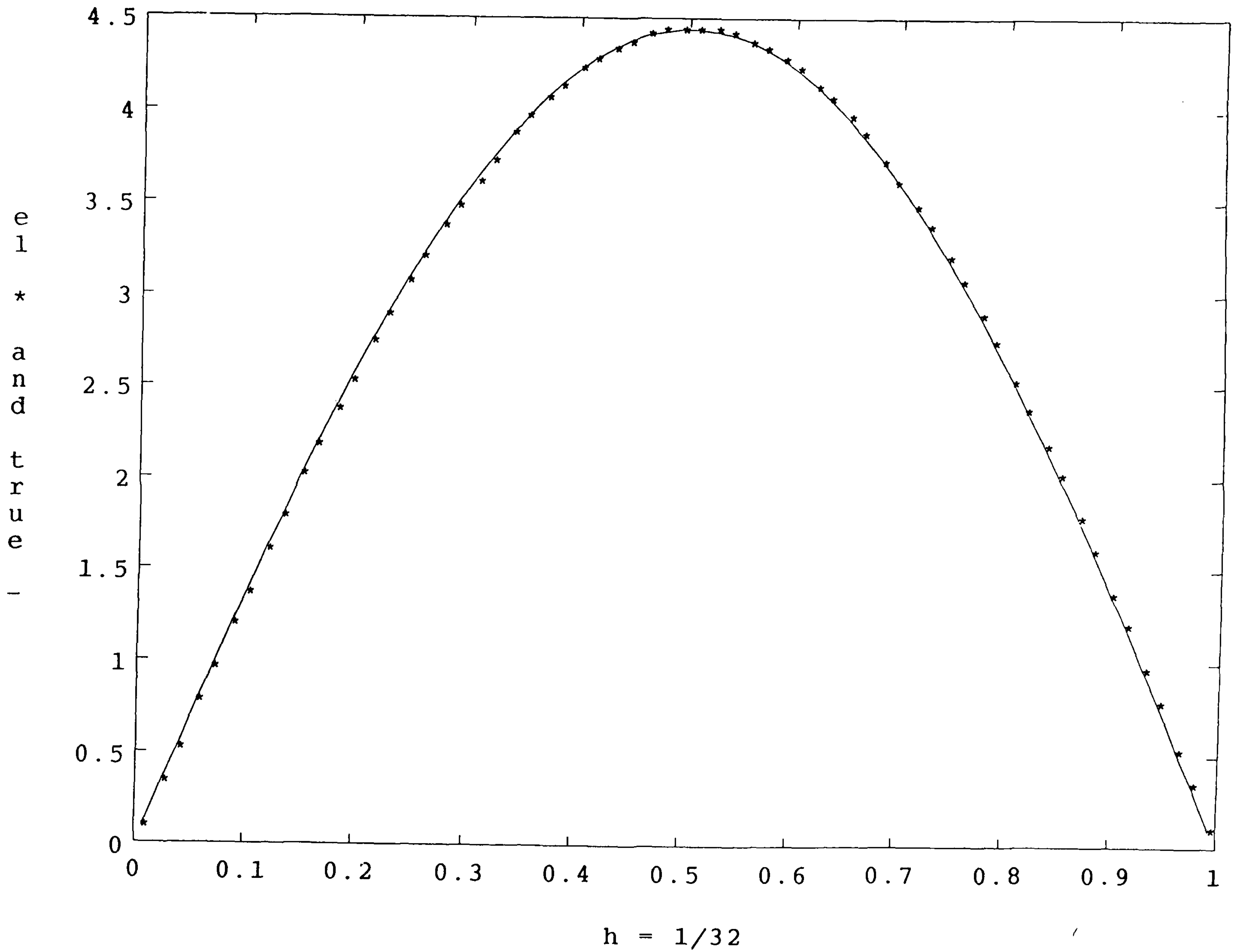
h =1/32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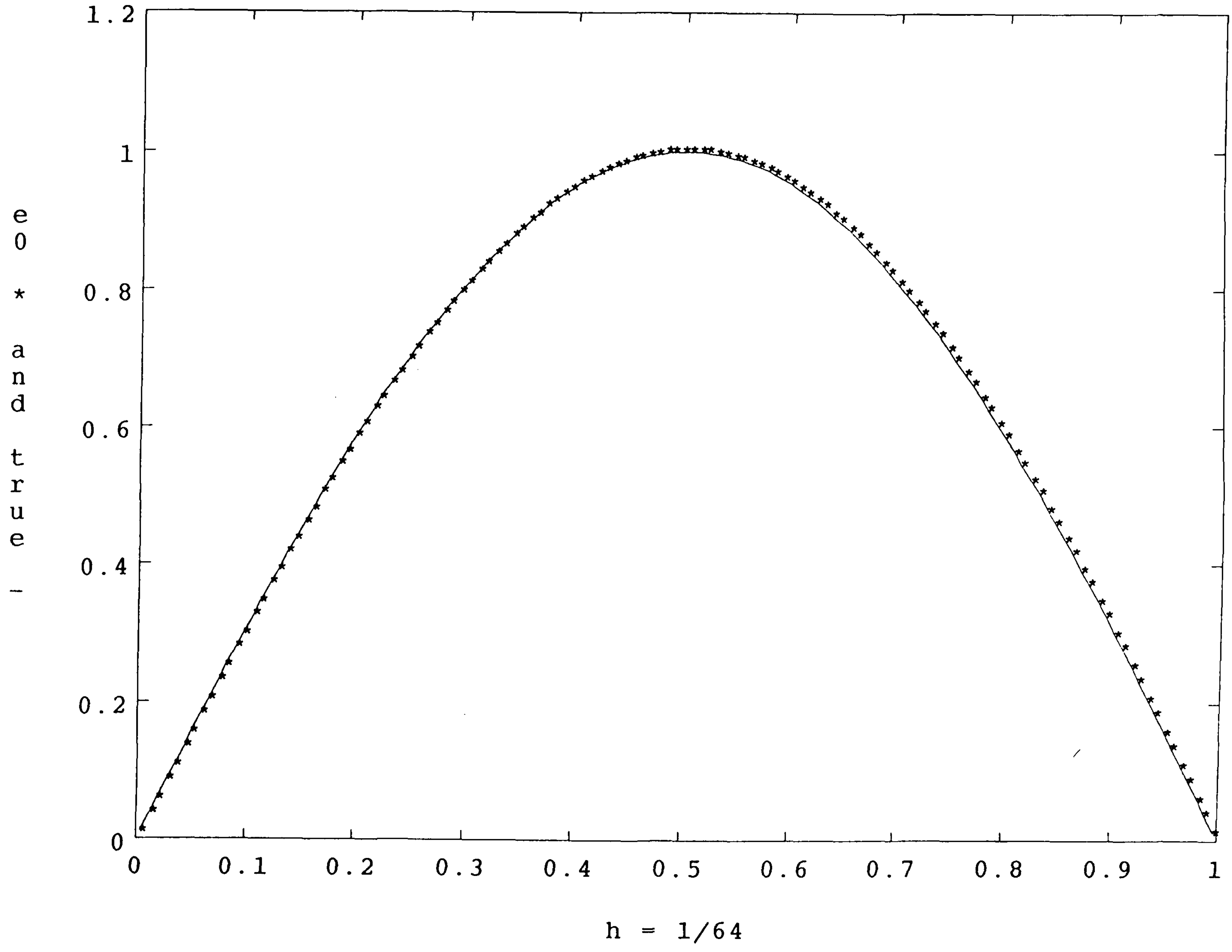
h = 1/32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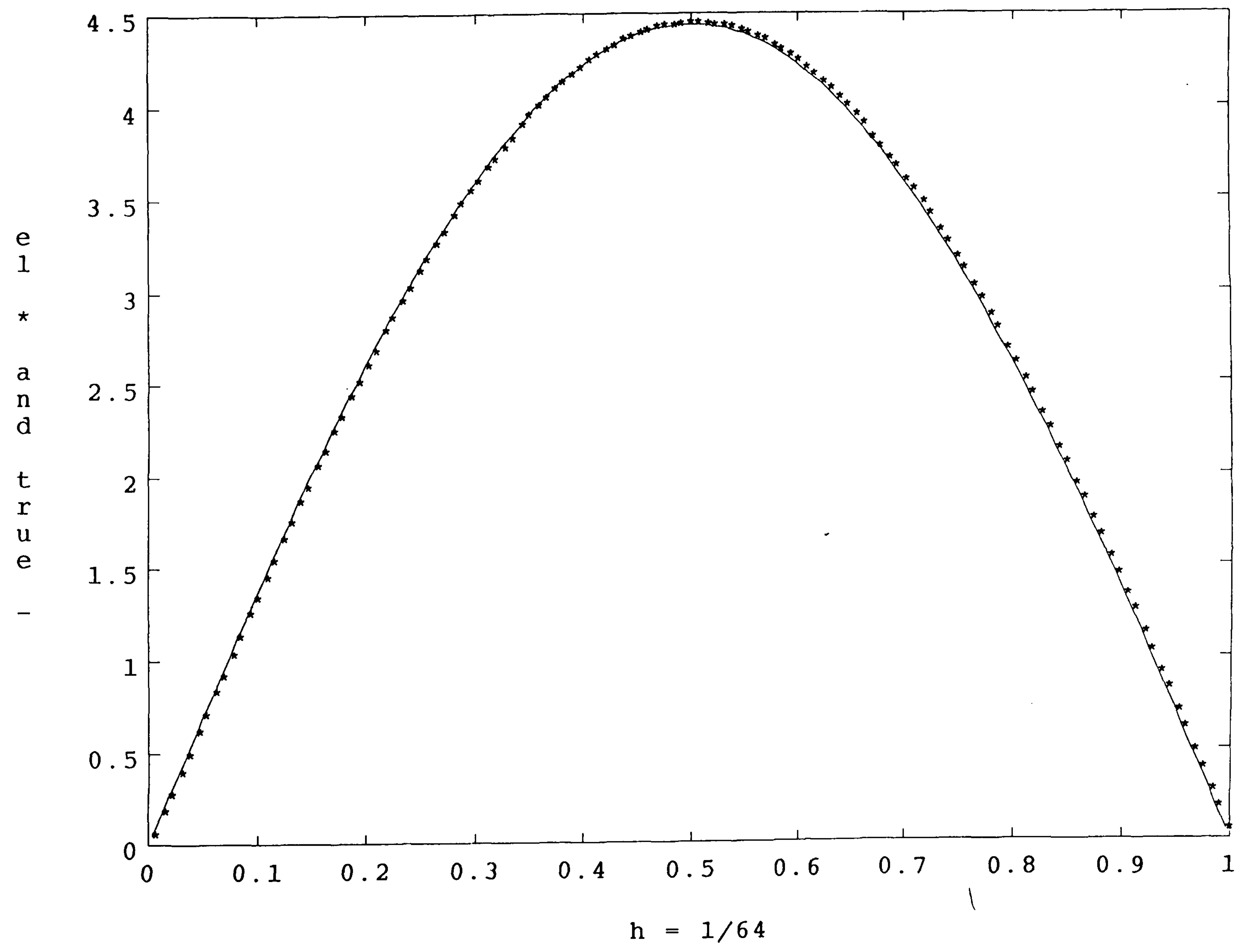
0.6
0.7
0.8
0.9
1
h = 1/64



e1 * and true –
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0.5
1
1.5
2
2.5
3
3.5
4
4.5
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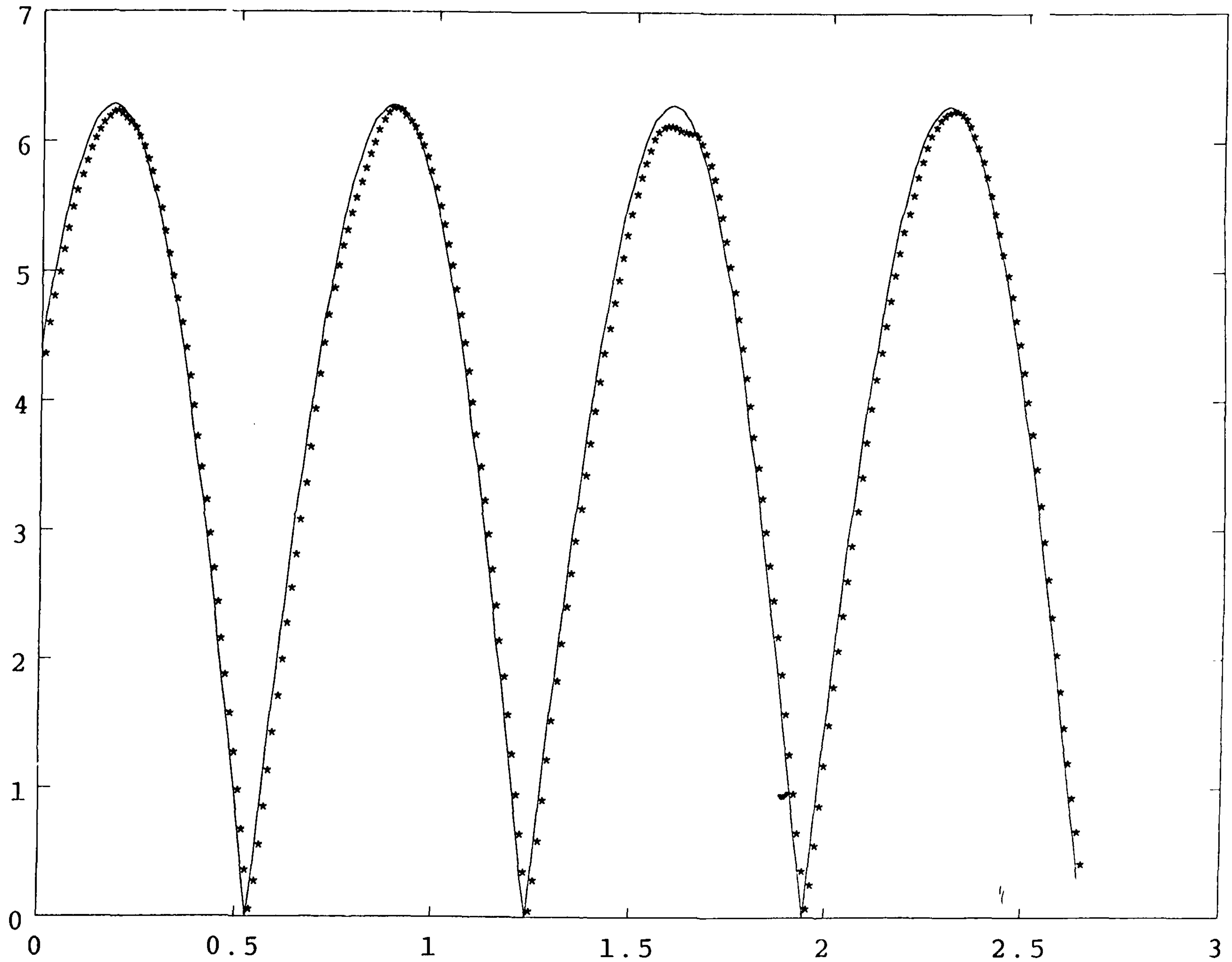
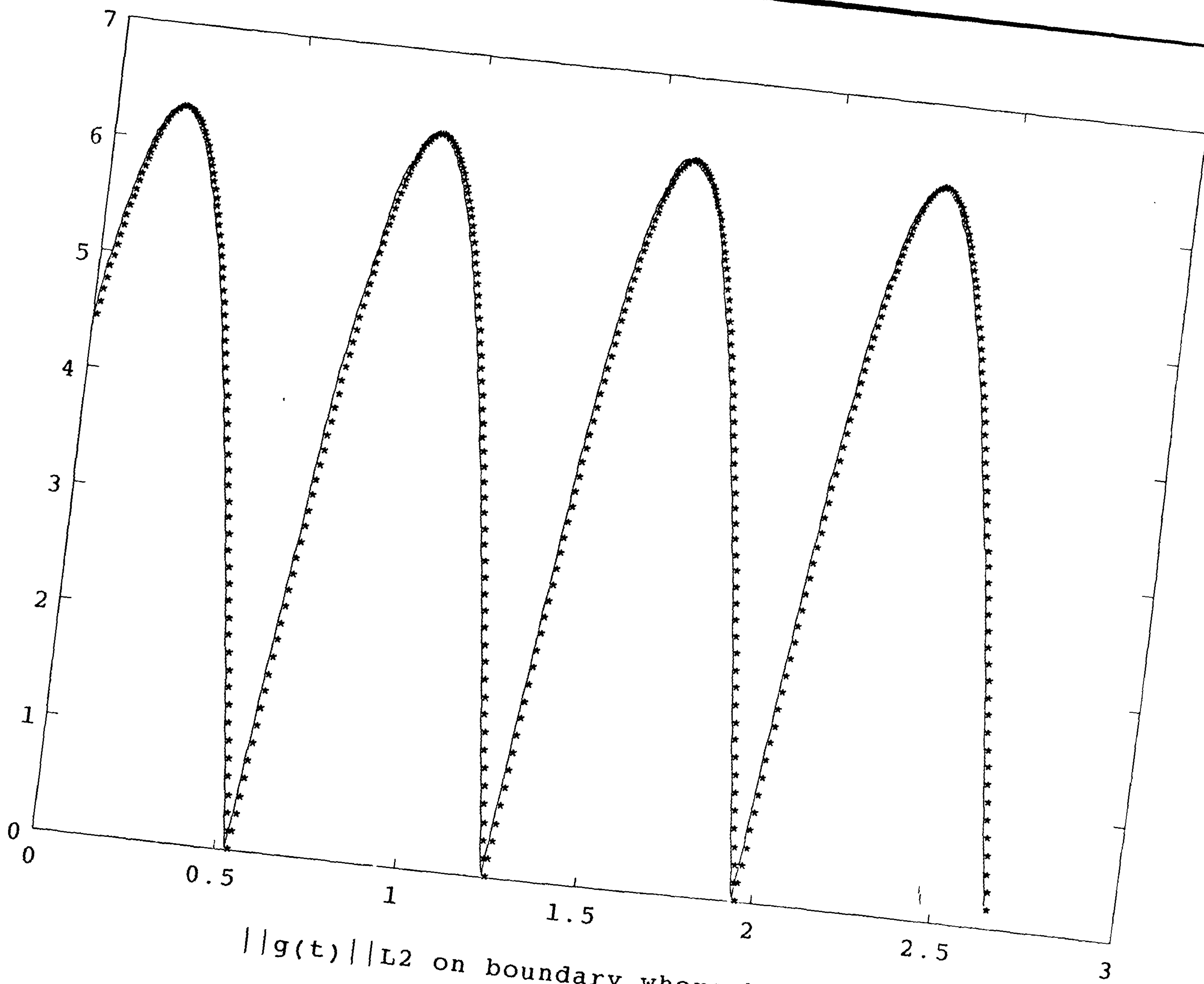
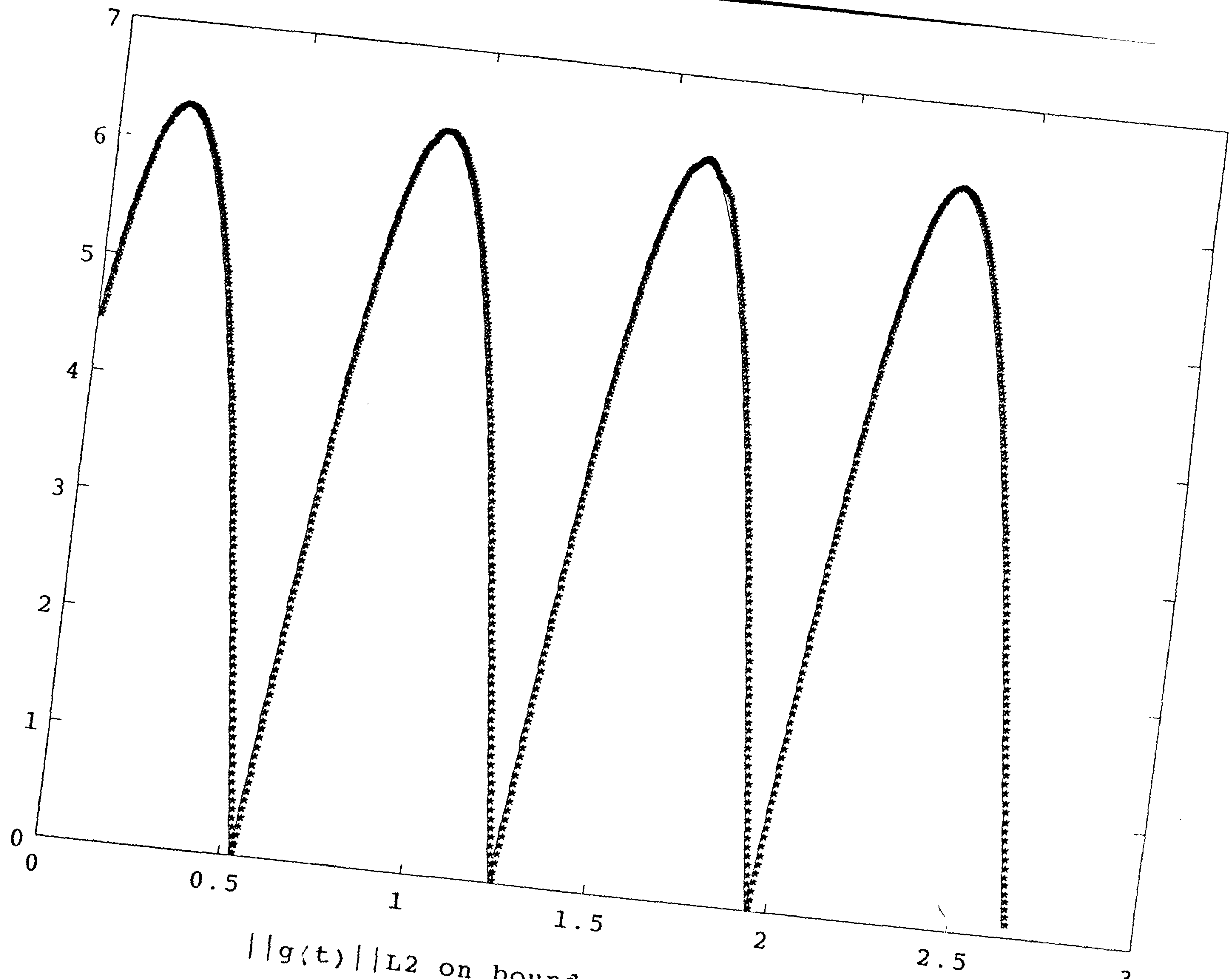
0.9
1
h = 1/64



||g(t)||L2 on boundary where h = 1/16



||g(t)||L2 on boundary where h = 1/32



||g(t)||L2 on boundary where h = 1/64

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Research Accomplishments under AFOSR - 89 - 0029

Under the Air Force grant AFSOR - 89 - 0029 support of one graduate student, Wendy Kinton, was provided. The research focus was on mixed finite element methods. Three manuscripts were written and have been accepted for publication; preprints are enclosed.

In particular the application of mixed finite element methods to the boundary controllability of the linear wave equation and domain decomposition in conjunction with mixed methods for elliptic partial differential equations were treated.

#1/3

ACCELERATION OF DOMAIN DECOMPOSITION ALGORITHMS FOR MIXED FINITE ELEMENTS BY MULTI-LEVEL METHODS

R. Glowinski*, W. Kinton**, M. F. Wheeler*

Research Report UH/MD-62

October 1989

* Department of Mathematics, University of Houston, Houston, Texas 77204

** Department of Mathematical Sciences, Rice University Houston, Texas 77251

Acceleration of Domain Decomposition
Algorithms for Mixed Finite Elements
by Multi-Level Methods

R. Glowinski*, W. Kinton**, and M. F. Wheeler*

Abstract. In this paper we consider the numerical solution of elliptic partial differential equations by a combination of domain decomposition algorithms, mixed finite element methods and multi-level procedures. The multi-level procedures are used to accelerate convergence of the algorithm which iteratively adjusts the matching conditions at the interfaces of the subdomains. Numerical results are included in this paper which exhibit improvements in convergence by applying this multi-level approach, compared to more traditional iterative methods.

*Department of Mathematics, University of Houston, Houston, Texas 77004, U.S.A.
**Department of Mathematical Sciences, Rice University, Houston, Texas 77251, U.S.A.

0. Introduction. In [1] Glowinski and Wheeler defined domain decomposition algorithms for solving mixed finite element approximations of elliptic problems with non-constant coefficients. A key result in [1] was the formulation of the matching conditions at the interfaces of the subdomains as variational problems defined over convenient trace space. These new problems were solved by conjugate gradient algorithms using simple preconditioners resulting in a $0(h^{-.5})$ number of iterations to achieve convergence. In this paper we shall discuss a procedure for accelerating the convergence of the above algorithms which is essentially based on a multi-level technique acting on the trace space associated to the interfaces.

In Section 1, we shall give some examples of elliptic problems originating from flow in porous media. Compared to more traditional solution methods the algorithm described in this paper have been quite successful as we shall demonstrate in Section 4. In Section 2 which follows closely [1] we shall recall the mixed variational formulation of elliptic problems, the mixed finite element approximations and the associated domain decomposition methods. In Section 3 we shall discuss a multilevel method to speed up convergence of the domain decomposition algorithms discussed in Section 2. Results of numerical experiments will be discussed in Section 4. Finally some mesh refinement methods well suited for domain decomposition and mixed finite element methods will be discussed in Section 5.

1. Motivation for Robust Elliptic Solvers.

In our first example we consider the pressure equation which arises from *miscible displacements* in porous media. The equation has the form

(1.1) $\quad u = -A \text{ grad } p \text{ in } \Omega,$

(1.2) $\quad \nabla \cdot u = q \text{ in } \Omega,$

(1.3) $\quad u \cdot \nu = 0 \text{ on } \partial\Omega,$

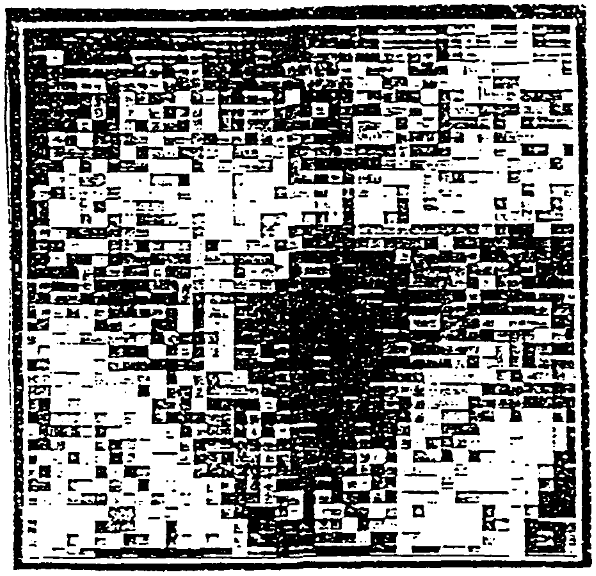
where

$$A = k(x, y)/\mu(c).$$

In this problem Ω is the flow region, u is the Darcy velocity, p is the pressure, q is a source or sinks term, k is the permeability of the porous media, μ is the viscosity of the concentration c of the fluid which is flowing through the porous media. In this example we use a permeability field and a form of the viscosity which has been previously obtained from laboratory experiments. In Figure 1.1, a visualization of A is shown. In this case we have

$$\min A = .810\times10^{-2} \text{ and } \max A = .282\times10^{-3},$$

implying that (1.1)-(1.3) is badly conditioned. However, as it will be seen with more detail in Section 4, we have been able to solve this problem, using domain decomposition, in less than 10 iterations.



Variation of coefficient A

Figure 1.1

2. Mixed Formulation of Elliptic Problems · Associated Finite Element Approximation and Domain Decomposition.

2.1 The Model Problem.

We consider on $\Omega \subset \mathbb{R}^n$ the following Neumann problem

$$
(2.1) \qquad \begin{cases} -\nabla \cdot A \nabla p = f \quad in\ \Omega, \\ \\ A \nabla p \cdot \nu = g \quad on\ \partial\Omega(=\Gamma), \end{cases}
$$

where ν is the outward normal vector. We assume the compatibility condition

$$
(2.2) \qquad \int_{\Omega} f\, dx + \int_{\Gamma} g\, d\Gamma = 0.
$$

Our formalism is motivated from flow in porous media where (2.1) is the pressure equation, but the method to be described applies to other branches of science and engineering. Also we have been considering the pure Neumann problem since it is the one occurring most frequently in applications. In fact, it is also the most difficult case.

2.2 A Mixed Variational Formulation of Problem (2.1)

Define u by

$$
(2.3) \qquad u = -A\nabla p.
$$

We then have

$$
(2.4) \qquad \nabla \cdot u - f = 0,
$$

and

$$(2.5) \qquad \nabla p = -A^{-1} u.$$

Multiplying (2.4) and (2.5) by q and v respectively, we obtain

$$(2.6) \qquad \int_\Omega (\nabla \cdot u - f) q dx = 0, \ \forall \ q \epsilon L^2(\Omega),$$

and

$$(2.7) \qquad \int_\Omega A^{-1} u \cdot v \ dx - \int_\Omega p \nabla \cdot v \ dx = 0, \ \forall v \epsilon V_o,$$

where

$$(2.8) \qquad V_o = \{ v \mid v \epsilon H(\Omega, \text{div}), \ v \cdot \nu = 0 \ \textit{on} \ \Gamma \}.$$

Here

$$(2.9) \qquad H(\Omega; \text{div}) = \{ v \ \epsilon (L^2(\Omega))^n \ \textit{and} \ \text{div} \ v \epsilon L^2(\Omega) \}.$$

Suppose $f \epsilon L^2(\Omega)$, $g \epsilon H^{-\frac{1}{2}}(\Gamma)$ and A is symmetric such that $A \epsilon (L^\infty(\Omega))^{n \times n}$ and

$$A(x) \xi \cdot \xi \geq \alpha |\xi|^2, \ \forall \xi \epsilon R^n, \text{ a. e. on } \Omega,$$

with α a positive constant.

If (2.2) holds then (2.1) has a unique solution on $H^1(\Omega)/R$ implying the uniqueness of u. An alternative formulation of (2.1) is given by

Find $p \epsilon L^2(\Omega)$, $u \epsilon H(\Omega; \text{div})$, *such that*

$$u\cdot\nu + g=0 \text{ on } \Gamma,$$

$$\int_\Omega (\nabla\cdot u - f)\, q\, dx = 0, \quad \forall\, q \in L^2(\Omega),$$

(2.10)

$$\int_\Omega A^{-1}\, u\cdot v\, dx - \int_\Omega p\, \nabla\cdot v\, dx = 0, \quad \forall\, v \in V_o.$$

2.3 Finite Element Approximation of Problem (2.10).

We denote by W^h and V^h finite dimensional subspaces of $L^2(\Omega)$ and $H(\Omega;\, \text{div})$, respectively. In addition we set $V_o^h = V^h \cap V_o$. We shall assume that div $V^h \subset W^h$.

It is natural then to approximate problem (2.1), using its mixed equivalent formulation, by

Find $p_h \in W^h$, $u_h \in V^h$ *satisfying*

$$\int_\Gamma (u_h\cdot\nu + g)\, v\cdot\nu\, d\Gamma = 0, \quad \forall v \in V^h,$$

(2.11)
$$\int_\Omega (\nabla\cdot u_h - f) q\, dx = 0, \quad \forall\, q \in W^h,$$

$$\int_\Omega A^{-1} u_h \cdot v dx - \int_\Omega p_h\, \nabla\cdot v\, dx = 0, \quad \forall\, v \in V_o^h.$$

Examples of particular finite element spaces for which (2.11) is well posed and for which lim $u_h \rightarrow u$ and lim $p_h \rightarrow p$ can be found in [2]. Additional convergence results including error estimates can be found in [3, 4].

2.4 Domain Decomposition Method for Problem (2.1), (2.11).

We follow here the notation and methodology developed in [1]. Considering first the continuous problem whose formula is much simpler we suppose that Ω has been decomposed in two subdomains Ω_1 and Ω_2. Figures 2.1a and 2.1b show such domain decompositions and corresponding notation,

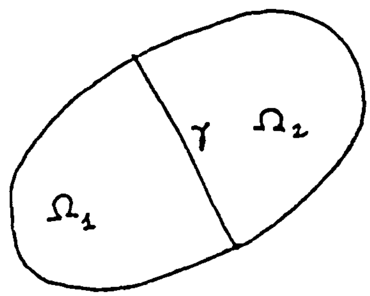


Figure 2.1a

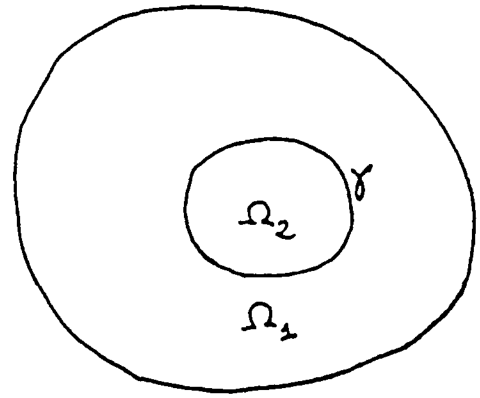


Figure 2.1b

If we denote by $\{p_i, u_i\}$ the restriction of $\{p, u\}$ to Ω_i there is equivalence between (2.10) and

(2.12)
$$\begin{cases} \int_{\Omega_i} (\nabla\cdot u_i - f)\, q_i dx = 0, \quad \forall\, q_i \epsilon L^2(\Omega_i), \\ \int_{\Omega_i} \left(A^{-1}\, u_i\cdot v_i \;-\; p_i \nabla\cdot v_i\right) dx = 0, \quad \forall v_i \epsilon V_{io},\ i=1,2, \end{cases}$$

(2.13)
$$u_i\cdot\nu_i + g = 0 \quad on\ \Gamma\cap\partial\Omega_i,\quad i=1,2,$$

(2.14)
$$\sum_{i=1}^{2} u_i\cdot\nu_i = 0 \quad on\ \gamma,$$

(2.15)
$$\sum_{i=1}^{2} \int_{\Omega_i} \left(A^{-1} u_i\cdot v - p_i \nabla\cdot v\right)\, dx = 0,\ \forall v \epsilon V_o,$$

with

$$V_{io}=\left\{v_i | v_i \epsilon H(\Omega_i, \text{div}),\ v_i \cdot \nu_i = 0 \ on\ \partial\Omega_i\right\}.$$

Since $V_o = V_{1o} \oplus V_{2o} \oplus V_{\gamma o}$ (where $V_{\gamma o}$ is a complementary subspace of $V_{1o} \oplus V_{2o}$ in V_o) it follows from (2.12) and (2.15) that (2.15) can be replaced by

$$(2.16) \qquad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot v - p_i \nabla \cdot v)\, dx = 0, \quad \forall v \epsilon V_{\gamma o}.$$

In addition to (2.12)-(2.16), $\{p_i, u_i\}$ must satisfy the compatibility condition

$$(2.17) \qquad \int_{\Omega_i} f\, dx + \int_{\partial\Omega_i \cap \Gamma} g\, d\Gamma + \int_{\gamma} u_i \cdot \nu_i\, d\gamma = 0.$$

From (2.12)-(2.16), the local solutions satisfy at the interface γ the matching conditions (2.14) and (2.16). From this observation we can generate two classes (at least) of iterative methods for solving problem (2.11) by domain decomposition. In both approaches we assume that one of the matching conditions is satisfied by an appropriate choice of boundary conditions on γ and we try iteratively to satisfy the other matching condition. In this paper we shall concentrate on the case where the balance given by (2.14) is satisfied; we try therefore to verify (2.16).

This leads to the introduction of a variational problem involving functional spaces defined on γ. Precisely such a functional space is $V^o_{\gamma o}$ defined by

$$(2.18) \qquad V^o_{\gamma o} = \{\mu | \mu\ \epsilon V_{\gamma o},\ \int_{\gamma} \mu \cdot \nu\, d\gamma = 0\}.$$

We define next a bilinear form $a(\cdot,\cdot)$ over $V^o_{\gamma o} \times V^o_{\gamma o}$ as follows:

Consider $\mu \epsilon V^o_{\gamma o}$; we associate to μ, $u_i(\mu)$ and $p_i(\mu)$ by solving

(2.19) $$\int_{\Omega_i} \nabla\cdot u_i(\mu)\, v_i dx=0, \quad \forall v_i \epsilon L^2(\Omega_i),$$

(2.20) $$\int_{\Omega_i} (A^{-1}\, u_i(\mu)\cdot v_i - p_i(\mu)\, \nabla\cdot v_i) dx=0, \quad \forall v_i \epsilon V_{io},$$

(2.21) $$u_i(\mu)\cdot \nu_i=0 \;\; on\ \Gamma\cap\partial\Omega_i,\ u_i(\mu)\cdot \nu_i = \mu\cdot\nu_i \;\; on\ \gamma.$$

Since $\int_{\partial\Omega_i} u_i(\mu)\cdot\nu_i\, d\Gamma_i=0$, the above problem is well posed in $H(\Omega_i, \text{div}) \times L^2(\Omega_i)/\mathbb{R}$. To insure uniqueness of $p_i(\mu)$ we enforce the conditions

(2.22) $$\int_{\Omega_1} p_1(\mu)\, dx=0\ , \ \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1}\, u_i(\mu)\cdot\Pi - p_i(\mu)\, \nabla\cdot\Pi)\, dx=0$$

where $\Pi\epsilon(V_{\gamma o}-V^o_{\gamma o})$. Finally we define $a(\cdot,\cdot)$ by

(2.23) $$a\Big(\mu,\mu'\Big) = \sum_{i=1}^{2} \int_{\Omega_i} \Big(A^{-1}u_i\,(\mu)\cdot\mu' - p_i(\mu)\, \nabla\cdot\mu'\Big)dx,\ \forall\ \mu'\epsilon V^o_{\gamma o}.$$

It has been shown in [1] that the bilinear form $a(\cdot\ ,\ \cdot)$ is symmetric and positive semi-definite over $V^o_{\gamma o}\times V^o_{\gamma o}$. Moreover, it is elliptic for the norm induced by $H(\Omega;\ \text{div})$ over the quotient space $V^o_{\gamma o}/\hat{R}$, where $\hat{R}$ is the equivalence relation defined by $\mu\ \hat{R}\ \mu' \leftrightarrow \Big(\mu-\mu'\Big)\cdot\nu=0$ on γ.

From the above result we can interpret (2.12)-(2.17) as a *linear* variational problem in $V^o_{\gamma o}$. To formulate this latter problem consider $\lambda_o\ \epsilon H(\Omega;\ \text{div})$ such that

(2.24) $$\lambda_o\cdot\nu+g=0 \;\; on\ \Gamma,$$

(2.25) $$\int_{\Omega_i} f\, dx + \int_{\Gamma\cap\partial\Omega_i} g\, d\Gamma + \int_{\gamma} \lambda_o\cdot\nu_i\, d\gamma=0,\ \forall\ i=1,2;$$

solve then for i=1, 2,

(2.26) $$\int_{\Omega_i} (\nabla\cdot u_{oi} - f) q_i \, dx = 0, \quad \forall q_i \epsilon L^2(\Omega_i),$$

(2.27) $$\int_{\Omega_i} (A^{-1} u_{oi}\cdot v_i - p_{oi}\nabla\cdot v_i) dx = 0, \ \forall \ v_i \epsilon V_{io},$$

(2.28) $$u_{oi}\cdot\nu_i + g = 0 \ \textit{on} \ \gamma\cap\partial\Omega_i,$$

(2.29) $$u_{oi}\cdot\nu_i = \lambda_o\cdot\nu_i \ \textit{on} \ \gamma.$$

The constants associated to the p_{oi} are adjusted as follows:

(2.30) $$\int_{\Omega_1} p_{o1} \, dx = 0,$$

(2.31) $$\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oi}\cdot\Pi - p_{oi} \ \nabla\cdot\Pi) \, dx = 0.$$

Let us now denote by u_o the element of $H(\Omega; \text{div})$ such that $u_o|_{\Omega_i} = u_{oi}$. If we define $\bar{u}$ by

(2.32) $$\bar{u} = u - u_o,$$

we clearly have $\bar{u}\epsilon V_o$. Denoting $\bar{\lambda}\epsilon V_{\gamma o}$ as the component of $\bar{u}$ in the decomposition $V_o = V_{1o}\oplus V_{2o}\oplus V_{\gamma o}$ we have from (2.17), (2.25), (2.28), (2.29) that

(2.33) $$\int_{\gamma} \bar{\lambda}\cdot\nu_i \, d\gamma = 0, \ \textit{i.e.} \ \bar{\lambda}\epsilon V^o_{\gamma o};$$

define similarly $\bar{p}_i$ by $\bar{p}_i = p_i - p_{oi}$.

We have then

$$(2.34) \qquad \int_{\Omega_i} \nabla \cdot \bar{u}_i q_i dx = 0, \quad \forall \; q_i \epsilon L^2(\Omega_i),$$

$$(2.35) \qquad \int_{\Omega_i} (A^{-1} \; \bar{u}_i \cdot v_i - \bar{p}_i \; \nabla \cdot v_i) dx = 0, \quad \forall v_i \epsilon V_{io},$$

$$(2.36) \qquad \bar{u}_i \cdot \nu_i = 0 \;\; on \; \partial\Omega_i \cap \Gamma, \; \bar{u}_i \cdot \nu_i = \bar{\lambda} \cdot \nu_i \text{ on } \gamma,$$

$$(2.37) \qquad \int_{\Omega_1} \bar{p}_1 \; dx = 0, \quad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} \bar{u}_i \cdot \Pi - \bar{p}_i \; \nabla \cdot \Pi) dx = 0.$$

It follows from (2.16) that

$$(2.38) \qquad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_i \cdot \mu - p_i \; \nabla \cdot \mu) \; dx = 0, \quad \forall \; \mu \; \epsilon V^o_{\gamma o}.$$

From the definition of $\bar{u}_i$, p_i and from (2.38) we obtain

$$(2.39) \qquad \sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} \bar{u}_i \cdot \mu - \bar{p}_i \; \nabla \cdot \mu) \; dx = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oi} \cdot \mu - p_{oi} \; \nabla \cdot \mu) \; dx, \quad \forall \mu V^o_{\gamma o}.$$

It follows from (2.23) and (2.33) that $\bar{\lambda}$ is the unique solution of the linear variational equation

$$(2.40) \qquad \begin{cases} \textit{Find } \bar{\lambda} \epsilon V^o_{\gamma o} \textit{ such that} \\ \\ a(\bar{\lambda}, \mu) = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{oi} \cdot \mu - p_{oi} \; \nabla \cdot \mu) dx, \quad \forall \; \mu \epsilon V^o_{\gamma o}. \end{cases}$$

In [1], we showed that the variational problem (2.40) can be approximated by a finite dimensional problem of the same nature, obtained by combining the mixed approximation of Section 2.3 with the domain decomposition principle of Section 2.4. In addition, a conjugate gradient method for solving this finite dimensional problem approximating (2.40) was discussed in detail in the above reference.

In the following Section 3, we shall describe multilevel techniques for solving the finite dimensional problem approximating (2.40); it can be seen as a multigrid method operating on the interface γ.

3. Multilevel Solution of Problem (2.40).

3.1. Domain Decomposition of the Discrete Problem.

Following Section 2.3, it is easily shown that the discrete mixed problem (2.11) is equivalent to finding $\{u_{h,i}, p_{h,i}\}$, i=1, 2, satisfying

$$(3.1) \qquad \int_{\Omega_i} (\nabla \cdot u_{h,i} - f)\, q_i \, dx = 0, \quad \forall q_i \in W_{h,i},$$

$$(3.2) \qquad \int_{\Omega_i} (A^{-1} u_{h,i} \cdot v_i - p_{h,i} \nabla \cdot v_i)\, dx = 0, \quad \forall v_i \in V_{o\,h,i}$$

$$(3.3) \qquad \int_{\partial\Omega_i \cap \Gamma} (u_{h,i} \cdot \nu + g) \; v \cdot \nu \, d\Gamma = 0, \quad \forall \; v_i \in V_{oh,i},$$

$$(3.4) \qquad \sum_{i=1}^{2} u_{h,i} \cdot \nu_i = 0 \quad on \; \gamma,$$

$$(3.5) \qquad \sum_{i=1}^{2} \int_{\Omega} (A^{-1} u_{h,i} \cdot v - p_{h,i} \; \nabla \cdot v)\, dx = 0, \quad \forall v \in V_{oh},$$

where $V_{oh,i}$ (resp. $W_{h,i}$) is equal to $V_{oh}|_{\Omega_i}$ (resp. $W_h|_{\Omega_i}$). As in the continuous case we associate to γ a complementary subspace $V_{oh,\gamma}$ of $V_{oh,1} \oplus V_{oh,2}$ in V_{oh}; that is

$$V_{oh} = V_{oh,1} \oplus V_{oh,2} \oplus V_{oh,\gamma}.$$

It follows from (3.1) and (3.2) that (3.5) can be replaced by

(3.6)
$$\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1} u_{h,i} \cdot v - p_{h,i} \nabla \cdot v)\, dx = 0, \quad \forall\ v \epsilon V_{oh,\gamma}.$$

In addition to (3.5) and (3.6) $\{u_{h,i}, p_{h,i}\}$ has to satisfy the compatibility conditions

(3.7)
$$\int_{\Omega_i} f\, dx + \int_{\partial\Omega_i \cap \Gamma} g d\Gamma + \int_{\gamma} u_{h,i} \cdot \nu_i d\gamma = 0,\ i=1, 2.$$

Finally we decompose $V_{oh,\gamma}$ as the direct sum,

(3.8)
$$V_{oh,\gamma} = V^{o}_{oh,\gamma} \oplus V^{\Pi}_{oh,\gamma}$$

where

(3.9)
$$V^{o}_{oh,\gamma} = \{z \epsilon V_{oh,\gamma} \mid \int_{\gamma} z \cdot \nu\ d\gamma = 0\},$$

and

(3.10)
$$V^{\Pi}_{o,h,\gamma} = \{t\Pi,\ t \epsilon \mathbb{R} \textit{ and } \Pi \epsilon\ V_{oh,\gamma} \textit{ with} \int_{\gamma} \Pi \cdot \nu\ d\gamma \neq 0\}.$$

3.2. Discretization of the Boundary Problem (2.40).

Following the development in Section 2.4, we approximate (2.40) by the following variational problem in $V^o_{oh,\gamma} \times V^o_{oh,\gamma}$:

$$(3.11)\qquad \begin{cases} \textit{Find } \bar{\lambda}_h \in V^o_{oh,\gamma} \textit{ such that} \\ a_h(\bar{\lambda}_h,\mu) = -\sum_{i=1}^{2} \int_{\Omega_i} (A^{-1}u_{oh,i} \cdot \mu - p_{oh,i}\, \nabla\cdot\mu)dx, \quad \forall \mu \in V^o_{oh,\gamma}, \end{cases}$$

where $\bar{\lambda}_h$, $u_{oh,i}$ and $p_{oh,i}$ are obtained as discrete analogues of $\bar{\lambda}$, u_{oi} and $p_{o,i}$ in Section 2.4 (see [1] for all the details).

3.3. Multilevel Algorithms for Solving Problem (3.11).

3.3.1. Synopsis

We first introduce a discretization parameters h_j to which we associate all the above discrete spaces. For simplicity we denote by Z^j the space $V^o_{oh_j,\gamma}$. We assume that the sequence $\{Z^j\}$ satisfies the following inclusion property

$$(3.12)\qquad Z^0 \subset Z^1 \subset \ldots \subset Z^J.$$

At level J (the finest level) we wish to solve problem (3.11) with $h=h_J$.

Before defining a multilevel algorithm for solving problem (3.11), we describe in the following Section 3.3.2 the solution of general variational problems by multilevel methods. The application to the specific problem (3.11) will be discussed in Section 3.3.3.

3.3.2. A Multi level Method for Linear Variational Problem in Hilbert Spaces.

Let V be a Hilbert space with $(\cdot\,,\cdot)$ as inner product and $\|\cdot\|_V$ the corresponding norm. We consider the following problem

$$
(3.13) \qquad \begin{cases} \textit{Find } u \in V \textit{ such that} \\ \\ a(u, v) = L(v), \quad \forall v \in V, \end{cases}
$$

where

(1) a: $V \times V \rightarrow \mathbb{R}$ is *bilinear*, *continuous* and *V-elliptic*,

(2) L: $V \rightarrow \mathbb{R}$ is *linear* and *continuous.*

We consider now a family of finite dimensional subspaces $V^0 \subset V^1 \subset V^2 \subset \ldots \subset V^J \subset V$. The idea here is to approximate (3.13) by

$$
(3.14) \qquad \begin{cases} \textit{Find } u^J \in V^J \textit{ such that} \\ \\ a_J(u^J, v) = L_J(v), \ \forall \ v \in V^J, \end{cases}
$$

where a_J and L_J are approximations to $a(\cdot,\cdot)$ and L respectively (for those applications associated to mixed finite element approximations, a_J and L_J are never the restrictions of $a(\cdot,\cdot)$ and L to $V \times V$ and V respectively).

The basic principle of all multilevel methods is to solve (3.14) using solutions of problems of the form (3.14) defined on V^j, $j=0, 1, \ldots J-1$. A classical way to handle this is to use a V-cycle multilevel method [5, 6, 7, 8]. For problem (3.14) the V-cycle with J levels takes the following form:

<u>Step 0</u>: *Suppose that* $u_n^J \in V^J$ *is known.*

<u>Step 1</u>: *Starting from* u_n^J, *iterate* ν_J *steps of some iterative method and call the result* u_n^{*J}.

<u>Step 2</u>: *Now for* $j=J-1, \ldots, 1$, *assuming that* u_n^{*j+1} *is known and starting from* 0 *perform* ν_j *steps of some iterative procedure for solving the following variational residual equation*

$$(3.15)\quad \begin{cases} a_j\,(u_n^j, v) = L_J(v) - \sum_{l=J}^{j+1} a_l\,(u_n^{*l}, v),\ v \in V^j, \\ u_j^n \in V^j. \end{cases}$$

Call u_n^{*j} *the result of this smoothing.*

Step 3: *For* $j=0$ *solve exactly the residual equation* (3.15). *Set* $u_n^{po} = u_n^o$.

Step 4: *For* $j=1, 2, \ldots, J$, *assuming* $u_n^{p\,j-1}$ *is known, take* $u_n^{p\,j-1} + u_n^{*j}$ *as an initial condition. Perform* μ_j *steps of some iterative procedure for solving* (3.15). *Call the result* u_n^{pj}.

Step 5: *Take* $u_{n+1}^J = u_n^{pJ}$.

3.3.3 Application of the V-cycle Method to the Solution of Problem (3.11).

Problem (3.11) is a particular case of problem (3.14). Thus, it can be solved by the multilevel method described in Section 3.3.2. Once the basic iterative methods involved in the V-cycle have been specified, thus applying the above multilevel method is canonical.

The numerical results discussed in Section 4 have been obtained using conjugate gradient as a smoother in Steps 1 and 2, taking $\nu_j=2$. For $j=0$ we also used conjugate gradient to obtain u_n^o. In Step 4 we employed one iteration of steepest descent.

The conjugate gradient algorithm for solving problem (3.11) is described in Section 4 of [1].

4. Numerical Results

In this section we shall present the results of numerical experiments where the mixed element/multi-level domain decomposition methods described in Section 2.3 have been applied to the solution of test problems. The examples considered here include both some standard cases as well as physical problems arising in flow in porous media, such as (1.1)-(1.3) of Section 1. In all our examples, the discrete problem (2.11) approximating the elliptic problem (2.1) has been obtained using for W^h and V^h the Raviart-Thomas mixed finite element spaces. A full description of these elements can be found in [1] and [2]; however for completeness we shall describe these spaces in the following Section 4.1.

4.1 Mixed Finite Element Approximations of Problem (2.1).

Let Ω be the rectangular domain $(0, x_L)\times(0, y_L)$ and let Δ_x: $0=x_0<x_1<\ldots<x_{N_x}=x_L$ and Δ_y: $0=y_0<y_1<\ldots<y_{N_y}=y_L$ define partitions of $[0, x_L]$ and $[0, y_L]$, respectively. For Δ a partition, define the piecewise polynomial space

$$M_s^r(\Delta)=\{v\epsilon C^s([0, L]):\ v \textit{ is a polynomial of degree } \leq r \textit{ on each subinterval of } \Delta\},$$

where $s=-1$ refers to the discontinuous functions. We define now the following approximations of $L^2(\Omega)$, $H(\Omega;\text{div})$ and V_o respectively

$$W_h^{s,r}=M_s^r(\Delta_x)\otimes M_s^r(\Delta_y),$$

$$V_h^{s,r}=\left[M_{s+1}^{r+1}(\Delta_x)\otimes M_s^r(\Delta_y)\right]\times\left[M_s^r(\Delta_x)\otimes M_{s+1}^{r+1}(\Delta_y)\right],$$

$$V_{h,o}^{s,r}=V_h^{s,r}\cap\left\{v:\ v\cdot\nu=0 \textit{ on } \partial\Omega\right\},$$

where $h=\max_{i,j}\left\{(x_{i+1}-x_i), (y_{j+1}-y_j)\right\}$. We remark that these spaces satisfy

$$\nabla\cdot v\ \epsilon W_h^{s,r},\ \forall\ v\epsilon V_h^{s,r} (\textit{i.e. } \nabla\cdot V_h^{s,r}\subset W_h^{s,r}).$$

In our numerical experiments we set $r=1$.

4.2. Solution of Standard Test Problems

Motivated by applications in reservoir engineering we are considering now the following class of test problems:

$$(4.1)\quad \begin{cases} -\nabla\cdot(A\nabla p)=\delta_{(1,0)}-\delta_{(0,1)}, \\ A\nabla p\cdot\nu=0 \textit{ on } \partial\Omega, \end{cases}$$

where $\Omega=(0, 1)^2$ and where A is defined by either

(i) $A = A_1 = I$,

or

(ii) $A = A_2 = \frac{1}{1+100(x^2+y^2)} I$,

or

(iii) $A = A_3 = \alpha I$, where $\alpha = 100$ if $0 \leq x \leq .5$ and $\alpha = 1$ if $.5 < x \leq 1$.

The partitionings of Ω used to implement the domain decomposition are those shown in Section 8 of [1]. In particular a (N_x, N_y) decomposition involves a partitioning into $N_x N_y$ rectangular subdomains whose edges are parallel to the coordinate axis.

Table 4.1 depicts the number of multi-level V cycles versus mesh and subdomain partitions:

Coefficient	h^{-1}	(#Subdomains, #V cycles)
A_1	20	(4, 6)
	40	(4, 6); (16, 7)
	80	(4, 9); (16, 8); (64, 7)
A_2	20	(4, 6)
	40	(4, 8); (16, 7)
	80	(4, 10); (16, 8); (64, 7)
A_3	20	(4, 7)
	40	(4, 6); (16, 7)
	80	(4, 10); (16, 8); (64, 7)

Number of Cycles versus Mesh Size and Subdomain Partition for the 3-Level V-Cycle.

Table 4.1

Interestingly the above table applies for the three cases (i)−(iii). We also observe that the number of grid points by subdomain is the same for the three decompositions considered and that the number of V cycles is practically independent of h despite the fact that the dimension of the interface problem is growing like h^{-1}.

To further illustrate the efficiency of the above methods we are providing in Table 4.2 below the dimensions of the various finite element and boundary spaces involved in our combined domain decomposition/mixed finite elements methodology (below, γ is defined by an $N \times M$ decomposition).

h^{-1}	Dim W^h	Dim V^h	Dim $V^o_{oh,\gamma}$
20	1600	3120	$40\,(N + M) - 79 - NM$
40	6400	12640	$80\,(N + M) - 159 - NM$
80	25600	50800	$160\,(N + M) - 319 - NM$

Dimension of the Discrete Spaces

Table 4.2

This insensibility to the smooth or fast variation of coefficient A over Ω is a remarkable property which shows that this methodology has attractive potential for the solutio of badly conditioned practical problem such as geostatistics problems arising in porous n: ·lia ([9, 10].)

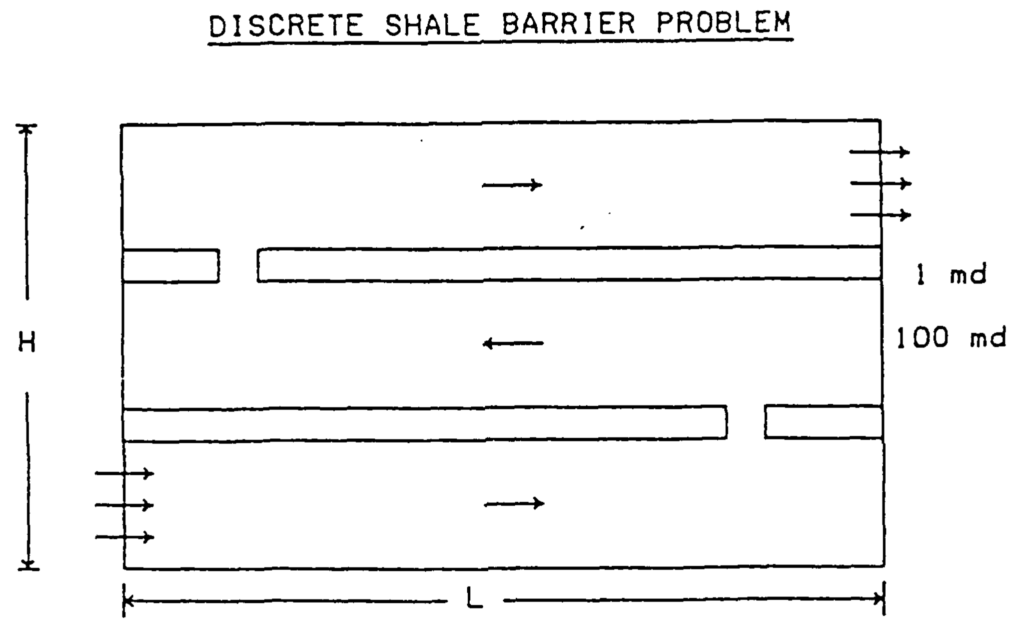
The above results represent a substantial improvement in terms of robu·!ness and speedup compared to the results obtained in [1] for the same test problems with the same grids and decompositions.

Another interesting property of the above methodology (already observed in [1]) is that the subdomain problems need not be solved exactly. We also observed, concerning the multilevel solution of the matching problem, that one to two V cycles are sufficient in practice to achieve the solution within truncation error; in particular, with $\nu_j = \mu_j = 2$ in the algorithm of Section 3.3.2, the initial residual is reduced by six orders of magnitude in six to seven iterations, the largest reduction taking place in the first V-cycle.

4.3. Solution of Real-Life Test Problems.

To be honest the test cases discussed here are more relevant to [1] since the domain decomposition methodology is exactly the one described in the above reference, i.e. without-yet-multilevel speedup. Nevertheless, we have inserted these problems because they are typical of real-life applications in petroleum reservoir engineering. Also they provide significant benchmarks for elliptic solvers of various types.

This first problem to be considered was communicated to us by petroleum reservoir engineers. It is a model for a discrete shale barrier and involves solving (1.1)-(1.3) where A is visualized in Figure 4.1, where we have used different scales for L and H since L is of the order of 300 feet and H is of the order of 20 feet implying an aspect ratio of 15. Also the thickness of the barrier is of order one foot. The ratio of permeability coefficients is 10^2.



Geometry of the Discrete Shale Barrier Problem

Figure 4.1

The arrows in Figure 4.1 indicate the flow direction.

Concerning the numerical solution of the above problem we have been using a 40×40 finite element grid and a (2, 2) domain decomposition. For comparison purposes we have treated the cases with aspect ratios 1 and 15.

Using the domain decomposition algorithm discussed in [1] we need 33 iterations if R=1 and 48 if R=15. We can expect the number of iterations to be practically independent of R once our V

In the same vein the second problem is also a real life problem (1.1)-(1.3) where $A=k(x, y)/\mu(c)$ and

$$\mu(c)=c\mu_1^{-1/4}+(1-c)\mu_2^{-1/4},$$

with $\mu_1,\mu_2>0$.

Applying the domain decomposition-mixed finite element methods of [1] to the above problem, with a 80×80 finite element grid and a (10, 10) domain decomposition, the solution was obtained in 9 conjugate gradient iterations. This represents a substantial improvement over a preconditioned conjugate gradient solution of the same discrete problem (without domain decomposition) since the convergence was requiring then about 150 iteration, (taking advantage of a good unital guess). Incidentally the lowest order Raviart-Thomas space ($r=0$ (in 4.1)) or cell-centered finite differences [11] do not work well on this type of problems due to the impossibility for these low order approximations to reproduce correctly flows which are not parallel to the coordinate axes; this drawback disappears if we chose $r=1$.

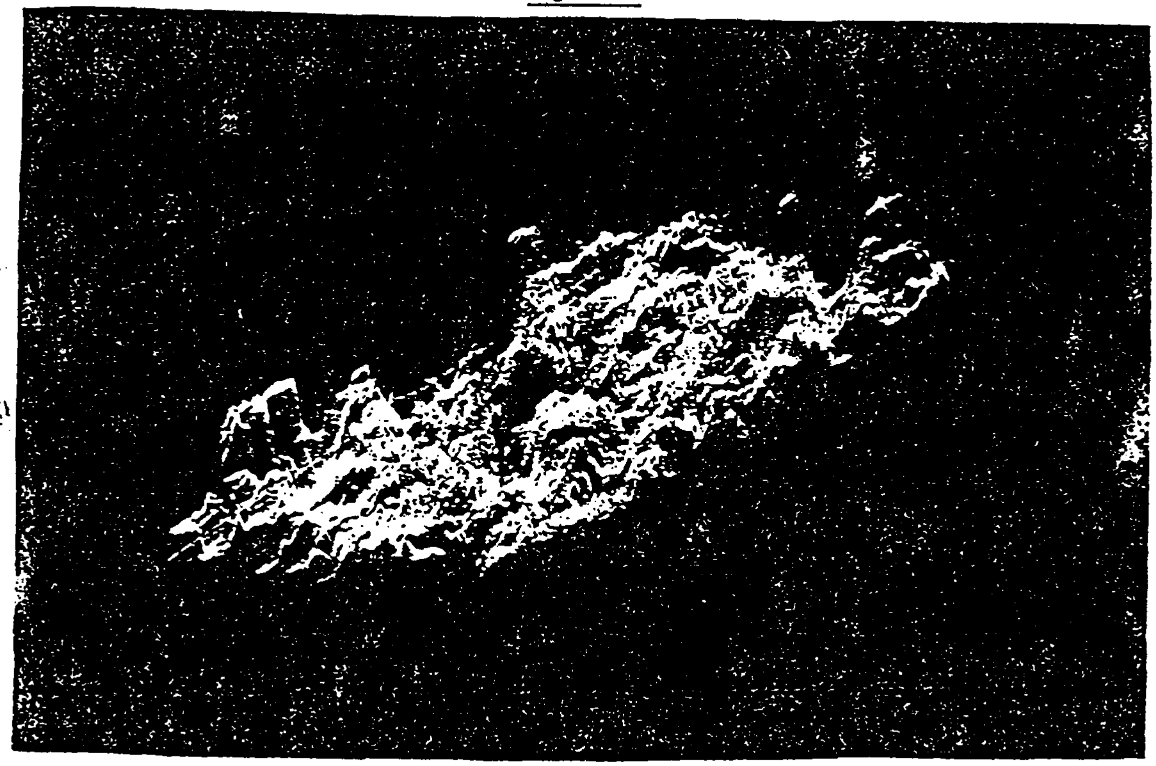
In Figure 4.2 we have visu..ized the permeability $k(x, y)$, this data was measured by researchers at Atlantic Richfield Corporation and kindly communicated to us. Similarly the function $A=k/\mu$ is visualized in Figure 4.3.

5. Mesh Refinements Via Domain Decomposition

Mesh refinements are necessary when strong gradients arise locally. In view of saving computer storage and avoiding complicated data structures it is interesting to incorporate local grid refinement over subdomains where the strong variations are arising and retain coarser grids elsewhere. The concept of domain decomposition provides an elegant and systematic way to implement the above ideas. In this section we would like to present a particular implementation of our scheme, new to our knowledge, relying again on a combination of Raviart-Thomas mixed finite element and domain decomposition methods.

Representation of k(x, y)

Figure 4.2



Representation of $A = k/\mu$.

Figure 4.3

5.1 Mesh Refinement Via a Modified Raneart-Thomas Mixed Finite Element Method

Consider the situation depicted in Figure 5.1 where a local refinement is necessary in a subregion Ω^* of Ω. The basic idea is to employ essentially mixed finite elements of Raviart-Thomas type inside and outside subregion Ω^*; the main issue here is clearly the matching between the "fine" and "coarse" approximations. To realize this matching we introduce the following finite dimensional spaces of mixed type.

Let $\Omega^*=(a^*, b^*)\times(c^*, d^*)$ and define Δ_x^* and Δ_y^* be partitions of $[a^*, b^*]$ and $[c^*, d^*]$, respectively. Generalizing the notation of Section 4.1, we denote by

$$W_{h^*}^{-1, r^*}(\Omega^*)=M_{-1}^{r^*}(\Delta x^*) \otimes M_{-1}^{r^*}(\Delta y^*), \tag{5.1}$$

$$V_{h^*}^{*-1, r^*}(\Omega^*)=(M_0^{r^*+1}(\Delta x^*) \otimes M_{-1}^{r^*}(\Delta y^*)) \times (M_{-1}^{r^*}(\Delta x^*) \otimes M_0^{r^*+1}(\Delta y^*)), \tag{5.2}$$

and

$$V_{h^*,0}^{*-1, r^*}(\Omega^*)=V_h^{*-1, r^*}(\Omega^*) \cap \{q : q\cdot\nu=0 \ on\ \partial\Omega^*\}. \tag{5.3}$$

Similarly we define the corresponding "coarse" spaces by

$$W_h^{-1, r}(\Omega-\Omega^*)=W_h^{-1, r}\Big|_{\Omega-\Omega^*}, \tag{5.4}$$

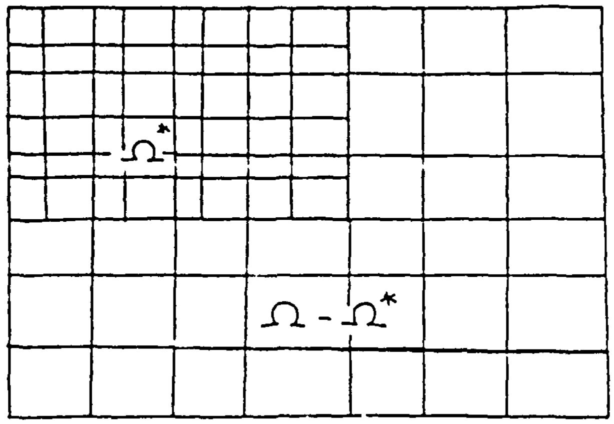


Figure 5.1

and

(5.5) $$V_h^{-1,r}(\Omega-\Omega^*)=V_h^{-1,r}|_{\Omega-\Omega^*},$$

with $W_h^{-1,r}$ and $V_h^{-1,r}$ as defined in Section 4.1. We set

(5.6) $$W_h^R=W_{h^*}^{-1,r^*}(\Omega^*)\cup W_h^{-1,r}(\Omega-\Omega^*),$$

(5.7) $$V_h^R=V_{h^*}^{-1,r^*}(\Omega^*)\cup V_h^{-1,r}(\Omega-\Omega^*),$$

(5.8) $$V_{h,0}^R=V_h^R\cap\{q:\ q\cdot\nu=0 \text{ on } \partial\Omega\}.$$

Strictly speaking W_h^R and V_h^R are not Raviart-Thomas spaces, however, they share the same approximating properties which include div $V_h^R\subset W_h^R$ and the order approximation is the same if $r^*=r$.

From a computational point of view this refinement technique is well suited for domain decomposition with Ω^* and Ω-Ω^* as subdomains.

The above approach is well suited for a multi-level solution of problem (2.1) in which we shall use different number of grid levels in the subdomains (usually more grid levels in the more refined regions). Domain decomposition allow a lot of flexibility by the fact that in one of the phases of their realization they decouple the computation to be done in each subdomain.

6. Conclusions

From the numerical results described in this paper the combination of mixed finite element, domain decomposition and multilevel methods discussed in Sections 2, 3 and 4 provides a robust, accurate and fast technique for solving elliptic problem with non-smooth coefficients like those arising in flow in porous media and other applications from Mechanics and Physics.

These methods are quite interesting from a parallel computing point of view since the ratio

$$\frac{\text{Work in Solving Subdomain Problems}}{\text{Communication Costs}}$$

is of order $0(h^{-1})$.

Here the communication involves the transfer of the boundary data at the subdomain interfaces.

We are presently cooperating with the computer scientists at the National Science Foundation Center for Research in Parallel Computation in the parallel implementation of the methods discussed in this paper.

Acknowledgement. We acknowledge the support of Cray Research, the National Science Foundation under Grants DMS8814841, DMS8822522 , and INT8612680 and the Air Force Office of Scientific Research under Grant AF0SR-89-0025 . Additional acknowledgements are given to Clint Dawson, Todd Dupont, David Moissis, and Steve Poole for help and suggestions and to Lasonya Jones for her diligence in processing this paper.

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Mixed Finite Element Methods For Time Dependent Problems: Application To Control

T. Dupont*, R. Glowinski**, W. Kinton***, M. F. Wheeler****

Research Report UH/MD-54

February, 1989

* University of Chicago
** University of Houston and INRIA
*** Rice University and University of Houston
**** University of Houston and Rice University

MIXED FINITE ELEMENT METHODS FOR TIME DEPENDENT PROBLEMS: APPLICATION TO CONTROL.

T. Dupont, University of Chicago

R. Glowinski, University of Houston and INRIA

W. Kinton, Rice University and University of Houston

M.F. Wheeler, University of Houston and Rice University

ABSTRACT

The main goal of this paper is to discuss mixed variational formulations for time dependent problems such as initial and boundary value problems for the heat and wave equations in a bounded domain Ω of $\mathbb{R}^N (N \geq 1)$. Then we shall use these formulations to derive mixed finite element approximations of the heat and wave equations. Finally, an application to an exact boundary controllability problem for the wave equation will be presented together with some numerical results. The techniques and application briefly considered here will be discussed with more details in a forthcoming paper.

INTRODUCTION

Mixed variational principles and the associated *finite element approximations* have proved to be very useful in order to derive accurate solution methods for boundary value problems for partial differential equations. This is particularly true for *elliptic problems* (see, e.g., [1], [2] and the references therein). A strong point of these techniques - compared to more traditional finite element methods - is that they give fairly *accurate approximations of the derivatives*; this last property is very interesting since in many problems one is more interested by the derivatives of a function than by the function itself. Mixed methods have also been applied to *time dependent problems* (see, e.g., [3]) but there are indeed very few published papers and applications where these methods have been used for time dependent problems compared to the more classical finite element methods. Motivated by *optimal control* applications (cf. [4], [5]) we shall briefly discuss in this short article the following topics:

(i) *Mixed variational formulations* for the *heat* and *wave equations* (Section 1.).

(ii) *Mixed finite element approximations* of the *heat* and *wave equations* (Section 2.).

(iii) An application to a *boundary control problem* for the *wave equation* (Section 3.).

1. MIXED VARIATIONAL FORMULATIONS FOR THE HEAT AND WAVE EQUATIONS.

1.1 *Formulation of the basic time dependent problems.*

Let Ω be a bounded domain of $\mathbb{R}^N (N \geq 1)$; we denote by Γ the boundary of Ω . Let T be a positive number (possibly equal to $+\infty$) ; we denote by Q and Σ the following sets of $\mathbb{R}^{N+1}$:

$$Q = \Omega \times (0,T), \Sigma = \Gamma \times (0,T).$$

We suppose now that physical phenomena are taking place on Ω , modelled by either the following *heat equation*

$$u_t - \Delta u = f \; in \; Q, \tag{1.1}$$

$$u = g \; on \; \Sigma, \tag{1.2}$$

$$u(x,0) = u_o(x) \; on \; \Omega, \tag{1.3}$$

or by the following *wave equation*

$$u_{tt} - \Delta u = f \; in \; Q, \tag{1.4}$$

$$u = g \; on \; \Sigma, \tag{1.5}$$

$$u(x,0) = u_o(x), \; u_t(x,0) = u_1(x) \; on \; \Omega. \tag{1.6}$$

In (1.1) - (1.6) we have

$$x = \{x_i\}_{i=1}^N, u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, \Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}.$$

It follows from, e.g. [6], [7], that each of the two above problems has a unique solution provided that the data f and g belong to well chosen functional spaces. Since this paper is *engineering oriented* we shall not go into the details of those (Sobolev type) spaces for which the above problems are well-posed (there will be however some exceptions).

1.2 *Mixed variational formulations for problems (1) – (1.3) and (1.4) – (1.6).*

The key idea is to take $\nabla u (\nabla = \{\frac{\partial}{\partial x_i}\}_{i=1}^N)$ as *master variable* ; we introduce therefore a new unknown $\mathbf{p}$ defined by

(1.7) $$\mathbf{p} = \nabla u (in\ Q).$$

Assuming that u and $\mathbf{p}$ are sufficiently smooth we obtain - *integrating by parts* with respect to the space variables - the following mixed variational formulations:

Mixed variational formulations of the heat equation (1.1) – (1.3) :

(1.8) $$\int_\Omega (u_t - \nabla \cdot \mathbf{p} - f) v dx = 0,\ \forall v \epsilon L^2(\Omega), a.e.\ on\ (0,T),$$

(1.9) $$\int_\Omega (\mathbf{p} \cdot \mathbf{q} + u \nabla \cdot \mathbf{q}) dx = \int_\Gamma g \mathbf{q} \cdot \mathbf{n} d\Gamma, \forall \mathbf{q} \epsilon H(\Omega, div), a.e.\ on\ (0,T),$$

(1.10) $$u(x,0) = u_o(x)\ on\ \Omega.$$

Mixed variational formulations of the wave equation (1.4) – (1.6) :

(1.11) $$\int_\Omega (u_{tt} - \nabla \cdot \mathbf{p} - f) v dx = 0, \forall v \epsilon L^2(\Omega), a.e. on\ (0,T),$$

1.12) $$\int_\Omega (\mathbf{p} \cdot \mathbf{q} + u \nabla \cdot \mathbf{q}) dx = \int_\Gamma g \mathbf{q} \cdot \mathbf{n} d\Gamma, \forall \mathbf{q} \epsilon H(\Omega, div), a.e.\ on\ (0,T),$$

(1.13) $$u(x,0) = u_o(x), u_t(x,0) = u_1(x). \Box$$

In (1.8) - (1.13), we have used the following notation: $\mathbf{y} \cdot \mathbf{z} = \sum_{i=1}^N y_i z_i, \forall \mathbf{y}, \mathbf{z} \epsilon \mathbb{R}^N$; $\mathbf{n}$ is the unit vector of the outward normal at Γ; $dx = dx_1 \cdots dx_N$ and finally

$$H(\Omega, div) = \{\mathbf{q} | \mathbf{q} \epsilon L^2(\Omega), \nabla \cdot \mathbf{q} \epsilon L^2(\Omega)\}.$$

2. MIXED FINITE ELEMENT APPROXIMATIONS OF THE HEAT AND WAVE EQUATIONS.

2.1 *Generalities.*

With h a *space discretization step*, we approximate $L^2(\Omega)$ and $H(\Omega, div)$ by V_h and Q_h , respectively. We suppose that $V_h \subset L^2(\Omega), Q_h \subset H(\Omega,\ div)$ and also that V_h and Q_h satisfy compatibility conditions implying convergence properties for the corresponding approximations (see e.g., [1], [2] for details); an important condition to be satisfied is:

$$\nabla \cdot Q_h \subset V_h. \tag{2.1}$$

In the particular case where Ω is a 2 dimensional polygonal whose boundary is the union of segments parallel to the coordinate axis, we associate to Ω a "partition" R_h such that

(i) $$R_h = \{K\}, \overline{\Omega} = \bigcup_{K \epsilon R_h} \overline{K},$$

(ii) Each K is a rectangle whose edges are parallel to the coordinate axis,

(iii) If K and $K' \epsilon R_h$, then $K \cap K' = \phi$, and either $\overline{K} \cap \overline{K}' = \phi$, or K and K' have only a whole edge or one vertex in common.

Following [1], [2] and [8] - [10], a *convergent* choice for V_h and Q_h, constructed from the above R_h, is given by:

$$V_h = \{v_h | v_h|_K \epsilon Q_k, \forall K \epsilon R_h\}, \tag{2.2}$$

$$\begin{cases} Q_h = \{\mathbf{q}_h | \mathbf{q}_h = \{q_{ih}\}_{i=1}^2, \mathbf{q}_h|_K \epsilon (P_{k+1} \otimes P_k) \times (P_k \otimes P_{k+1}), \\ \forall K \epsilon R_h; q_{ih} \textit{ is continuous along the edges} \\ \textit{of } R_h \textit{ parallel to } 0x_{i+1}\}; \end{cases} \tag{2.3}$$

in (2.2), (2.3), k is a nonnegative integer, $Q_k = P_k \otimes P_k, P_s$ is the space of the polynomials in one variable of degree $\leq s$, and $i+1$ has to be taken modulo 2. With such a choice for V_h and Q_h , condition (2.1) is clearly satisfied.

2.2 *Discretization of the heat equation* (1.1) – (1.3).

Semi – Discretization in space :

Using the spaces V_h and Q_h we shall "space discretize" (1.1) - (1.3), via (1.8) - (1.10) as follows:

Find a pair $\{u_h(t), \mathbf{p}_h(t)\} \epsilon V_h \times Q_h$, *a.e. on* $(0,T)$, *such that*

$$(2.4) \qquad \int_\Omega \left(\frac{\partial u_h}{\partial t} - \nabla \cdot \mathbf{p}_h - f_h\right) v_h dx = 0, \forall v_h \epsilon V_h, \ a.e. \ on \ (0,T),$$

$$(2.5) \qquad \int_\Omega (\mathbf{p}_h \cdot \mathbf{q}_h + u_h \nabla \cdot \mathbf{q}_h) dx = \int_\Gamma g_h \mathbf{q}_h \cdot \mathbf{n} d\Gamma; \forall \mathbf{q}_h \epsilon Q_h, a.e. \ on \ (0,T),$$

$$(2.6) \qquad u_h(0) = u_{oh}.$$

In (2.4) - (2.6), f_h, g_h and u_{oh} are convenient approximations of f, g and u_o, respectively (we can take, for example, u_{oh} as the L^2-projection of u_o on V_h) .

The above approximation is not practical since we still have to solve an ordinary differential system, or to be more precise a system, coupling ordinary differential equations and (linear) algebraic equations.

Full Discretization in space – time : Concentrating (for simplicity) on the *backward Euler* time discretization of (2.4) - (2.6) we finally obtain the following system of difference - algebraic equations (with $\Delta t(> 0)$ a *time discretization* step):

For $n \geq 0$, *find* $\{u_h^{n+1}, \mathbf{p}_h^{n+1}\} \epsilon V_h \times Q_h$ *such that*

$$(2.7) \qquad u_h^o = u_{oh},$$

$$(2.8) \qquad \int_\Omega \left(\frac{u_h^{n+1} - u_h^n}{\Delta t} - \nabla \cdot \mathbf{p}_h^{n+1} - f_h^{n+1}\right) v_h dx = 0, \forall v_h \epsilon V_h,$$

$$(2.9) \qquad \int_\Omega \left(\mathbf{p}_h^{n+1} \cdot \mathbf{q}_h + u_h^{n+1} \nabla \cdot \mathbf{q}_h\right) dx = \int_\Gamma g_h^{n+1} \mathbf{q}_h \cdot \mathbf{n} d\Gamma, \forall \mathbf{q}_h \epsilon Q_h.$$

From a practical point of view, we can easily eliminate u_h^{n+1} from (2.8), using the fact that $\nabla \cdot \mathbf{q}_h \epsilon V_h$; we obtain then the following *linear variational equation* satisfied by $\mathbf{p}_h^{n+1}$:

$$(2.10) \qquad \begin{cases} \int_\Omega (\Delta t \nabla \cdot \mathbf{p}_h^{n+1} \nabla \cdot \mathbf{q}_h + \mathbf{p}_h^{n+1} \cdot \mathbf{q}_h) dx = \int_\Gamma g_h^{n+1} \mathbf{q}_h \cdot \mathbf{n} d\Gamma \\ - \int_\Omega (u_h^n + \Delta t f_h^{n+1}) \nabla \cdot \mathbf{q}_h dx, \forall \mathbf{q}_h \epsilon Q_h; \mathbf{p}_h^{n+1} \epsilon Q_h. \end{cases}$$

Solving (2.10) can be done by a *direct method* - such as *Cholesky's* since the bilinear form in (2.10) is *symmetric* and *positive definite* - or by a *conjugate gradient algorithm* (see, for example, [11]). Once $\mathbf{p}_h^{n+1}$ is known, computing u_h^{n+1} from (2.8) is straightforward.

Similarly, instead of backward Euler, we could have used schemes such as forward Euler, Crank - Nicholson, multisteps, Runge - Kutta, ...

2.3 *Discretization of the wave equation* (1.4) – (1.6).

Starting from the following variant of (2.4) - (2.6): *Find a pair*

$$\{u_h(t), \mathbf{p}_h(t)\} \epsilon V_h \times Q_h, a.e. on (0,T), \text{ such that}$$

(2.11) $$\int_\Omega \left(\frac{\partial^2 u_h}{\partial t^2} - \nabla \cdot \mathbf{p}_h - f_h\right) v_h dx = 0, \forall v_h \epsilon V_h, \ a.e. \ on (0,T),$$

(2.12) $$\int_\Omega (\mathbf{p}_h \cdot \mathbf{q}_h + u_h \nabla \cdot \mathbf{q}_h) dx = \int_\Gamma g_h \mathbf{q}_h \cdot \mathbf{n} d\Gamma, \forall \mathbf{q}_h \epsilon Q_h, \ a.e. \ on (0,T),$$

(2.13) $$u_h(0) = u_{oh}, \frac{\partial u_h}{\partial t}(0) = u_{1h},$$

we can fully discretize the wave problem (1.4) - (1.6) by the following variant of the usual second order accurate, explicit finite difference discretization scheme of the wave equation:

Assuming that, for $n \geq 0, u_h^n, \mathbf{p}_h^n$ *and* u_h^{n-1} *are known compute first* u_h^{n+1} *as the solution of*

(2.14) $$\int_\Omega \left(\frac{u_h^{n+1} + u_h^{n-1} - 2u_h^n}{|\Delta t|^2} - \nabla \cdot \mathbf{p}_h^n - f_h^n\right) v_h dx = 0, \forall v_h \epsilon V_h; u_h^{n+1} \epsilon V_h,$$

and then $\mathbf{p}_h^{n+1}$ *as the solution of*

(2.15) $$\int_\Omega \mathbf{p}_h^{n+1} \cdot \mathbf{q}_h dx = \int_\Gamma g_h^{n+1} \mathbf{q}_h \cdot \mathbf{n} d\Gamma - \int_\Omega u_h^{n+1} \nabla \cdot \mathbf{q}_h dx, \forall \mathbf{q}_h \epsilon Q_h; \mathbf{p}_h^{n+1} \epsilon Q_h.$$

A most important step is clearly the *initialization* of scheme (2.14), (2.15); assuming that f, g, u_o, u_1 are sufficiently smooth we shall proceed as follows: compute $u_h^o, u_h^{-1}, \mathbf{p}_h^o$ and u_h^1

(2.16) $$u_h^o = u_{oh}, \quad u_h^1 = u_h^{-1} + 2\Delta t u_{1h},$$

$$(2.17) \qquad \begin{cases} \mathbf{p}_h^o \in Q_h, \\ \int_\Omega \mathbf{p}_h^o \cdot \mathbf{q}_h dx = \int_\Gamma g_h^o \mathbf{q}_h \cdot \mathbf{n} d\Gamma - \int_\Omega u_{oh} \nabla \cdot \mathbf{q}_h dx, \forall \mathbf{q}_h \in Q_h. \end{cases}$$

As shown in [12], $u_h(t)$ and $\mathbf{p}_h(t)$ will converge to $u(t)$ and $\nabla u(t)(u$: solution of (1.4) - (1.6)) as h and $\Delta t \to 0$ if a *stability condition* such as

$$(2.18) \qquad \Delta t \leq Ch$$

is satisfied.

Second order, unconditionally stable implicit variants of the above scheme can be obtained; they will discussed in a following paper, together with applications to boundary control of the wave equation.

3. APPLICATION TO AN EXACT CONTROLLABILITY PROBLEM FOR THE WAVE EQUATION, VIA DIRICHLET BOUNDARY CONTROLS.

3.1 *Formulation of the boundary control problem.*

We follow here [4], [5]; we consider then a phenomenon taking place in Ω and modelled by the wave equation (we keep the notation of Section 1):

$$(3.1) \qquad u_{tt} - \Delta u = 0 \text{ in } Q,$$

with the *initial conditions*

$$(3.2) \qquad u(x,0) = u_o(x), u_t(x,o) = u_1(x) \text{ in } \Omega.$$

The problem here is to find g defined over $\sum(= \Gamma \times (0,T))$ such that the following *final conditions*

$$(3.3) \qquad u(x,T) = 0,\ u_t(x,T) = 0 \text{ on } \Omega$$

hold if one has

$$(3.4) \qquad u = g \text{ on} \Sigma$$

as *boundary condition.*

It has been proved by several authors (see [4], [5], [13] for references) that such a g exists provided that T is *sufficiently large* (the lower bound of the T's for which (3.3) holds, $\forall u_o, u_1$, is - not surprisingly - of the order of diameter (Ω)).

3.2 Calculation of an exact Dirichlet control via the HilbertUniquenessMethod of J. L. Lions

In [4], [5], J.L. Lions has introduced and analyzed a systematic way for constructing Dirichlet controls for which (3.3) holds. The construction technique is systematic and based on the *Hilbert Uniqueness Method* (HUM) to be briefly discussed below. From now on, we suppose that

$$u_o \epsilon L^2(\Omega), u_1 \epsilon H^{-1}(\Omega)(= (H_o^1(\Omega)'), \tag{3.5}$$

where

$$H_o^1(\Omega) = \{v | v \epsilon L^2(\Omega), \frac{\partial v}{\partial x_i} \epsilon L^2(\Omega), \forall i = 1, \cdots N, v = 0 \text{ on } \Gamma\},$$

$H^{-1}(\Omega)$ is the *dual space* of $H_o^1(\Omega)$,

and we define E and E' by

$$E = H_o^1(\Omega) \times L^2(\Omega), E' = H^{-1}(\Omega) \times L^2(\Omega). \tag{3.6}$$

Next we define an operator $\Lambda \epsilon L(E, E')$ as follows:

(i) *Take* $\mathbf{e} = \{e_o, e_1\} \epsilon E$;

(ii) *Integrate from* 0 *to* T :

$$\phi_{tt} - \Delta\phi = 0 \; in Q, \tag{3.7$_1$}$$

$$\phi = 0 \; on \sum, \tag{3.7$_2$}$$

$$\phi(x,0) = e_o(x), \phi_t(x,0) = e_1(x) \; on \; \Omega. \tag{3.7$_3$}$$

(iii) *Integrate from* T *to* 0 :

$$\psi_{tt} - \Delta\psi = 0 \; in Q, \tag{3.8$_1$}$$

(3.8)$_2$ $$\psi = \frac{\partial \phi}{\partial n} \text{ on } \sum,$$

(3.8)$_3$ $$\psi(x,T) = 0, \psi_t(x,T) = 0 \text{ on } \Omega.$$

(iv) take

(3.9) $$\Lambda \mathrm{e} = \{\psi_t(0), -\psi(0)\},$$

where $\psi(0)$ *(resp.* $\psi_t(0)$ *) is the function* $x \to \psi(x,0)$ *(resp.* $x \to \psi_t(x,0)$*).*

It follows from J.L. Lions [4], [5] that $\Lambda \epsilon L(E, E'), \forall T > 0$; moreover, if T is sufficiently large ($T >$ diameter (Ω)) then Λ is a *strongly elliptic* operator from E onto E'. In addition to these properties, Λ is *self-adjcint* and satisfies (with obvious notation):

(3.10) $$\langle \Lambda \mathrm{e}, \mathrm{e}' \rangle = \int_{\sum} \frac{\partial \phi}{\partial n} \frac{\partial \phi'}{\partial n} d\Gamma dt, \forall \mathrm{e}, \mathrm{e}' \epsilon E;$$

in (3.10), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E' and E which satisfies

$$\langle \Lambda \mathrm{e}, \mathrm{e}' \rangle = \int_{\Omega} (\Lambda \mathrm{e}) \cdot \mathrm{e}' dx$$

if $\Lambda \mathrm{e}$ is sufficiently smooth.

Application to the exact boundary controllability of the wave equation :

(i) *Solve*

(3.11) $$\Lambda \mathrm{e} = \{u_1, -u_o\}.$$

(ii) *Solve* (3.7), *taking for* e , *in* (3.7)$_3$, *the solution of* (3.11).

(iii) *Take* $g = \frac{\partial \phi}{\partial n}$ *on* $\sum$.

If T is *sufficiently large*, it follows - from the properties of Λ - that (3.11) has a *unique* solution in E ; we have (cf. [4], [5]) $g \epsilon L^2(\sum)$, and the corresponding solution of (3.8) satisfies (3.1) - (3.4), implying that g is a Dirichlet boundary control for which the exact controllability property (3.3) holds. Actually, of all the Dirichlet boundary control for which exact controllability holds, the one obtained by HUM, i.e. by solving (3.11) is the only one of *minimal norm* in $L^2(\sum)$, as shown in [4], [5]. From the properties of Λ , problem (3.11) can be solved by a *conjugate gradient algorithm* operating in space E ; such an algorithm is described in [13], [14], together with conforming finite finite element implementations of it.

3.3 *Mixed formulation of the boundary control problem.*

In fact, we shall describe a mixed formulation of problem (3.11):

Assuming that the initial data u_o and u_1 are sufficiently smooth, so that we can use integral representations, the problem is now to find a triple $\{e_o, \mathbf{p}_o, e_1\}$ satisfying

$$(3.12)\qquad \begin{cases} \{e_o, \mathbf{p}_o\}\epsilon W_o, e_1 \epsilon L^2(\Omega); \forall \{v_o, \pi_o\}\epsilon W_o, v_1 \epsilon L^2(\Omega) \text{ we have} \\ \int_\Omega (\psi_t(0)v_o - \psi(0)v_1)dx = \int_\Omega (u_1 v_o - u_o v_1)dx, \end{cases}$$

where in (3.12):

(i) The space W_o is defined by

$$(3.13)\qquad \begin{cases} W_o = \{\{v_o, \pi_o\} | v_o \epsilon L^2(\Omega), \pi_o \epsilon (L^2(\Omega))^N . \int_\Omega (\pi_o \cdot \mathbf{q} + v_o \nabla \cdot \mathbf{q})dx = 0, \\ \forall \mathbf{q} \epsilon H(\Omega, div)\}; \end{cases}$$

it can be shown that

$$\{v_o, \pi_o\}\epsilon W_o \leftrightarrow v_o \epsilon H_o^1(\Omega), \pi_o = \nabla v_o.$$

(ii) $\psi(0)$ and $\psi_t(0)$ are obtained from $e_o, \mathbf{p}_o, e_1$ as follows:

Integrate from 0 to T the *mixed formulated* following wave equation (cf. Section 2):

$$(3.14)_1 \qquad \int_\Omega (\phi_{tt} - \nabla \cdot \mathbf{p})v dx = 0, \forall v \epsilon L^2(\Omega),\ a.e.\ on (0,T),$$

$$(3.14)_2 \qquad \int_\Omega (\mathbf{p} \cdot \mathbf{z} + \phi \nabla \cdot \mathbf{z})dx = 0, \forall \mathbf{z} \epsilon H(\Omega, div),\ a.e.\ on\ (0,T),$$

$$(3.14)_3 \qquad \phi(x,0) = e_o(x), \phi_t(x,0) = e_1(x)\ on\ \Omega;$$

then from T to 0 (using the fact that $\frac{\partial \phi}{\partial n} = \mathbf{p} \cdot \mathbf{n}$ *on* $\sum$):

$$(3.15)_1 \int_\Omega (\psi_{tt} - \nabla \cdot \mathbf{q})v dx = 0, \forall v \epsilon L^2(\Omega),\ a.e.\ on (0,T),$$

$$(3.15)_2 \int_\Omega (\mathbf{q} \cdot \mathbf{z} + \psi \nabla \cdot \mathbf{z})dx = \int_\Gamma \mathbf{p} \cdot \mathbf{n}\ \mathbf{z} \cdot \mathbf{n} d\Gamma, \forall \mathbf{z} \epsilon H(\Omega,\ div),\ a.e.\ on (0,T),$$

$$(3.15)_3\, \psi(x,T) = 0, \psi_t(x,T) = 0\ on\ \Omega.$$

An easy calculation will show that (with obvious notation):

(3.16)
$$\begin{cases} \int_\Omega (\psi_t(0)e'_o - \psi(0)e'_1)dx = \int_{\Sigma} \mathbf{p}\cdot\mathbf{n}\mathbf{p}'\cdot\mathbf{n}d\Gamma dt, \\ \forall\{e_o, \pi_o; e_1\}, \{e'_o, \pi'_o; e'_1\} \epsilon W_o \times L^2(\Omega). \end{cases}$$

From (3.16) it appears that the bilinear form occuring in (3.12) is *symmetric* and *positive semi definite* ; actually, for T sufficiently large it is *strongly elliptic* (coercive) over $(W_o \times L^2(\Omega))^2$. From these properties, problem (3.12) can be solved by a *conjugate gradient algorithm* operating in $W_\sigma \times L^2(\Omega)$; such an algorithm is described in Section 3.4.

3.4 *Conjugate gradient solution of problem (3.12).*

3.4.1. *Generalities.*

Problem (3.12) is a particular case of

(3.17)
$$\textit{Find } u\epsilon V \textit{ such that} a(u,v) = L(v), \forall v \epsilon V,$$

where in (3.17):

(i) V is an *Hilbert space*, equipped with the scalar product $(\cdot,\cdot)$, and the corresponding norm $||\cdot||$.

(ii) $a : V \times V \to \mathbb{R}$ is bilinear, continuous and V- elliptic (i.e. $\exists\alpha > 0$ such that $a(v,v) \geq \alpha||v||^2, \forall v\epsilon V$).

(iii) $L : V \to \mathbb{R}$ is linear and continuous.

It is well known (cf., e.g., [15, Appendix 1]) that under the above hypotheses, problem (3.17) has a unique solution. If in addition to (i) - (iii), the bilinear form $a(\cdot,\cdot)$ is *symmetric* then problem (3.17) is equivalent to the following *minimization* one

(3.18)
$$\begin{cases} u\epsilon V, \\ J(u) \leq J(v), \forall v\epsilon V, \end{cases}$$

with $J(v) = \frac{1}{2}a(v,u) - L(v)$. Problem (3.17), (3.18) can then be solved by the following *conjugate gradient algorithm*:

Initialization

(3.19) $$u^o \epsilon V \text{ is given.}$$

Solve then

(3.20) $$\begin{cases} g^o \epsilon V, \\ (g^o, v) = a(u^o, v) - L(v), \forall v \epsilon V. \end{cases}$$

If $g^o = 0$, *or is "small", take* $u = u^o$ *; if not, set*

(3.21) $$w^o = g^o.\square$$

Now for $n \geq 0$ *, suppose that* u^n, g^n, w^n, *are known with* $w^n \neq 0$ *; define then* $u^{n+1}, g^{n+1}, w^{n+1}$ *as follows:*

Descent : *Compute*

(3.22) $$\rho_n = ||g^n||^2 / a(w^n, w^n),$$

and

(3.23) $$u^{n+1} = u^n - \rho_n w^n.$$

Test of the convergence and updating the descent direction : *Solve*

(3.24) $$\begin{cases} g^{n+1} \epsilon V, \\ (g^{n+1}, v) = (g^n, v) - \rho_n a(w^n, v), \forall v \epsilon V. \end{cases}$$

If $g^{n+1} = 0$ *- or is small-take* $u = u^{n+1}$ *; if not compute*

(3.25) $$\gamma_n = ||g^{n+1}||^2 / ||g^n||^2,$$

and update w^n *by*

(3.26) $$w^{n+1} = g^{n+1} + \gamma_n w^n.\square$$

Do $n = n + 1$ *and go to* (3.22).

The above algorithm converges, $\forall u^o \epsilon V$, and we have (cf. [16]):

$$\|u^n - u\| \leq C\|u^o - u\|\left(\frac{\sqrt{\nu_a}-1}{\sqrt{\nu_a}+1}\right)^n, \tag{3.27}$$

where C is a constant, and where the *condition number* ν_a is given by

$$\nu_a = \sup_{v \epsilon S} a(v,v) / \inf_{v \epsilon S} a(v,v), \tag{3.28}$$

with $S = \{v | v \epsilon V, \|v\| = 1\}$.

3.4.2 *Application to the solution of problem(3.12)*

Since problem (3.12) is a particular problem (3.17), with $V = W_o \times L^2(\Omega)$, it can be solved by the conjugate gradient algorithm (3.19) - (3.26). An important practical issue is the proper choice of the scalar product to be used over $W_o \times L^2(\Omega)$. A fairly convenient one is provided by

$$\begin{cases} \int_\Omega (v_o v_o' + \pi_o \cdot \pi_o' + v_1 v_1')dx, \\ \forall \{v_o, \pi_o; v_1\}, \{v_o', \pi_o'; v_1'\} \epsilon W_o \times L^2(\Omega). \end{cases} \tag{3.29}$$

Applying algorithm (3.19) - (3.26) to the solution of problem (3.12), with $W_o \times L^2(\Omega)$ equipped with the scalar product (3.29), we obtain the following algorithm:

Initialization :

$$\{e_o^o, \mathrm{p}_o^o\} \epsilon W_o, e_1^o \epsilon L^2(\Omega) \ \textit{are given.} \tag{3.30}$$

Integrate then from 0 *to* T *the wave equation*

$$\int_\Omega (\phi_{tt}^o - \nabla \cdot \mathrm{p}^o) v dx = 0, \forall v \epsilon L^2(\Omega), \ a.e. \ on \ (0,T), \tag{3.31$_1$}$$

$$\int_\Omega (\mathrm{p}^o \cdot \mathrm{z} + \phi^o \nabla \cdot \mathrm{z}) dx = 0, \forall \mathrm{z} \epsilon H(\Omega, div), \ a.e. \ on \ (0,T), \tag{3.31$_2$}$$

$$\phi^o(0) = e_o^o, \phi_t^o(0) = e_1^o. \tag{3.31$_3$}$$

Then from T to 0 :

(3.32)$_1$ $$\int_\Omega (\psi^o_{tt} - \nabla \cdot \mathbf{q}^o) v dx = 0, \forall v \epsilon L^2(\Omega), \; a.e. \; on \; (0,T),$$

3.32)$_2$ $$\int_\Omega (\mathbf{q}^o \cdot \mathbf{z} + \psi^o \nabla \cdot \mathbf{z}) dx = \int_\Gamma \mathbf{p}^o \cdot \mathbf{n} \, \mathbf{z} \cdot \mathbf{n} d\Gamma, \forall \mathbf{z} \epsilon H(\Omega, div),$$

a.e. on $(0,T)$,

(3.32)$_3$ $$\psi^o(T) = 0, \psi^o_t(T) = 0.$$

Compute then $\{g^o_o, \pi g^o_o\}$ *and* g^o_1 *as follows:*

Solve the mixed elliptic problem:

Find $\{g^o_o, \pi g^o_o\} \epsilon W_o$ *such that*

(3.33)$_1$ $$\int_\Omega (g^o_o - \nabla \cdot \pi g^o_o) v dx = \int_\Omega (\psi^o_t(0) - u_1) v dx, \forall v \epsilon L^2(\Omega),$$

(3.33)$_2$ $$\int_\Omega (\pi g^o_o \cdot \mathbf{q} + g^o_o \nabla \cdot \mathbf{q}) dx = 0, \forall \mathbf{q} \epsilon H(\Omega, div),$$

and then

(3.34) $$g^o_1 = u_o - \psi^o(0).$$

If $\{g^o_o, \pi g^o_o\} = \{0,0\}, g^o_1 = 0$ *, or are small, take* $\mathbf{p}^o \cdot \mathbf{n}|_\Sigma$ *as boundary control; if not, set*

(3.35) $$\{w^o_o, \pi w^o_o; w^o_1\} = \{g^o_o, \pi g^o_o; g^o_1\}. \square$$

Then for $n \geq 0$ *, assuming that* $\{e^n_o, \mathbf{p}^n_o\}, e^o_1, \phi^n, \psi^n, \{g^n_o, \pi g^n_o\}, g^n_1, \{w^n_o, \pi w^n_o\}, w^n_1$ *are known, we compute* $\{e^{n+1}_o, \mathbf{p}^{n+1}_o\}, e^{n+1}_1, \phi^{n+1}, \psi^{n+1}, \{g^{n+1}_o, \pi g^{n+1}_o, \},$

$g^{n+1}_1, \{w^{n+1}_o, \pi w^{n+1}_o\}, w^{n+1}_1,$ *as follows :*

<u>*Descent :*</u>

Integrate from 0 to T

(3.36)$_1$ $$\int_\Omega (\overline{\varphi}^n_{tt} - \nabla \cdot \overline{\mathbf{p}}^n) v dx = 0, \forall v \epsilon L^2(\Omega), a.e. \; on \; (0,T),$$

(3.36)$_2$ $$\int_\Omega (\overline{\mathbf{p}}^n \cdot \mathbf{z} + \overline{\phi}^n \nabla \cdot \mathbf{z}) dx = 0, \forall \mathbf{z} \epsilon H(\Omega, div), \; a.e. \; on \; (0,T),$$

(3.36)$_3$ $$\overline{\phi}^n(0) = w^n_o, \overline{\phi}^n_t(0) = w^n_1.$$

Then from T to 0 :

$$(3.37)_1 \qquad \int_\Omega (\overline{\psi}^n_{tt} - \nabla \cdot \overline{\mathrm{q}}^n) v dx = 0, \forall v \epsilon L^2(\Omega), \quad a.e. \text{ on } (0,T),$$

$$(3.37)_2 \qquad \int_\Omega (\overline{\mathrm{q}}^n \cdot \mathbf{z} + \overline{\psi}^n \nabla \cdot \mathbf{z}) dx = \int_\Gamma \overline{\mathrm{p}}^n \cdot \mathrm{n} \mathbf{z} \cdot \mathrm{n} d\Gamma, \forall \mathbf{z} \epsilon H(\Omega, div), \quad a.e. \text{ on } (0,T),$$

$$(3.37)_3 \qquad \overline{\psi}^n(T) = 0, \overline{\psi}^n_t(T) = 0$$

Solve now the mixed elliptic problem :

Find $\{\overline{g}^n_o, \pi\overline{g}^n_o\} \epsilon W_o$ *such that*

$$(3.38)_1 \qquad \int_\Omega (\overline{g}^n_o - \nabla \cdot \pi\overline{g}^n_o) v dx = \int_\Omega \overline{\psi}^n_t(0) v dx, \forall v \epsilon L^2(\Omega),$$

$$(3.38)_2 \qquad \int_\Omega (\pi\overline{g}^n_o \cdot \mathrm{q} + \overline{g}^n_o \nabla \cdot \mathrm{q}) dx = 0, \forall \mathrm{q} \epsilon H(\Omega, div),$$

and set

$$(3.39) \qquad \overline{g}^n_1 = -\overline{\psi}^n(0).$$

Compute now

$$(3.40) \qquad \begin{cases} \rho_n = \dfrac{\int_\Omega (|g^n_o|^2 + |\pi g^n_o|^2 + |g^n_1|^2) dx}{\int_\Omega (w^n_o \overline{\psi}^n_t(0) - w^n_1 \overline{\psi}^n(0)) dx} \\ \quad = \dfrac{\int_\Omega (|g^n_o|^2 + |\pi g^n_o|^2 + g^n_1|^2) dx}{\int_\Omega (\overline{g}^n_o w^n_o + \pi\overline{g}^n_o \cdot \pi w^n_o + \overline{g}^n_1 w^n_1) dx} \end{cases}$$

and then

$$(3.41) \qquad \{e^{n+1}_o, \mathrm{p}^{n+1}_o, e^{n+1}_1\} = \{e^n_o, \mathrm{p}^n_o, e^n_1\} - \rho_n \{w^n_o, \pi w^n_o, w^n_1\},$$

$$(3.42) \qquad \{\phi^{n+1}, \mathrm{p}^{n+1}\} = \{\phi^n, \mathrm{p}^n\} - \rho_n \{\overline{\phi}^n, \overline{\mathrm{p}}^n\},$$

$$(3.43) \qquad \{\psi^{n+1}, \mathrm{q}^{n+1}\} = \{\psi^n, \mathrm{q}^n\} - \rho_n \{\overline{\psi}^n, \overline{\mathrm{q}}^n\},$$

$$(3.44) \qquad \{g^{n+1}_o, \pi g^{n+1}_o, g^{n+1}_1\} = \{g^n_o, \pi g^n_o, g^n_1,\} - \rho_n \{\overline{g}^n_o, \pi\overline{g}^n_o, \overline{g}^n_1,\}.$$

Test of the convergence. New descent Direction :

If $\{g_o^{n+1}, \pi g_o^{n+1}, g_1^{n+1}\} = \{0,0,0\}$ *- or is small - take* $\mathbf{p}^{n+1}\cdot\mathbf{n}|_{\Sigma}$ *as boundary control; if not compute*

$$\gamma_n = \frac{\int_\Omega(|g_o^{n+1}|^2 + |\pi g_o^{n+1}|^2 + |g_1^{n+1}|^2)dx}{\int_\Omega(|g_o^n|^2 + |\pi g_o^n|^2 + |g_1^n|^2)dx} \tag{3.45}$$

and then

$$\{w_o^{n+1}, \pi w_o^{n+1}, w_1^{n+1}\} = \{g_o^{n+1}, \pi g_o^{n+1}, g_1^{n+1}\} + \gamma_n\{w_o^n, \pi w_o^n, w_1^n\}. \tag{3.46}$$

Do $n = n + 1$ and go to (3.36).

Remark 3.1 : Problems (3.33) and (3.38) are particular cases of

$$\int_\Omega (u - \nabla\cdot\mathbf{p})v\,dx = \int_\Omega fv\,dx, \forall v \epsilon L^2(\Omega), \tag{3.47$_1$}$$

$$\int_\Omega (\mathbf{p}\cdot\mathbf{q} + u\nabla\cdot\mathbf{q})dx = 0, \forall \mathbf{q}\epsilon H(\Omega, div), \tag{3.47$_2$}$$

which is the mixed formulation of the following *Dirichlet problem*

$$-\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma. \tag{3.48}$$

Observing that $\nabla\cdot\mathbf{q}\epsilon L^2(\Omega), \forall\mathbf{q}\epsilon H(\Omega, div)$, we can eliminate u from $(3.47)_1, (3.47)_2$ to obtain that $\mathbf{p}$ satisfies (if $f\epsilon L^2(\Omega)$) :

$$\begin{cases} \mathbf{p}\epsilon H(\Omega, div), \\ \int_\Omega(\nabla\cdot\mathbf{p}\nabla\cdot\mathbf{q} + \mathbf{p}\cdot\mathbf{q})dx = -\int_\Omega f\nabla\cdot\mathbf{q}\,dx, \forall\mathbf{q}\epsilon H(\Omega, div). \end{cases} \tag{3.49}$$

Solving (3.49) (in fact its discrete variants) is fairly easy and can be done by conjugate gradient algorithms (see, e.g., [9] for details). Once $\mathbf{p}$ is known, one obtains easily u from $(3.47)_1$. Combining the above algorithm with the mixed finite element approximations and time discretization schemes of the wave equation discussed in Section 2 is (almost) straightforward; this issue will be discussed in a forthcoming paper.

3.4.3. *Numerical experiments.*

The mixed finite element approximation and time discretization schemes of the wave equation, described in Section 2, have been combined to algorithm (3.30) - (3.46), to solve problem (3.11) when $\Omega = (0,1) \times (0,1)$ and $T = 2\sqrt{2}$. Using the Fourier series techniques described in [13] we have computed those initial data u_o and u_1 for which the solution $\mathbf{e}(= \{e_o, e_1\})$ of (3.11) is given by

$$e_o(x_1, x_2) = \sin \pi x_1 \sin \pi x_2, \; e_1 = \pi\sqrt{2} e_o. \tag{3.50}$$

We have used the mixed finite element approximations of Section 2, with $k = 1$ and R_h the regular partition of Ω associated to the vertices $\{ih, jh\}$ with $0 \leq i, j \leq N$, N being an integer such that $Nh = 1$; we have taken $N = 16, 32, 64$. The time discretization of the various wave equations involved in the calculations was obtained using the (conditionally stable) explicit scheme described in Section 2. Obtaining the (approximate) values of the control $\frac{\partial \phi}{\partial n} = \mathbf{p} \cdot \mathbf{n}$ on $\sum$, was quite easy since the values of the fluxes (i.e. of the normal components of $\mathbf{p}_h$), at the element interfaces and at the boundary Γ , are the natural *degrees of freedom* for the functions belonging to the finite dimensional space Q_h approximating $H(\Omega, div)$.

For $h = 1/16$(resp.$1/32, 1/64$) the finite dimensional variant of algorithm (3.30) - (3.46) converges in 48 (resp. 72, 119) iterations (the number of iterations varies - approximately - like $\sqrt{N}$). These numbers are much higher than those obtained in [13], where the space approximation was achieved by a conforming finite element method, coupled to a biharmonic Tychonoff regularization to eliminate spurious oscillations. On the other hand, using, as in the present paper, mixed finite element approximations, it is not necessary to use regularization to obtain very good numerical results, as shown in Figures 3.1 (a), (b), (c) (N=6), 3.2 (a), (b), (c) (N=32), 3.3(a), (b), (c) (N=64).

Figures (a) (resp. (b)) show the variation of the exact (-) and computed $(\cdots) e_o$ (resp. e_1), for $0 \leq x_1 \leq 1, x_2 = \cdot 5$. Figures (c) show the variation on $(0, T)$ of the

$L^2(\Gamma)-$ norm of the exact and approximate boundary controls.

All the above calculations have been done on a CRAY X-MP supercomputer.

4. CONCLUSION.

In this paper we have discussed the application of *mixed finite element methods* to the numerical solution of direct or inverse problems for time dependent equations. These mixed methods are robust and accurate. They are however more complicated to implement than the traditional finite element methods. Indeed many important issues remain concerning the practical use of the mixed methods considered here, such as speeding up calculations by multigrid and/or domain decomposition methods (cf. [10]); we intend to investigate them in the near future.

ACKNOWLEDGEMENT:

The authors would like to thank C. Lawson for helpful comments and suggestions, J. Cassidy for her processing of the manuscript of this paper, Caltech and the Fairchild Foundation since this paper was written when the second author was visiting Caltech as a Fairchild Scholar. The support of CRAY Research is also acknowledged.

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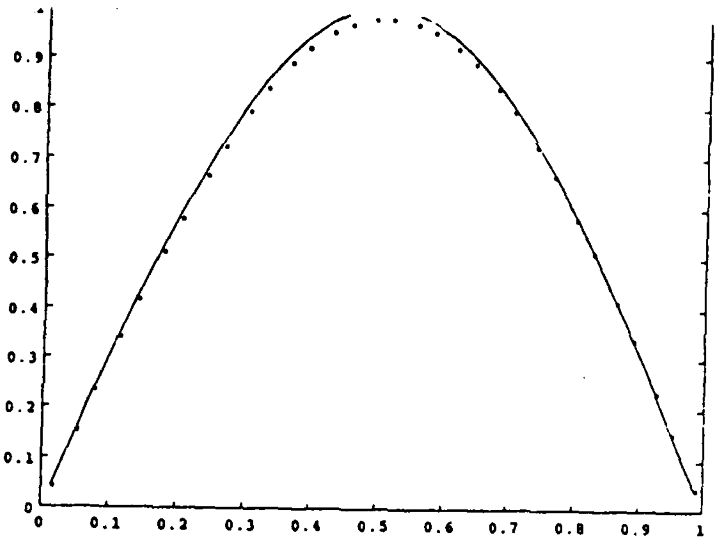

FIGURE 3.1(a)

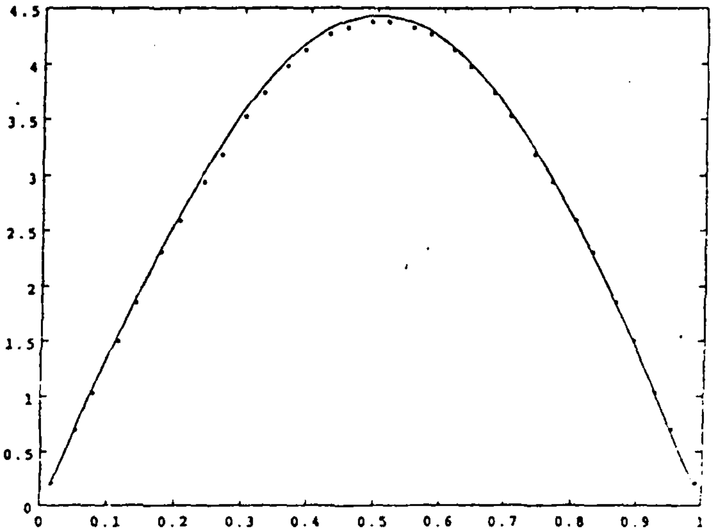

FIGURE 3.1(b)

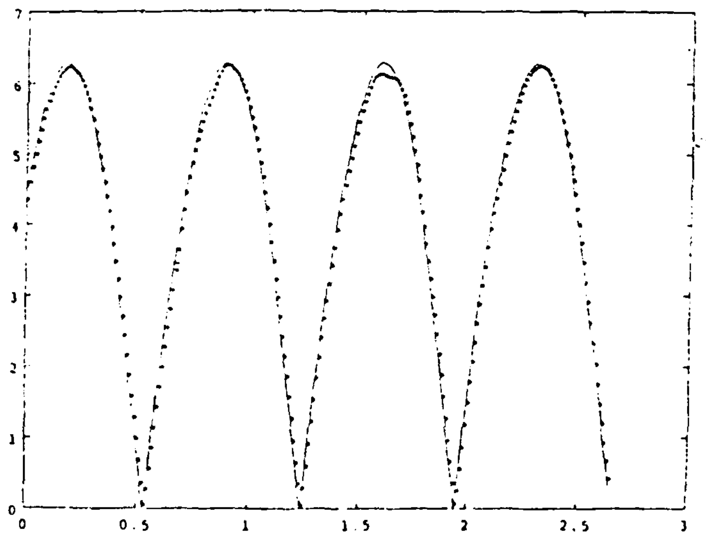

FIGURE 3.1(c)

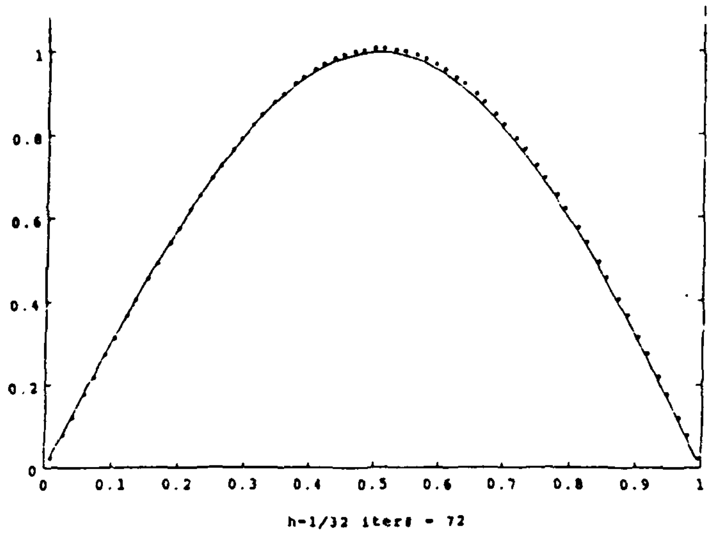

FIGURE 3.2 (a)

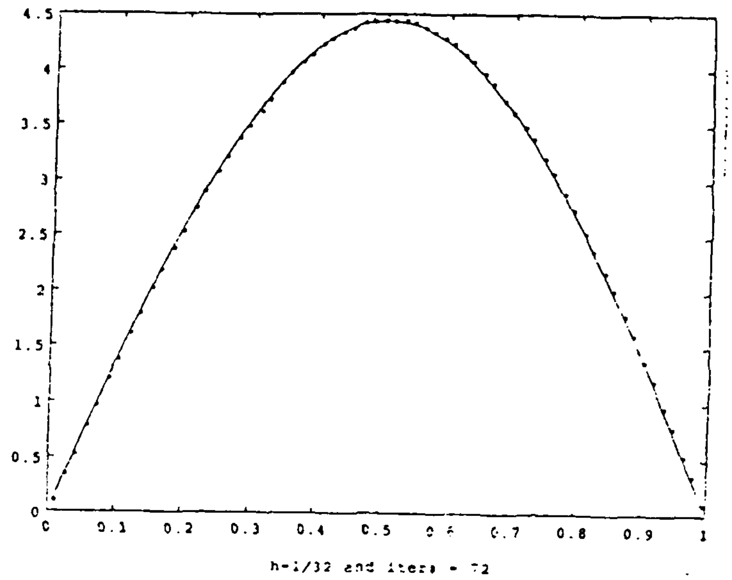

FIGURE 3.2 (b)

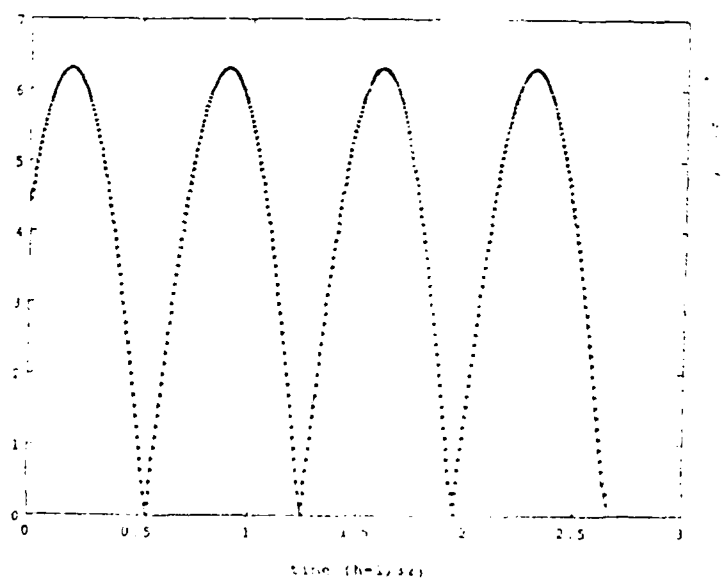

FIGURE 3.2(c)

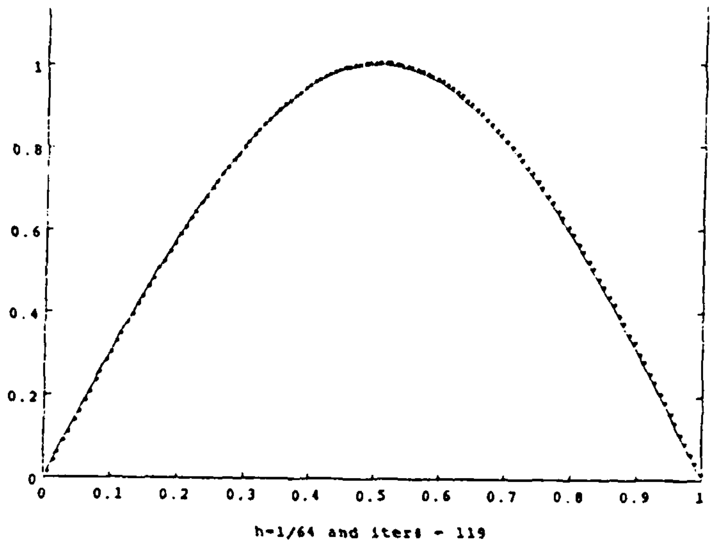

FIGURE 3.3 (a)

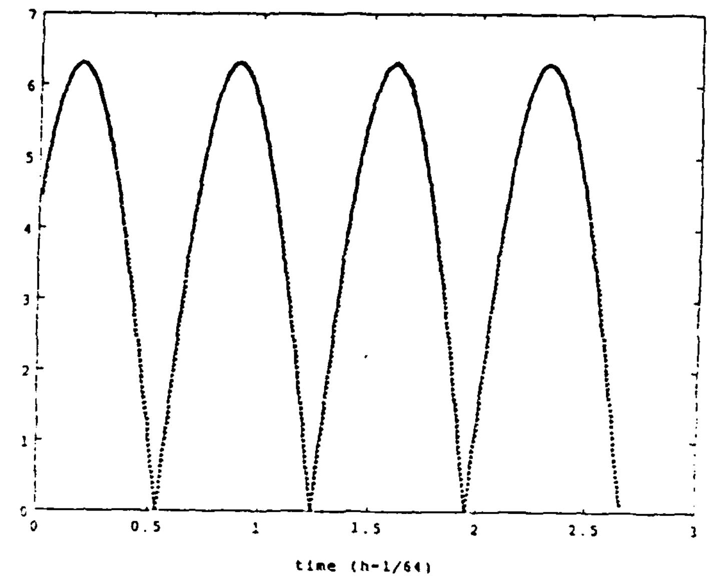

FIGURE 3.3 (b)

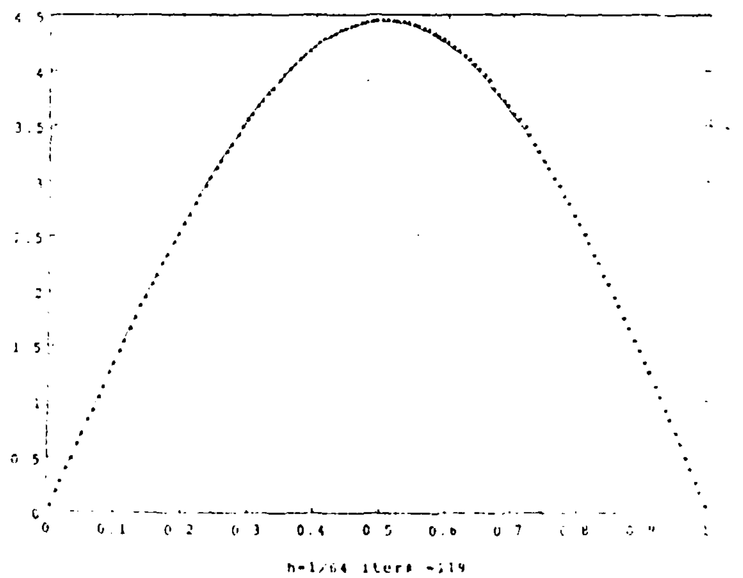

FIGURE 3.3(c)