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A mixed finite element method for beam and frame problems

R. L. Taylor, F. C. Filippou, A. Saritas, F. Auricchio

Abstract In this work we consider solutions for the Euler-Bernoulli and Timoshenko theories of beams in which material behavior may be elastic or inelastic. The formulation relies on the integration of the local constitutive equation over the beam cross section to develop the relations for beam resultants. For this case we include axial, bending and shear effects. This permits consideration in a direct manner of elastic and inelastic behavior with or without shear deformation.

A finite element solution method is presented from a three-field variational form based on an extension of the Hu–Washizu principle to permit inelastic material behavior. The approximation for beams uses equilibrium satisfying axial force and bending moments in each element combined with discontinuous strain approximations. Shear forces are computed as derivative of bending moment and, thus, also satisfy equilibrium. For quasi-static applications no interpolation is needed for the displacement fields, these are merely expressed in terms of nodal values. The development results in a straight forward, variationally consistent formulation which shares all the properties of so-called flexibility methods. Moreover, the approach leads to a shear deformable formulation which is free of locking effects – identical to the behavior of flexibility based elements.

The advantages of the approach are illustrated with a few numerical examples.

Keywords Inelastic beam, Finite elements, Mixed method, Shear deformation

1 Introduction

The development of computational models for beam bending problems dates from the earliest days of structural analysis and the literature is too extensive to fully cite here. Mike Crisfield considered solution of beam problems from many perspectives as indicated in approaches contained in his books [1–3] and papers with co-workers [4–8]. Most of his work was for finite displacement applications – many using co-rotational formulations for which he was well known. Here we would like to remember him for his pioneering work in this field of endeavor.

In a displacement formulation, the nonlinear strain-displacement relations are postulated and polynomial interpolation functions are used for the displacement approximation [9–11]. Because the postulated displacement interpolation functions are approximate in nonlinear material and geometric behavior, each structural member needs to be discretized into several elements in order to capture the actual variation of deformations along its axis. This fine discretization results in a large number of degrees of freedom in the final numerical model of the structure, thus, reducing the computational efficiency of this approach. Alternatively, higher order displacement interpolation functions can be used. This approach results in several internal degrees of freedom that need to be condensed out during the element state determination. Even so, mesh discretization is often required for accuracy. With both of these approaches numerical instabilities are not uncommon, particularly under cyclic loading conditions. Problems with the displacement formulation of beam elements encouraged researchers to seek a solution with force interpolation functions. One of the earlier studies in the field of structural analysis is by Menegotto and Pinto [12] who interpolated both section deformations and section flexibilities. Backlund [13] proposed a hybrid beam element for the analysis of elasto–plastic plane frames with large displacements. In this study the flexibility matrix is determined from an assumed distribution of forces along the element. However, this method also uses displacement interpolation functions corresponding to linear curvature and constant axial strain distribution for the determination of section deformations from end displacements. Large displacement effects are taken into account by updating the element geometry. The paper does not provide details on the numerical implementation of the element, which is critical to the approach. A later study by Mahasuverachai and Powell [14] proposed flexibility-dependent shape functions that are continuously updated during the analysis. This study is followed by the flexibility-based element of Kaba and Mahin [15] and its later improvement by Zeris and Mahin [16]. However, the latter studies lack a consistent framework for the

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Dedicated to the memory of Prof. Mike Crisfield, for his cheerfulness and cooperation as a colleague and friend over many years.

formulation and, thus, suffer from limitations and numerical problems. The first study to provide a consistent formulation for a force-based element and its numerical implementation in a general purpose computer program is the work of Ciampi and Carlesimo [17]. An independent attempt in the same direction is reported by Carol and Murcia [18] who proposed a hybrid frame element for nonlinear material and second-order plane frame analysis. Second-order effects are accounted for, but the use of a linear strain-displacement relation limits the formulation to relatively small deformations. At about the same time Kondoh and Atluri [19] used an assumed-stress approach to derive the tangent stiffness of a plane frame element under general loading. The element is assumed to undergo arbitrarily large rigid rotations but small axial stretch and relative (non-rigid) point-wise rotations. They show that the tangent stiffness can be derived explicitly if a plastic-hinge method is used. Shi and Atluri [20] extended these ideas to three-dimensional frames. Details of the force-formulation of Ciampi are expounded and refined in several studies published subsequently [21–27]. With the work of Ayoub and Filippou [28, 29] attempts are undertaken to generalize the formulation to mixed methods with independent interpolation of force and displacement variables for applications with displacement dependent equilibrium equations. Finally, the study by Souza [30] formulates a force-based element under nonlinear geometry and nonlinear material response on the basis of the variational framework of the Hellinger–Reissner principle. An independent attempt in this direction is reported by Hjelmstad and Taciroglu [31] and for steel-concrete composite beams by Limkatanyu and Spacone [32]. These attempts, however, lack full variational consistency.

The purpose of this study is to formulate a beam element within the generalized and variationally consistent framework of the Hu–Washizu principle.

2 Formulation with section integration

We present here a beam formulation in which integration of local constitutive equations is carried out on each cross section. We start at the local level where we assume that displacements vary linearly over each cross section. Accordingly, for the Euler–Bernoulli theory in two dimensions we have

$$\begin{aligned} u_1(x,y) &= u(x) - y w_{,x} \\ u_2(x,y) &= w(x) \end{aligned} \tag{1}$$

where $(\cdot)_{,x}$ denotes differentiation with respect to x. This displacement field results in the axial strain expression

$$\epsilon_1(x,y) = u_{,x} - y w_{,xx} = \epsilon(x) - y\chi(x) \tag{2}$$

with all other strains being zero.

If we consider the effects of only the stress σ_1 we can identify the axial force resultant as

$$N(\epsilon_1) = \int_A \sigma_1(\epsilon) \mathrm{d}A \tag{3}$$

and the bending moment resultant as

$$M(\epsilon_1) = -\int_A y\sigma_1(\epsilon)\mathrm{d}A \ . \tag{4}$$

Equilibrium of the beam requires

$$\frac{\partial N}{\partial x} + b_x = 0 \tag{5}$$

in the axial direction and

$$\frac{\partial^2 M}{\partial x^2} + b_y = 0 \tag{6}$$

in the transverse direction. Here b_x and b_y are loadings per unit length of beam in the x and y directions, respectively.

2.1 Constitutive behavior

For linear elastic behavior the stress is deduced from

$$\sigma_1 = E\epsilon_1 \ . \tag{7}$$

If the beam axis is placed at the centroid where

$$\int_A \mathrm{d}A = A; \quad \int_A y\,\mathrm{d}A = 0; \quad \int_A y^2\,\mathrm{d}A = I \tag{8}$$

we obtain from (3) and (4) the expressions for resultants as

$$\begin{aligned} N(\epsilon_1) &= EAu_{,x} = EA\epsilon(x) \\ M(\epsilon_1) &= EIw_{,xx} = EI\chi(x) \ . \end{aligned} \tag{9}$$

In the sequel we shall also consider an inelastic form in which an elasto–plastic model is given by

$$\sigma_1 = E(\epsilon_1 - \epsilon^p), \quad \dot{\epsilon}^p = \dot{\gamma} f_{,\sigma_1} \quad \text{and} \quad f = f(\sigma_1, H) \le 0$$

in which ϵ^p is the plastic strain, γ is the consistency parameter and f is a yield function in terms of stress and hardening parameter H. Equation (10) may be integrated in time using a backward Euler scheme to obtain an incremental form for use in numerical calculations. The solution may be obtained locally using a return-map algorithm [33, 34]. The solution for the stress is then inserted into the resultant equations to compute the internal force resultants. Linearization can provide a means of computing the elasto-plastic modulus which can also be used to compute section tangent stiffness properties.

The force resultants on each cross-section are computed from Eqs. (3) and (4). Performing a linearization on the resultant equations gives

$$\begin{aligned} \mathrm{d}N(\epsilon_1) &= \int_A \mathrm{d}\sigma_1(\epsilon)\mathrm{d}A \\ \mathrm{d}M(\epsilon_1) &= -\int_A y\,\mathrm{d}\sigma_1(\epsilon)\mathrm{d}A \ . \end{aligned} \tag{11}$$

The linearization of the stress involves both the axial strain and change in curvature as

$$\mathrm{d}\sigma_1 = E_\mathrm{T}[\mathrm{d}\epsilon(x) - y\,\mathrm{d}\chi(x)] \tag{12}$$

where E_T is the tangent modulus from the constitutive equation. Insertion of (12) into (11) then gives

$$\begin{bmatrix} \mathrm{d}N \\ \mathrm{d}M \end{bmatrix} = \left(\int_A \begin{bmatrix} 1 \\ -y \end{bmatrix} E_\mathrm{T} [1 \quad -y] \mathrm{d}A \right) \begin{bmatrix} \mathrm{d}u_{,x} \\ \mathrm{d}w_{,xx} \end{bmatrix} \tag{13}$$

Note that the behavior may become coupled when both axial and bending deformations occur at a cross-section and E_T is variable.

2.2
Three field variational formulation

The approach we now present is based on the use of a three-field (displacement, strain, stress) formulation based on the Hu–Washizu variational principle. For an elastic material with stress σ_1 and strain ϵ_1 the Hu–Washizu principle may be written as

$$\Pi_{\mathrm{hw}}(\sigma_1, \epsilon_1, \mathbf{u}) = \int_\Omega W(\epsilon_1) \mathrm{d}\Omega + \int_\Omega \sigma_1 \left[\frac{\partial u_1}{\partial x_1} - \epsilon_1 \right] \mathrm{d}\Omega - \Pi_{\mathrm{ext}} \tag{14}$$

In (14) $W(\epsilon_1)$ is the stored energy function from which stresses are computed as

$$\sigma_1 = \frac{\partial W}{\partial \epsilon_1} \tag{15}$$

and Π_{ext} is the potential for the body and boundary loading.

Setting the variation of Eq. (14) to zero yields

$$\delta\Pi_{\mathrm{hw}} = \int_\Omega \delta\epsilon_1 \left[\frac{\partial W}{\partial \epsilon_1} - \sigma_1 \right] \mathrm{d}\Omega + \int_\Omega \delta\sigma_1 \left[\frac{\partial u_1}{\partial x_1} - \epsilon_1 \right] \mathrm{d}\Omega + \int_\Omega \frac{\partial \delta u_1}{\partial x_1} \sigma_1 \, \mathrm{d}\Omega - \delta\Pi_{\mathrm{ext}} = 0 \tag{16}$$

When the term involving a derivative on δu_1 is integrated by parts and combined with the boundary terms the above functional includes all equations for solution of static problems in one-dimensional elasticity.

To permit solution of inelastic constitutive forms, we replace the term involving the variation of the stored energy by

$$\int_\Omega \delta\epsilon_1 \frac{\partial W}{\partial \epsilon_1} \mathrm{d}\Omega \Rightarrow \int_\Omega \delta\epsilon_1 \hat{\sigma}_1(\epsilon_1) \mathrm{d}\Omega \tag{17}$$

in which $\hat{\sigma}_1(\epsilon_1)$ denotes a stress computed from any constitutive model in terms of specified strains, strain rates (plasticity), or functional of strain (viscoelasticity). In this form we can directly introduce the beam approximations for all of the field variables, thus affording a very general variational formulation basis.

2.2.1
Beam formulation

Let us now apply the Hu–Washizu functional to the solution of beam problems. We first write Eq. (16) with the aid of (17) for the beam approximations. To accomplish this we assume that the displacements and strain over the cross section are given by (1) and (2) to obtain

$$\begin{aligned} \delta\Pi_{\mathrm{hw}} = & \int_L \left\{ \delta\epsilon \left(\int_A [\hat{\sigma}_1(\epsilon,\chi) - \sigma_1] \mathrm{d}A \right) - \delta\chi \left(\int_A y[\hat{\sigma}_1(\epsilon,\chi) - \sigma_1] \mathrm{d}A \right) \right\} \mathrm{d}x \\ & + \int_L \left\{ \int_A \delta\sigma_1 \mathrm{d}A \left[u_{,x} - \epsilon \right] - \int_A y \delta\sigma_1 \mathrm{d}A \left[w_{,xx} - \chi \right] \right\} \mathrm{d}x \\ & + \int_L \left\{ \delta u_{,x} \int_A \sigma_1 \mathrm{d}A - \delta w_{,xx} \int_A y \sigma_1 \mathrm{d}A \right\} \mathrm{d}x \\ & - \int_L \left[\delta u b_x + \delta w b_y \right] \mathrm{d}x - \delta\Pi_{bc} \end{aligned} \tag{18}$$

Introducing the definitions given in Eqs. (3) and (4) we may write (18) as

$$\begin{aligned} \delta\Pi_{\mathrm{hw}} = & \int_L \left\{ \delta\epsilon[\hat{N}(\epsilon,\chi) - N] + \delta\chi[\hat{M}(\epsilon,\chi) - M] \right\} \mathrm{d}x \\ & + \int_L \left\{ \delta N \left[u_{,x} - \epsilon \right] + \delta M \left[w_{,xx} - \chi \right] \right\} \mathrm{d}x \\ & + \int_L \left\{ \delta u_{,x} N + \delta w_{,xx} M \right\} \mathrm{d}x \\ & - \int_L \left[\delta u \, b_x + \delta w \, b_y \right] \mathrm{d}x - \delta\Pi_{bc} \end{aligned} \tag{19}$$

2.2.2
Finite element approximation

The solution of the three field form of the beam problem given by (19) provides considerable flexibility in choice of approximating functions. In general we need to ensure that the number of terms taken for each variable satisfy consistency and stability conditions. An essential requirement is the mixed patch test count condition [9]. Considering a solution in two-dimensions where the displacement degrees of freedom at each node are

$$\tilde{\mathbf{a}}^\alpha = (\tilde{u}^\alpha, \tilde{w}^\alpha, \tilde{w}^\alpha_{,x}); \quad \alpha = 1, 2 \tag{20}$$

there are six degrees of freedom for each element. For this case we must have three rigid body modes of displacement and three straining ones. Considering the functional form given by Eq. (19) we can show that the conditions of approximation for stress and strain for this case must satisfy the mixed patch test count conditions [9]

$$\begin{aligned} n_\epsilon &\geq n_N \geq 1 \\ n_\chi &\geq n_M \geq 2 \end{aligned} \tag{21}$$

where n_N, n_M are the number of unknown element parameters in N, M and n_ϵ, n_χ are the number of unknown element parameters in ϵ, χ, respectively. The approximations for each of the variables are commonly taken as continuous polynomials within each element; however, we shall find that considerable advantages arise by using discontinuous or discrete (quadrature) approximation for the strains.

For the finite element approximation we consider a typical element of length $h = x^2 - x^1$. We begin by integrating by parts all terms with derivatives on displacements. The terms involving u and δu become

$$\int_h \{\delta N u_{,x} + \delta u_{,x} N\} \mathrm{d}x = -\int_h \{\delta N_{,x} u + \delta u N_{,x}\} \mathrm{d}x + \{\delta N\, u + \delta u\, N\}|_{\Gamma_h} \qquad (22)$$

where Γ_h is the left and right boundary of the element. If we assume approximations such that

$$N_{,x} + b_x = 0 \quad \text{and} \quad \delta N_{,x} = 0 \ . \qquad (23)$$

and add the b_x loading term to (22) we obtain

$$\int_h \{\delta N\, u_{,x} + \delta u_{,x} N\} \mathrm{d}x - \int_h \delta u\, b_x \, \mathrm{d}x = \{\delta N\, u + \delta u\, N\}|_{\Gamma_h} \qquad (24)$$

Similarly, we can integrate by parts the terms involving derivatives on w to obtain

$$\begin{aligned} \int_h \{\delta M\, w_{,xx} + \delta w_{,xx} M\} \mathrm{d}x. \\ = \int_h \{\delta M_{,xx} w + \delta w M_{,xx}\} \mathrm{d}x + \{\delta M\, w_{,x} + \delta w_{,x} M\}|_{\Gamma_h} \\ - \{\delta M_{,x}\, w + \delta w\, M_{,x}\}|_{\Gamma_h} \end{aligned} \qquad (25)$$

If we now assume approximations for M and δM that satisfy

$$M_{,xx} + b_y = 0 \quad \text{and} \quad \delta M_{,xx} = 0 \qquad (26)$$

then from (25) we obtain

$$\begin{aligned} \int_h \{\delta M\, w_{,xx} + \delta w_{,xx} M\} \mathrm{d}x - \int_h \delta w\, b_y \, \mathrm{d}x \\ = \{\delta M\, w_{,x} + \delta w_{,x} M\}|_{\Gamma_h} - \{\delta M_{,x} w + \delta w\, M_{,x}\}|_{\Gamma_h} \ . \end{aligned} \qquad (27)$$

Introducing Eqs. (24) and (27) into (19) we obtain the reduced variational functional

$$\begin{aligned} \delta\Pi_{hw} = \int_L \{\delta\epsilon[\hat{N}(\epsilon, \chi) - N] + \delta\chi[\hat{M}(\epsilon, \chi) - M]\} \mathrm{d}x \\ - \int_L \{\delta N \epsilon + \delta M \chi\} \mathrm{d}x + \{\delta N\, u + \delta u\, N\}|_{\Gamma_h} \\ + \{\delta M\, w_{,x} + \delta w_{,x} M\}|_{\Gamma_h} \\ - \{\delta M_{,x} w + \delta w\, M_{,x}\}|_{\Gamma_h} - \delta\Pi_{bc} = 0 \end{aligned} \qquad (28)$$

which is the form from which we will make our approximations.

Displacement approximation

We note that in the form given in (28) no interpolation for u or w is needed in each element. We merely use their nodal values together with nodal values of the first derivative of w (i.e., the usual nodal values for a displacement element with 2 nodes).

Although to this point we do not need an approximation for the displacement within the element, there are occasions for which one is needed. One case is merely for graphical display of the final displaced shape and others are for beams on elastic foundations, transient analysis, and non-linear geometric behavior. There is no fully consistent means to recover the displacements from the variational formulation presented above. Indeed, any field which satisfies the end conditions (20) is sufficient. Consequently, we will rely on computing the displacement by a double integration of the curvature field over the element. This approach is discussed in previous work (e.g., see [26]) and is used here to plot deformed shapes.

Resultant approximations

To satisfy (23) and (26) we will write the approximation for force resultants as

$$\begin{aligned} N &= \tilde{N} + N_p(\xi) \\ \delta N &= \delta\tilde{N} \end{aligned} \qquad (29)$$

and bending moment resultants as

$$\begin{aligned} M &= \tfrac{1}{2}(1 - \xi)\tilde{M}^1 + \tfrac{1}{2}(1 + \xi)\tilde{M}^2 + M_p(\xi) \\ \delta M &= \tfrac{1}{2}(1 - \xi)\delta\tilde{M}^1 + \tfrac{1}{2}(1 + \xi)\delta\tilde{M}^2 \end{aligned} \qquad (30)$$

where $-1 \le \xi \le 1$, $\tilde{N}$, $\tilde{M}^1$ and $\tilde{M}^2$ are element parameters and N_p and M_p are particular solutions for specified non-zero b_x or b_y. For example, if b_x and b_y are constant within each element suitable forms are

$$N_\mathrm{p} = -\tfrac{1}{2} b_x h \xi \quad \text{and} \quad M_\mathrm{p} = \tfrac{1}{8} b_y h^2 (1 - \xi^2) \ . \qquad (31)$$

For simplicity, we use (31) for N_p and M_p in the remaining development. The shape functions for unknown parameters in the axial force and bending moment are shown in Fig. 1.

Expanding the boundary terms in (28) we obtain

$$\begin{aligned} \{\delta N u + \delta u N\}|_{\Gamma_h} = \delta\tilde{N}(\tilde{u}^2 - \tilde{u}^1) + (\delta\tilde{u}^2 - \delta\tilde{u}^1)\tilde{N} \\ - (\delta\tilde{u}^2 + \delta\tilde{u}^1)\tfrac{1}{2} b_x h \end{aligned} \qquad (32)$$

for axial loading terms and

$$\begin{aligned} \{\delta M w_{,x} + \delta w_{,x} M\}|_{\Gamma_h} \\ = \delta\tilde{M}^2\tilde{\theta}^2 - \delta\tilde{M}^1\tilde{\theta}^1 + \delta\tilde{\theta}^2\tilde{M}^2 - \delta\tilde{\theta}^1\tilde{M}^1 \end{aligned} \qquad (33)$$

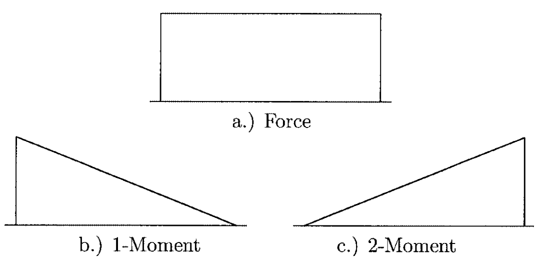

Fig. 1. Beam force and moment shape functions

$$\{\delta M_{,x}w + \delta w M_{,x}\}\big|_{\Gamma_h} = \frac{1}{h}\left[\delta\tilde{M}^1 - \delta\tilde{M}^2\right]\left[\tilde{w}^1 - \tilde{w}^2\right] + \frac{1}{h}\left[\delta\tilde{w}^1 - \delta\tilde{w}^2\right]\left[\tilde{M}^1 - \tilde{M}^2\right] - \left[\delta\tilde{w}^1 + \delta\tilde{w}^2\right]\frac{1}{2}b_y h \qquad (34)$$

for bending moment terms, where $\theta = \partial w/\partial x$.

The product terms between the stress and strains are considered next. For interpolation of the strains ϵ and χ we can use discontinuous piecewise constant functions where we take

$$\epsilon = \sum_\alpha N^e_\alpha \tilde{\epsilon}^\alpha \qquad \chi = \sum_\alpha N^e_\alpha \tilde{\chi}^\alpha \qquad (35)$$

with typical N^e_α as shown in Fig. 2. The integration may be conveniently carried out by defining the N^e_α as Lagrange polynomials with reference to quadrature points and approximating the integrals with a single point evaluation. In this case the $\tilde{\epsilon}^\alpha$ and $\tilde{\chi}^\alpha$ are merely amplitudes at the quadrature points. When multiplied by the strain parameters for each part and superimposed a typical strain distribution in an element is shown in Fig. 3.

The line integrals in Eq. (28) are now approximated as

$$\int_L \delta\epsilon[\hat{N}(\epsilon,\chi) - N]\mathrm{d}x \approx \sum_l \delta\tilde{\epsilon}^l[\hat{N}(\tilde{\epsilon}^l,\tilde{\chi}^l) - \tilde{N} - N_p]W_l$$

$$\int_L \delta\chi[\hat{M}(\epsilon,\chi) - M]\mathrm{d}x \approx \sum_l \delta\tilde{\chi}^l\left[\hat{M}(\tilde{\epsilon}^l,\tilde{\chi}^l) - \tfrac{1}{2}(1-\xi_l)\tilde{M}^1 - \frac{1}{2}(1+\xi_l)\tilde{M}^2 - M_p\right]W_l$$

$$\int_L \delta N\epsilon\,\mathrm{d}x \approx \sum_l \delta\tilde{N}\tilde{\epsilon}^l W_l$$

$$\int_L \delta M\chi\,\mathrm{d}x \approx \sum_l [\tfrac{1}{2}(1-\xi_l)\delta\tilde{M}^1 + \tfrac{1}{2}(1+\xi_l)\delta\tilde{M}^2]\tilde{\chi}^l W_l \qquad (36)$$

where ξ_l denotes one quadrature point for each discontinuous function and W_l denotes a quadrature weight and length.

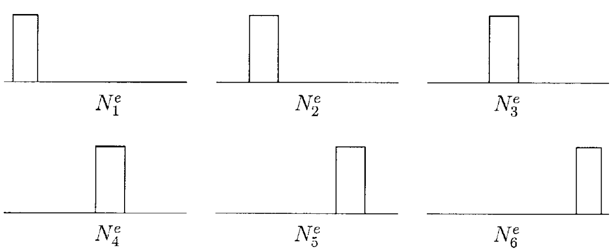

Fig. 2. Beam discontinuous strain shape functions

As an alternative to (35), we can use continuous shape functions with their definition point coinciding with Gauss–Lobbato (or other quadrature type) points as shown for a 4-point case in Fig. 4. Combining the function for strain as

$$\epsilon = \sum_\alpha N^e_\alpha \tilde{\epsilon}^\alpha$$

we obtain a continuous function in each element as shown in Fig. 5.

Matrix expression for equations

Assembling the above approximations, Eq. (19) may be written in matrix form as:

$$\delta\Pi = \begin{Bmatrix} \delta\tilde{\mathbf{a}} \\ \delta\tilde{\mathbf{q}} \\ \delta\tilde{\mathbf{e}}^l \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{0} & \mathbf{H}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{H} & \mathbf{0} & -\mathbf{b}_l^{\mathrm{T}} \\ \mathbf{0} & -\mathbf{b}_l & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{q}} \\ \tilde{\mathbf{e}}^l \end{Bmatrix} - \begin{Bmatrix} \mathbf{F} \\ \mathbf{0} \\ \mathbf{s}_l^p - \hat{\mathbf{s}}_l \end{Bmatrix} \right) = 0 \qquad (37)$$

where l denotes evaluation at a quadrature point,

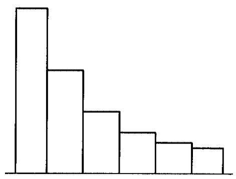

Fig. 3. Typical discontinuous strain distribution in a beam element

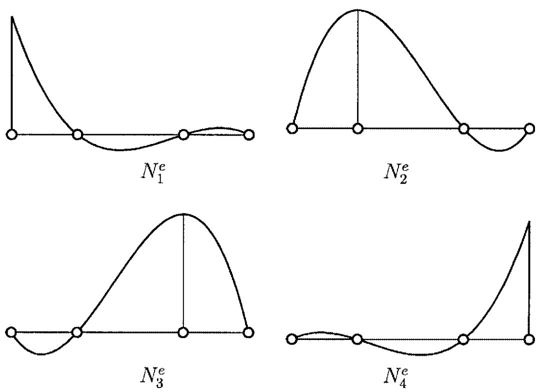

Fig. 4. Beam continuous strain shape functions

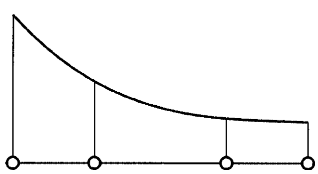

Fig. 5. Typical continuous strain distribution in a beam element

$$\mathbf{H} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/h & -1 & 0 & -1/h & 0 \\ 0 & -1/h & 0 & 0 & 1/h & 1 \end{bmatrix} \quad (38)$$

and

$$\mathbf{b}_l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(1-\xi_l) & \frac{1}{2}(1+\xi_l) \end{bmatrix} \; . \quad (39)$$

The particular solution is given as

$$\mathbf{s}_l^p = \begin{Bmatrix} N_\mathrm{p} W_l \\ M_\mathrm{p} W_l \end{Bmatrix} \quad (40)$$

and the constitutive equation evaluation by

$$\hat{\mathbf{s}}_l = \begin{Bmatrix} \hat{N} W_l \\ \hat{M} W_l \end{Bmatrix} \; . \quad (41)$$

Applying a linearization to (37) gives the incremental form for a Newton solution process as

$$\begin{Bmatrix} \delta\tilde{\mathbf{a}} \\ \delta\tilde{\mathbf{q}} \\ \delta\tilde{\mathbf{e}}^l \end{Bmatrix}^\mathrm{T} \left(\begin{bmatrix} \mathbf{0} & \mathbf{H}^\mathrm{T} & \mathbf{0} \\ \mathbf{H} & \mathbf{0} & -\mathbf{b}_l^\mathrm{T} \\ \mathbf{0} & -\mathbf{b}_l & \mathbf{k}_{ll} \end{bmatrix} \begin{Bmatrix} d\tilde{\mathbf{a}} \\ d\tilde{\mathbf{q}} \\ d\tilde{\mathbf{e}}^l \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_a \\ \mathbf{R}_q \\ \mathbf{R}_{e^l} \end{Bmatrix} \right) \quad (42)$$

where "d" is an increment, the residual expression is given by

$$\begin{Bmatrix} \mathbf{R}_a \\ \mathbf{R}_q \\ \mathbf{R}_{e^l} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} - \mathbf{H}^\mathrm{T}\tilde{\mathbf{q}} \\ \mathbf{b}_l^\mathrm{T}\tilde{\mathbf{e}}^l - \mathbf{H}\tilde{\mathbf{a}} \\ \mathbf{b}_l\tilde{\mathbf{q}} + \mathbf{s}_l^p - \hat{\mathbf{s}}_l \end{Bmatrix} \quad (43)$$

and the tangent matrix by

$$\mathbf{k}_{ll} = \int_{A(\xi_l)} \begin{bmatrix} 1 \\ -y \end{bmatrix} E_\mathrm{T}(\xi_l, y) [1 \quad -y] \mathrm{d}A \quad (44)$$

The forces $\mathbf{F}$ are nodal values computed from the particular solutions N_p and M_p and any applied concentrated forces at the nodes.[1]

Solution strategy

It should be noted in the above formulation that continuity between elements is enforced only for displacement degrees of freedom. Forces and strains may be discontinuous between elements. Thus, the parameters for forces and strains may be eliminated at the element level resulting in a stiffness matrix for displacement parameter determination. The elimination may be performed in two steps:

1. Eliminate section strain components separately, resulting in

   $$d\tilde{\mathbf{e}}^l = \mathbf{k}_{ll}^{-1}[\mathbf{b}_l d\tilde{\mathbf{q}} + \mathbf{R}_{e^l}]$$

   where from (43) we have

   $$\mathbf{R}_{e^l} = \mathbf{b}_l\tilde{\mathbf{q}} + \mathbf{s}_l^p - \hat{\mathbf{s}}_l \; .$$

   Substitute into the remaining equations to obtain

   $$\begin{Bmatrix} \delta\tilde{\mathbf{a}} \\ \delta\tilde{\mathbf{q}} \end{Bmatrix}^\mathrm{T} \left(\begin{bmatrix} \mathbf{0} & \mathbf{H}^\mathrm{T} \\ \mathbf{H} & -\mathbf{f} \end{bmatrix} \begin{Bmatrix} d\tilde{\mathbf{a}} \\ d\tilde{\mathbf{q}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_a \\ \bar{\mathbf{R}}_q \end{Bmatrix} \right)$$

   where

   $$\mathbf{f} = \sum_l \mathbf{b}_l^\mathrm{T}\mathbf{k}_{ll}^{-1}\mathbf{b}_l$$

   is the element flexibility and

   $$\bar{\mathbf{R}}_q = \mathbf{R}_q + \sum_l \mathbf{b}_l^\mathrm{T}\mathbf{k}_{ll}^{-1}\mathbf{R}_{e^l}$$

   is a modified stress residual. Given an increment $d\tilde{\mathbf{a}}$ the second of the above equations may be solved for increments in $\tilde{\mathbf{q}}$.[2]
2. Eliminate the stress parameters for each element giving[3]

   $$d\tilde{\mathbf{q}} = \mathbf{f}^{-1}\left[\mathbf{H}d\tilde{\mathbf{a}} - \bar{\mathbf{R}}_q\right]$$

   When the result of the above two steps is substituted into the remaining equation set we obtain

   $$\bar{\mathbf{K}}d\tilde{\mathbf{a}} = \bar{\mathbf{R}}_a$$

   where an element stiffness is given as

   $$\bar{\mathbf{K}} = \mathbf{H}^\mathrm{T}\mathbf{f}^{-1}\mathbf{H}$$

   and a modified element residual by

   $$\bar{\mathbf{R}}_a = \mathbf{R}_a + \mathbf{H}^\mathrm{T}\mathbf{f}^{-1}\bar{\mathbf{R}}_q$$

The resulting stiffness and residual now may be assembled into the global equations in an identical manner to any displacement formulation. We note that at convergence the $\mathbf{R}_q$ and $\mathbf{R}_{e^l}$ residuals are zero for each element and, thus, the $\mathbf{R}_a$ residual becomes the usual element residual on equilibrium. During iteration steps, however, the strain residuals $\mathbf{R}_{e^l}$ in general will not be zero (except for linear problems).

The update strategy for the parameters may be carried out as follows:

1. For solution at time t_{n+1} assume the state at the previous step t_n is known.[4] For the first iteration step $j = 0$, set $\tilde{\mathbf{a}}_{n+1}^{(0)} = \tilde{\mathbf{a}}_n$, $\tilde{\mathbf{q}}_{n+1}^{(0)} = \tilde{\mathbf{q}}_n$ and $\tilde{\mathbf{e}}_{n+1}^{(0)} = \tilde{\mathbf{e}}_n$.
2. Form element matrix and residual for state at iteration j.
3. Condense arrays as described above and assemble global stiffness and residual for the nodal parameters $\tilde{\mathbf{a}}$.
4. Solve equation system

   $$\mathbf{K}d\tilde{\mathbf{a}} = \mathbf{R}$$

   and update solution

   $$\tilde{\mathbf{a}}_{n+1}^{j+1} = \tilde{\mathbf{a}}_{n+1}^{j} + d\tilde{\mathbf{a}}$$
5. For each element determine $d\tilde{\mathbf{q}}$ and $d\tilde{\mathbf{e}}$ and update the stress and strain parameters

[1] Concentrated forces applied at an element interior are used to compute solutions for N_p and/or M_p.

[2] In solution of highly non-linear problems we sometimes find it necessary to subincrement the displacement increments $d\tilde{\mathbf{a}}$ in order to converge the solution for the $\tilde{\mathbf{q}}$.

[3] For non-linear materials it may be desirable to use a singular valued decomposition and construct an inverse or pseudo-inverse to avoid numerical precision problems [35].

[4] In some definitions we delete the individual cross-section identifier l when defining the $\tilde{\mathbf{e}}^l$ to avoid cumbersome notation.

$$\tilde{\mathbf{q}}_{n+1}^{j+1} = \tilde{\mathbf{q}}_{n+1}^{j} + d\tilde{\mathbf{q}}$$
$$\tilde{\mathbf{e}}_{n+1}^{j+1} = \tilde{\mathbf{e}}_{n+1}^{j} + d\tilde{\mathbf{e}}$$

6. Check convergence on global residual for the displacements $\mathbf{R}$ and each element residual $\mathbf{R}_q$ and $\mathbf{R}_{e^l}$.
   (a) If converged: set $n = n + 1$ and go to Step 1.
   (b) If not converged: set $j = j + 1$ and go to Step 2.

Generally, it is more efficient to compute and perform the update step for stress and strain just before the next element matrix and residual are computed for iteration $j + 1$. This requires one to either recompute the element matrix and residual at the values of the j iteration or store the matrices for later use. In either case, however, it is necessary to save the parameters for stress, $\tilde{\mathbf{q}}$, and strain $\tilde{\mathbf{e}}$ for each element.

2.3
Beam formulation with shear deformation

Let us now apply the Hu–Washizu functional to the solution of beam problems which include shearing deformation. The displacement approximation for a beam which includes the primary effect of shear deformation is given by

$$u_1(x, y) = u(x) - y\theta(x) \quad \text{and} \quad u_2 = w(x) \tag{45}$$

where θ is the rotation of the beam cross-section and γ is the average cross-section shearing strain. In this case we have two strain components at each point in the beam which are given by

$$\begin{aligned} \epsilon_1 &= \frac{\partial u}{\partial x} - y\frac{\partial \theta}{\partial x} = \epsilon(x) - y\chi(x) \\ \gamma_{12} &= \frac{\partial w}{\partial x} - \theta = \gamma(x) \ . \end{aligned} \tag{46}$$

We first modify Eq. (14) to include σ_{12} and γ_{12} for the shear effects. We then assume strain distributions given by (46), define the shear resultant by

$$V = \int_A \tau \, \mathrm{d}A \tag{47}$$

and integrate over the cross section to obtain

$$\begin{aligned} \delta\Pi_{hw} &= \int_L \left\{ \delta\epsilon[\hat{N}(\epsilon,\gamma,\chi) - N] + \delta N\left[\frac{\partial u}{\partial x} - \epsilon\right] + \frac{\partial \delta u}{\partial x}N \right\} \mathrm{d}x \\ &+ \int_L \left\{ \delta\chi[\hat{M}(\epsilon,\gamma,\chi) - M] + \delta M\left[\frac{\partial \theta}{\partial x} - \chi\right] + \frac{\partial \delta\theta}{\partial x}M \right\} \mathrm{d}x \\ &+ \int_L \left\{ \delta\gamma[\hat{V}(\epsilon,\gamma,\chi) - V] + \delta V\left[\frac{\partial w}{\partial x} - \theta - \gamma\right] \right\} \mathrm{d}x \\ &+ \int_L \left\{ \left[\frac{\partial \delta w}{\partial x} - \delta\theta\right] V \right\} \mathrm{d}x \\ &- \int_L \left[\delta u b_x + \delta w b_y\right] \mathrm{d}x + \delta\Pi_{bc} \end{aligned} \tag{48}$$

Finite element approximation

We again use the mixed patch test count condition as a guide to construct finite element approximations. Considering a 2-node element in which the displacement degrees of freedom at each node are

$$\mathbf{a}^\alpha = (\tilde{u}^\alpha, \tilde{w}^\alpha, \tilde{\theta}^\alpha); \quad \alpha = 1, 2 \tag{49}$$

there are again six degrees of freedom for each element (Note that now the $\tilde{\theta}^\alpha$ is a parameter describing the rotation of the beam cross section and, in general, is not equal to $w^\alpha_{,x}$). For the case with shear deformation we must have three rigid body modes of displacement and three straining ones. Considering the functional form given by Eq. (48), the conditions to approximate stress resultants and strain functions must satisfy the mixed patch test count conditions

$$\begin{aligned} n_\epsilon &\ge n_N \ge 1 \\ n_\gamma &\ge n_V \ge 1 \\ n_\chi &\ge n_M \ge 2 \end{aligned} \tag{50}$$

where n_N, n_V, n_M are the number of unknown element parameters in N, V, M and n_ϵ, n_γ, n_χ are the number of unknown element parameters in ϵ, γ, χ, respectively. By using force solutions which satisfy equilibrium we will be able to satisfy these conditions easily.

For the finite element approximation we again consider a typical element of length $h = x^2 - x^1$ and integrate by parts all terms with derivatives on displacements. The terms involving u and δu again yield (22) and we assume approximations which satisfy (23). Including the element force b_x we obtain (24) again.

Similarly, we can integrate by parts the terms involving derivatives on θ to obtain

$$\int_h \left\{ \delta M\frac{\partial \theta}{\partial x} + \frac{\partial \delta\theta}{\partial x}M \right\} \mathrm{d}x = -\int_h \left\{ \frac{\partial \delta M}{\partial x}\theta + \delta\theta\frac{\partial M}{\partial x} \right\} \mathrm{d}x + \{\delta M\theta + \delta\theta M\}|_{\Gamma_h} \tag{51}$$

and those involving derivatives on w to obtain

$$\int_h \left\{ \delta V\frac{\partial w}{\partial x} + \frac{\partial \delta w}{\partial x}V \right\} \mathrm{d}x = -\int_h \left\{ \frac{\partial \delta V}{\partial x}w + \delta w\frac{\partial V}{\partial x} \right\} \mathrm{d}x + \{\delta V w + \delta w V\}|_{\Gamma_h} \tag{52}$$

We assume that the approximations for M, V and δM, δV satisfy the equilibrium relations

$$\begin{aligned} &\frac{\partial V}{\partial x} + b_y = 0 \quad \text{and} \quad \frac{\partial \delta V}{\partial x} = 0 \\ &\frac{\partial M}{\partial x} + V = 0 \quad \text{and} \quad \frac{\partial \delta M}{\partial x} + \delta V = 0 \ . \end{aligned} \tag{53}$$

We then combine (52) and (53) with the remaining terms for w and θ in (48) to obtain

$$\int_h \left\{ \delta M \frac{\partial \theta}{\partial x} + \frac{\partial \delta\theta}{\partial x} M \right\} \mathrm{d}x - \int_h \{\delta\theta V + \delta V \theta\} \mathrm{d}x + \int_h \left\{ \delta V \frac{\partial w}{\partial x} + \frac{\partial \delta w}{\partial x} V \right\} \mathrm{d}x - \int_h \delta w b_y \, \mathrm{d}x = \{\delta M\theta + \delta\theta M\}|_{\Gamma_h} + \{\delta V w + \delta w V\}|_{\Gamma_h} \tag{54}$$

Introducing the above into (48) we obtain

$$\begin{aligned} \delta\Pi_{hw} = & \int_L \left\{ \delta\epsilon[\hat{N}(\epsilon,\gamma,\chi) - N] - \delta N \epsilon \right\} \mathrm{d}x \\ & + \int_L \{\delta\chi[\hat{M}(\epsilon,\gamma,\chi) - M] + \delta M \chi\} \mathrm{d}x \\ & + \int_L \{\delta\gamma[\hat{V}(\epsilon,\gamma,\chi) - V] + \delta V \gamma\} \mathrm{d}x \\ & + \{\delta N u + \delta u N\}|_{\Gamma_h} + \{\delta M\theta + \delta\theta M\}|_{\Gamma_h} \\ & + \{\delta V w + \delta w V\}|_{\Gamma_h} + \delta\Pi_{bc} \end{aligned} \tag{55}$$

which is the form from which we will make the approximations. We note that in this form, subject to the conditions imposed on N, V and M, no interpolation for u, w or θ is needed in each element. We merely use their nodal values (i.e., the usual nodal values for a shear deformable displacement formulation element with 2 nodes). We can again use the approximation for force and bending moment resultants given by (29) and (30). We also note that the shear in each element may be computed from moment equilibrium as

$$V = -\frac{\partial M}{\partial x} = \frac{1}{h}(M_1 - M_2) + V_p \quad \text{with } V_p = -\frac{\partial M_\mathrm{p}}{\partial x} \tag{56}$$

and, thus, it is not necessary to add additional force parameters to the element.

From Eq. (55) we again obtain the axial load terms given by Eq. (32) and for bending moments and shears

$$\{\delta M\theta + \delta\theta M\}|_{\Gamma_h} = \delta\tilde{M}^2\tilde{\theta}^2 - \delta\tilde{M}^1\tilde{\theta}^1 + \delta\tilde{\theta}^2\tilde{M}^2 - \delta\tilde{\theta}^1\tilde{M}^1 \tag{57}$$

$$\begin{aligned} \{\delta V w + \delta w V\}|_{\Gamma_h} = & \frac{1}{h}\left[\delta\tilde{M}^1 - \delta\tilde{M}^2\right]\left[\tilde{w}^2 - \tilde{w}^1\right] \\ & + \frac{1}{h}\left[\delta\tilde{w}^2 - \delta\tilde{w}^1\right]\left[\tilde{M}^1 - \tilde{M}^2\right] \\ & + \delta\tilde{w}^2 V_p^2 - \delta\tilde{w}^1 V_p^1 \end{aligned} \tag{58}$$

where V_p^1 and V_p^2 are values of the particular solution for shear at the 1 and 2 ends, respectively. These boundary terms may be written in matrix form (38) and we note that no differences arise by including shear deformation.

The product terms between the stress and strains are considered next. For interpolation of the strain parts ϵ, γ and χ we can use discontinuous piecewise constant functions given by Eq. (35) and evaluated at a single quadrature point or continuous shape functions with their definition point coinciding with the Gauss–Lobbato (or other quadrature type) point as shown for the 4-point case in Fig. 4.

The line integrals in Eq. (55) are again approximated as

$$\begin{aligned} & \int_L \delta\epsilon[\hat{N}(\epsilon,\gamma,\chi) - N]\mathrm{d}x \\ & \quad \approx \sum_l \delta\tilde{\epsilon}^l[\hat{N}(\tilde{\epsilon}^l,\tilde{\gamma}^l,\tilde{\chi}^l) - \tilde{N} - N_\mathrm{p}]W_l \\ & \int_L \delta\chi[\hat{M}(\epsilon,\gamma,\chi) - M]\mathrm{d} \\ & \quad \approx \sum_l \delta\tilde{\chi}^l\big[\hat{M}(\tilde{\epsilon}^l,\tilde{\gamma}^l,\tilde{\chi}^l) - \tfrac{1}{2}(1-\xi_l)\tilde{M}^1 \\ & \qquad - \tfrac{1}{2}(1+\xi_l)\tilde{M}^2 - M_\mathrm{p}\big]W_l \\ & \int_L \delta\gamma[\hat{V}(\epsilon,\gamma,\chi) - V]\mathrm{d}x \\ & \quad \approx \sum_l \delta\tilde{\gamma}^l[\hat{V}(\tilde{\epsilon}^l,\tilde{\gamma}^l,\tilde{\chi}^l) - \tilde{V} - V_p]W_l \\ & \int_L \delta N \epsilon \, \mathrm{d}x \approx \sum_l \delta\tilde{N}\tilde{\epsilon}^l W_l \\ & \int_L \delta M \chi \, \mathrm{d}x \approx \sum_l [\tfrac{1}{2}(1-\xi_l)\delta\tilde{M}^1 + \tfrac{1}{2}(1+\xi_l)\delta\tilde{M}^2]\tilde{\chi}^l W_l \\ & \int_L \delta V \gamma \, \mathrm{d}x \approx \sum_l \delta\tilde{V}\tilde{\gamma}^l W_l \end{aligned} \tag{59}$$

where ξ_l denotes one quadrature point for each function and W_l denotes a quadrature weight and length. All terms except those involving $\hat{N}$, $\hat{V}$ and $\hat{M}$ may be written in matrix form as

$$\delta\tilde{\epsilon}^l\tilde{N} + \delta\tilde{\chi}^l[\tfrac{1}{2}(1-\xi_l)\tilde{M}^1 + \tfrac{1}{2}(1+\xi_l)\tilde{M}^2] + \delta\tilde{\gamma}^l\tilde{V} = (\delta\tilde{\mathbf{e}}^l)^\mathrm{T}\mathbf{b}_l\tilde{\mathbf{q}} \tag{60}$$

and

$$\delta\tilde{N}\tilde{\epsilon}^l + [\tfrac{1}{2}(1-\xi_l)\delta\tilde{M}^1 + \tfrac{1}{2}(1+\xi_l)\delta\tilde{M}^2]\tilde{\chi}^l + \delta\tilde{V}\tilde{\gamma}^l = \delta\tilde{\mathbf{q}}^\mathrm{T}\mathbf{b}_l^\mathrm{T}\tilde{\mathbf{e}}^l \tag{61}$$

where $\tilde{\mathbf{e}}^l = (\tilde{\epsilon}^l,\ \tilde{\chi}^l,\ \tilde{\gamma}^l)^\mathrm{T}$, $\quad \tilde{\mathbf{q}} = (\tilde{N},\ \tilde{M}^1,\ \tilde{M}^2)^\mathrm{T}$ and

$$\mathbf{b}_l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(1-\xi_l) & \frac{1}{2}(1+\xi_l) \\ 0 & \frac{-1}{h} & \frac{1}{h} \end{bmatrix} . \tag{62}$$

Here a difference with the formulation without shear deformation arises from the addition of a third row in the $\mathbf{b}_l$ array and the inclusion of $\tilde{\gamma}^l$ in the definition of $\tilde{\mathbf{e}}^l$.

Applying a linearization to the equations equivalent to (37) results in the element expression

$$\begin{bmatrix} \delta\tilde{\mathbf{a}} \\ \delta\tilde{\mathbf{q}} \\ \delta\tilde{\mathbf{e}}^l \end{bmatrix}^\mathrm{T} \left(\begin{bmatrix} \mathbf{0} & \mathbf{H}^\mathrm{T} & \mathbf{0} \\ \mathbf{H} & \mathbf{0} & -\mathbf{b}_l^\mathrm{T} \\ \mathbf{0} & -\mathbf{b}_l & \mathbf{k}_{ll} \end{bmatrix} \begin{bmatrix} d\tilde{\mathbf{a}} \\ d\tilde{\mathbf{q}} \\ d\tilde{\mathbf{e}}^l \end{bmatrix} = \begin{bmatrix} \mathbf{R}_a \\ \mathbf{R}_q \\ \mathbf{R}_{e^l} \end{bmatrix} \right) \tag{63}$$

where "d" is an increment and the residual expression is given by

$$\begin{bmatrix} \mathbf{R}_a \\ \mathbf{R}_q \\ \mathbf{R}_{e^l} \end{bmatrix} = \begin{bmatrix} \mathbf{F} - \mathbf{H}^{\mathrm{T}}\tilde{\mathbf{q}} \\ \mathbf{b}_l^{\mathrm{T}}\tilde{\mathbf{e}}^l - \mathbf{H}\tilde{\mathbf{a}} \\ \mathbf{b}_l\tilde{\mathbf{q}} + \mathbf{s}_l^p - \hat{\mathbf{s}}_l \end{bmatrix} . \tag{64}$$

The constitutive equation terms are

$$\hat{\mathrm{s}}_l = \begin{Bmatrix} \hat{N} W_l \\ \hat{M} W_l \\ \hat{V} W_l \end{Bmatrix} \quad \text{and} \quad \mathrm{s}_l^p = \begin{Bmatrix} N^p W_l \\ M^p W_l \\ V^p W_l \end{Bmatrix} \tag{65}$$

and the tangent matrix for a decoupled bending-shear behavior becomes

$$\mathrm{k}_{ll} = \int_A \begin{bmatrix} 1 & 0 \\ -y & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_{\mathrm{T}}(y) & 0 \\ 0 & G_{\mathrm{T}}(y) \end{bmatrix} \begin{bmatrix} 1 & -y & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{d}A \tag{66}$$

where E_{T} and G_{T} are tangent Young's and shear modulus, respectively. Coupling between the behavior merely adds off diagonals to the modulus array.

The modifications to include shear behavior are minimal and the solution strategy for the model is identical to that already presented for the case without transverse shearing strains.

3 Numerical examples

3.1 Simply supported beam with uniform loading

Consider a simply supported beam under uniformly distributed load of intensity q with length L, elastic properties E and G, cross sectional area A and moment of inertia I as shown in Fig. 6. The exact displacement at mid-span for the Euler–Bernoulli theory is

$$w^{\mathrm{E}}_{\max} = \frac{5\,q\,L^4}{384\,EI} \tag{67}$$

and including the effects of shear deformation (Timoshenko beam theory) is

$$w^{\mathrm{T}}_{\max} = \frac{5\,q\,L^4}{384\,EI} + \frac{q\,L^2}{8\,\kappa\,GA} \tag{68}$$

where κ is the shear correction factor. Similarly the cross-section rotation at the left support for the two theories is the same and is given by

$$\theta^{\mathrm{E}}_{\max} = -\frac{q\,L^3}{24\,EI} = \theta^{\mathrm{T}}_{\max} \ . \tag{69}$$

For the comparison with the element presented above we consider a rectangular cross section with linear elastic material. The properties are: $E = 10^6$, $\nu = 0.25$, $q = 1$, $\kappa = 5/6$, $h = b = 1$. Using symmetry, one half of the beam is modeled with one element based on the theory given above. This problem has been analyzed by Reddy [36] where it is shown that several elements are required along the length to get satisfactory answers using standard displacement approaches. We note that our approach uses shape functions which involve no material parameters, contrary to the displacement shape functions proposed by Reddy to avoid shear locking. Table 1 shows that the solution at the nodes is exact for the present development.

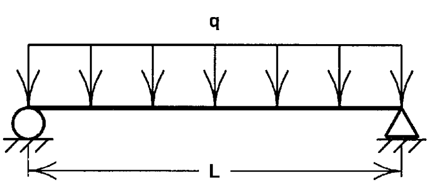

Fig. 6. Uniformly loaded, simply supported beam

3.2 Frame structure

The frame structure considered by Reddy [36] is analyzed using the beam element developed above. The only modification from that presented previously is the need to transform the member from local coordinates (where the theory is developed) to global coordinates. This standard operation is described in any text on structural analysis. The geometry for the frame is shown in Fig. 7. Cross-section properties are: $A = 10\,\mathrm{in}^2$, $I = 10\,\mathrm{in}^4$, $E = 10^6$ psi, $\nu = 0.3$, and $\kappa = 5/6$. Our model for the frame consists of three elements: one for the vertical column and two for the inclined beam. The results for the displacement at point B are compared with the exact solution (given in [36]) in Table 2. We note that the results are exact at this point. Moreover, the force distribution obtained is also exact. The ability of the mixed formulation given here to produce correct results with and without shear and no shear locking is clearly evident.

3.3 Simply supported beam with point load

As a final problem we consider a simply supported beam with a central point load as shown in Fig. 8. To show the

Table 1. Simply supported beam: $w_{\max} = w(L/2)$ and $\theta_0 = (\theta 0)$

	$L/h = 10$		$L/h = 100$	
	$w_{\max} \times 10^2$	$\theta_0 \times 10^3$	$w_{\max} \times 10^{-2}$	θ_0
Exact (no shear)	0.15625	−0.50000	0.15625	−0.50000
Exact (with shear)	0.16000	−0.50000	0.15629	−0.50000
Present (no shear)	0.15625	−0.50000	0.15625	−0.50000
Present (shear)	0.16000	−0.50000	0.15629	−0.50000

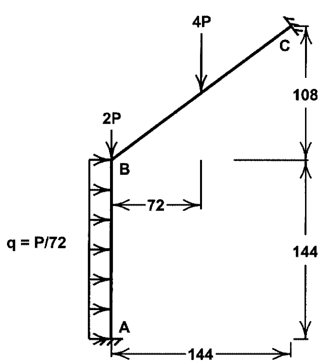

Fig. 7. Frame structure

Table 2. Displacement at B for frame structure

	Displacement/P at B $\times 10^4$		
	u_B	w_B	θ_B
Exact (no shear)	0.83904	−0.68124	−0.96098
Exact (with shear)	0.83898	−0.68123	−0.96206
Present (no shear)	0.83904	−0.68124	−0.96098
Present (shear)	0.83898	−0.68123	−0.96206

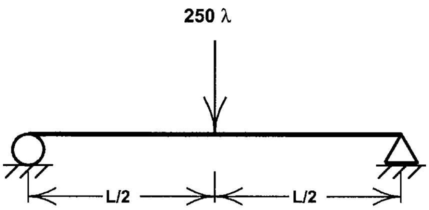

Fig. 8. Simply supported beam with central point load

Table 3. Properties for inelastic beam

Length	$L = 180$
Depth	$h = 10$
Width	$b = 10$
Elastic modulus	$E = 29000$
Yield stress	$\sigma_y = 50$
Hardening modulus	$H_{iso} = 290$ (1%)

advantages of the formulation presented here we allow the beam to have elasto–plastic behavior. The entire beam is modeled with two elements (one element for each symmetric half); five Gauss-Lobatto curvature stations per element are used along the beam axis. A rectangular cross-section is considered with 10 Gauss–Lobatto points through the depth to permit modeling of the spread of the plastic zone. No shear deformation is included. The properties for the analysis are:

The central load is allowed to vary using a load control strategy [3]. For the comparison we also consider the solution using a standard displacement model with cubic Hermite polynomial shape functions. Solutions for two, four, and eight elements for the length are used (one, two and four on each half length). In Fig. 9 we show the force-displacement relation, deformed shape and distribution of moment and curvature along the length of the beam at the last computed load state for each analysis. The displacement model permits only linear change, whereas the mixed model presented here allows for arbitrary change at each axial station used (5 in the present case). The superiority of the mixed form is evident in both the force-displacement, the computed deformed shape and the moment and the curvature distribution.

4 Closure

In this work we have presented a three-field variational formulation for beams. The presentation is restricted to two-dimensional, small-displacement theory and includes

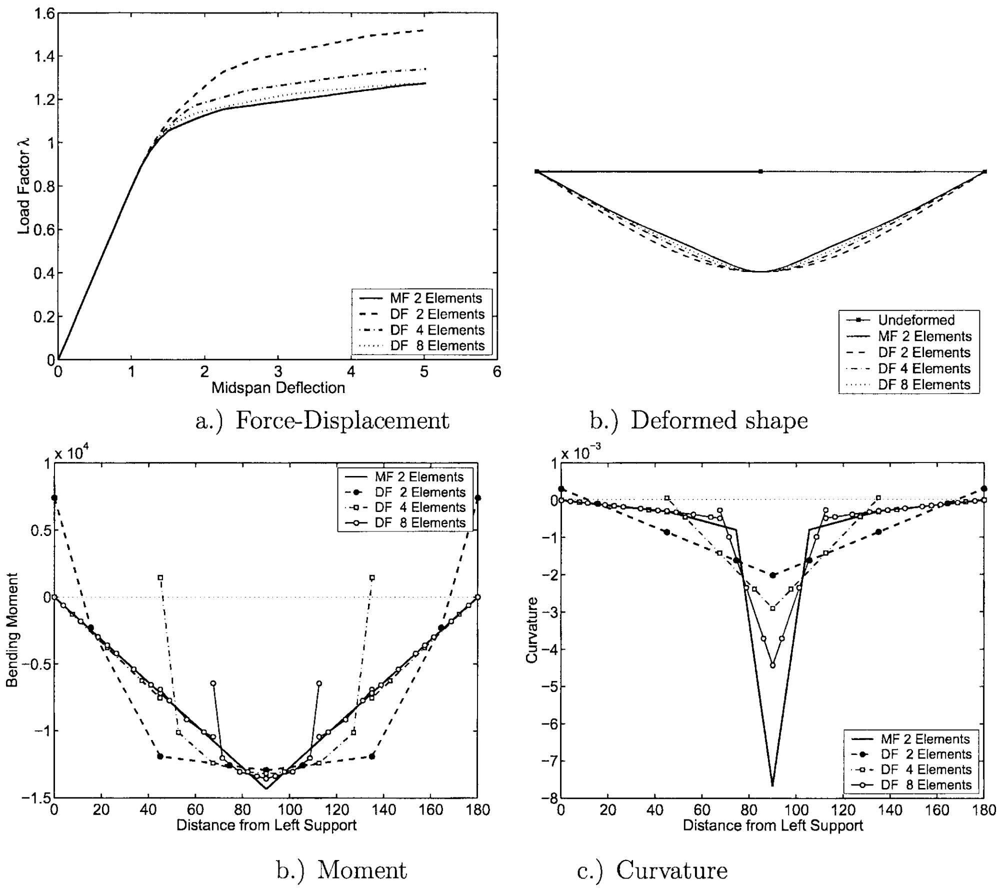

Fig. 9. Simply supported, point loaded beam – inelastic solution. DF = Displacement formulation; MF = Mixed formulation

the effects of shear deformation. Shear deformation can be readily included without danger of shear locking and, thus, behavior is independent of the number of integration points along the axis of each element. Non-linear material behavior is included by integrating the resultants for axial force, shear force, and bending moment over the member cross-section.

The extension to three dimensions is straightforward. Geometric nonlinearity may be included with the approach presented by Sousa [30] for linear and nonlinear material response. In that study full geometric-nonlinearity for large displacements is included with the co-rotational formulation that Crisfield was so instrumental in developing and refining over the years.

The numerical examples demonstrate the advantages of the mixed approach over results from traditional displacement based formulations – especially for very coarse mesh discretizations.

In closing, we again wish to remember our late colleague Mike Crisfield and the motivation he has instilled in us to pursue our development. Mike's wry wit and insights into computational mechanics will be sorely missed!

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