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A MIXED PROBLEM FOR HYPERBOLIC EQUATIONS OF SECOND ORDER WITH NON-HOMOGENEOUS NEUMANN TYPE BOUNDARY CONDITION

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1. Introduction

In this paper we again consider a mixed problem for hyperbolic equations of second order. The domain Ω and the equations are same ones in the previous paper [3]. Let S be a sufficiently smooth compact hypersurface in R^n and let Ω be the interior or exterior domain of S.

Consider the hyperbolic equation of second order

$$
(1.1) \qquad L[u] = \frac{\partial^2}{\partial t^2}u + a_1(x, t:D)\frac{\partial}{\partial t}u + a_2(x, t:D)u = f
$$

$$
a_1(x, t:D) = \sum_{i=1}^{n} 2h_i(x, t)\frac{\partial}{\partial x_i} + h(x, t)
$$

$$
a_2(x, t:\ D) = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\ \frac{\partial}{\partial x_j}\right) + \sum_{j=1}^{n} b_j(x, t)\frac{\partial}{\partial x_j} + c(x, t)
$$

where the coefficients belong to $\mathcal{B}^2(\Omega\times[0, \infty))$[1]. We assume that $a_2(x, t:D)$ is an elliptic operator satisfying

$$
(1.2) \qquad \sum_{i,j=1}^{n} a_{ij}(x, t)\,\xi_i\xi_j \geqslant d\sum_{i=1}^{n}\xi_i^2 \qquad (d>0)
$$

$$
a_{ij}(x, t) = a_{ji}(x, t)
$$

for all $(x, t)\in\Omega\times[0,\infty)$ and $\xi=(\xi_1, \xi_2, \cdots, \xi_n)\in R^n$, and that $h_i(x, t)$ $(i=1, 2, \dots, n)$ are real-valued.

Let $\sigma_1(s, t)$ be a sufficiently smooth real-valued function defined on $S\times[0, \infty)$ such that for some constant $\varepsilon_0>0$

1) $\mathcal{B}^k(\omega)$, ω being an open set, is the set of all functions defined in ω such that their partial derivatives of order $\leq k$ all exist and are continuous and bounded.

$$\sigma_1(s, t) \leqslant \langle h(s, t), \nu\rangle - \mathcal{E}_0 \tag{1.3}$$

holds where

$$\langle h(s, t), \nu\rangle = \sum_{i=1}^{n} h_i(s, t)\nu_i$$

$\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ is the unit outer normal of S at $s \in S$. Then we consider the following boundary operator

$$B(t) = \frac{\partial}{\partial n_t} - \sigma_1(s, t)\frac{\partial}{\partial t} + \sigma_2(s, t) \tag{1.4}$$

where

$$\frac{\partial}{\partial n_t} = \sum_{i,j=1}^{n} a_{ij}(s, t)\nu_i \frac{\partial}{\partial x_j},$$

$\sigma_2(s, t)$ is a sufficiently smooth function defined on $S \times [0, \infty)$.

Our problem is to obtain $u(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$[2], for given initial data $\{u_0(x), u_1(x)\}$, the second member $f(x, t)$ and the boundary data $g(s, t)$, satisfying

$$\begin{cases} \text{(i)} \quad L[u] = f(x, t) \quad \text{in} \quad \Omega \times (0, T) \\ \text{(ii)} \quad u(x, 0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \\ \text{(iii)} \quad B(t)u(x, t) = g(s, t) \quad \text{on} \quad S \times [0, T). \end{cases} \tag{1.5}$$

In the previous paper [3] the boundary condition was taken as $\sigma_1(s, t) = \langle h(s, t), \nu\rangle$ and $g(s, t) \equiv 0$. But in our treatment it seems to be difficult to show the existence of solution without the assumption that the coefficients of the principal part of L are independent of t on S or that $\langle h(s, t), \nu\rangle = 0$ on S.

By exchanging the boundary condition as (1.3) (1.4), the existence of solution can be proved without any additional assumption about L.

Now we state Theorems:

Theorem 1. *Given* $\{u_0(x), u_1(x)\} \in H^2(\Omega) \times H^1(\Omega)$, $f(x, t) \in \mathcal{E}_t^1(L^2(\Omega))$ *and* $g(s, t) \in \mathcal{E}_t^0(H^{1/2}(S)) \cap \mathcal{E}_t^1(L^2(S))$, *if the compatibility condition at* $t=0$, *namely*

$$\frac{\partial}{\partial n_0}u_0(x) - \sigma_1(s, 0)u_1(x) + \sigma_2(s, 0)u_0(x) = g(s, 0) \quad \textit{on } S \tag{1.6}$$

is satisfied then there exists one and only one solution $u(x, t)$ *of* (1.5) *such that*

$$u(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega)).$$

2) $u(x, t) \in \mathcal{E}_t^k(E)$ means that $u(x, t)$ is k times continuously differentiable in t as E-valued function.

When the coefficients of L are sufficiently smooth the solution $u(x, t)$ becomes regular according to the regularity of given functions:

Theorem 2. *Suppose that the coefficients of L belong to $\mathcal{B}^{m+2}(\Omega\times[0, T])$ and*

$$\{u_0(x), u_1(x)\}\in H^{m+2}(\Omega)\times H^{m+1}(\Omega)$$

$$(1.7)\qquad f(x, t)\in\mathcal{E}_t^0(H^m(\Omega))\cap\mathcal{E}_t^1(H^{m-1}(\Omega))\cap\cdots\cdots\cap\mathcal{E}_t^{m-1}(H^1(\Omega))\cap\mathcal{E}_t^{m+1}(L^2(\Omega))$$

$$(1.8)\qquad g(s, t)\in\mathcal{E}_t^0(H^{m+1/2}(S))\cap\mathcal{E}_t^1(H^{m-1/2}(S))\cap\cdots\cdots\cap\mathcal{E}_t^m(H^{1/2}(S))\cap\mathcal{E}_t^{m+1}(L^2(S)),$$

then if the compatibility condition of order m[3] *is satisfied the solution $u(x, t)$ of* (1.5) *satisfies*

$$(1.9)\qquad u(x, t)\in\mathcal{E}_t^0(H^{m+2}(\Omega))\cap\mathcal{E}_t^1(H^{m+1}(\Omega))\cap\cdots\cdots\cap\mathcal{E}_t^{m+1}(H^1(\Omega))\cap\mathcal{E}_t^{m+2}(L^2(\Omega)).$$

We treat this problem as the following equivalent system

$$(1.10)\qquad \frac{d}{dt}U(t)=\mathcal{A}(t)U(t)+F(t)$$

$$(1.11)\qquad \mathcal{B}(t)U(t)=g(s, t)$$
$$U(0)=U_0,$$

where

$$U(t)=\begin{bmatrix}u(x, t)\\ u'(x, t)\end{bmatrix},\qquad F(t)=\begin{bmatrix}0\\ f(x, t)\end{bmatrix},$$

and

$$\mathcal{A}(t)=\begin{bmatrix}0 & 1\\ -a_2(x, t:D) & -a_1(x, t:D)\end{bmatrix}$$

the operator from $H^2(\Omega)\times H^1(\Omega)$ into $H^1(\Omega)\times L^2(\Omega)$ and

$$\mathcal{B}(t)=\begin{bmatrix}\frac{\partial}{\partial n_t}+\sigma_2(s, t) & \sigma_1(s, t)\end{bmatrix}$$

the operator from $H^2(\Omega)\times H^1(\Omega)$ into $H^{1/2}(S)$.

In our treatment, the energy inequality for the solution with non zero boundary data plays an essential role. The L^2-estimate of the solution by the boundary data has not been derived excepting the case of two independent variables [12].

3) This condition is stated in Section 5.

In Appendix it is seen that the condition (1.3) is necessary to hold Theorem 1 and the energy inequality (3.2) simultaneously.

Our method to prove Theorem 1 is as follows: at first we show Theorem 1 when the coefficients of L are independent of t, then in general case we make use of the method of Cauchy's polygonal line. The proof of Theorem 2 is essentially same as the proof of the regularity of the solution in the previous paper.

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2. Notations and lemmas

In this section we introduce some norms in the spaces $H^1(\Omega)\times L^2(\Omega)$ and $H^2(\Omega)\times H^1(\Omega)$, and show some basic properties of $\mathcal{A}(t)$ and $\mathcal{B}(t)$, because for our treatment it is convenient to make use some of suitable norms attached to the operators $\mathcal{A}(t)$ and $\mathcal{B}(t)$.

We denote by E_i ($i=1, 2, 3,\cdots$) the space $H^i(\Omega)\times H^{i-1}(\Omega)$ whose norm is denoted by $|||\cdot|||_i$, i. e.

$$|||U|||_i^2 = ||u||_{i,L^2(\Omega)}^2+||v||_{i-1,L^2(\Omega)}^2 \tag{2.1}$$

for $U=\{u, v\}\in H^i(\Omega)\times H^{i-1}(\Omega)$.

Let us remark that

$$\Big(\sum_{i,j=1}^{n}\Big(a_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big)+(u, u)\Big)^{1/2}$$

gives an equivalent norm in $H^1(\Omega)$ from the condition (1.2).

Denote by $\mathcal{H}(t)$ the space $H^1(\Omega)\times L^2(\Omega)$ equipped with the following norm, which is equivalent to $|||\cdot|||_1$,

$$||U||^2_{\mathcal{H}(t)} = (U, U)_{\mathcal{H}(t)} =\sum_{i,j=1}^{n}\Big(a_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big) +(u, u)+(v, v) \tag{2.2}$$

for $U=\{u, v\}\in H^1(\Omega)\times L^2(\Omega)$.

Lemma 2.1. *There exists a constant $M>0$ such that for all $U\in E_2$*

$$\frac{1}{M}|||U|||_2^2\leqslant||\mathcal{A}(t)U||^2_{\mathcal{H}(t)}+||U||^2_{\mathcal{H}(t)}+\langle\mathcal{B}(t)U\rangle^2_{1/2}\leqslant M|||U|||_2^2 \tag{2.3}$$

where $\langle\cdot\rangle_{1/2}$ denotes the norm of the space $H^{1/2}(S)$.

Proof. It is easily seen that $||\mathcal{A}(t)U||_{\mathcal{H}(t)}$ and $\langle\mathcal{B}(t)U\rangle_{1/2}$ are bounded by const. $|||U|||_2$, then it is enough to show only the left-hand side inequality of (2.3).

In order to drive this enequality we make use of the well known apriori estimate concerning the elliptic operator $a_2(x, t : D)$

$$(2.4) \qquad |u||^2_{2,L^2(\Omega)} \leqslant K\left(||a_2(x, t:D)u||^2 + ||u||^2 + \left\langle\frac{\partial}{\partial n_t}u\right\rangle^2_{1/2}\right).$$

Let us put $\mathcal{A}(t)U = F = \{f, g\}$, this means

$$\begin{cases} v = f \\ -a_2u - a_1v = g \,, \end{cases}$$

from which it follows immediately that

$$a_2u = -g - a_1f$$
$$\frac{\partial u}{\partial n_t} = \sigma_1(s, t)f - \sigma_2(s, t)u + \mathcal{B}(t)U \,.$$

The application of the estimate (2.4) to the above relation gives

$$\begin{aligned}(2.5) \qquad ||u||^2_{2,L^2(\Omega)} &\leqslant K(||g + a_1f||^2 + ||u||^2 + \langle\sigma_1f - \sigma_2u + \mathcal{B}(t)U\rangle^2_{1/2}) \\ &\leqslant K \text{ const. } (||g||^2 + ||f||^2_{1,L^2(\Omega)} + ||u||^2 + \langle f\rangle^2_{1/2} \\ &\quad + \langle\mathcal{B}(t)U\rangle_{1/2} + \langle u\rangle^2_{1/2}) \\ &\leqslant K \text{ const. } (||F||^2_{\mathcal{H}(t)} + ||U||^2_{\mathcal{H}(t)} + \langle\mathcal{B}(t)U\rangle^2_{1/2}).\end{aligned}$$

Of course

$$||v||_{1,L^2(\Omega)} = ||f||_{1,L(\Omega)} \leqslant \text{const. } ||F||_{\mathcal{H}(t)} \,.$$

By combining these estimates the desired inequality follows. Q.E.D.

As the immediate consequence of the above lemma if we define $|||U|||_{\mathcal{H}(t)}$ by the relation

$$(2.5) \qquad |||U|||^2_{\mathcal{H}(t)} = ||\mathcal{A}(t)U||^2_{\mathcal{H}(t)} + ||U||^2_{\mathcal{H}(t)} + \langle\mathcal{B}(t)U\rangle^2_{1/2}$$

$|||U|||_{\mathcal{H}(t)}$ gives an equivalent norm in E_2.

$\mathcal{D}(t)$ denotes the subset of E_2 of all the elements such that $\mathcal{B}(t)U = 0$ on S.

Lemma 2.2. *There exists a constant $c > 0$ such that for any $U \in E_2$ the following estimate holos*

$$\begin{aligned}(2.6) \qquad &(\mathcal{A}(t)U, U)_{\mathcal{H}(t)} + (U, \mathcal{A}(t)U)_{\mathcal{H}(t)} \\ &\qquad \leqslant c\{(U, U)_{\mathcal{H}(t)} + \langle\mathcal{B}(t)U\rangle^2_{1/2}\} \,.\end{aligned}$$

Proof. $(\mathcal{A}(t)U, U)_{\mathcal{H}(t)}+(U, \mathcal{A}(t)U)_{\mathcal{H}(t)}$

$$=\sum_{i,j=1}^{n}\left(a_{ij}(x, t)\frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_j}\right)+(v, u)+(-a_2u-a_1v, v)$$

$$+\sum_{i,j=1}^{n}\left(a_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j}\right)+(u, v)+(v, -a_2u-a_1v)$$

by integration by parts

$$=\int_S v\overline{\frac{\partial u}{\partial n_t}}dS+\int_S\frac{\partial u}{\partial n_t}\bar{v}\,dS-2\int_S\langle h, \nu\rangle v\bar{v}\,dS$$

$$+2\operatorname{Re}\left[(u, v)-\left(\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu, v\right)+\left(\left(\sum_{i=1}^{n}\frac{\partial h_i}{\partial x_i}-h\right)v, v\right)\right]$$

$$=I+II$$

$$I=\int_S v\cdot\overline{\left(\frac{\partial u}{\partial n_t}-\sigma_1v+\sigma_2u\right)}dS+\int_S\left(\frac{\partial u}{\partial n_t}-\sigma_1v+\sigma_2u\right)\bar{v}dS$$

$$+2\int_S(\sigma_1-\langle h, \nu\rangle)|v|^2dS-\int_S(\sigma_2u\bar{v}+v\overline{\sigma_2u})dS$$

$$=2\operatorname{Re}\int_S v\overline{\mathcal{B}U}dS+2\int_S(\sigma_1-\langle h, \nu\rangle)|v|^2dS-2\operatorname{Re}\int_S\sigma_2u\bar{v}\,dS$$

since $\sigma_1(s, t)-\langle h, \nu\rangle\leqslant-\varepsilon_0$

$$\leqslant 2\operatorname{Re}\int_S v\overline{\mathcal{B}U}dS-2\varepsilon_0\int_S|v|^2dS-2\operatorname{Re}\int_S\sigma_2u\bar{v}\,dS$$

$$\leqslant\varepsilon_0\int_S|v|^2dS+\frac{1}{\varepsilon_0}\int_S|\mathcal{B}U|^2dS-2\varepsilon_0\int_S|v|^2dS$$

$$+\varepsilon_0\int_S|v|^2dS+\frac{1}{\varepsilon_0}\int_S|\sigma_2u|^2dS$$

$$=\frac{1}{\varepsilon_0}\left[\int_S|\mathcal{B}U|^2dS+\int_S|\sigma_2u|^2dS\right]$$

$$\leqslant\text{const.}\ [\langle\mathcal{B}U\rangle^2+\|U\|^2_{\mathcal{H}(t)}]\,.$$

Evidently

$$|II|\leqslant 2\|u\|\,\|v\|+\text{const.}\ \|u\|_1\,\|v\|+\text{const.}\ \|v\|^2$$

$$\leqslant\text{const.}\ (\|u\|_1^2+\|v\|^2)\leqslant\text{const.}\ \|U\|^2_{\mathcal{H}(t)}\,.$$

From these estimates for I and II, we obtain the inequality (2.6). Q.E.D.

Corollary. *For all $U\in\mathcal{D}(t)$ we have*

$$\|(\lambda I-\mathcal{A}(t))U\|_{\mathcal{H}(t)}\geqslant(\lambda-c)\|U\|_{\mathcal{H}(t)} \tag{2.7}$$

if $\lambda>c$.

Proof. $\|(\lambda I-\mathcal{A}(t))U\|^2_{\mathcal{H}(t)}$

$$= ((\lambda I-\mathcal{A}(t))U,\ (\lambda I-\mathcal{A}(t))U)_{\mathcal{H}(t)}$$

$$\geqslant \lambda^2\|U\|^2_{\mathcal{H}(t)}-\lambda\{(\mathcal{A}(t)U,\ U)_{\mathcal{H}(t)}+(U,\ \mathcal{A}(t)U)_{\mathcal{H}(t)}\}$$

from (2.6) and $\mathcal{B}(t)U=0$

$$\geqslant \lambda^2\|U\|^2_{\mathcal{H}(t)}-c\lambda\|U\|^2_{\mathcal{H}(t)}$$

$$= (\lambda-c)^2\|U\|^2_{\mathcal{H}(t)}+c(\lambda-c)\|U\|^2_{\mathcal{H}(t)}\,.$$

Thus (2.7) is obtained. Q.E.D.

Lemma 2.3. *There exists a constant $\lambda_0>0$ such that for any $\lambda>\lambda_0$ $\lambda I-\mathcal{A}(t)$ is a bijective mapping from $\mathcal{D}(t)$ onto $\mathcal{H}(t)$. And if we denote by $(\lambda I-\mathcal{A}(t))^{-1}$ the inverse of the above mapping the following estimate holds:*

$$\|(\lambda I-\mathcal{A}(t))^{-1}\|_{\mathcal{H}(t)}\leqslant\frac{1}{\lambda-\lambda_0}\,. \tag{2.8}$$

Proof. Consider an equation in U

$$(\lambda I-\mathcal{A}(t))U = F\,,\quad U\in\mathcal{D}(t)\,,\quad F\in\mathcal{H}(t)\,. \tag{2.9}$$

Namely

$$\lambda u-v=f$$

$$a_2u+(\lambda+a_1)v=g\,,$$

where $f\in H^1(\Omega)$, $g\in L^2(\Omega)$. The substitution of the first relation into the second gives

$$a_\lambda u\equiv(a_2+\lambda a_1+\lambda^2)u = (\lambda+a_1)f+g\in L^2(\Omega)\,, \tag{2.10}$$

and since $\mathcal{B}U=0$, u satisfies the boundary condition

$$\frac{\partial u}{\partial n_t}-\lambda\sigma_1u+\sigma_2u= -\sigma_1f\,. \tag{2.11}$$

Conversely, if $u\in H^2(\Omega)$ satisfies (2.10) and (2.11), by defining $v=\lambda u-f$, we see that $U=\{u,\ v\}$ is the solution of (2.9).

Hence the solvability of (2.9) means the existence of the solution $u\in H^2(\Omega)$ of the boundary value problem of the elliptic equation containing the parameter

$$a_\lambda u=f \qquad \text{in}\ \ \Omega \tag{2.12}$$

$$\left(\frac{\partial}{\partial n_t}-\lambda\sigma_1+\sigma_2\right)u = -\sigma_1h \qquad \text{on}\ \ S \tag{2.13}$$

for any $f\in L^2(\Omega)$ and $h\in H^1(\Omega)$. To prove this, consider the quadratic form for

$\varphi, \psi \in H^1(\Omega)$

$$a_\lambda(\varphi, \psi) = \sum_{i,j=1}^{n}\left(a_{ij}(x, t)\frac{\partial\varphi}{\partial x_i}, \frac{\partial\psi}{\partial x_j}\right) + \left(\sum_{i=1}^{n} b_i(x, t)\frac{\partial\varphi}{\partial x_i} + c(x, t)\varphi, \psi\right) + \lambda\left\{\left(\sum_{i=1}^{n} h_i\frac{\partial\varphi}{\partial x_i}, \psi\right) + \left(\varphi, -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}h_i\,\psi\right) + (h\varphi, \psi)\right\} + \lambda^2(\varphi, \psi) + \lambda\int_S (\langle h, \nu\rangle - \sigma_1)\varphi\bar{\psi}\,dS + \int_S \sigma_2\varphi\bar{\psi}\,dS. \tag{2.14}$$

By taking account of (1.2) and (1.3), it follows for $\lambda > 0$

$$\mathrm{Re}\, a_\lambda(\varphi, \varphi) \geqslant d\sum_{i=1}^{n}\left\|\frac{\partial\varphi}{\partial x_i}\right\| - c_1\lambda\|\varphi\|^2 + \lambda\varepsilon_0\int_S |\varphi|^2 dS - \int_S \sigma_2|\varphi|^2 dS + \lambda^2\|\varphi\|^2, \tag{2.15}$$

then for some constant $\lambda_0 > 0$ if $\lambda > \lambda_0$ we have

$$\mathrm{Re}\, a_\lambda(\varphi, \varphi) \geqslant d\|\varphi\|^2_{1, L^2(\Omega)},$$

which assures the existence of the solution $u \in H^1(\Omega)$ of the variational equation

$$a_\lambda(u, \varphi) = (f, \varphi) - \int_S \sigma_1 h\bar{\varphi}\,dS \quad \forall\varphi \in H^1(\Omega) \tag{2.16}$$

for any $f \in L^2(\Omega)$, $h \in H^1(\Omega)$. This shows that u satisfies (2.12) in the sense of $\mathcal{D}'(\Omega)$. Moreover with the aid of the theory of the regularity we see $u \in H^2(\Omega)$, which implies u is the solution of (2.12) and (2.13), therefore the solvability (2.9) is shown.

The last part of Lemma is led immediately by combining the solvability of (2.9) and the corollary of the previous lemma. Q.E.D.

Lemma 2.4. *Let t_0 be any fixed point in $[0, T]$. Suppose that $F(t) \in \mathcal{D}(t_0)$ for all $t \in [0, T]$ and $F(t)$, $\mathcal{A}(t_0)F(t)$ are continuous in $\mathcal{H}(t_0)$, then for any $U_0 \in \mathcal{D}(t_0)$ there exists one and only one solution of the equation*

$$\begin{cases} \dfrac{d}{dt}U(t) = \mathcal{A}(t_0)U(t) + F(t) \\ U(0) = U_0 \end{cases} \tag{2.17}$$

such that $U(t) \in \mathcal{D}(t_0)$ for all $t \in [0, T]$ and $U(t) \in \mathcal{E}_t^1(\mathcal{H}(t_0))$.

Proof. This is an immediate consequence of the application of Hille-Yosida's theorem to the operator $\mathcal{A}(t_0)$ in $\mathcal{H}(t_0)$. In virture of Lemma 2.3 it suffices to show only that $\mathcal{D}(t_0)$ is dense in $\mathcal{H}(t_0)$. And this follows from $(\lambda I - \mathcal{A}(t_0))^{-1}\mathcal{H}(t_0)=\mathcal{D}(t_0)$ and (2.6)[4]. Q.E.D.

3. Energy inequality

We derive the energy inequality for the solution with non-homogeneous boundary condition. This inequality plays an essential role in the proof of the existence of the solution since even for the zero boundary data if the coefficients of the principal part of L depend on t on the boundary S, our proof needs the existence of the solution for non-zero boundary data related to the equation with coefficients independent of t.

At first we show the following inequality

Lemma 3.1. *Let* $u(x, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$ *and*

$$L[u(x, t)] = f(x, t)$$
$$B(t)u(x, t) = g(s, t)\,,$$

then the inequality

$$\|U(t)\|^2_{\mathcal{H}(t)}\leqslant e^{ct}\Big(\|U(0)\|^2_{\mathcal{H}(t)}+\int_0^t\|f(x, \tau)\|^2 d\tau + c\int_0^t\langle g(s, \tau)\rangle^2 d\tau\Big) \tag{3.1}$$

holds where $U(t)=\{u(x, t), u'(x, t)\}$.

Proof.
$$\frac{d}{dt}\|U(t)\|^2_{\mathcal{H}(t)}=(U'(t), U(t))_{\mathcal{H}(t)}+(U(t), U'(t))_{\mathcal{H}(t)}+(U(t), U(t))_{\dot{\mathcal{H}}(t)}$$
$$= (\mathcal{A}(t)U(t), U(t))_{\mathcal{H}(t)}+(U(t), \mathcal{A}(t)U(t))_{\mathcal{H}(t)}+(F(t), U(t))_{\mathcal{H}(t)}+(U(t), F(t))_{\mathcal{H}(t)}+(U(t), U(t))_{\dot{\mathcal{H}}(t)}$$

where

$$(U, U)_{\dot{\mathcal{H}}(t)} =\sum_{i,j=1}^n\Big(a'_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big)$$

for $U=\{u, v\}\in\mathcal{H}(t)$. Evidently

$$|(U(t), U(t))_{\dot{\mathcal{H}}(t)}| \leqslant\text{const.}\ \|U(t)\|^2_{\mathcal{H}(t)}\,.$$

And

$$|(U(t), F(t))_{\mathcal{H}(t)}| \leqslant\|F(t)\|^2_{\mathcal{H}(t)}+\|U(t)\|^2_{\mathcal{H}(t)}\,.$$

Then by using the inequality (2.6) for $U(t)$ and the above two estimates, we get

4) See T. Kato, Perturbation theory of linear operator, Springer (1966), page 277.

$$\frac{d}{dt}||U(t)||^2_{\mathcal{H}(t)} \leqslant c[||U(t)||^2_{\mathcal{H}(t)} + \langle \mathcal{B}(t)U(t)\rangle^2] + ||F(t)||^2_{\mathcal{H}(t)},$$

from which it follows

$$||U(t)||^2_{\mathcal{H}(t)} \leqslant e^{ct}[||U(0)||^2_{\mathcal{H}(0)} + c\int_0^t \langle \mathcal{B}(t)U(t)\rangle^2 dt + \int_0^t ||F(\tau)||^2_{\mathcal{H}(\tau)} d\tau].$$

Since $||F(\tau)||^2_{\mathcal{H}(\tau)} = ||f(x, \tau)||^2$ and $\mathcal{B}(t)U(t) = g(s, t)$ (3.1) is shown. Q.E.D.

Now we prove the desired energy inequality

Proposition 3.1. *Let* $u(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$ *for* $t \in [0, T+\delta_0]$ $(\delta_0 > 0)$. *If*

$$L[u(x, t)] = f(x, t) \in \mathcal{E}_t^1(L^2(\Omega))$$
$$B(t)u(x, t) = g(s, t) \in \mathcal{E}_t^0(H^{1/2}(S)) \cap \mathcal{E}_t^1(L^2(S)),$$

then the energy inequality

$$\begin{aligned}(3.2)\qquad &||u(x, t)||^2_{2, L^2(\Omega)} + ||u'(x, t)||^2_{1, L^2(\Omega)} + ||u''(x, t)||^2_{L^2(\Omega)} \\ &\leqslant c(T)[||u(x, 0)||^2_{2, L^2(\Omega)} + ||u'(x, 0)||^2_{1, L^2(\Omega)} \\ &\quad + ||f(x, 0)||^2_{L^2(\Omega)} + \int_0^t ||f'(x, \tau)||^2 d\tau + \int_0^t \langle g'(s, \tau)\rangle^2 d\tau \\ &\quad + \sup_{0 \leqslant \tau \leqslant t} \langle g(s, \tau)\rangle_{1/2}{}^2]\end{aligned}$$

holds for $t \in [0, T]$, *where* $c(T)$ *does not depend on* $u(x, t)$.

Proof. Assume that

$$u'(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega)).$$

Then $U'(t)$ satisfies

$$\frac{d}{dt}U'(t) = \mathcal{A}(t)U'(t) + \mathcal{A}'(t)U(t) + F'(t)$$
$$\mathcal{B}(t)U'(t) = -\mathcal{B}'(t)U(t) + g'(s, t).$$

Now by applying (3.1) for $U'(t)$ we have

$$||U'(t)||^2_{\mathcal{H}(t)} \leqslant e^{ct}[||U'(0)||^2_{\mathcal{H}(0)} + \int_0^t ||\mathcal{A}'(\tau)U(\tau) + F'(\tau)||^2_{\mathcal{H}(\tau)} d\tau + c\int_0^t \langle -\mathcal{B}'(\tau)U(\tau) + g'(s, \tau)\rangle^2 d\tau].$$

Remark that

$$\begin{aligned}||\mathcal{A}'(t)U(t)||^2_{\mathcal{H}(t)} &= ||a_2'(x,t:D)u(x,t)+a_1'(x,t:D)u'(x,t)||^2_{L^2(\Omega)}\\ &\leqslant \text{const.}\,(||u(x,t)||^2_{2,L^2(\Omega)}+||u'(x,t)||^2_{1,L^2(\Omega)})\\ &= \text{const.}\,|||U(t)|||^2_2\end{aligned}$$

$$\begin{aligned}\langle\mathcal{B}'(t)U(t)\rangle^2 &= \left\langle\left(\frac{\partial}{\partial n_t}\right)'u(x,t)-\sigma_1'(s,t)u'(s,t)+\sigma_2'(s,t)u(x,t)\right\rangle^2\\ &\leqslant \text{const.}\,(||u(x,t)||^2_{2,L^2(\Omega)}+||u'(x,t)||^2_{1,L^2(\Omega)})\\ &= \text{const.}|||U(t)|||^2_2\,,\end{aligned}$$

and

$$\begin{aligned}||U'(0)||^2_{\mathcal{H}(0)} &= ||\mathcal{A}(0)U(0)+F(0)||^2_{\mathcal{H}(0)}\\ &\leqslant 2(||\mathcal{A}(0)U(0)||^2_{\mathcal{H}(0)}+||F(0)||^2_{\mathcal{H}(0)})\\ &\leqslant \text{const.}\,(|||U(0)|||^2_2+||f(x,0)||^2)\,.\end{aligned}$$

Thus we get for some constant $c_1>0$, which is independent of u,

$$\begin{aligned}||U'(t)||^2_{\mathcal{H}(t)} \leqslant c_1e^{ct}\Big[&|||U(0)|||^2_2+||f(x,0)||^2_{L^2(\Omega)}\\ &+\int_0^t\{|||U(\tau)|||^2_2+||f'(x,\tau)||^2\}\,d\tau\\ &+\int_0^t\{|||U(\tau)|||^2_2+\langle g'(s,\tau)\rangle^2\}\,d\tau\Big]\end{aligned}\tag{3.3}$$

From inequality (2.3)

$$\begin{aligned}|||U(t)|||^2_2&+||U'(t)||^2_{\mathcal{H}(t)} \leqslant M\{\,||\mathcal{A}(t)U(t)||^2_{\mathcal{H}(t)}+||U(t)||^2_{\mathcal{H}(t)}\\ &+\langle\mathcal{B}(t)U(t)\rangle_{1/2}{}^2\}+||U'(t)||^2_{\mathcal{H}(t)}\\ &= M\{||U'(t)-F(t)||^2_{\mathcal{H}(t)}+||U(t)||^2_{\mathcal{H}(t)}+\langle\mathcal{B}(t)U(t)\rangle_{1/2}{}^2\}\\ &+||U'(t)||^2_{\mathcal{H}(t)}\\ &\leqslant(2M+1)\{||U'(t)||^2_{\mathcal{H}(t)}+||U(t)||^2_{\mathcal{H}(t)}+||f(t)||^2+\langle g(s,t)\rangle_{1/2}{}^2\}\end{aligned}$$

by inserting the estimates (3.1) and (3.3)

$$\begin{aligned}&\leqslant(2M+1)c_1e^{ct}\Big[|||U(0)|||^2_2+||f(0)||^2\\ &+\int_0^t\{|||U(\tau)|||^2_2+||f'(\tau)||^2\}\,d\tau+\int_0^t\{|||U(\tau)|||^2_2+\langle g'(s,\tau)\rangle^2\}\,d\tau\\ &+||U(0)||^2_{\mathcal{H}(0)}+\int_0^t||f(x,\tau)||^2\,d\tau+\int_0^t\langle g(s,\tau)\rangle^2\,d\tau\\ &+||f(t)||^2+\langle g(s,t)\rangle_{1/2}{}^2\Big]\,.\end{aligned}$$

And by using the obvious estimates

$$||f(x,t)||^2\leqslant 2T(||f(x,0)||^2+\int_0^t||f'(x,\tau)||^2\,d\tau)$$

$$\langle g(s,t)\rangle^2\leqslant 2T(\langle g(s,0)\rangle^2+\int_0^t\langle g'(s,\tau)\rangle^2\,d\tau)\,.$$

the following inequality holds for some constant c_1'

$$\begin{aligned}&|||U(t)|||_2^2+||U'(t)||^2_{\mathcal{H}(t)}\\&\leqslant c_1'[|||U(0)|||_2^2+||f(0)||^2+\int_0^t||f'(x,\tau)||^2d\tau+\int_0^t\langle g'(s,\tau)\rangle^2d\tau\\&\quad+\sup_{0\leqslant\tau\leqslant t}\langle g(s,\tau)\rangle_{1/2}{}^2+\int_0^t|||U(\tau)|||_2^2d\tau]\end{aligned}$$

for all $t\in[0, T]$.

At this point we make use of the following Lemma:

Lemma. *Let $\gamma(t)$ and $\rho(t)$ be defined on $[0, a]$ $(a>0)$ and non-negative. If $\gamma(t)$ is summable on $[0, a]$ and $\rho(t)$ is non-decreasing, and*

$$\gamma(t)\leqslant c\int_0^t\gamma(s)\,ds+\rho(t)$$

holds for all $t\in[0, a]$, then we get

$$\gamma(t)\leqslant e^{ct}\rho(t)\,.$$

Then (3.2) is led by applying Lemma by taking as

$$\begin{aligned}\gamma(t)&=|||U(t)|||_2^2+||U'(t)||^2_{\mathcal{H}(t)}\\\rho(t)&=c_1'[|||U(0)|||_2^2+||f(0)||^2+\int_0^t||f'(x,\tau)||^2d\tau\\&\quad+\int_0^t\langle g'(s,\tau)\rangle^2d\tau+\sup_{0\leqslant\tau\leqslant t}\langle g(s,\tau)\rangle_{1/2}{}^2]\,,\end{aligned}$$

here we take $c(T)$ as $c_1'e^{c_1'T}$.

To remove the additional assumption that $u'(x, t)$ is again in $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$, we use the mollifier with respect to t, which is the following one: Let $\varphi(t)$ be C^∞-function with a support contained in $[-2, -1]$ such that $\varphi(t)\geqslant 0$ and $\int_{-\infty}^{\infty}\varphi(t)\,dt=1$. Then we define $\varphi_{\delta}\underset{(t)}{*}$ by

$$u_\delta(x,t)=(\varphi_\delta\underset{(t)}{*}u)(x,t)=\int\varphi_\delta(t-\tau)\,u(x,\tau)\,d\tau$$

for $u(x, t)\in L^2(\Omega\times(0, T+\delta_0))$, where

$$\varphi_\delta(t)=\frac{1}{\delta}\varphi\Big(\frac{t}{\delta}\Big)\,.$$

Remark that $u_\delta(x, t)\in\mathcal{E}_t^\infty(L^2(\Omega))$ and $\varphi_\delta\underset{(t)}{*}$ commutes with $\dfrac{\partial}{\partial t}$ if $t\in[0, T]$ and $0<\delta<\dfrac{\delta_0}{2}$.

Applying $\varphi_{\delta(t)}^{*}$ to (1.1) and (1.4), we get

$$L[u_\delta]=f_\delta-C_\delta u \tag{3.4}$$

$$B(t)u_\delta=g_\delta-\Gamma_\delta u \tag{3.5}$$

where

$$(C_\delta u)(x,t)=[\varphi_{\delta(t)}^{*},a_1(x,t:D)]\frac{\partial u}{\partial t}+[\varphi_{\delta(t)}^{*},a_2(x,t:D)]u$$

$$(\Gamma_\delta u)(x,t)=[\varphi_{\delta(t)}^{*},\frac{\partial}{\partial n_t}]u-[\varphi_{\delta(t)}^{*},\sigma_1(s,t)]\frac{\partial u}{\partial t}+[\varphi_{\delta(t)}^{*},\sigma_2(s,t)]u$$

for all $t\in[0,T]$ if $0<\delta<\frac{\delta_0}{2}$.

Since $u_\delta(x,t)\in\mathcal{E}_t^\infty(H^2(\Omega))$, we can apply the just obtained result for $u_\delta(x,t)$ then

$$\begin{aligned}&||u_\delta(x,t)||_2^2+||u_\delta'(x,t)||_1^2+||u_\delta''(x,t)||^2\\&\leqslant c(T)\Big[||u_\delta(x,0)||_2^2+||u_\delta'(x,0)||_1^2+||f_\delta(x,0)||^2\\&+\int_0^t||f_\delta'(x,\tau)||^2d\tau+||(C_\delta u)(x,0)||^2\\&+\int_0^t\left\|\frac{\partial}{\partial\tau}(C_\delta u)(x,\tau)\right\|^2d\tau+\int_0^t\langle g_\delta'(s,\tau)\rangle^2d\tau\\&+\sup_{0\leqslant\tau\leqslant t}\langle g_\delta(x,\tau)\rangle_{1/2}{}^2+\int_0^t\left\langle\frac{\partial}{\partial\tau}(\Gamma_\delta u)(s,\tau)\right\rangle^2d\tau\\&+\sup_{0\leqslant\tau\leqslant t}\langle(\Gamma_\delta u)(s,\tau)\rangle_{1/2}{}^2\Big].\end{aligned} \tag{3.6}$$

Evidently we have for all $t\in[0,T]$

$$\begin{aligned}||u_\delta(x,t)||_2&\to||u(x,t)||_2\\||u_\delta'(x,t)||_1&\to||u'(x,t)||_1\\||u_\delta''(x,t)||&\to||u''(x,t)||\\||f_\delta'(x,t)||&\to||f'(x,t)||\\||f_\delta(x,t)||&\to||f(x,t)||\\\langle g_\delta'(s,t)\rangle&\to\langle g'(s,t)\rangle\\\langle g_\delta(s,t)\rangle_{1/2}&\to\langle g(s,t)\rangle_{1/2}\end{aligned}$$

when δ tends to zero. Moreover we have

$$(3.7)\quad \begin{cases} \|(C_\delta u)(x, 0)\| \to 0 \\ \int_0^T \left\| \frac{\partial}{\partial \tau}(C_\delta u)x, \tau) \right\|^2 d\tau \to 0 \\ \sup_{0\leqq \tau \leqq T} \langle (\Gamma_\delta u)(s, \tau)\rangle_{1/2}{}^2 \to 0 \\ \int_0^T \left\langle \frac{\partial}{\partial \tau}(\Gamma_\delta u)(s, \tau)\right\rangle^2 d\tau \to 0 \end{cases}$$

when δ tends to zero. To show the second, in the view of the explicit form of $C_\delta u$, it is enough to show the following fact: Let $a(x, t) \in \mathscr{B}^2(\Omega \times [0, T+\delta_0])$ and $v(x, t) \in L^2(\Omega \times [0, T+\delta_0])$.

Then by putting

$$\psi_\delta(x, t) = \frac{\partial}{\partial t}\{[\varphi_{\delta(\overset{*}{t})}, a(x, t)]\, v(x, t)\}$$

we have

$$\int_0^T \|\psi_\delta(x, t)\|^2 dt \to 0$$

when $\delta \to 0$. From

$$\psi_\delta(x, t) = \int \frac{\partial}{\partial \tau}\{\varphi_\delta(t-\tau)[a(x, \tau)-a(x, t)]\}[v(x, \tau)-v(x, t)]\, d\tau$$
$$+\int \varphi_\delta(t-\tau)[a(x, \tau)-a(x, t)]\, v(x, \tau)\, d\tau$$

the desired property of ψ_δ is led. The first one can be shown more easily.

Thus the passage to the limit of (3.6) when $\delta \to 0$ proves Proposition.

Q.E.D.

4. Existence of the solution (Proof of Theorem 1)

In the case where the coefficients of L depend on t, we use the method of Cauchy's polygonal line[5], for which it is need of the existence and a certain estimate of the solution for non zero boundary data in the case where the coefficients are independent of t.

Let us denote by $L(t_0)$ and $B(t_0)$ the operators

$$L(t_0) = \frac{\partial^2}{\partial t^2} + a_0(x, t_0 : D)\frac{\partial}{\partial t} + a_2(x, t_0 : D)$$

$$B(t_0) = \frac{\partial}{\partial n_{t_0}} - \sigma_1(s, t_0)\frac{\partial}{\partial t} + \sigma_2(s, t_0)$$

5) See, for example, Mizohata [8] Chapter 6.

respectively. Now we shall treat the existence of the solution for $L(t_0)$ and $B(t_0)$.

Proposition 4.1. *Given* $\{u_0(x), u_1(x)\}\in E_2$, $f(x, t)\in\mathcal{E}_t^1(L^2(\Omega))$ *and* $g(s, t)\in \mathcal{E}_t^0(H^{1/2}(S))\cap\mathcal{E}_t^2(L^2(S))$, *if the compatibility condition at* $t=0$

$$\frac{\partial}{\partial n_{t_0}}u_0(x)-\sigma_1(s, t_0)u_1(x)+\sigma_2(s, t_0)u_0(x)=g(s, 0) \qquad on\ S \tag{4.1}$$

is satisfied there exists one and only one solution $u(x, t)$ *of the mixed problem*

$$\begin{cases} L(t_0)[u(x, t)]=f(x, t) & in\ \ \Omega\times(0, T) \\ B(t_0)u(x, t)=g(s, t) & on\ \ S\times[0, T) \\ u(x, 0)=u_0(x),\ \dfrac{\partial u}{\partial t}(x, 0)=u_1(x) \end{cases} \tag{4.2}$$

such that $u(x, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$.

Proof. When $g(s, t)\equiv 0$, (4.1) means $\{u_0(x), u_1(x)\}\in\mathcal{D}(t_0)$. Then if $f(x, t)\in \mathcal{E}_t^0(\mathcal{D}_{L^2}^1(\Omega))$, Lemma 2.4 assures the existence of solution $U(t)$ of (2.17), thus $u(x, t)$ the first component of $U(t)$ is a solution of (4.2) in $\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$ and $u(x, t)\in H^2(\Omega)$. We can see that $U(t)\in\mathcal{E}_t^0(E_2)$ from

$$U(t)\in\mathcal{D}(t_0)\,,$$
$$\mathcal{A}(t_0)U(t)=\frac{d}{dt}U(t)-F(t)\in\mathcal{E}_t^0(\mathcal{H}(t_0))$$

and by using the inequality (2.3), thus $u(x, t)\in\mathcal{E}_t^0(H^2(\Omega))$ is shown. The condition $f(x, t)\in\mathcal{E}_t^0(\mathcal{D}_{L^2}^1(\Omega))$ is removed with the aid of the fact that $\mathcal{E}_t^1(\mathcal{D}_{L^2}^1(\Omega))$ is dense in $\mathcal{E}_t^1(L^2(\Omega))$ and the energy inequality (3.2).

When $g(s, t)\not\equiv 0$, at first assume that $g(s, t)$ is sufficiently smooth, then we can construct a function $w(x, t)\in\mathcal{E}_t^3(H^2(\Omega))$ such that

$$B(t_0)w(x, t)=g(s, t)\,.$$

Then the obtained result assures the existence of a function

$$v(x, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$$

satisfying

$$L(t_0)[v(x, t)]=f(x, t)-L(t_0)[w(x, t)]$$
$$v(x, 0)=u_0(x)-w(x, 0)$$
$$\frac{\partial v}{\partial t}(x, 0)=u_1(x)-\frac{\partial w}{\partial t}(x, 0)$$
$$B(t_0)v(x, t)=0$$

since $\{u_0-w(x, 0), u_1-w'(x, 0)\}\in\mathscr{D}(t_0)$, $f(x, t)-L(t_0)[w(x, t)]\in\mathcal{E}_t^1(L^2(\Omega))$. Thus $u(x, t)=v(x, t)+w(x, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$ is a solution of the mixed problem (4.2). Then if we know the existence of a sequence of initial data $\{u_{k0}(x), u_{k1}(x)\}\in E_2$ and sufficiently smooth boundary data $g_k(s, t)$ such that

$$\mathscr{B}(t_0)\{u_{k0}(x), u_{k1}(x)\} = g_k(s, 0) \tag{4.3}$$

$$\{u_{k0}(x), u_{k1}(x)\} \to \{u_0(x), u_1(x)\} \quad \text{in} \quad E_2 \tag{4.4}$$

$$g_k(s, t) \to g(s, t) \quad \text{in} \quad \mathcal{E}_t^0(H^{1/2}(S))\cap\mathcal{E}_t^1(L^2(S)), \tag{4.5}$$

Proposition is proved. Indeed $u_k(x, t)$ the solution of (4.2) for $\{u_{k0}(x), u_{k1}(x)\}$ and $g_k(s, t)$ exists and the sequence $u_k(x, t)$ $(k=1, 2,\cdots)$ is a Cauchy sequence in $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$, this is seen by applying (3.2) for $u_k(x, t)-u_l(x, t)$. Then the limit of $u_k(x, t)$ is the required solution of (4.2) for $\{u_0(x), u_1(x)\}$ and $g(s, t)$.

Now let us show the existence of such $\{u_{k0}(x), u_{k1}(x)\}$ and $g_k(s, t)$. Since sufficiently smooth functions in $\mathcal{E}_t^1(H^{1/2}(S))$ are dense in $\mathcal{E}_t^0(H^{1/2}(S))\cap\mathcal{E}_t^1(L^2(S))$, there exists a sequence $g_k(s, t)$ of sufficiently smooth functions in $\mathcal{E}_t^1(H^{1/2}(S))$ which tends to $g(s, t)$ in $\mathcal{E}_t^0(H^{1/2}(S))\cap\mathcal{E}_t^1(L^2(S))$. Of course

$$\langle g_k(s, 0)-g(s, 0)\rangle_{1/2} \to 0 \quad \text{when} \quad k\to\infty . \tag{4.6}$$

If Ω is the interior of S the boundary value problem

$$\begin{cases} (a_2(x, t_0 : D)+\lambda_0)u = 0 & \text{in} \quad \Omega \\ \left(\dfrac{\partial}{\partial n_{t_0}}+\sigma_2(s, 0)\right)u = q(s) & \text{on} \quad S \end{cases} \tag{4.7}$$

has a unique solution $u\in H^2(\Omega)$ for any $q(s)\in H^{1/2}(S)$ for a large $\lambda_0>0$ and the following estimates holds

$$\|u\|_{2,L^2(\Omega)}^2 \leqslant K\langle q(s)\rangle_{1/2}^2 . \tag{4.8}$$

Then if we take $u_{k0}=u_0(x)+\tilde{u}_k(x)$, $u_{k1}(x)=u_1(x)$, where $\tilde{u}_k(x)$ is the solution of (4.7) for taking $q(s)=g_k(s, 0)-g(s, 0)$, they are the required ones, for (4.4) follows from

$$\|u_{k0}(x)-u_0(x)\|_{2,L^2(\Omega)}^2 \leqslant K\langle g_k(s, 0)-g(s, 0)\rangle_{1/2}^2 ,$$

and (4.3) and (4.5) are evident.

When Ω is the exterior of S, let S_1 be a sufficiently smooth hypersurface containing S in its interior and denote by Ω_1 the domain surrounded by S and S_1. $\alpha(x)$ be a C^∞-function such that $\alpha(x)\equiv 1$ near S and $\alpha(x)\equiv 0$ near and outside of S_1. Consider

$$\begin{cases} (-a_1(x, t_2 : D)+\lambda_0)u = 0 & \text{in } \Omega_1 \\ \left(\frac{\partial}{\partial n_{t_0}}+\sigma_2(s, t_0)\right)u = q(s) & \text{on } S \\ \left(\frac{\partial}{\partial n_{t_0}}+\sigma_2(s, t_0)\right)u = 0 & \text{on } S_1 \end{cases}$$

instead of (4.7). Then we have

$$\|u\|^2_{2, L^2(\Omega_1)} \leqslant K\langle q(s)\rangle_{1/2}^2$$

therefore

$$\|\alpha u\|^2_{2, L^2(\Omega)} \leqslant K'\langle q(s)\rangle_{1/2}^2 .$$

Hence

$$u_{k0}(x)=u_1(x)+\alpha(x)\tilde{u}_k(x)$$
$$u_{k1}(x) = u_1(x)$$

are the required ones. Thus Proposition is proved. Q.E.D.

Proposition 4.2. *Let $u(x, t)\in\mathcal{E}_t^0(H^1(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$ for $t\geqslant t_0$. If*

$$L(t_0)[u(x, t)] = f(x, t)\in\mathcal{E}_t^1(\mathcal{D}^1_{L^2}(\Omega))$$
$$B(t_0)u(x, t) = g(s, t)\in\mathcal{E}_t^1(H^{1/2}(S)) ,$$

then for $t\geqslant t_0$ the estimate

$$\text{(4.8)} \qquad |||U(t)|||^2_{\mathcal{H}(t_0)} \leqslant e^{c_2(t-t_0)}[|||U(t_0)|||^2_{\mathcal{H}(t_0)}+\int_{t_0}^t |||F(\tau)|||^2_{\mathcal{H}(t_0)}\,d\tau + c_0\int_{t_0}^t \{\langle g(s, \tau)\rangle_{1/2}^2+\langle g'(s, \tau)\rangle_{1/2}^2\}\,d\tau]$$

holds where c_2, c_0 do not depend on u and t_0.

Proof. Apply Lemma 3.1 to this case taking as $t=t_0$ being an initial plane. Here the coefficients of the operators are independent of t. We have

$$\text{(4.9)} \qquad \|U(t)\|^2_{\mathcal{H}(t_0)} \leqslant e^{c(t-t_0)}[\|U(t_0)\|^2_{\mathcal{H}(t_0)}+\int_{t_0}^t \|F(\tau)\|^2_{\mathcal{H}(t_0)}\,d\tau + c_1\int_{t_0}^t \langle g(s, \tau)\rangle^2\,d\tau] .$$

Now suppose that $u'(x, t)$ is also in $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$. Then

$$\mathcal{A}(t_0)U(t) = \frac{d}{dt}U(t)-F(t)\in\mathcal{E}_t^0(E_2)\cap\mathcal{E}_t^1(E_1)$$

$$\mathcal{B}(t_0)(\mathcal{A}(t_0)U(t)) = \mathcal{B}(t_0)\left(\frac{d}{dt}U(t)-F(t)\right)$$
$$= \mathcal{B}(t_0)\left(\frac{d}{dt}\,U(t)\right)=\frac{d}{dt}\mathcal{B}(t_0)U(t) = g'(s, t) ,$$

since $F(t)\in\mathcal{D}(t_0)$ for all t. By applying again Lemma 3.1 for $\mathcal{A}(t_0)U(t)$ it follows that

$$\|\mathcal{A}(t_0)U(t)\|^2_{\mathcal{H}(t_0)}\leqslant e^{c(t-t_0)}[\|\mathcal{A}(t_0)U(t)\|^2_{\mathcal{H}(t_0)} +\int_{t_0}^t\|\mathcal{A}(t_0)F(\tau)\|^2_{\mathcal{H}(t_0)}\,d\tau+c_1\int_{t_0}^t\langle g'(s,\tau)\rangle^2\,d\tau]\,. \tag{4.10}$$

$$|||U(t)|||^2_{\mathcal{H}(t_0)}=\|U(t)\|^2_{\mathcal{H}(t_0)}+\|\mathcal{A}(t_0)U(t)\|^2_{\mathcal{H}(t_0)}+\langle\mathcal{B}(t_0)U(t)\rangle_{1/2}^2$$

by inserting (4.9) and (4.10)

$$\begin{aligned}
&\leqslant e^{c(t-t_0)}[\|U(t_0)\|^2_{\mathcal{H}(t_0)}+\|\mathcal{A}(t_0)U(t)\|^2_{\mathcal{H}(t_0)}\\
&\quad+\int_{t_0}^t\|F(\tau)\|^2_{\mathcal{H}(t_0)}\,d\tau+\int_{t_0}^t\|\mathcal{A}(t_0)F(\tau)\|^2_{\mathcal{H}(t_0)}\,d\tau\\
&\quad+c_1\int_{t_0}^t\langle g(s,\tau)\rangle^2\,d\tau+c_1\int_{t_0}^t\langle g'(s,\tau)\rangle^2\,d\tau\\
&\quad+\langle\mathcal{B}(t_0)U(t_0)\rangle_{1/2}^2-\langle\mathcal{B}(t_0)U(t_0)\rangle_{1/2}^2+\langle\mathcal{B}(t_0)U(t)\rangle_{1/2}^2]\\
&=e^{c(t-t_0)}[|||U(t_0)|||^2_{\mathcal{H}(t_0)}+\int_{t_0}^t|||F(\tau)|||^2_{\mathcal{H}(t_0)}\,d\tau\\
&\quad+c_1\int_{t_0}^t\langle g(s,\tau)\rangle^2\,d\tau+c_1\int_{t_0}^t\langle g'(s,\tau)\rangle^2\,d\tau\\
&\quad+\langle g(s,t)\rangle_{1/2}^2-\langle g(s,t_0)\rangle_{1/2}^2]\,.
\end{aligned}$$

And

$$\begin{aligned}
&\langle g(s,t)\rangle_{1/2}^2-\langle g(s,t_0)\rangle_{1/2}^2\\
&=\int_{t_0}^t\frac{d}{d\tau}\langle g(s,\tau)\rangle_{1/2}^2\,d\tau\\
&=\int_{t_0}^t 2\langle g(s,\tau)\rangle_{1/2}\langle g'(s,\tau)\rangle_{1/2}\,d\tau\\
&\leqslant\int_{t_0}^t\{\langle g(s,\tau)\rangle_{1/2}^2+\langle g'(s,\tau)\rangle_{1/2}^2\}\,d\tau\,.
\end{aligned}$$

Then if we take $c_0\geqslant c_1+1$, (4.8) follows.

To remove the additional condition that

$$u'(x,t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$$

we make use of the mollifier as used in the proof of Proposition 3.1 and by taking account of the fact $\varphi_{\delta}(\overset{*}{t})$ commutes with $L(t_0)$, $B(t_0)$, the proof will be carried out without any difficulty.

Hereafter we denote $\langle g(s,\tau)\rangle_{1/2}^2+\langle g'(s,\tau)\rangle_{1/2}^2$ by $\langle\!\langle g(s,\tau)\rangle\!\rangle^2$.

Lemma 4.1. *There exists a constant $c_0>0$ such that for any t, $t'\in[0, T]$ and $U\in E_2$ the following estimate*

$$|||U|||^2_{\mathcal{H}(t')}\leqslant(1+c_0|t-t'|)|||U|||^2_{\mathcal{H}(t)} \tag{4.11}$$

holds.

Proof. Remark that for all $U \in E_1$

$$(4.12) \qquad \|U\|^2_{\mathcal{H}(t')} \leqslant (1+c_0'|t-t'|)\|U\|^2_{\mathcal{H}(t)}$$

holds since

$$\begin{aligned}
\|U\|^2_{\mathcal{H}(t')}-\|U\|^2_{\mathcal{H}(t)} &= \sum_{i,j=1}^{n}\Big((a_{ij}(x,t')-a_{ij}(x,t))\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big)\\
&\leqslant \text{const. } |t'-t|\cdot\|u\|^2_{1,L^2(\Omega)}\\
&\leqslant \text{const. } |t'-t|\cdot\|U\|^2_{\mathcal{H}(t)}\,.
\end{aligned}$$

$$\begin{aligned}
|||U|||^2_{\mathcal{H}(t')} &= \|U\|^2_{\mathcal{H}(t')}+\|\mathcal{A}(t')U\|^2_{\mathcal{H}(t')}+\langle\mathcal{B}(t')U\rangle_{1/2}^2\\
&\leqslant(1+c_0'|t'-t|)(\|U\|^2_{\mathcal{H}(t)}+\|\mathcal{A}(t')U\|^2_{\mathcal{H}(t)}+\langle\mathcal{B}(t')U\rangle_{1/2}^2)\\
&\leqslant(1+c_0'|t'-t|)\{\|U\|^2_{\mathcal{H}(t)}+\|\mathcal{A}(t)U\|^2_{\mathcal{H}(t)}+\langle\mathcal{B}(t)U\rangle_{1/2}^2\\
&\quad+2\|\mathcal{A}(t)U\|_{\mathcal{H}(t)}\|(\mathcal{A}(t')-\mathcal{A}(t))U\|_{\mathcal{H}(t)}\\
&\quad+\|(\mathcal{A}(t')-\mathcal{A}(t))U\|^2_{\mathcal{H}(t)}+2\langle\mathcal{B}(t)U\rangle_{1/2}\langle(\mathcal{B}(t')-\mathcal{B}(t))U\rangle_{1/2}\\
&\quad+\langle(\mathcal{B}(t')-\mathcal{B}(t))U\rangle_{1/2}^2\}\\
&\leqslant(1+c_0'|t'-t|)\{|||U|||^2_{\mathcal{H}(t)}+c_0''|t'-t|\,|||U|||^2_{\mathcal{H}(t)}\\
&\qquad\qquad+c_0''^2|t'-t|^2|||U|||^2_{\mathcal{H}(t)}\}\,,
\end{aligned}$$

here we used the estimates

$$\begin{aligned}
\|(\mathcal{A}(t')-\mathcal{A}(t))U\|^2_{\mathcal{H}(t)} &\leqslant \text{const. } |t'-t|^2|||U|||^2_2\\
&\leqslant \text{const. } M|t'-t|^2|||U|||^2_{\mathcal{H}(t)}\\
\langle(\mathcal{B}(t')-\mathcal{B}(t))U\rangle_{1/2}^2 &\leqslant \text{const. } |t'-t|^2|||U|||^2_{\mathcal{H}(t)}\,.
\end{aligned}$$

Thus we get (4.11) by taking $c_0=c_0'+2(c_0''T)^2$. Q.E.D.

Under these preparations we prove the existence of the solution for zero initial data.

Suppose that

$$\begin{aligned}
&f(x,t)\in\mathcal{E}^1_t(\mathcal{D}^1_{L^2}(\Omega))\\
&g(s,t)\in\mathcal{E}^1_t(H^{1/2}(S))\\
&g(s,0)=0\,.
\end{aligned}$$

Let

$$\Delta_k: t_0=0<t_1<t_2<\cdots<t_k=T$$

be the subdivision of $[0, T]$ into k equal parts. $u_k(x, t)$ is the Cauchy's polygonal line for this subdivision, which is constructed as follows:
Let $u_{k0}(x, t)$, defined for $t\in[t_0, t_1]$, be the solution of

$$\begin{aligned}
&L(t_0)[u_{k0}(x,t)]=f(x,t) &&\text{in}\quad \Omega\times(t_0,t_1]\\
&B(t_0)u_{k0}(x,t)=g(s,t) &&\text{on}\quad S\times[t_0,t_1]\\
&u_{k0}(x,t_0)=u'_{k0}(x,t_0)=0\,,
\end{aligned}$$

and for $i\geqslant 1$ $u_{ki}(x, t)$, defined for $t\in[t_i, t_{i+1}]$, be the solution of

$$
\begin{aligned}
&L(t_i)[u_{ki}(x, t)] = f(x, t) \qquad \text{in} \quad \Omega\times(t_i, t_{i+1}]\\
&B(t_i)u_{ki}(x, t) = g(s, t)+\frac{t_{i+1}-t}{t_{i+1}-t_i}[(B(t_i)-B(t_{i-1}))u_{ki-1}(t)]_{t=t_i}\\
&u_{ki}(x, t_i) = u_{ki-1}(x, t_i)\\
&u'_{ki}(x, t_i) = u'_{ki-1}(x, t_i)\,.
\end{aligned}
$$

Then $u_k(x, t)$ is defined for $t\in[0, T]$ by

$$u_k(x, t) = u_{ki}(x, t) \qquad \text{if} \quad t\in[t_i, t_{i+1}]\,.$$

The existence of such $u_{ki}(x, t)$ ($i=0, 1, \cdots, k-1$) is assured by Proposition 4.1, since the compatibility condition (4.1) is satisfied at each t_i. Consequently we find

$$u_k(x, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega)) \qquad \text{for} \quad t\in[0, T]$$

and

$$u_k(x, t)\in\mathcal{E}_t^2(L^2(\Omega)) \qquad \text{if} \quad t\neq t_i\,,$$

from which it follows that

$$u_k(x, t)\in H^2(\Omega\times(0, T))\,.$$

Now we shall show that for some constant $K>0$

$$\|u_k(x, t)\|_{2,L^2(\Omega\times(0,T))}\leqslant K \tag{4.13}$$

holds for all k. Let

$$U_{ki} = \{u_{ki}(x, t), u'_{ki}(x, t)\}\,.$$

We get

$$\langle\!\langle B(t_i)u_{ki}(x, t)\rangle\!\rangle^2\leqslant 2\langle\!\langle g(s, t)\rangle\!\rangle_{1/2}{}^2+\text{const.}\,|||U_{ki-1}(t_i)|||^2{}_{\mathcal{H}(t_i)} \tag{4.14}$$

if $t\neq t_i$ by combining these estimates

$$
\begin{aligned}
\langle B(t_i)u_{ki}(x, t)\rangle_{1/2}{}^2 &\leqslant 2\langle g(s, t)\rangle_{1/2}{}^2+2\langle(\mathcal{B}(t_i)-\mathcal{B}(t_{i-1}))\,U_{ki-1}(t_i)\rangle_{1/2}{}^2\\
&\leqslant 2\langle g(s, t)\rangle_{1/2}{}^2+\text{const.}\,|t_i-t_{i-1}|^2|||U_{ki-1}(t_i)|||_{\mathcal{H}(t_i)}{}^2\\
\langle B(t_i)u'_{ki}(x, t)\rangle_{1/2}{}^2 &\leqslant 2\langle g'(s, t)\rangle_{1/2}{}^2+\frac{2}{(t_i-t_{i-1})^2}\langle(\mathcal{B}(t_i)-\mathcal{B}(t_{i-1}))U_{ki-1}(t_i)\rangle_{1/2}{}^2\\
&\leqslant 2\langle g'(s, t)\rangle_{1/2}{}^2+\text{const.}\,|||U_{ki-1}(t_i)|||^2{}_{\mathcal{H}(t_i)}\,.
\end{aligned}
$$

In order to derive (4.13) we shall show the following:

$$(4.15)\qquad |||U_{ki}(t)|||^2_{\mathcal{H}(t_i)} \leqslant e^{ct}\Big(1+c_0\frac{T}{k}\Big)^{3i} \times\Big[\int_0^t |||F(\tau)|||^2_{\mathcal{H}(t_i)}\,d\tau+\int_0^t 2c_1 \ll g(s.\ \tau)\gg^2 d\tau\Big]$$

for $t\in[t_i, t_{i+1}]$ $(i=0, 1, 2, \cdots, k-1)$. For $i=0$, this is nothing but the inequality (4.8). Suppose that (4.15) holds for $i-1$, then by taking t as t_i, it follows

$$(4.16)\qquad |||U_{ki-1}(t_i)|||^2_{\mathcal{H}(t_{i-1})} \leqslant e^{ct_i}\Big(1+c_0\frac{T}{k}\Big)^{3(i-1)} \times\Big[\int_0^{t_i} |||F(\tau)|||^2_{\mathcal{H}(t_{i-1})}\,d\tau+2c_1\int_0^{t_i} \ll g(s,\ \tau)\gg^2 d\tau\Big].$$

And

$$(4.17)\qquad |||U_{ki}(t_i)|||^2_{\mathcal{H}(t_i)} = |||U_{ki-1}(t_i)|||^2_{\mathcal{H}(t_i)}$$

by (4.11)

$$\leqslant\Big(1+c_0\frac{T}{k}\Big)|||U_{ki-1}(t_i)|||^2_{\mathcal{H}(t_{i-1})}$$

by using (4.16)

$$\leqslant e^{ct_i}\Big(1+c_0\frac{T}{k}\Big)^{3i-2}\Big[\int_0^{t_i} |||F(\tau)|||^2_{\mathcal{H}(t_{i-1})}\,d\tau+2c_1\int_0^{t_i} \ll g(s,\ \tau)\gg^2 d\tau\Big]$$

by using again (4.11)

$$\leqslant e^{ct_i}\Big(1+c_0\frac{T}{k}\Big)^{3i-1}\Big[\int_0^{t_i} |||F(\tau)|||^2_{\mathcal{H}(t_i)}\,d\tau+2c_1\int_0^{t_i} \ll g(s,\ \tau)\gg^2 d\tau\Big].$$

Taking account of (4.14) the application (4.8) to U_{ki} gives

$$|||U_{ki}(t)|||^2_{\mathcal{H}(t_i)} \leqslant e^{c(t-t_i)}\Big[\,|||U_{ki}(t_i)|||^2_{\mathcal{H}(t_i)}+\int_{t_i}^t |||F(\tau)|||^2_{\mathcal{H}(t_i)}\,d\tau +2c_1\int_{t_i}^t\{\ll g(s,\ \tau)\gg^2+c''|||U_{ki}(t_i)|||^2_{\mathcal{H}(t_i)}\}\,d\tau\Big]$$

$$\leqslant e^{c(t-t_i)}\Big[\Big(1+\frac{2c_1c''T}{k}\Big)|||U_{ki}(t_i)|||^2_{\mathcal{H}(t_i)} +\int_{t_i}^t |||F(\tau)|||^2_{\mathcal{H}(t_i)}\,d\tau+2c_1\int_{t_i}^t \ll g(s,\ \tau)\gg^2 d\tau\Big]$$

by (4.17)

$$\leqslant e^{c(t-t_i)}\Big[\Big(1+\frac{2c''c_0T}{k}\Big)e^{ct_i}\Big(1+\frac{c_0T}{k}\Big)^{3i-1} \cdot\Big\{\int_0^{t_i} |||F(\tau)|||^2_{\mathcal{H}(t_i)}\,d\tau+\int_0^{t_i} 2c_1 \ll g(s,\ \tau)\gg^2 d\tau\Big\}$$

$$+\int_{t_i}^{t}|||F(\tau)|||^2_{\mathcal{H}(t_i)}d\tau+\int_{t_i}^{t}2c_1\ll g(s,\tau)\gg^2 d\tau\Big]$$
$$\leqslant e^{ct}\Big(1+\frac{c_0T}{k}\Big)^{3i}\Big[\int_0^t|||F(\tau)|||^2_{\mathcal{H}(t_i)}d\tau+2c_1\int_0^t\ll g(s,\tau)\gg^2 d\tau\Big].$$

Hence by the mathematical induction (4.15) is proved. From (4.15) by using (2.3), there exists $M''>0$ such that

$$|||U_k(t)|||_2^2\leqslant M''\Big[\int_0^t|||F(\tau)|||_2^2 d\tau+\int_0^t\ll g(s,\tau)\gg^2 d\tau\Big] \tag{4.18}$$

holds for $t\in[0, T]$, where M'' is independent of k, $f(x, t)$ and $g(s, t)$, and since

$$\Big|\Big|\Big|\frac{d}{dt}U_k(t)\Big|\Big|\Big|_1^2\leqslant \text{const.}\,(|||U_k(t)|||_2^2+|||F(t)|||_1^2) \tag{4.19}$$

holds excepting $t=t_i$. By combining (4.18) and (4.19), (4.13) is shown.

Thus $\{u_k(x, t)\}$ $(k=1, 2, \cdots)$ is a bounded set in $H^2(\Omega\times(0, T))$, consequently weakly compact. Therefore there exists a subsequence k_p $(p=1, 2, \cdots)$ of k and $u(x, t)\in H^2(\Omega\times(0, T))$ such that

$$u_{k_p}\longrightarrow u \text{ weakly in } H^2(\Omega\times(0, T))$$

when p increases infinitely. It is easily seen that $u(x, t)$ satisfies

$$L[u(x, t)]=f(x, t)\qquad \text{in the sense of } \mathcal{D}'(\Omega\times(0, T)) \tag{4.20}$$

$$B(t)u(x, t)=g(s, t)\qquad \text{in the sense of } H^{1/2}(S\times(0, T)). \tag{4.21}$$

Since (4.20) is shown by the same observation as that of Cauchy problem, we only show (4.21).

The mapping

$$H^2(\Omega\times(0, T))\ni w\rightarrow B(t)w\in H^{1/2}(S\times(0, T))$$

is strongly continuous, then the weak convergence of u_{k_p} to u in $H^2(\Omega\times(0, T))$ implies that $B(t)u_{k_p}$ converges to $B(t)u$ weakly in $H^{1/2}(S\times(0, T))$. On the other hand from

$$\langle B(t)u_{k_p}(x, t)-g(s, t)\rangle_{1/2}{}^2\leqslant \text{const.}\Big(\frac{T}{k_p}\Big)^2$$

we see $B(t)u_{k_p}(x, t)$ converges strongly to $g(s, t)$ in $L^2(S\times(0, T))$. Thus we get (4.21).

Remark that for some M'

$$(4.22)\qquad \sum_{\substack{j+|\alpha|\leq 2\\ j\leq 1}}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial t}\right)^{j}u(x,t)\right\|^2_{L^2(\Omega\times(0,\tau))}\leqslant M'\tau^2$$

holds since from (4.18) we have for all k

$$\sum_{\substack{j+|\alpha|\leq 2\\ j\leq 1}}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial t}\right)^{j}u_k(x,t)\right\|^2_{L^2(\Omega\times(0,\tau))}$$

$$=\sum_{\substack{j+|\alpha|\leq 2\\ j\leq 1}}\int_0^{\tau}\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial t}\right)^{j}u_k(x,t)\right\|^2_{L^2(\Omega)}dt\leqslant M'\tau^2$$

Now we prove that $u(x, t)$ is the solution of (1.5) in

$$\mathcal{E}^0_t(H^2(\Omega))\cap\mathcal{E}^1_t(H^1(\Omega))\cap\mathcal{E}^2_t(L^2(\Omega))$$

by exchanging the values of $u(x, t)$ on a set measure 0 if necessary.

Put

$$u_\delta(x,t)=(\varphi_{\delta}\underset{(t)}{*}u)(x,t)\in\mathcal{E}^\infty_t(H^2(\Omega))$$

then

$$\left(\frac{\partial}{\partial t}\right)^j u_\delta(x,t)=\left(\varphi_{\delta}\underset{(t)}{*}\left(\frac{\partial}{\partial t}\right)^j u\right)(x,t)\qquad (j=1,2)$$

for $t\in[0, T-\delta_0]$ if $0<\delta<\frac{\delta_0}{2}$.

We get

$$(4.23)\qquad u_\delta(x,0)\to 0 \quad\text{in}\quad H^2(\Omega)$$

$$(4.24)\qquad u_\delta'(x,0)\to 0 \quad\text{in}\quad H^1(\Omega)$$

when $\delta\to 0$. Indeed, (4.22) means

$$\int_0^t\left\|\left(\frac{\partial}{\partial t}\right)^j u(x,t)\right\|^2_{2-j,L^2(\Omega)}dt\leqslant M't^2\qquad (j=0,1)\,,$$

then

$$\left\|\left(\frac{\partial}{\partial t}\right)^j u_\delta(x,0)\right\|^2_{2-j,L^2(\Omega)}=\left\|\int\varphi_\delta(-\tau)\left(\frac{\partial}{\partial\tau}\right)^j u(x,\tau)\,d\tau\right\|^2_{2-j,L^2(\Omega)}$$

$$\leqslant\left(\int\varphi_\delta(-\tau)\left\|\left(\frac{\partial}{\partial\tau}\right)^j u(x,\tau)\right\|_{2-j,L^2(\Omega)}d\tau\right)^2$$

$$\leqslant\int\varphi_\delta(-\tau)\,d\tau\cdot\int\varphi_\delta(-\tau)\left\|\left(\frac{\partial}{\partial\tau}\right)^j u(x,\tau)\right\|^2_{2-j,L^2(\Omega)}d\tau$$

$$\leqslant\text{const.}\,\frac{1}{\delta}\int_\delta^{2\delta}\left\|\left(\frac{\partial}{\partial\tau}\right)^j u(x,\tau)\right\|^2_{2-j,L^2(\Omega)}d\tau$$

$$\leqslant\text{const.}\,M'\cdot\delta\,,$$

this shows (4.23) and (4.24).

Applying $\varphi_{\delta(t^*)}$ to both sides of (4.20) and (4.21) we get

$$L[u_\delta(x, t)] = f_\delta(x, t) - (C_\delta u)(x, t)$$
$$B(t) u_\delta(x, t) = g_\delta(x, t) - (\Gamma_\delta u)(x, t)$$

and if $0<\delta<\frac{\delta_0}{2}$, when $t \in [0, T-\delta_0]$ $C_\delta u$ has the form

$$\sum_{|\alpha|+j\leq 2} [\varphi_{\delta(t^*)}, a_{\alpha j}(x, t)] \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial t}\right)^j u .$$

By applying (3.2) to $u_{\delta'}(x, t) - u_\delta(x, t)$ it follows

$$\begin{aligned}
(4.25) \qquad & \|u_\delta(t)-u_{\delta'}(t)\|_2^2 + \|u_\delta'(t)-u_{\delta'}'(t)\|_1^2 + \|u_\delta''(t)-u_{\delta'}''(t)\|^2 \\
& \leq c(T)[\|u_\delta(0)-u_{\delta'}(0)\|_2^2 + \|u_\delta'(0)-u_{\delta'}'(0)\|_1^2 \\
& + \|f_\delta(0)-f_{\delta'}(0)\|^2 + \int_0^t \|f_\delta'(x, \tau)-f_{\delta'}'(x, \tau)\|^2 d\tau \\
& + \int_0^t \langle g_\delta'(x, \tau)-g_{\delta'}'(x, \tau)\rangle^2 d\tau + \sup_{0\leq\tau\leq t} \langle g_\delta(x, \tau)-g_{\delta'}(x, \tau)\rangle_{1/2}{}^2 \\
& + \|(C_\delta u)(x, 0)-(C_{\delta'} u)(x, 0)\|^2 + \\
& + \int_0^t \|\frac{\partial}{\partial\tau}(C_\delta u)(x, \tau) - \frac{\partial}{\partial\tau}(C_{\delta'} u)(x, \tau)\|^2 d\tau \\
& + \int_0^t \left\langle \frac{\partial}{\partial\tau}(\Gamma_\delta u)(s, \tau) - \left(\frac{\partial}{\partial\tau}\right)(\Gamma_{\delta'} u)(s, \tau)\right\rangle^2 d\tau \\
& + \sup_{0\leq\tau\leq t} \langle (\Gamma_\delta u)(s, \tau)-(\Gamma_{\delta'} u)(s, \tau)\rangle_{1/2}{}^2] .
\end{aligned}$$

Recall that (3.7) holds for any $u(x, t) \in H^2(\Omega\times(0, T))$. Then by using (3.7), (4.23) and (4.24) we find that the right-hand side of (4.25) tends to zero when δ, δ' tend to zero. This implies $u_\delta(x, t)$ is a Cauchy sequence in $\mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$, therefore the limit $u(x, t)$ is also in $\mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^2(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$. Evidently $u(x, 0) = u'(x, 0) = 0$. Thus we get

Lemma 4.2. *Given* $f(x, t) \in \mathcal{E}_t^1(\mathcal{D}_{L^2}^1(\Omega))$ *and* $g(s, t) \in \mathcal{E}_t^1(H^{1/2}(S))$, *if* $g(s, 0)=0$, *the mixed problem*

$$L[u(x, t)] = f(x, t)$$
$$B(t) u(x, t) = g(s, t)$$
$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0$$

has a unique solution in the space

$$\mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega)) .$$

Lemma 4.2 can be extended for any $f(x, t)\in\mathcal{E}_t^1(L^2(\Omega))$ and $g(s, t)\in \mathcal{E}_t^0(H^{1/2}(S))\cap\mathcal{E}_t^1(L^2(S))$ such that $g(s, 0)=0$ if we take account of the fact that $f(x, t)$ is approximated by functions in $\mathcal{E}_t^1(\mathcal{D}_{L^2}^1(\Omega))$ and $g(s, t)$ by functions in $\mathcal{E}_t^1(H^{1/2}(S))$ vanishing at $t=0$. Then

Proposition 4.3. *Given* $f(x, t)\in\mathcal{E}_t^1(L^2(\Omega))$ *and* $g(s, t)\in\mathcal{E}_t^0(H^{1/2}(S))\cap \mathcal{E}_t^1(L^2(S))$, *if* $g(s, 0)=0$ *the mixed problem* (1.5) *has a unique solution satisfying*

$$u(x, 0) = u'(x, 0) = 0$$
$$u(x, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))\,.$$

Now we prove Theorem 1. Assume that

$$\{u_0(x),\ u_1(x)\}\in H^3(\Omega)\times H^2(\Omega)$$

and the compatibility condition at $t=0$ is satisfied.

Set

$$w(x, t) = u_0(x)+tu_1(x)\,.$$

Let $v(x, t)$ be the solution of

$$L[v(x, t)] = f(x, t)-L[w(x, t)]$$
$$B(t)v(x, t) = g(s, t)-B(t)w(x, t)$$
$$v(x, 0) = v'(x, 0) = 0\,.$$

The existence of such solution is assured by Proposition 4.3 since $L[w(x, t)]\in \mathcal{E}_t^1(L^2(\Omega))$, $B(t)w(x, t)\in\mathcal{E}_t^1(H^{1/2}(S))$ and

$$g(s, 0)-[B(t)w(x, t)]_{t=0} = g(s, 0)-\mathcal{B}(0)U_0 = 0$$

Then

$$u(x, t) = v(x, t)+w(x, t)$$

is the required solution of Theorem 1. At this point by using again the energy inequality (3.2) and the density of $H^3(\Omega)\times H^2(\Omega)$ in $H^2(\Omega)\times H^1(\Omega)$ the additional condition that $\{u_0(x), u_1(x)\}\in H^3(\Omega)\times H^2(\Omega)$ is removed. Indeed, for $\{u_0(x), u_1(x)\}\in H^2(\Omega)\times H^1(\Omega)$ we choose $\{u_{k0}(x), u_{k1}(x)\}\in H^3(\Omega)\times H^2(\Omega)$ such that

$$\{u_{k0}(x),\ u_{k1}(x)\} \to \{u_0(x),\ u_1(x)\} \qquad \text{in} \quad E_2$$

and set

$$g_k(s, t) = g(s, t)+\mathcal{B}(0)\{u_{k0}(x),\ u_{k1}(x)\}-g(s, 0)\,.$$

Then $g_k(s, t)$ converges to $g(s, t)$ in $\mathcal{E}_t^0(H^{1/2}(S))\cap\mathcal{E}_t^1(L^2(S))$ and $g_k(s, t) =\mathcal{B}(0)\{u_{k0}(x), u_{k1}(x)\}$. The just obtained result assures the existence of the solution $u_k(x, t)$ for $\{u_{k0}(x), u_{k1}(x)\}$ and $g_k(s, t)$. It is found that $u_k(x, t)$ is a

Cauchy sequence in $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$ by applying (3.2) for $u_k(x, t)-u_l(x, t)$. Then its limit $u(x, t)$ is the solution for $\{u_0(x),u_1(x)\}$ and $g(s, t)$. Thus the proof is completed.

5. Regularity of the solution (Proof of Theorem 2)

The solution of this problem becomes more regular according to the regularities of the initial data, the second member and the boundary data. Of course they must satisfy the compatibility condition of higher order because this equation is hyperbolic. Here we describe *the compatibility condition of order* m[6]: *Suppose the given functions are that of Theorem 2. Define successively* $u_p(x)$ $(p=2, 3,\cdots, m+1)$ *by*

$$(5.1)\qquad u_p(x)=-\sum_{k=0}^{p-2}\binom{p-2}{k}\{a_1^{(k)}(x, 0:D)u_{p-k-1}(x)+a_2^{(k)}(x, 0:D)u_{p-k-2}(x)\}+f^{(p-2)}(x, 0),$$

evidently $u_p(x)\in H^{m+2-p}(\Omega)$. *Then the following relations hold*

$$(5.2)\qquad \sum_{k=0}^{p}\binom{p}{k}\mathcal{B}^{(k)}(0)\{u_{p-k}(x),\ u_{p-k+1}(x)\}=g^{(p)}(x, 0)\quad for\quad p=0, 1, 2,\dots, m.$$

At first we prove

$$(5.3)\qquad u(x, t)\in\mathcal{E}_t^m(H^2(\Omega))\cap\mathcal{E}_t^{m+1}(H^1(\Omega))\cap\mathcal{E}_t^{m+2}(L^2(\Omega)).$$

Consider the solution $\tilde{u}(x, t)$ of the problem

$$(5.4)\qquad \begin{cases} \text{(i)}\quad L[\tilde{u}^{(m)}(x, t)]=-\sum_{k=1}^{m}\binom{m}{k}L^{(k)}[\tilde{u}^{(m-k)}(x, t)]+f^{(m)}(x, t) \\ \text{(ii)}\quad B(t)\tilde{u}^{m)}(x, t)=-\sum_{k=1}^{m}\binom{m}{k}B^{(k)}(t)\,\tilde{u}^{(m-k)}(x, t)+g^{(m)}(s, t) \\ \text{(iii)}\quad \tilde{u}^{(p)}(x, 0)=u_p(x)\qquad (p=0, 1, 2,\cdots, m+1) \end{cases}$$

The existence of such $\tilde{u}(x, t)$ is shown by the method of successive approximation with the aid of Theorem 1 and the energy inequality (3.2). Define $\tilde{u}_j(x, t)$ $(j=1, 2, \cdots)$ successively as follows: Let $v_j(x, t)$ be the solution

$$(5.5)\qquad L[v_j]=-\sum_{k=1}^{m}\binom{m}{k}L^{(k)}[\tilde{u}_{j-1}^{(m-k)}(x, t)]+f^{(m)}(x, t)$$

$$(5.6)\qquad B(t)v_j=-\sum_{k=1}^{m}\binom{m}{k}B^{(k)}(t)\tilde{u}_{j-1}^{(m-k)}(x, t)+g^{(m)}(s, t)$$

$$(5.7)\qquad v_j(x, 0)=u_m(x),\ v_j'(x, 0)=u_{m+1}(x)$$

6) The condition (1.6) is the compatibility condition of order 0.

and $\tilde{u}_j(x, t)$ is defined by

$$\tilde{u}_j(x, t) = u_0 + tu_1(x) + \cdots + \frac{t^{m-1}u_{m-1}(x)}{(m-1)!} + \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} v_j(x, \tau)\, d\tau . \tag{5.8}$$

Here we take

$$\tilde{u}_0(x, t) = u_0(x) + tu_1(x) + \cdots + \frac{t^{m-1}u_{m-1}(x)}{(m-1)!} . \tag{5.9}$$

Let us see that $\tilde{u}_j(x, t)$ can be defined successively in $\mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega))$. When $j=1$, since $\tilde{u}_0(x, t) \in \mathcal{E}_t^\infty(H^3(\Omega))$, $f^{(m)}(x, t) \in \mathcal{E}_t^1(L^2(\Omega))$ and $g^{(m)}(s, t) \in \mathcal{E}_t^0(H^{1/2}(S)) \cap \mathcal{E}_t^1(L^2(S))$ it suffices to make sure the compatibility condition of order 0. Indeed, if we take account of (5.9) the compatibility condition for $v_1(x, t)$ is nothing but (5.2) for $p=m$. Thus the existence $v_1(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$ is shown, then $\tilde{u}_1(x, t) \in \mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega))$. Now suppose that $\tilde{u}_{j-1}(x, t) \in \mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega))$. Then the right-hand sides of (5.5) and (5.6) are in $\mathcal{E}_t^1(L^2(\Omega))$ and $\mathcal{E}_t^0(H^{1/2}(S)) \cap \mathcal{E}_t^1(L^2(S))$ respectively, and the compatibility condition of order m assures the compatibility condition of order 0 for $v_j(x, t)$. Thus the existence of $v_j(x, t)$ in $\mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$, therefore $\tilde{u}_j(x, t)$ in $\mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega))$ are derived. By the mathematical induction we see that $\tilde{u}_j(x, t)$ can be defined for all j.

Next we show that $\tilde{u}_j(x, t)$ is a Causchy sequence in $\mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+1}(L^2(\Omega))$. By applying (3.2) for $v_{j+1}(x, t) - v_j(x, t)$ it follows

$$\begin{aligned} &\|v_{j+1}(x, t) - v_j(x, t)\|_2^2 + \|v'_{j+1}(x, t) - v_j'(x, t)\|_1^2 \\ &\qquad + \|v''_{j+1}(x, t) - v_j''(x, t)\|^2 \\ &\leqslant c(T)\Big[\Big\| \sum_{k=1}^m \binom{m}{k} L^{(k)}[\tilde{u}_j^{(m-k)}(x, t) - \tilde{u}_{j-1}^{(m-k)}(x, t)]_{t=0}\Big\|^2 \\ &\quad + \int_0^t \Big\| \sum_{k=1}^m \binom{m}{k} \frac{\partial}{\partial\tau}\{L^{(k)}(\tilde{u}_j^{(m-k)}(x, \tau) - \tilde{u}_{j-1}^{(m-k)}(x, \tau))\}\Big\|^2 d\tau \\ &\quad + \int_0^t \Big\langle \sum_{k=1}^m \binom{m}{k} \frac{\partial}{\partial\tau}\{B^{(k)}(\tau)(\tilde{u}_j^{(m-k)}(x, \tau) - \tilde{u}_{j-1}^{(m-k)}(x, \tau))\}\Big\rangle^2 d\tau \\ &\quad + \sup_{0\leqslant\tau\leqslant t} \Big\langle \sum_{k=1}^m \binom{m}{k} B^{(k)}(\tau)\{\tilde{u}_j^{(m-k)}(x, \tau) - \tilde{u}_{j-1}^{(m-k)}(x, \tau)\}\Big\rangle_{1/2}^2\Big] \\ &\qquad\qquad \text{for} \quad j = 1, 2, 3, \cdots. \end{aligned} \tag{5.10}$$

Remark that if $k \leqslant m+1-i$

$$\tilde{u}_j^{(k)}(x, t) - \tilde{u}_{j-1}^{(k)}(x, t) = \int_0^t \frac{(t-\tau)^{m+1-k-i}}{(m+1-k-i)!}\{v_j^{(2-i)}(x, \tau) - v_{j-1}^{(2-i)}(x, \tau)\}\, d\tau$$

and

$$\tilde{u}_j^{(m+2-i)}(x, t) - \tilde{u}_{j-1}^{(m+2-i)}(x, t) = v_j^{(2-i)}(x, t) - v_{j-1}^{(2-i)}(x, t) .$$

Then we find if $k\leqslant m+1-i$

$$\|\tilde{u}_j^{(k)}(x, t)-\tilde{u}_{j-1}^{(k)}(x, t)\|_i^2\leqslant\text{const.}\left|\int_0^t\|v_j^{(2-i)}(x, \tau)-v_{j-1}^{(2-i)}(x, \tau)\|_i d\tau\right|^2$$
$$\leqslant\text{const.}\ T\int_0^t\|v_j^{(2-i)}(x, \tau)-v_{j-1}^{(2-i)}(x, \tau)\|_i^2 d\tau$$

and

$$\|\tilde{u}_j^{(m+2-i)}(x, t)-\tilde{u}_{j-1}^{(m+2-i)}(x, t)\|_i^2$$
$$=\|v_j^{(2-i)}(x, t)-v_{j-1}^{(2-i)}(x, t)\|_i^2 .$$

From these estimates (5.10) is led to

$$\|v_{j+1}(x, t)-v_j(x, t)\|_2^2+\|v'_{j+1}(x, t)-v_j'(x, t)\|_1^2$$
$$+\|v''_{j+1}(x, t)-v_j''(x, t)\|^2$$
$$\leqslant\text{const.}\int_0^t[\|v_j(x, \tau)-v_{j-1}(x, \tau)\|_2^2+\|v_j'(x, \tau)-v'_{j-1}(x, \tau)\|_1^2$$
$$+\|v''_j(x, \tau)-v''_{j-1}(x, \tau)\|^2]\,d\tau ,$$

from which it follows that

$$\sum_{j=1}^{\infty}[\|v_{j+1}(x, t)-v_j(x, t)\|_2^2+\|v'_{j+1}(x, t)-v_j'(x, t)\|_1^2$$
$$+\|v''_{j+1}(x, t)-v_j'(x, t)\|]\leqslant\sum_{j=1}^{\infty}K_0\frac{(KT)^j}{j!}$$

for all $t\in[0, T]$. This assures the convergence of $v_j(x, t)$ in $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$, therefore that of $\tilde{u}_j(x, t)$ in $\mathcal{E}_t^m(H^2(\Omega))\cap\mathcal{E}_t^{m+1}(H^1(\Omega))\cap\mathcal{E}_t^{m+2}(L^2(\Omega))$. Then the limit of $\tilde{u}_j(x, t)$, which is denoted by $\tilde{u}(x, t)$, is the solution of (5.4). This is derived by the passage to the limit of (5.5) and (5.6) and the definition of $\tilde{u}_j(x, t)$.

(5.5) and (5.6) are

$$\frac{d^m}{dt^m}(L[\tilde{u}(x, t)])=f^{(m)}(x, t)$$
$$\frac{d^m}{dt^m}(B(t)\tilde{u}(x, t))=g^{(m)}(x, t) .$$

Our definition of $\tilde{u}^{(p)}(x, 0)=u_p(x)$ is taken as

$$\left[\frac{d^k}{dt^k}(L[\tilde{u}(x, t)])\right]_{t=0}=f^{(k)}(x, 0)\qquad(k=0, 1,2 ,\cdots, m)$$

and the compatibility condition of order m means that

$$\left[\frac{d^k}{dt^k}(B(t)\tilde{u}(x, t))\right]_{t=0}=g^{(k)}(s, 0)\qquad(k=0, 1, 2 ,\cdots, m) .$$

Thus $\tilde{u}(x, t) \in \mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega))$ is a solution of problem (1.5). The uniqueness of solution assures (5.3).

To derive the regularity with respect to x, we make use the following Lemma:

Lemma.[7] *Suppose that the coefficients of $a_2(x, t: D)$ belong to $\mathcal{B}^{p+k}(\Omega \times [0, T])$ and*

$$(5.11) \qquad u(x, t) \in H^{p+2}(\Omega) \quad \textit{for all } t \textit{ and } \in \mathcal{E}_t^k(L^2(\Omega))$$

$$(5.12) \qquad a_2(x, t: D) u(x, t) = q(x, t) \in \mathcal{E}_t^k(H^p(\Omega))$$

$$(5.13) \qquad \frac{\partial}{\partial n_t} u(x, t) = r(s, t) \in \mathcal{E}_t^k(H^{p+1/2}(S))$$

where $p \geqslant 0$, $k \geqslant 0$ then

$$(5.14) \qquad u(x, t) \in \mathcal{E}_t^k(H^{p+2}(\Omega)) .$$

Now let us prove Theorem 2. From (5.3) it follows

$$(5.15) \qquad a_2(x, t: D)\, u(x, t) = -\frac{\partial^2}{\partial t^2} u - a_1 \frac{\partial}{\partial t} u + f(x, t) \in \mathcal{E}_t^{m-1}(H^1(\Omega))$$

$$(5.16) \qquad \frac{\partial}{\partial n_t} u(x, t) = \sigma_1 \frac{\partial u}{\partial t} - \sigma_2 u + g(s\ t) \in \mathcal{E}_t^{m-1}(H^{1+1/2}(S)) .$$

Of course from these relations $u(x, t) \in H^3(\Omega)$ for all t, then the application of Lemma by taking $p=1$, $k=m-1$ proves

$$u(x, t) \in \mathcal{E}_t^{m-1}(H^3(\Omega)) .$$

If $m \geqslant 1$, it turns out the right-hand side of (5.15) $\in \mathcal{E}_t^{m-2}(H^2(\Omega))$ and that of (5.16) $\in \mathcal{E}_t^{m-2}(H^{2+1/2}(S))$ by the just obtained result. And $u(x, t) \in H^4(\Omega)$ holds for all t, then by applying Lemma oncemore by taking $p=2$, $k=m-2$, we get

$$u(x, t) \in \mathcal{E}_t^{m-2}(H^4(\Omega)) .$$

Repeating this process step by step, finally Theorem 2 is proved for any m.

Appendix

We show that the condition (1.3) is necessary for the treatment in L^2-sense of the mixed problem with nonhomogeneous boundary condition.

Let $\Omega = \{(x, y);\ x>0,\ -\infty < y < \infty\}$, $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ and $B = \frac{\partial}{\partial x}$.

7) This is Lemma 3.5 of our previous paper [3].

Consider the mixed problem

$$(P)\qquad \begin{cases} L[u]=0 & \text{in } \Omega\times(0,T) \\ [Bu]_{x=0}=g(y,t) \\ u(x,y,0)=\dfrac{\partial u}{\partial t}(x,y,0)=0\,. \end{cases}$$

When $g(y,t)$ is sufficiently smooth and its support is contained in $t>\varepsilon_0$ the solution $u(x,y,t)$ of (P) exists uniquely in $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega)\cap\mathcal{E}_t^2(L^2(\Omega))$ and it is also smooth. This is derived from the existence theorem for the homogeneous boundary condition with the aid of the construction of a sufficiently smooth function $v(x,y,t)$ such that $[Bv]_{x=0}=g(y,t)$. But concerning the problem (P) the energy inequality of the type (3.2) is never held. We show the following:

Theorem A. *Whatever we choose T and C the following energy inequality*

$$\text{(A.1)}\qquad \|u(x,y,t)\|_{2,L^2(\Omega)}^2+\left\|\frac{\partial u}{\partial t}(x,y,t)\right\|_{1,L^2(\Omega)}^2+\left\|\frac{\partial^2 u}{\partial t^2}(x,y,t)\right\|_{L^2(\Omega)}^2$$
$$\leqslant C\left\{\int_0^t\left(\|g(y,s)\|_{1,L^2(\partial\Omega)}^2+\left\|\frac{\partial g}{\partial t}(y,s)\right\|_{L^2(\partial\Omega)}^2\right)ds+\sup_{0\leqslant s\leqslant t}\|g(y,s)\|_{1,L^2(\partial\Omega)}^2\right\}$$

for all $t\in[0,T]$
never holds.

At first we note some lemmas without proof.

Lemma A.1. *Let k_1 and k_2 be constants such that $k_1>k_2>0$. Put $\tau=\mu+i\nu$. Then for all*

$$\eta\in[n+k_1,\ n+2k_1]$$
$$\nu\in[n-k_2,\ n+k_2]$$

we have

$$\text{(A.2)}\qquad \sqrt{|\eta^2+\tau^2|}\leqslant c_1\sqrt{n}+c_2\,,$$

where c_1 and c_2 are positive constants depending on k_1, k_2 and μ.

Lemma A.2. *Let η be real and* $\mathrm{Re}\,\tau>0$. *For any $f(x)\in L^2(R_+)$ and g a complex number, there exists one and only one solution $u(x)$ in $H^2(R_+)$ of the boundary value problem*

$$\begin{cases} \left(\left(\frac{1}{i}\frac{d}{dx}\right)^2+\eta^2+\tau^2\right)u(x)=f(x) & x>0 \\ \frac{1}{i}\frac{d}{dx}u(x)\Big|_{x=0}=g\,, \end{cases}$$

and $u(x)$ *is given in the form*

$$u(x)=g\frac{e^{i\xi_+\cdot x}}{\xi_+}+w(x) \tag{A.3}$$

where $w(x)$ *is the Fourier inverse image of*

$$\hat{w}(\xi)=\frac{1}{\xi^2+\eta^2+\tau^2}(\hat{f}(\xi)+\hat{p}(\xi))$$

here
$$\hat{p}(\xi)=-\frac{\operatorname{Im}\xi_+}{\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(\xi)}{\xi-\xi_-}d\xi\cdot\frac{1}{\xi-\xi_+}\,,$$

ξ_+ (ξ_-) *is the square root with positive* (*negative*) *imaginary part of* $-(\eta^2+\tau^2)$.

We prove Theorem A by contradiction. Assume that (A.1) holds for some C and T. Let $h(t)$ be a sufficiently smooth function with support contained in $[\delta_0, T_0]$ $(0<\delta_0<T_0<T)$, and we choose $k_2>0$ as

$$\int_{-k_2}^{k_2}|\mathcal{F}[e^{-\gamma t}h(t)](\eta)|^2d\eta\geqslant\frac{1}{2}\int_{\delta_0}^{T_0}|e^{-t\gamma}h(t)|^2dt \tag{A.4}$$

holds where γ is a fixed positive constant. Next take k_1 as $k_1>k_2$. Let $\hat{k}(\eta)$ be a sufficiently smooth function whose support contained in $[k_1, 2k_1]$ and denote its Fourier inverse image by $k(y)$. Put

$$g_n(y, t)=e^{iny}k(y)\,e^{int}h(t)\,,$$

evidently $g_n(y, t)\in\mathcal{E}_t^\infty(H^1(R))$.

Denote by $u_n(x,y, t)$ the solution of (P) for $f\equiv 0$, $g=g_n$, i.e. $u_n(x, y, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$ satisfies

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)u_n(x, y, t)=0 \\ \frac{\partial u_n}{\partial x}(0, y, t)=g_n(y, t) \\ u_n(x, y, 0)=\frac{\partial u_n}{\partial t}(x, y, 0)=0\,. \end{cases}$$

By employing the notations of

$$E_n(t) = ||u_n(x, y, t)||^2_{2,L^2(\Omega)} + ||u_n'(x, y, t)||^2_{1,L^2(\Omega)} + ||u_n''(x, y, t)||^2_{L^2(\Omega)}$$

$$e_n(t) = \int_0^t \{||g_n(y, t)||^2_{1,L^2(\partial\Omega)} + ||g_n'(y, t)||^2_{L^2(\partial\Omega)}\} dt + \sup_{0\leqslant s\leqslant t} ||g_n(y, s)||^2_{1,L^2(\partial\Omega)},$$

it follows from (A.1)

(A.5)
$$E_n(t) \leqslant Ce_n(t).$$

Let $\alpha(t)$ be a function in $C^\infty(R)$ such that

$$\alpha(t) = \begin{cases} 1, & t < T_0 \\ 0, & t > T_0' \end{cases}$$

where $T_0 < T_0' < T$. Then

(A.6)
$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\alpha(t)u_n(x, y, t) = 2\frac{d\alpha}{dt}(t)\frac{\partial u_n}{\partial t}(x, y, t) + \frac{d^2\alpha}{dt^2}(t)u_n(x, y, t)$$

(A.7)
$$\frac{\partial}{\partial x}(\alpha(t)u_n(x, y, t))\Big|_{x=0} = \alpha(t)g_n(y, t) = g_n(y, t)$$

$$\alpha(0)u_n(x, y, 0) = \frac{\partial}{\partial t}(\alpha(t)u_n(x, y, t))\Big|_{t=0} = 0.$$

Put

$$v_n(x, y, t) = \alpha(t)u_n(x, y, t)$$

$$f_n(x, y, t) = 2\frac{d\alpha}{dt}(t)\frac{\partial u_n}{\partial t}(x, y, t) + \frac{d^2\alpha}{dt^2}(t)u_n(x, y, t),$$

and evidently we have

(A.8)
$$||v_n(x, y, t)||^2_2 + ||v_n'(x, y, t)||^2_1 + ||v_n''(x, y, t)^2||^2 \leqslant \text{const. } E_n(t)$$

(A.9)
$$||f_n(x, y, t)||^2_1 + ||f_n'(x, y, t)||^2 \leqslant \text{const. } E_n(t).$$

The equations (A.6) and (A.7) are transformed into the following after Fourier-Laplace transformation

$$\begin{cases} \left(\left(\frac{1}{i}\frac{d}{dx}\right)^2 + \eta^2 + \tau^2\right)\hat{v}_n(x, \eta, \tau) = \hat{f}_n(x, \eta, \tau) \\ \frac{1}{i}\frac{d}{dx}\hat{v}_n(x, \eta, \tau)\Big|_{x=0} = \frac{1}{i}\hat{g}_n(\eta, \tau), \end{cases}$$

where $\tau = \gamma + i\nu$. The application of Lemma A.2 for each (η, τ) gives

(A.10)
$$\hat{v}_n(x, \eta, \tau) = e^{i\xi_+(\eta,\tau)x}\frac{\hat{g}_n(\eta, \tau)}{i\xi_+(\eta, \tau)}+w_n(x, \eta, \tau)$$
$$= v_n^{(1)}(x, \tau, \tau)+w_n(x, \eta, \tau)\,.$$

Now

$$\int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta\int_0^{\infty} |\eta^2 v_n^{(1)}(x, \eta, \tau)|^2 dx$$
$$= \int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta\frac{\pi}{\mathrm{Im}\ \xi_+(\eta, \tau)}\frac{\eta^4}{|\eta^2+\tau^2|}|\hat{k}(\eta-n)|^2|h(\gamma+i(\nu-n))|^2$$
$$\geqslant \int_{n-k_2}^{n+k_2} d\nu\int_{n+k_1}^{n+2k_1} d\eta\frac{\pi\eta^4}{\mathrm{Im}\ \xi_+(\eta, \tau)|\eta^2+\tau^2|}\,|\hat{k}(\eta-n)|^2|\hat{h}(\gamma+i(\nu-n))|^2$$

by using Lemma A.1

$$\geqslant \frac{\pi(n+k_1)^4}{(c_1\sqrt{n}+c_2)^3}\int_{n-k_2}^{n+k_2}|\hat{h}(\gamma+i(\nu-n))|^2 d\nu\int_{n+k_1}^{n+2k_1}|(\hat{k}(\eta-n)|^2 d\eta$$
$$\geqslant \frac{\pi n^4}{(c_1\sqrt{n}+c_2)^3}\int_{-k_2}^{k_2}|\hat{h}(\gamma)+i\nu)|^2 d\,\nu\int_{k_1}^{2k_1}|\hat{k}(\eta)|^2 d\eta$$

from (A.4) and the choice of $\hat{k}(\eta)$

$$\geqslant \frac{\pi n^4}{(c_1\sqrt{n}+c_2)^3}||k||^2_{L^2(R)}\cdot\frac{1}{2}\int_{\delta_0}^{T_0}|e^{-\gamma t}h(t)|^2 dt\,.$$

Therefore we get for some constant $c_0>0$

(A.11)
$$\int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta|\eta^2 v_n^{(1)}(x, \eta, \tau)|^2 dx \geqslant \frac{n^4}{(c_1\sqrt{n}+c_2)^3}c_0\,.$$

And we estimate the second term of (A.10).

$$\int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta\int_{-\infty}^{\infty} d\xi|\eta^2\hat{w}_n(\xi, \eta, \tau)|^2$$
$$\leqslant \sup_{(\xi,\eta,\nu)\in R^3}\left|\frac{\eta}{\xi^2+\eta^2+\tau^2}\right|^2$$
$$\times 2\int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta\int_{-\infty}^{\infty} d\xi(|\eta\hat{f}_n(\xi, \eta, \tau)|^2+|\eta\hat{p}_n(\xi, \eta, \tau)|^2)$$
$$\leqslant \text{const.}\int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta\int_{-\infty}^{\infty} d\xi(|\eta\hat{f}_n(\xi, \eta, \tau)|^2+|\eta\hat{p}_n(\xi, \eta, \tau)|^2)^{8)}\,.$$

$$\int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta\int_{-\infty}^{\infty} d\xi|\eta\hat{p}_n(\xi, \eta, \tau)|^2$$
$$= \int_{-\infty}^{\infty} d\nu\int_{-\infty}^{\infty} d\eta\left|\eta\frac{\mathrm{Im}\ \xi_+(\eta, \tau)}{\pi}\right|^2\left|\int_{-\infty}^{\infty}\frac{\hat{f}_n(\xi, \eta, \tau)}{\xi-\xi_-(\eta, \tau)}d\xi\right|^2\int_{-\infty}^{\infty}\frac{d\xi}{|\xi-\xi_+(\eta, \tau)|^2}$$

---

8) $\sup_{(\xi,\eta,\nu)\in R^3}\left|\frac{\eta}{\xi^2+\eta^2+(\gamma+i\nu)^2}\right| \leqslant \frac{\text{const.}}{\gamma}$

$$\leqslant \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\eta \left| \eta \frac{\operatorname{Im} \xi_+(\eta, \tau)}{\pi} \right|^2 \int_{-\infty}^{\infty} \frac{d\xi}{|\xi - \xi_-(\eta, \tau)|^2} \int_{-\infty}^{\infty} |\hat{f}_n(\xi, \eta, \tau)|^2 d\xi \cdot \int_{-\infty}^{\infty} \frac{d\xi}{|\xi - \xi_+(\eta, \tau)|^2}$$

$$= \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} |\eta \hat{f}_n(\xi, \eta, \tau)|^2 d\xi .$$

Then it follows

$$\text{(A.12)} \quad \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi |\eta^2 \hat{w}_n(\xi, \eta, \tau)|^2$$

$$\leqslant \text{const.} \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} |\eta \hat{f}_n(\xi, \eta, \tau)|^2 d\xi$$

$$\leqslant \text{const.} \int_0^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^{\infty} \left| e^{-\gamma t} \frac{\partial}{\partial y} f_n(x, y, t) \right|^2 dt$$

$$\leqslant \text{const.} \int_{T_0}^{T_0'} \|f_n(x, y, t)\|_1^2 \, dt$$

$$\leqslant \text{const.} \int_{T_0}^{T_0'} E_n(t) \, dt \leqslant \text{const. } e_n(T) .$$

And now

$$\int_{-\infty}^{\infty} dy \int_0^{\infty} dx \int_0^{T} \left| e^{-\gamma t} \frac{\partial^2}{\partial y^2} v_n(x, y, t) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} d\eta \int_0^{\infty} dx \int_{-\infty}^{\infty} |\eta^2 \hat{v}_n(x, \eta, \tau)|^2 d\nu$$

$$\geqslant \int_{\infty-}^{\infty} d\eta \int_{-\infty}^{\infty} d\nu \int_0^{\infty} dx \cdot \frac{1}{2} \left| \eta^2 v_n^{(1)}(x, \eta, \tau) \right|^2$$

$$-3 \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\nu \int_0^{\infty} dx |\eta^2 w_n(x, \eta, \tau)|^2$$

from the estimates (A.11) and (A.12)

$$\geqslant \frac{n^4}{(c_1\sqrt{n} + c_2)^3} \cdot c_0 - \text{const. } e_n(T) .$$

On the other hand by taking account of (A.5)

$$\int_{-\infty}^{\infty} dy \int_0^{\infty} dx \int_0^{\infty} \left| e^{-\gamma t} \frac{\partial^2}{\partial y^2} v_n(x, y, t) \right|^2 dt$$

$$\leqslant \text{const.} \int_0^{T} E_n(t) \, dt$$

$$\leqslant \text{const. } T \, e_n(T) .$$

Thus it follows that for some c_3, c_2

$$c_3\, e_n(T) \geqslant \frac{n^4}{(c_1\sqrt{n}+c_2)_{\varepsilon}} \cdot c_0 - c_4\, e_n(T)\,,$$

then

$$(c_3+c_4)\, e_n(T) \geqslant \frac{n^4}{(c_0\sqrt{n}+c_3)^3} c_0\,. \tag{A.13}$$

From the definition of $g_n(y, t)$ it is easily seen that

$$e_n(T) \leqslant \text{const. } n^2\,.$$

Then (A.13) leads that it holds for all n

$$(c_3+c_4)\text{ const. } n^2 \geqslant \frac{n^4}{(c_1\sqrt{n}+c_2)^3} c_0\,.$$

This is a contradiction. Thus Therem A is proved. Q.E.D.

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