A MIXED PROBLEM FOR THE INFINITY LAPLACIAN VIA TUG-OF-WAR GAMES

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Abstract. In this paper we prove that a function $u \in \mathcal{C}(\overline{\Omega})$ is the continuous value of the Tug-of-War game described in [19] if and only if it is the unique viscosity solution to the infinity laplacian with mixed boundary conditions

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases}$$

By using the results in [19], it follows that this viscous PDE problem has a unique solution, which is the unique *absolutely minimizing Lipschitz extension* to the whole $\overline{\Omega}$ (in the sense of [2] and [19]) of the Lipschitz boundary data $F : \Gamma_D \to \mathbb{R}$.

1. Introduction

A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome). Recently, these type of games have been used in connection with some PDE problems, see [6], [15], [18], [19].

For the reader's convenience, let us first describe briefly the game introduced in [19] by Y. Peres, O. Schramm, S. Sheffield and D. Wilson. Consider a bounded domain $\Omega \subset \mathbb{R}^n$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$. Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\epsilon(x_0)} \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\epsilon(x_{k-1})} \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function*. In more general models, it is considered also a *running payoff* $f(x)$ defined in Ω, which represents the reward (respectively, the cost) at each intermediate state x, and gives rise to nonhomogeneous problems. We will assume throughout the paper that $f \equiv 0$. This procedure yields a sequence of game states $x_0, x_1, x_2, \ldots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now we want to give a precise definition of the *value of the game*. To this end we have to introduce some notation and put the game into its normal or strategic form

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(see [18] and [16]). The initial state $x_0 \in \overline{\Omega} \setminus \Gamma_D$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in \overline{B_\epsilon(0)}$ which is announced to the other player; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)}$. Then, the new state $x_1 \in \overline{B_\epsilon(x_0)}$ (namely, the current state plus the action) is selected according to a probability distribution $p(\cdot|x_0, a_0)$ in $\overline{\Omega}$ which, in our case, is given by the fair coin toss. At stage k, knowing the history $h_k = (x_0, a_0, x_1, a_1, \dots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), each player i chooses an action a_k^i. If the game terminated at time $j < k$, we set $x_m = x_j$ and $a_m = 0$ for $j \le m \le k$. The current state x_k and the profile $a_k = \{a_k^1, a_k^2\}$ determine the distribution $p(\cdot|x_k, a_k)$ (again given by the fair coin toss) of the new state x_{k+1}.

Denote $H_k = (\overline{\Omega} \setminus \Gamma_D) \times \left(\overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)} \times \overline{\Omega}\right)^k$, the set of *histories up to stage* k, and by $H = \bigcup_{k \ge 1} H_k$ the set of all histories. Notice that H_k, as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \dots, a_{k-1}, x_k, \dots)$ endowed with the product topology. Then, the final payoff for Player I, i.e. F, induces a Borel-measurable function on H_∞. A *pure strategy* $S_i = \{S_i^k\}_k$ for Player i, is a sequence of mappings from histories to actions, namely, a mapping from H to $\overline{B_\epsilon(0)}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\epsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\epsilon(0)}$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}^{x_0}_{S_I, S_{II}}$ on the space of plays H_∞. We denote by $\mathbb{E}^{x_0}_{S_I, S_{II}}$ the corresponding expectation.

Then, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, we can define the *ϵ-value of the game for Player I* as

$$u_I^\epsilon(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I, S_{II}),$$

while the ϵ-value of the game for Player II is defined as

$$u_{II}^\epsilon(x_0) = \inf_{S_{II}} \sup_{S_I} V_{x_0,II}(S_I, S_{II}).$$

In some sense, $u_I^\epsilon(x_0), u_{II}^\epsilon(x_0)$ are the least possible outcomes that each player expects to get when the ϵ-game starts at x_0. Notice that, as in [19], we penalize severely the games that never end.

If $u_I^\epsilon = u_{II}^\epsilon := u_\epsilon$, we say that *the game has a value*. In [19] it is shown that, under very general hypotheses, that are fulfilled in the present setting, the ϵ-Tug-of-War game has a value.

All these $\epsilon-$values are Lipschitz functions with respect to the discrete distance d^ϵ, see [19] (but in general they are not continuous), which converge uniformly when $\epsilon \to 0$. The uniform limit as $\epsilon \to 0$ of the game values u_ϵ is called *the continuous value* of the game that we will denote by u. Indeed, see [19], it turns out that u is

a viscosity solution to the problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \Gamma_D, \end{cases} \tag{1}$$

where $\Delta_\infty u = |\nabla u|^{-2} \sum_{ij} u_{x_i} u_{x_i x_j} u_{x_j}$ is the $1-$homogeneous infinity laplacian (see Section 2 for a discussion about the actual definition at points where $\nabla u(x)$ vanishes). Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) appear naturally as limits of $p-$harmonic functions (solutions to $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$) and have applications to optimal transport problems, image processing, etc. For limits as $p \to \infty$ for $p-$laplacian type problems we refer to [3], [7], [10], [11] and references therein.

When $\Gamma_D \equiv \partial\Omega$, it is known that problem (1) has a unique viscosity solution, (as proved in [12]; see also [5], [8], and in a more general framework, [19]). Moreover, it is the unique AMLE (absolutely minimal Lipschitz extension) of $F : \Gamma_D \to \mathbb{R}$ in the sense that $Lip_U(u) = Lip_{\partial U \cap \Omega}(u)$ for every open set $U \subset \overline{\Omega} \setminus \Gamma_D$. AMLE extensions were introduced by Aronsson in [2], see the survey [3] for more references and applications of this subject.

However, when $\Gamma_D \neq \partial\Omega$ the PDE problem (1) is incomplete, since there is a missing boundary condition on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Our main concern is to find the boundary condition that completes the problem.

Assuming that Γ_N is regular, in the sense that the normal vector field $\vec{n}(x)$ is well defined and continuous for all $x \in \Gamma_N$, we prove that it is in fact the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n}(x) = 0\,, \quad x \in \Gamma_N.$$

Let us point out that no regularity is needed on the Dirichlet part Γ_D, but the boundary data F has to be Lipschitz continuous.

On the other hand, instead of using the beautiful and involved proof based on game theory arguments, written in [19], we give an alternative proof of the property $-\Delta_\infty u = 0$ in Ω, by using direct viscosity techniques, perhaps more natural in this context. The key point in our proof is the *Dynamic Programming Principle*, which, in some sense, plays the role of the mean property for harmonic functions in the infinity-harmonic case. This principle turns out to be an important qualitative property of the approximations of infinity-harmonic functions, and is the main tool to construct convergent numerical methods in this kind of problems; see [17].

We have the following result,

Theorem 1. *Let $u(x)$ be the continuous value of the Tug-of-War game introduced in* [19]. *Assume that $\partial\Omega = \Gamma_N \cup \Gamma_D$, where Γ_N is of class C^1, and F is a Lipschitz function defined on Γ_D.*

Then,

i) *$u(x)$ is a viscosity solution to the mixed boundary value problem*

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases} \tag{2}$$

ii) *Reciprocally, assume that* Ω *verifies that for every* $z \in \overline{\Omega}$ *and every* $x^* \in \Gamma_N$ $z \neq x^*$ *that*

$$\left\langle \frac{x^* - z}{|x^* - z|}; n(x^*) \right\rangle > 0.$$

Then, if $u(x)$ *is a viscosity solution to* (2)*, it coincides with the unique continuous value of the game.*

The hypothesis imposed on Ω in part ii) holds whenever Γ_N is strictly convex. The first part of the theorem comes as a consequence of the Dynamic Programming Principle read in the viscosity sense. To prove the second part we will use that the continuous value of the game is determined by the fact that it enjoys comparison with quadratic functions in the sense described in [19].

We have found a PDE problem, (2), which allows to find both the continuous value of the game and the AMLE of the Dirichlet data F (which is given only on a subset of the boundary) to $\overline{\Omega}$. To summarize, we point out that a complete equivalence holds, in the following sense:

Theorem 2. *It holds*

$$u \text{ is AMLE of } F \Leftrightarrow u \text{ is the value of the game.} \Leftrightarrow u \text{ solves } (2).$$

The first equivalence was proved in [19] and the second one is just Theorem 1.

Another consequence of Theorem 1 is the following:

Corollary 3. *There exists a unique viscosity solution to* (2)*.*

The existence of a solution is a consequence of the existence of a continuous value for the game together with part i) in the previous theorem, while the uniqueness follows by uniqueness of the value of the game and part ii).

Note that to obtain uniqueness we have to invoke the uniqueness of the game value. It should be desirable to obtain a direct proof (using only PDE methods) of existence and uniqueness for (2) but it is not clear how to find the appropriate perturbations near Γ_N to obtain uniqueness (existence follows easily by taking the limit as $p \to \infty$ in the mixed boundary value problem problem for the $p-$laplacian).

Remark 4. Corollary 3 allows to improve the convergence result given in [11] for solutions to the Neumann problem for the $p-$laplacian as $p \to \infty$. The uniqueness of the limit holds under weaker assumptions on the data (for example, Ω strictly convex).

The rest of the paper is devoted to the proof of Theorem 1. In Section 2 we prove part i) of the theorem and in Section 3 we prove part ii).

2. The continuous value of the game is a viscosity solution to the mixed problem

As we have already mentioned in the Introduction, it is shown in [19] that the continuous value of the game u is infinity harmonic within Ω and, in the case that $\Gamma_D = \partial\Omega$, it satisfies a Dirichlet boundary condition $u = F$ on $\partial\Omega$.

In this paper, we are concerned with the case in which $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_N \neq \emptyset$. Our aim in the present section is to prove that u satisfies an homogeneous

Neumann boundary condition on Γ_N, namely

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D, \end{cases} \tag{3}$$

in the viscosity sense, where

$$\Delta_\infty u(x) = \begin{cases} \Big\langle D^2u(x)\dfrac{\nabla u(x)}{|\nabla u(x)|}, \dfrac{\nabla u(x)}{|\nabla u(x)|}\Big\rangle, & \text{if } \nabla u(x) \neq 0, \\ \lim\limits_{y\to x} \dfrac{2\big(u(y)-u(x)\big)}{|y-x|^2}, & \text{otherwise.} \end{cases} \tag{4}$$

In defining $\Delta_\infty u$ we have followed [19]. Let us point out that it is possible to define the infinity laplacian at points with zero gradient in an alternative way, as in [13]. However, it is easy to see that both definitions are equivalent.

To motivate the above definition, notice that $\Delta_\infty u$ is the second derivative of u in the direction of the gradient. In fact, if u is a C^2 function and we take a direction v, then the second derivative of u in the direction of v is

$$D_v^2u(x) = \left.\frac{d^2}{dt^2}\right|_{t=0} u(x+tv) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) v_i v_j.$$

If $\nabla u(x) \neq 0$, we can take $v = \dfrac{\nabla u(x)}{|\nabla u(x)|}$, and get $\Delta_\infty u(x) = D_v^2 u(x)$.

In points where $\nabla u(x) = 0$, no direction is preferred, and then expression (4) arises from the second-order Taylor's expansion of u at the point x,

$$\frac{2(u(y)-u(x))}{|y-x|^2} = \Big\langle D^2u(x)\frac{y-x}{|y-x|}, \frac{y-x}{|y-x|}\Big\rangle + o(1).$$

We say that, at these points, $\Delta_\infty u(x)$ is defined if $D^2u(x)$ is the same in every direction, that is, if the limit $\dfrac{(u(y)-u(x))}{|y-x|^2}$ exists as $y \to x$.

Because of the singular nature of (4) in points where $\nabla u(x) = 0$, we have to restrict our class of test functions. We will denote

$$S(x) = \big\{\phi \in \mathcal{C}^2 \text{ near } x \text{ for which } \Delta_\infty \phi(x) \text{ has been defined}\big\},$$

this is, $\phi \in S(x)$ if $\phi \in \mathcal{C}^2$ in a neighborhood of x and either $\nabla\phi(x) \neq 0$ or $\nabla\phi(x) = 0$ and the limit

$$\lim_{y\to x} \frac{2\big(\phi(y)-\phi(x)\big)}{|y-x|^2},$$

exists.

Now, using the above discussion of the infinity laplacian, we give the precise definition of viscosity solution to (3) following [4].

Definition 5. Consider the boundary value problem (3). Then,

(1) A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in S(x_0)$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \Gamma_D$,

$$F(x_0) \leq \phi(x_0);$$

if $x_0 \in \Gamma_N$, the inequality

$$\max\left\{\langle n(x_0), \nabla\phi(x_0)\rangle,\ -\Delta_\infty\phi(x_0)\right\} \geq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$-\Delta_\infty\phi(x_0) \geq 0,$$

with $\Delta_\infty\phi(x_0)$ given by (4).

(2) An upper semi-continuous function u is a subsolution if for every $\phi \in S(x_0)$ such that $u-\phi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \Gamma_D$,

$$F(x_0) \geq \phi(x_0);$$

if $x_0 \in \Gamma_N$, the inequality

$$\min\left\{\langle n(x_0), \nabla\phi(x_0)\rangle,\ -\Delta_\infty\phi(x_0)\right\} \leq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$-\Delta_\infty\phi(x_0) \leq 0,$$

with $\Delta_\infty\phi(x_0)$ given by (4).

(3) Finally, u is a viscosity solution if it is both a super- and a subsolution.

Proof of part i) *of Theorem 1.* The starting point is the following Dynamic Programming Principle, which is satisfied by the value of the $\epsilon-$game (see [19]):

$$2u_\epsilon(x) = \sup_{y\in\overline{B_\epsilon(x)}\cap\bar{\Omega}} u_\epsilon(y) + \inf_{y\in\overline{B_\epsilon(x)}\cap\bar{\Omega}} u_\epsilon(y) \qquad \forall x \in \bar{\Omega}\setminus\Gamma_D, \tag{5}$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x.

Let us check that u (a uniform limit of u_ϵ) is a viscosity supersolution to (3). To this end, consider a function $\phi \in S(x_0)$ such that $u-\phi$ has a strict local minimum at x_0, this is,

$$u(x) - \phi(x) > u(x_0) - \phi(x_0), \quad x \neq x_0.$$

Without loss of generality, we can suppose that $\phi(x_0) = u(x_0)$. Let us see the inequality that these test functions satisfy, as a consequence of the Dynamic Programming Principle.

Let $\eta(\epsilon) > 0$ such that $\eta(\epsilon) = o(\epsilon^2)$. By the uniform convergence of u_ϵ to u, there exist a sequence $x_\epsilon \to x_0$ such that

$$u_\epsilon(x) - \phi(x) \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon), \tag{6}$$

for every x in a fixed neighborhood of x_0.

From (6), we deduce

$$\sup_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} u_\epsilon(y) \geq \max_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} \phi(y) + u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon)$$

and

$$\inf_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} u_\epsilon(y) \geq \min_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} \phi(y) + u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon).$$

Then, we have from (5)

$$2\phi(x_\epsilon) \geq \max_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} \phi(y) + \min_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} \phi(y) - 2\eta(\epsilon). \tag{7}$$

The above expression can be read as a *Dynamic Programming Principle in the viscosity sense.*

It is clear that the uniform limit of u_ϵ, u, verifies

$$u(x) = F(x) \qquad x \in \Gamma_D.$$

In $\overline{\Omega}\setminus\Gamma_D$ there are two possibilities: $x_0\in\Omega$ and $x_0\in\Gamma_N$. In the former case we have to check that

$$-\Delta_\infty\phi(x_0)\geq 0, \tag{8}$$

while in the latter, what we have to prove is

$$\max\left\{\frac{\partial\phi}{\partial n}(x_0), -\Delta_\infty\phi(x_0)\right\}\geq 0. \tag{9}$$

CASE A. Our aim is to prove $-\Delta_\infty\phi(x_0)\geq 0$. Notice that this is a consequence of the results in [19], nevertheless the elementary arguments below provide an alternative proof using only direct viscosity techniques.

First, assume that $x_0\in\Omega$. If $\nabla\phi(x_0)\neq 0$ we proceed as follows.

Since $\nabla\phi(x_0)\neq 0$ we also have $\nabla\phi(x_\epsilon)\neq 0$ for ϵ small enough.

In the sequel, $x_1^\epsilon, x_2^\epsilon\in\bar\Omega$ will be the points such that

$$\phi(x_1^\epsilon)=\max_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega}\phi(y)\qquad\text{and}\quad\phi(x_2^\epsilon)=\min_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega}\phi(y).$$

We remark that $x_1^\epsilon, x_2^\epsilon\in\partial B_\epsilon(x_\epsilon)$. Suppose to the contrary that there exists a subsequence $x_1^{\epsilon_j}\in B_{\epsilon_j}(x_{\epsilon_j})$ of maximum points of ϕ. Then, $\nabla\phi(x_1^{\epsilon_j})=0$ and, since $x_1^{\epsilon_j}\to x_0$ as $\epsilon_j\to 0$, we have by continuity that $\nabla\phi(x_0)=0$, a contradiction. The argument for x_2^ϵ is similar.

Hence, since $\overline{B_\epsilon(x_\epsilon)}\cap\partial\Omega=\emptyset$, we have

$$x_1^\epsilon=x_\epsilon+\epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}+o(1)\right],\quad\text{and}\quad x_2^\epsilon=x_\epsilon-\epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}+o(1)\right] \tag{10}$$

as $\epsilon\to 0$. This can be deduced from the fact that, for ϵ small enough ϕ is approximately the same as its tangent plane.

In fact, if we write $x_1^\epsilon=x_\epsilon+\epsilon v^\epsilon$ with $|v^\epsilon|=1$, and we fix any direction w, then the Taylor expansion of ϕ gives

$$\phi(x_\epsilon)+\langle\nabla\phi(x_\epsilon),\epsilon v^\epsilon\rangle+o(\epsilon)=\phi(x_1^\epsilon)\geq\phi(x_\epsilon+\epsilon w)$$

and hence

$$\langle\nabla\phi(x_\epsilon),v^\epsilon\rangle+o(1)\geq\frac{\phi(x_\epsilon+\epsilon w)-\phi(x_\epsilon)}{\epsilon}=\langle\nabla\phi(x_\epsilon),w\rangle+o(1)$$

for any direction w. This implies

$$v^\epsilon=\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}+o(1).$$

Now, consider the Taylor expansion of second order of ϕ

$$\phi(y)=\phi(x_\epsilon)+\nabla\phi(x_\epsilon)\cdot(y-x_\epsilon)+\frac{1}{2}\langle D^2\phi(x_\epsilon)(y-x_\epsilon),(y-x_\epsilon)\rangle+o(|y-x_\epsilon|^2)$$

as $|y-x_\epsilon|\to 0$. Evaluating the above expansion at the point at which ϕ attains its minimum in $\overline{B_\epsilon(x_\epsilon)}$, x_2^ϵ, we get

$$\phi(x_2^\epsilon)=\phi(x_\epsilon)+\nabla\phi(x_\epsilon)(x_2^\epsilon-x_\epsilon)+\frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon-x_\epsilon),(x_2^\epsilon-x_\epsilon)\rangle+o(\epsilon^2),$$

as $\epsilon\to 0$.

Evaluating at its symmetric point in the ball $\overline{B_\epsilon(x_\epsilon)}$, that is given by

$$\tilde x_2^\epsilon=2x_\epsilon-x_2^\epsilon \tag{11}$$

we get

$$\phi(\tilde x_2^\epsilon)=\phi(x_\epsilon)-\nabla\phi(x_\epsilon)(x_2^\epsilon-x_\epsilon)+\frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon-x_\epsilon),(x_2^\epsilon-x_\epsilon)\rangle+o(\epsilon^2).$$

Adding both expressions we obtain

$$\phi(\tilde{x}_2^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) = \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

We observe that, by our choice of x_2^ϵ as the point where the minimum is attained,

$$\phi(\tilde{x}_2^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) \leq \max_{y\in\overline{B_\epsilon(x)}\cap\bar{\Omega}} \phi(y) + \min_{y\in\overline{B_\epsilon(x)}\cap\bar{\Omega}} \phi(y) - 2\phi(x_\epsilon) \leq \eta(\epsilon).$$

Therefore

$$0 \geq \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

Note that from (10) we get

$$\lim_{\epsilon\to 0} \frac{x_2^\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x_0).$$

Then we get, dividing by ϵ^2 and passing to the limit,

$$0 \leq -\Delta_\infty\phi(x_0).$$

Now, if $\nabla\phi(x_0) = 0$ we can argue exactly as above and moreover, we can suppose (considering a subsequence) that

$$\frac{(x_2^\epsilon - x_\epsilon)}{\epsilon} \to v_2 \qquad \text{as } \epsilon \to 0,$$

for some $v_2 \in \mathbb{R}^n$. Thus

$$0 \leq -\left\langle D^2\phi(x_0)v_2, v_2\right\rangle = -\Delta_\infty\phi(x_0)$$

by definition, since $\phi \in S(x_0)$.

CASE B. Suppose that $x_0 \in \Gamma_N$. There are four sub-cases to be considered depending on the direction of the gradient $\nabla\phi(x_0)$ and the distance of the points x_ϵ to the boundary.

CASE 1: If either $\nabla\phi(x_0) = 0$, or $\nabla\phi(x_0) \neq 0$ and $\nabla\phi(x_0)\perp n(x_0)$, then

$$\frac{\partial\phi}{\partial n}(x_0) = 0 \quad \Rightarrow \quad \max\left\{\frac{\partial\phi}{\partial n}(x_0), -\Delta_\infty\phi(x_0)\right\} \geq 0, \tag{12}$$

where

$$\Delta_\infty\phi(x_0) = \lim_{y\to x_0} \frac{2\big(\phi(y) - \phi(x_0)\big)}{|y - x_0|^2}$$

is well defined since $\phi \in S(x_0)$.

CASE 2: $\liminf_{\epsilon\to 0} \dfrac{\operatorname{dist}(x_\epsilon, \partial\Omega)}{\epsilon} > 1$, and $\nabla\phi(x_0) \neq 0$.

Since $\nabla\phi(x_0) \neq 0$ we also have $\nabla\phi(x_\epsilon) \neq 0$ for ϵ small enough. Hence, since $B_\epsilon(x_\epsilon) \cap \partial\Omega = \emptyset$, we have, as before,

$$x_1^\epsilon = x_\epsilon + \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right], \quad \text{and} \quad x_2^\epsilon = x_\epsilon - \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right]$$

as $\epsilon \to 0$. Notice that both $x_1^\epsilon, x_2^\epsilon \to \partial B_\epsilon(x_\epsilon)$. This can be deduced from the fact that, for ϵ small enough ϕ is approximately the same as its tangent plane.

Then we can argue exactly as before (when $x_0 \in \Omega$) to obtain that

$$0 \leq -\Delta_\infty\phi(x_0).$$

CASE 3: $\limsup_{\epsilon\to 0} \dfrac{\operatorname{dist}(x_\epsilon, \partial\Omega)}{\epsilon} \leq 1$, and $\nabla\phi(x_0) \neq 0$ points inwards Ω.

In this case, for ϵ small enough we have that $\nabla\phi(x_\epsilon) \neq 0$ points inwards as well. Thus,

$$x_1^\epsilon = x_\epsilon + \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right] \in \Omega,$$

while $x_2^\epsilon \in \overline{\Omega} \cap \overline{B_\epsilon}(x_\epsilon)$. Indeed,

$$\frac{|x_2^\epsilon - x_\epsilon|}{\epsilon} = \delta_\epsilon \leq 1.$$

We have the following first-order Taylor's expansions,

$$\phi(x_1^\epsilon) = \phi(x_\epsilon) + \epsilon|\nabla\phi(x_\epsilon)| + o(\epsilon),$$

and

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)\cdot(x_2^\epsilon - x_\epsilon) + o(\epsilon),$$

as $\epsilon \to 0$. Adding both expressions, we arrive at

$$\phi(x_1^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) = \epsilon|\nabla\phi(x_\epsilon)| + \nabla\phi(x_\epsilon)\cdot(x_2^\epsilon - x_\epsilon) + o(\epsilon).$$

Using (7) and dividing by $\epsilon > 0$,

$$0 \geq |\nabla\phi(x_\epsilon)| + \nabla\phi(x_\epsilon)\cdot\frac{(x_2^\epsilon - x_\epsilon)}{\epsilon} + o(1)$$

as $\epsilon \to 0$. We can write

$$0 \geq |\nabla\phi(x_\epsilon)|\cdot(1 + \delta_\epsilon\cos\theta_\epsilon) + o(1)$$

where

$$\theta_\epsilon = \text{angle}\left(\nabla\phi(x_\epsilon), \frac{(x_2^\epsilon - x_\epsilon)}{\epsilon}\right).$$

Letting $\epsilon \to 0$ we get

$$0 \geq |\nabla\phi(x_0)|\cdot(1 + \delta_0\cos\theta_0),$$

where $\delta_0 \leq 1$, and

$$\theta_0 = \lim_{\epsilon\to 0}\theta_\epsilon = \text{angle}\left(\nabla\phi(x_0), v(x_0)\right),$$

with

$$v(x_0) = \lim_{\epsilon\to 0}\frac{x_2^\epsilon - x_\epsilon}{\epsilon}.$$

Since $|\nabla\phi(x_0)| \neq 0$, we find out $(1 + \delta_0\cos\theta_0) \leq 0$, and then $\theta_0 = \pi$ and $\delta_0 = 1$. Hence

$$\lim_{\epsilon\to 0}\frac{x_2^\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x_0), \tag{13}$$

or what is equivalent,

$$x_2^\epsilon = x_\epsilon - \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right].$$

Now, consider $\tilde{x}_2^\epsilon = 2x_\epsilon - x_2^\epsilon$ the symmetric point of x_2^ϵ with respect to x_ϵ. We go back to (7) and use the Taylor expansions of second order,

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),$$

and

$$\phi(\tilde{x}_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon), (\tilde{x}_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),$$

to get

$$\begin{aligned} 0 \geq & \min_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} \phi(y) + \max_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar{\Omega}} \phi(y) - 2\phi(x_\epsilon) \\ \geq & \ \phi(x_2^\epsilon) + \phi(\tilde{x}_2^\epsilon) - 2\phi(x_\epsilon) \\ = & \ \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \nabla\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle \\ & + \frac{1}{2}\langle D^2\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon), (\tilde{x}_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2), \\ = & \ \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2), \end{aligned}$$

by the definition of $\tilde{x}_2^\epsilon$. Then, we can divide by ϵ^2 and use (13) to obtain

$$-\Delta_\infty\phi(x_0) \geq 0.$$

CASE 4: $\limsup_{\epsilon\to 0} \dfrac{\operatorname{dist}(x_\epsilon, \partial\Omega)}{\epsilon} \leq 1$, and $\nabla\phi(x_0) \neq 0$ points outwards Ω.

In this case we have

$$\frac{\partial\phi}{\partial n}(x_0) = \nabla\phi(x_0)\cdot n(x_0) \geq 0,$$

since $n(x_0)$ is the exterior normal at x_0 and $\nabla\phi(x_0)$ points outwards Ω. Thus

$$\max\left\{\frac{\partial\phi}{\partial n}(x_0), -\Delta_\infty\phi(x_0)\right\} \geq 0,$$

and we conclude that u is a viscosity supersolution of (3).

It remains to check that u is a viscosity subsolution of (3). This fact can be proved in an analogous way, taking some care in the choice of the points where we perform Taylor expansions. In fact, instead of taking (11) we have to choose

$$\tilde{x}_1^\epsilon = 2x_\epsilon - x_1^\epsilon,$$

that is, the reflection of the point where the maximum in the ball $\overline{B_\epsilon(x_\epsilon)}$ of the test function is attained.

This ends the proof. □

3. A solution to the mixed problem enjoys comparison with quadratic functions.

In this section we will assume the following hypothesis on the domain Ω.

Hypothesis *For every $z \in \overline{\Omega}$ and every $x^* \in \Gamma_N$, $z \neq x^*$ we have*

$$\left\langle \frac{x^* - z}{|x^* - z|}; n(x^*)\right\rangle > 0.$$

Note that this holds, for example, if Ω is strictly convex.

We want to prove that a viscosity solution to

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D, \end{cases} \tag{14}$$

enjoys comparison with quadratic functions from above and below. By the results of [19] this turns out to be equivalent to be the unique continuous value for the Tug-of-War game obtained as the limit of the u_ϵ.

Let us recall the definition of comparison with quadratic functions given in [19].

Definition 6. Let $Q(r) = ar^2 + br + c$, with $a, b, c, r \in \mathbb{R}$. Let $z \in \overline{\Omega}$, we call the function

$$\varphi(x) = Q(|x - z|)$$

a *quadratic distance function.*

We say that a quadratic distance function is $*-$increasing on $V \subset \overline{\Omega}$ if either

1) $z \notin V$ and for every $x \in V$, we have $Q'(|x - z|) > 0$, or

2) $z \in V$ and $b = 0$ and $a > 0$.

Similarly, we say that a function φ is $*-$decreasing on V if $-\varphi$ is $*-$increasing on V.

(1) We say that u enjoys comparison with quadratic functions from above if for every $V \subset \overline{V} \subset U$ and a $*-$increasing quadratic function φ in V with quadratic term $a \leq 0$ then the inequality $\varphi \geq u$ on the relative boundary $\partial V \cap \Omega$ implies $\varphi \geq u$ in V.

(2) Analogously, we say that u enjoys comparison with quadratic functions from below if for every $V \subset \overline{V} \subset U$ and a $*-$decreasing quadratic function φ in V with quadratic term $a \geq 0$ then the inequality $\varphi \leq u$ on the relative boundary $\partial V \cap \Omega$ implies $\varphi \leq u$ in V.

We split our arguments in two lemmas.

Lemma 7. *If u is a solution to* (14) *then u enjoys comparison with quadratic functions from above.*

Proof. Take φ a $*-$increasing quadratic function in $V \subset \overline{\Omega} \setminus \Gamma_D$ with $a \leq 0$ and such that $\varphi \geq u$ on $\partial V \cap \Omega$. We have to show that $\varphi \geq u$ in V.

First, observe that we can assume that $\varphi > u$ on $\partial V \cap \Omega$. If the conclusion is valid for that kind of functions then just take $\varphi + k$ and then the limit as $k \to 0$ to get the conclusion for any φ with $\varphi \geq u$ on $\partial V \cap \Omega$. Therefore, assume that $\varphi > u$ on $\partial V \cap \Omega$.

Now, we argue by contradiction and assume that there is a point $x_0 \in V$ with $u(x_0) > \varphi(x_0)$. Take

$$\varphi_\delta(x) = \varphi(x) - \delta|x - z|^2$$

with δ small in order to have $\varphi_\delta > u$ on $\partial V \cap \Omega$ and

$$\max_V (u - \varphi_\delta) = u(x^*) - \varphi_\delta(x^*) > 0.$$

Notice that $x^* \in V \setminus (\partial V \cap \Omega)$.

We have two possibilities:

CASE A If $x^* \in V \cap \Omega$, since u is a viscosity subsolution to $-\Delta_\infty u = 0$ in Ω, we have that,

$$-\Delta_\infty \varphi_\delta(x^*) \leq 0. \tag{15}$$

On the other hand, since $a \leq 0$ we get (just by differentiation of the explicit expression of a quadratic function)

$$-\Delta_\infty \varphi_\delta(x^*) = -2a + 2\delta > 0.$$

This contradicts (15).

CASE B If $x^* \in V \cap \partial\Omega$, since u is a viscosity subsolution to (14), we have that

$$\min\left\{ \frac{\partial \varphi_\delta}{\partial n}(x^*),\ -\Delta_\infty \varphi_\delta(x^*) \right\} \leq 0.$$

Therefore, we have

$$\frac{\partial \varphi_\delta}{\partial n}(x^*) \leq 0 \qquad \text{or} \qquad -\Delta_\infty \varphi_\delta(x^*) \leq 0. \tag{16}$$

On the other hand, using again the fact that $a \leq 0$, we obtain, as before,

$$-\Delta_\infty \varphi_\delta(x^*) > 0. \tag{17}$$

Now, we argue as follows, since φ is a $*-$increasing quadratic function on $V \subset \overline{\Omega} \setminus \Gamma_D$ (see Definition 6) we have that either

1) $z \notin V$ and for every $x \in V$, we have $Q'(|x-z|) > 0$, or

2) $z \in V$ and $b = 0$ and $a > 0$.

Note that since $a \leq 0$ the second case, 2), is not possible.

Therefore, in the first case, 1), we have that

$$\begin{aligned} \frac{\partial \varphi_\delta}{\partial n}(x^*) &= \frac{\partial \varphi}{\partial n}(x^*) - \delta \frac{\partial |x-z|^2}{\partial n}(x^*) \\ &= \Big(Q'(|x^*-z|) - 2\delta |x^*-z|\Big)\Big\langle \frac{x^*-z}{|x^*-z|}, n(x^*)\Big\rangle > 0, \end{aligned} \tag{18}$$

choosing δ smaller if necessary. We are using here that $Q'(|x^*-z|) > 0$ and that

$$\Big\langle \frac{x^*-z}{|x^*-z|}, n(x^*)\Big\rangle > 0.$$

Inequalities (17) and (18) contradict (16). The proof is now complete. □

Remark 8. We have only used that u is a viscosity subsolution to (14) to prove this lemma.

In an analogous way (but using only that u is a viscosity supersolution) we can prove that,

Lemma 9. *If u is a solution to* (14) *then u enjoys comparison with quadratic functions from below.*

Proof. It is analogous to the proof of the previous lemma. □

Remark 10. It is possible to relax the geometric assumption on Ω in Lemmas 7 and 9 to allow convex domains with flat pieces of boundary if we assume that, for every $z \in \overline{\Omega}$ and every $x^* \in \Gamma_N$, $z \neq x^*$, we have

$$\Big\langle \frac{x^*-z}{|x^*-z|}, n(x^*)\Big\rangle \geq 0, \tag{19}$$

and, for all $V \subset \overline{\Omega} \setminus \Gamma_D$ there exists $p \in \mathbb{R}^n$, $|p| = 1$, such that $\langle p, n(x^*)\rangle > 0$ for all $x^* \in \Gamma_N \cup V$ for which (19) holds with an equality. Then, the proof of the Lemmas, can be carried out as above considering the perturbation

$$\tilde{\varphi}_\delta(x) = \varphi(x) - \delta|x-z|^2 + \delta^2 \langle p, (x-z)\rangle$$

(for p as in the above hypothesis) instead of φ_δ.

From these two lemmas and the results of [19] we can easily deduce that any viscosity solution to (14) has to be the unique continuous value of the Tug-of-War game.

Proof of part ii) *of Theorem 1.* The previous two lemmas show that a viscosity solution to (14) has comparison with quadratic functions from above and below, hence, by the results of [19], it is the unique continuous value of the game. □

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