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A MODAL EXTENSION OF FIRST ORDER CLASSICAL LOGIC–Part II

We define the semantics of the modal predicate logic introduced in Part I and prove its soundness and strong completeness with respect to appropriate structures. These semantical tools allow us to give a simple proof that the *main conservation requirement* articulated in Part I, Section 1, is met as it follows directly from Theorem 5.1 below. Section numbering is consecutive to that of Part I. The bibliography at the end applies only to Part II.

We will freely use notation and results from Part I. Moreover, in what follows $\forall A$ will denote the canonical universal closure of A, that is, $(\forall y_1)\cdots(\forall y_n)A$ where $y_1,\ldots,y_n$ are all the free variables of A in alphabetical order. Thus $\forall A$ is the same expression as A if the latter is closed. We may also abbreviate $(\forall y_1)\cdots(\forall y_n)$ by $(\forall \vec{y})$. In general, $\vec{a}$ denotes $a_1,\ldots,a_n$, where n is either unimportant or is clear from the context.

5. The Main Conservation Requirement

Theorem 5.1. *If A is a wff and $\mathcal{T}$ is a classical theory, then $\vdash_{\mathcal{T}} \Box A$ implies that $\mathcal{T} \vdash A$, classically. The converse also holds by the derived rule "WN" (cf. Part I, Metatheorem 4.2).*

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As for the conservation requirement–that "for classical $\mathcal{T}, A$ and B, we have $\vdash_{\mathcal{T}} \Box A \to \Box B$ iff $\mathcal{T} + A \vdash B$ classically"–assume the left hand side of the iff. Now using MP and adding a redundant axiom (A), we obtain $\mathcal{T} \cup \Box\mathcal{T} \cup \{A, \Box A\} \vdash \Box B$, that is, $\vdash_{\mathcal{T}+A} \Box B$. The right hand side follows by 5.1. The converse is as easy, applying the deduction theorem on the modal deduction $\vdash_{\mathcal{T}+A} B$ to obtain $\vdash_{\mathcal{T}} \forall A \to \Box A \to B$. An application of WN followed by the use of axiom (M1) yields $\vdash_{\mathcal{T}} \Box\forall A \to \Box\Box A \to \Box B$. Now tautological implication along with axioms (M2) and (M3) rest the case.

Theorem 5.1 holds, as it immediately follows from the following two lemmata. A consequence of 5.1 is that if $\mathcal{T}$ is consistent classically, then $\mathcal{T} \cup \Box\mathcal{T}$ is so modally. For if the latter proves everything, it must prove $\Box\bot$. But then $\mathcal{T}$ must prove $\bot$ classically, and hence prove everything by tautological implication.

LEMMA 5.2. *If A is a wfmf and $\mathcal{T}$ is a classical theory, then $\vdash_{\mathcal{T}} \Box A$ implies that $\vdash_{\mathcal{T}} A$.*

Note that the lemma above is claiming less than Theorem 5.1: In the lemma, A is a wfmf, and the proof implied by the expression $\vdash_{\mathcal{T}} A$ is still within the modal system, possibly using nonlogical axioms from $\mathcal{T} \cup \Box\mathcal{T}$.

LEMMA 5.3. *If A is a wff and $\mathcal{T}$ is a classical theory, then $\vdash_{\mathcal{T}} A$ modally implies that $\mathcal{T} \vdash A$ classically.*

The lemmata follow easily by semantical considerations that we briefly outline here. In the interest of brevity we will rely on known facts from the literature. In particular we are influenced by the notation and approach in [2], which we adapt here to first order theories such as those defined in Part I. Our semantics is appropriate for the presence of axiom (M3) and for our syntactic choice that $\Box A$ is always a sentence.

DEFINITION 5.4. A *pointed Kripke frame* is a triple $\mathcal{F} = (W, R, \alpha_0)$, where W is a nonempty set of objects–usually called "worlds"–R is a *transitive* relation on W, and $\alpha_0 \in W$ is R-minimum, that is, $(\forall\beta \in W)(\alpha_0 = \beta \vee \alpha_0 R \beta)$. ∎

Definition 5.5. A *Kripke structure* for a modal language L is a pair $\mathsf{M} = \big(\mathcal{F}, \{(M_\alpha, \mathcal{I}_\alpha) : \alpha \in W\}\big)$ where $\mathcal{F} = (W, R, \alpha_0)$ is a pointed frame and, for each α, M_α is a nonempty set of individuals and $\mathcal{I}_\alpha$ is an *interpretation mapping* with the following properties:

(i) For every constant c in L and $\alpha \in W$, $\mathcal{I}_\alpha(c) \in M_\alpha$,

(ii) For every function f of arity $n > 0$ in L and $\alpha \in W$, $\mathcal{I}_\alpha(f)$ is a total function $M_\alpha^n \to M_\alpha$,

(iii) For every predicate P of arity $n > 0$ in L and $\alpha \in W$, $\mathcal{I}_\alpha(P)$ is a subset of M_α^n,

(iv) For every propositional variable q in L and $\alpha \in W$, $\mathcal{I}_\alpha(q)$ is a member of $\{\mathbf{t}, \mathbf{f}\}$. By $\mathbf{t}$ and $\mathbf{f}$ we mean the metamathematical truth values "true" and "false" respectively. ∎

We extend semantics to arbitrary terms and formulae by performing the Henkin trick, that is, importing all the individuals of M_α into L as new constants (cf. [1], [3]). To simplify notation, we will use the same name, "c", to name an informal constant in M_α, and its formal counterpart that was imported into L.

Let us denote the language so extended by $L(M_\alpha)$. Then we extend the mapping $\mathcal{I}_\alpha$ to all closed terms and formulae of $L(M_\alpha)$ as follows:

Definition 5.6. [Extending $\mathcal{I}_\alpha$]
(1) By induction on *closed* terms over $L(M_\alpha)$ we define:

(a) For every $\alpha \in W$ and constant c in $L(M_\alpha)$, we let $\mathcal{I}_\alpha(c)$ be the same as in (i) of Definition 5.5 if $c \in L$. Else it is c itself. That is, imported individuals translate as themselves in every world.

(b) If $t = f(t_1, \ldots, t_n)$ and the t_i are closed terms of $L(M_\alpha)$, then

$$\mathcal{I}_\alpha(t) = \mathcal{I}_\alpha(f)\big(\mathcal{I}_\alpha(t_1), \ldots, \mathcal{I}_\alpha(t_n)\big)$$

(2) For each $\alpha \in W$ we define by induction on *closed* formulae of $L(M_\alpha)$:

(A) $\mathcal{I}_\alpha(\bot) = \mathbf{f}$ and $\mathcal{I}_\alpha(\top) = \mathbf{t}$.

(B) If t_i are closed terms of $L(M_\alpha)$ and P is an n-ary predicate, then

$$\mathcal{I}_\alpha\big(P(t_1, \ldots, t_n)\big) = \mathcal{I}_\alpha(P)\big(\mathcal{I}_\alpha(t_1), \ldots, \mathcal{I}_\alpha(t_n)\big)$$

(C) If t and s are closed terms of $L(M_\alpha)$, then $\mathcal{I}_\alpha(t = s) = \mathbf{t}$ iff $\mathcal{I}_\alpha(t) = \mathcal{I}_\alpha(s)$.

(D) For any closed formula A of $L(M_\alpha)$, $\mathcal{I}_\alpha\big(\neg A\big) = \mathbf{t}$ iff $\mathcal{I}_\alpha(A) = \mathbf{f}$.

(E) For any closed formula $(\forall x)A$ of $L(M_\alpha)$, it is $\mathcal{I}_\alpha\big((\forall x)A\big) = \mathbf{t}$ iff, for all $c \in M_\alpha$, it is $\mathcal{I}_\alpha(A[x := c]) = \mathbf{t}$

(F) For any formula A of $L(M)$, closed or not, it is $\mathcal{I}_\alpha\big(\Box A\big) = \mathbf{t}$ iff, for all β such that $\alpha R \beta$, it is $\mathcal{I}_\beta(\forall A) = \mathbf{t}$

(G) For any closed formulae A and B of $L(M_\alpha)$, we have $\mathcal{I}_\alpha(A \vee B) = \mathbf{t}$ iff $\mathcal{I}_\alpha(A) = \mathbf{t}$ or $\mathcal{I}_\alpha(B) = \mathbf{t}$. ∎

Definition 5.7. Let $\mathsf{M} = \big(\mathcal{F}, \{(M_\alpha, \mathcal{I}_\alpha) : \alpha \in W\}\big)$ be a structure for L, where $\mathcal{F} = (W, R, \alpha_0)$ and A a wfmf of L. We say that A is *true* in M at α iff $\mathcal{I}_\alpha(\forall A) = \mathbf{t}$. We say that M is a Kripke model of A, and write $\models_{\mathsf{M}} A$, iff A is true at α_0 in M. If Γ is a set of formulae over L, we say that M is a Kripke model of Γ, in symbols $\models_{\mathsf{M}} \Gamma$, iff M is a Kripke model of every A in Γ. The symbol $\Gamma \models A$ is that for semantic implication. It means that every (Kripke) model of Γ is also a model of A. ∎

We have not defined modal *validity*, a notion that we will not employ anywhere.

One can easily prove that all the axioms in $\Lambda \cup \Box\Lambda$ are true in all Kripke structures M and at all α in each such structure. We briefly verify two interesting ones: First, consider $\Box A \to \Box\Box A$ for an arbitrary wfmf A and fix a $\mathsf{M} = \big(\mathcal{F}, \{(M_\alpha, \mathcal{I}_\alpha) : \alpha \in W\}\big)$. By Definition 5.6 $\big($(D), (F) and (G)$\big)$, we have two cases to consider: One, if $\mathcal{I}_\alpha(\Box A) = \mathbf{f}$ (recall that $\Box A$ is closed), then $\mathcal{I}_\alpha(\Box A \to \Box\Box A) = \mathbf{t}$. Suppose then that $\mathcal{I}_\alpha(\Box A) = \mathbf{t}$. Then

$$\mathcal{I}_\beta(A[\vec{x} := \vec{c}]) = \mathbf{t} \text{ for all } \beta \text{ satisfying } \alpha R \beta \text{ and all } \vec{c} \text{ in} M_\beta \qquad (1)$$

where $\vec{x}$ is the list of all the free variables of A. Assume now that $\mathcal{I}_\alpha(\Box\Box A) = \mathbf{f}$. Then for some β such that $\alpha R \beta$, we have $\mathcal{I}_\beta(\Box A) = \mathbf{f}$. This implies the existence of a γ with $\beta R \gamma$, and a $\vec{c}$ in M_γ such that $\mathcal{I}_\gamma(A[\vec{x} := \vec{c}]) = \mathbf{f}$. But $\alpha R \gamma$ by transitivity of R, and we have just contradicted (1).

Next, we verify that $\mathcal{I}_\alpha(\Box A \to \Box(\forall x)A) = \mathbf{t}$. Again, assume (the interesting case) $\mathcal{I}_\alpha(\Box A) = \mathbf{t}$. Thus, $\mathcal{I}_\beta((\forall \vec{y})(\forall x)A) = \mathbf{t}$ for all β satisfying $\alpha R \beta$, where $x, \vec{y}$ is the list of all the free variables of A. By (F) in 5.6, $\mathcal{I}_\alpha(\Box(\forall x)A) = \mathbf{t}$. It is as easy to check that all the other logical axioms are true at all α and also to prove that the two rules of inference preserve truth. Thus we have soundness:

PROPOSITION 5.8. [Soundness] *Let $\mathcal{T}$ be any theory. Then for any wfmf A, $\mathcal{T} \vdash A$ implies $\mathcal{T} \models A$. In particular, $\vdash_{\mathcal{T}} A$ implies $\mathcal{T} \cup \Box\mathcal{T} \models A$.*

PROPOSITION 5.9. [Completeness]*Let $\mathcal{T}$ be any theory. Then for any wfmf A, $\mathcal{T} \models A$ implies $\mathcal{T} \vdash A$. In particular, $\mathcal{T} \cup \Box\mathcal{T} \models A$ implies $\vdash_{\mathcal{T}} A$.*

For a detailed sketch of proof of 5.9 see the next section. We can now prove the two key lemmata of this section.

PROOF [of Lemma 5.2] Assume hypothesis, and also $\not\vdash_{\mathcal{T}} A$. Let $\mathsf{M} = \big(\mathcal{F}, \{(M_\alpha, \mathcal{I}_\alpha) : \alpha \in W\}\big)$ be a model of $\mathcal{T} \cup \Box\mathcal{T}$ such that $\not\models_{\mathsf{M}} A$, that is,

$$\mathcal{I}_{\alpha_0}(\forall A) = \mathbf{f} \tag{1}$$

Let α_{-1} be a new world and consider a new frame $\mathcal{F}' = \big(W', R', \alpha_{-1}\big)$, where $W' = W \cup \{\alpha_{-1}\}$ and $R' = R \cup (\{\alpha_{-1}\} \times W)$.

We now build a structure $\mathsf{M}' = \big(\mathcal{F}', \{(M'_\alpha, \mathcal{I}'_\alpha) : \alpha \in W'\}\big)$ where $M'_\alpha = M_\alpha$, $\mathcal{I}'_\alpha = \mathcal{I}_\alpha$ for $\alpha \in W$, while $M'_{\alpha_{-1}} = M_{\alpha_0}$ and $\mathcal{I}'_{\alpha_{-1}} = \mathcal{I}_{\alpha_0}$. Thus, $\models_{\mathsf{M}'} \mathcal{T} \cup \Box\mathcal{T}$, but $\mathcal{I}'_{\alpha_{-1}}(\Box A) = \mathbf{f}$ by (1), that is, $\mathcal{T} \cup \Box\mathcal{T} \not\models \Box A$, contradicting hypothesis by soundness. ■

PROOF [of Lemma 5.3] Assume hypothesis, and let $\mathsf{M} = (M, \mathcal{I})$ be a classical model of $\mathcal{T}$.[2] Consider the frame $\mathcal{F} = (\{0\}, \emptyset, 0)$ (one world, "0", and empty R). We now form the Kripke structure $\mathsf{M}' = \big(\mathcal{F}, \{(M_0, \mathcal{I}_0)\}\big)$ where $M_0 = M$, and $\mathcal{I}_0 = \mathcal{I}$. Clearly, M' is a model of $\mathcal{T} \cup \Box\mathcal{T}$ in the sense of Definition 5.7. Thus, by soundness, we have $\mathcal{I}_0(\forall A) = \mathbf{t}$. It is easy to verify that $\mathcal{I}_0(\forall A) = \mathcal{I}(\forall A)$, hence A is true in M, classically. The latter being an arbitrary classical model of $\mathcal{T}$, we have that A is classically derivable from $\mathcal{T}$. ■

6. The Completeness of M^3

In this section we outline the proof of completeness of M^3 with respect to pointed Kripke structures. There are some standard steps in the proof for which we refer the reader to the literature (e.g., [1], [2], [3]) to avoid labouring over the well-known. Thus we start with a consistent set of wfmf

[2] If $\mathcal{T}$ has no classical models, then it is inconsistent, hence $\mathcal{T}$ proves A classically anyway.

$\mathcal{T}$ and an arbitrary enumerable set M. We fix an enumeration $m_0, m_1, \ldots$ of M and also fix the two enumerations below:

$$A_0, A_1, A_2, \ldots \text{ of all closed wfmf over } L(M) \tag{1}$$

$$\mathtt{F}_1, \mathtt{F}_2, \ldots \text{ of all closed wfmf over } L(M) \text{ of the form } (\exists x)A \tag{2}$$

We can now define by recursion a sequence $\Gamma_0, \Gamma_1, \ldots$ in two stages: First, let $\Gamma_0 = \mathcal{T}$ and then

$$\Delta_n = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if } \Gamma_n \nvdash \neg A_n \\ \Gamma_n \cup \{\neg A_n\} & \text{otherwise} \end{cases}$$

Finally, we let

$$\Gamma_{n+1} = \begin{cases} \Delta_n \cup \{A[x := a]\} & \text{if } \Delta_n \vdash \mathtt{F}_{n+1} \text{ where } \mathtt{F}_{n+1} \text{ is } (\exists x)A \\ \Delta_n & \text{otherwise} \end{cases} \tag{3}$$

In (3) we choose the so-called *Henkin constant* a so that $a = m_i$ where i is smallest such that m_i does not occur in any of $A_0, \ldots, A_n, \mathtt{F}_1, \ldots, \mathtt{F}_{n+1}$.

It is standard folklore in the classical setting (cf. [3]) that Δ_n is consistent if Γ_n is, and Γ_{n+1} is consistent if Δ_n is. The proofs carry over unchanged to the modal case. Now setting $\Gamma = \bigcup_{n \geq 0} \Gamma_n$ we can state:

LEMMA 6.1. *Let $\mathcal{T}$ be a consistent set of wfmf over the language L, and let M be an enumerable set. Then there is a consistent Henkin completion Γ of $\mathcal{T}$ over $L(M)$. That is, a set of wfmf over $L(M)$ such that*

(i) $\mathcal{T} \subseteq \Gamma$

(ii) Γ is consistent

(iii) (Maximality) For any sentence A over $L(M)$, if $A \notin \Gamma$, then $\neg A$ is in Γ

(iv) (Henkin property) If Γ proves the sentence $(\exists x)A$ then it also proves $A[x := a]$ for some $a \in M$.

Maximality and consistency directly imply that such completions are deductively closed: If $\Gamma \vdash A$ for a sentence A over $L(M)$, then $A \in \Gamma$.

Our insistence to allow constants and functions in the language makes us work harder. We now need to cut down Γ so that it "distinguishes constants". Once again we defer to [3] for the details and we simply state:

LEMMA 6.2. [Main Semantic Lemma] *Let $\mathcal{T}$ be a consistent set of wfmf over the language L, and let M be an enumerable set. Then there is a finite or enumerable subset N of M, and a consistent Henkin completion Γ of $\mathcal{T}$ over $L(N)$ that distinguishes constants. That is, a set of wfmf over $L(N)$ such that*

(i) $\mathcal{T} \subseteq \Gamma$

(ii) Γ *is consistent*

(iii) (Maximality) For any sentence A over $L(N)$, one of A and $\neg A$ is in Γ

(iv) (Henkin property) If Γ proves the sentence $(\exists x)A$ over $L(N)$, then it also proves $A[x := a]$ for some $a \in N$

(v) (Distinguishing constants) If $a \neq b$ is (metamathematically) true in N, then $\Gamma \vdash \neg a = b$.

We are near our goal. We prove the *consistency theorem* first, that a consistent $\mathcal{T}$ must have a Kripke model M. We show how to construct M.

By 6.2 there is a countable set N, and a set of formulae Γ that is a consistent Henkin completion of $\mathcal{T}$ that moreover distinguishes constants. We fix one such Γ. We will build a pointed Kripke frame using Γ as our "α_0". Our proof outline follows the proof given for the propositional case in [2]. In principle, a world will be any consistent Henkin completion–in the sense of 6.2–of the logical axiom set $\Lambda \cup \Box\Lambda$. We fine tune this by keeping just those worlds that are accessible from Γ. Thus we define the accessibility relation first: For a set of formulae Δ we define

$$\Delta\Box = \{\forall A : \Box A \in \Delta\} \tag{4}$$

We now define the relation R for any two consistent Henkin completions of $\Lambda \cup \Box\Lambda$:

$$\Delta R \Sigma \text{ stands for } \Delta\Box \subseteq \Sigma \tag{5}$$

We easily check that R is transitive: Suppose $\Delta R \Delta' R \Delta''$ and let $\forall A \in \Delta\Box$. We want $\forall A \in \Delta''$. Indeed,

$\Box A \in \Delta$	implies $\Box\Box A \in \Delta$	(Δ is a completion of the logical axioms)
	implies $\Box A \in \Delta\Box$	(note that $\Box A$ being closed, $\forall\Box A$ is $\Box A$)
	implies $\Box A \in \Delta'$	
	implies $\forall A \in \Delta'\Box$	
	implies $\forall A \in \Delta''$	

We can now set $W = \{\Gamma\} \cup \{\Delta : \Gamma R \Delta\}$ and $\mathcal{F} = (W, R, \alpha_0)$ with $\alpha_0 = \Gamma$. For each world α (alias for some Δ) in W we let N_α denote a finite or enumerable set "N" as per Lemma 6.2. Our next task is to define a structure $\mathsf{N} = \big(\mathcal{F}, \{(N_\alpha, \mathcal{I}_\alpha) : \alpha \in W\}\big)$, that is, a model of $\mathcal{T}$. For each world α we define $\mathcal{I}_\alpha$ as follows:

$$\text{For each Boolean variable} q, \quad \mathcal{I}_\alpha(q) = \mathbf{t} \text{ iff } q \in \alpha \tag{6}$$

$$\text{For each } n\text{-ary predicate } P, \text{ and } \vec{a}_n \text{ in } N_\alpha, \mathcal{I}_\alpha\big(P(\vec{a}_n)\big) = \mathbf{t} \text{ iff } P(\vec{a}_n) \in \alpha \tag{7}$$

The Henkin and the "distinguishing constants" properties help to define $\mathcal{I}_\alpha$ for closed terms t over $L(N_\alpha)$, for each $\alpha \in W$, and prove $\alpha \vdash t = \mathcal{I}_\alpha(t)$ for such t (cf. [3]). This leads to

$$\mathcal{I}_\alpha\big(P(t_1, \ldots, t_n)\big) = \mathbf{t} \text{ iff } P(t_1, \ldots, t_n) \in \alpha \tag{7$'$}$$

for all predicates of arity n and closed terms t_i over $L(N_\alpha)$. We now claim

Lemma 6.3. *For each $\alpha \in W$ and each closed A over $L(N_\alpha)$,*

$$\mathcal{I}_\alpha(A) = \mathbf{t} \textit{ iff } A \in \alpha \tag{8}$$

Proof. The proof is by induction on formulae. For the basis, the cases P (including $=$) and q are (7$'$) and (6) respectively. The cases $\bot$ and $\top$ follow by soundness and 5.6 since α contains $\top$ but not $\bot$. We look at the interesting cases:

$\underline{A \text{ is } B \vee C}$: If $\mathcal{I}_\alpha(B \vee C) = \mathbf{t}$, then, say, $\mathcal{I}_\alpha(B) = \mathbf{t}$. By I.H., $B \in \alpha$, hence $\alpha \vdash A$, therefore $A \in \alpha$. Conversely, if $A \in \alpha$, then $B \in \alpha$ or $C \in \alpha$ (and we are done using the I.H.) Indeed, if $B \notin \alpha$ *and* $C \notin \alpha$, then $(\neg B) \in \alpha$ and $(\neg C) \in \alpha$ by maximality, rendering α inconsistent.

$\underline{A \text{ is } (\forall x)B}$: If $\mathcal{I}_\alpha\big((\forall x)B\big) = \mathbf{t}$, then $\mathcal{I}_\alpha\big(B[x := b]\big) = \mathbf{t}$ for all $b \in N_\alpha$. By I.H.,

$$B[x := b] \in \alpha, \text{ for all } b \in N_\alpha \tag{9}$$

We claim that $(\forall x)B \in \alpha$. If not, then $\big(\neg(\forall x)B\big) \in \alpha$ as before. That is, $\big((\exists x)\neg B\big) \in \alpha$; hence $\neg B[x := h]$ is in α for some $h \in N_\alpha$ by the Henkin property. This contradicts (9) by the consistency of α. Conversely, say $(\forall x)B \in \alpha$. Hence $\alpha \vdash (\forall x)B$ and thus (axiom (2)) $\alpha \vdash B[x := b]$, for all

$b \in N_\alpha$, from which we get (9). By the I.H., $\mathcal{I}_\alpha\big(B[x := b]\big) = \mathbf{t}$ for all $b \in N_\alpha$, hence $\mathcal{I}_\alpha\big((\forall x)B\big) = \mathbf{t}$.

$\underline{A \text{ is } \Box B}$: Let $\Box B \in \alpha$. Then $\forall B \in \alpha\Box$. It follows that if $\alpha R \beta$, then $\forall B \in \beta$, hence $\beta \vdash \forall B$, therefore (axiom (2)) $\beta \vdash B[\vec{x} := \vec{b}]$ for all b_i in N_β, where $\vec{x}$ is the list of all free variables in B. By earlier remarks, all the sentences $B[\vec{x} := \vec{b}]$ are in β, hence $\mathcal{I}_\beta\big(B[\vec{x} := \vec{b}]\big) = \mathbf{t}$ by I.H. and thus $\mathcal{I}_\beta\big(\forall B\big) = \mathbf{t}$. Therefore, β being arbitrary satisfying $\alpha R \beta$, we have $\mathcal{I}_\alpha\big(\Box B\big) = \mathbf{t}$.

For the converse we argue contrapositively: Let $\Box B \notin \alpha$. Thus $\forall B \notin \alpha\Box$. We next claim that

$$\alpha\Box \nvdash \forall B \tag{10}$$

If not, the deduction theorem yields $\vdash A_1 \to A_2 \to \ldots \to A_r \to \forall B$, for some A_i in $\alpha\Box$ (all the A_i are of the form $\forall C$). Hence $\vdash \Box A_1 \to \Box A_2 \to \ldots \to \Box A_r \to \Box\forall B$, from which (and $\Box A_i \in \alpha$)[3] we get $\alpha \vdash \Box\forall B$ by modus ponens. This yields $\alpha \vdash \Box B$ by $\Box$-monotonicity and axiom (2), thus $\Box B \in \alpha$, contradicting the assumption. With (10) established, let $\vec{x}$ be the list of all free variables in B and γ be a consistent Henkin completion of $\{\neg(\forall\vec{x})B\} \cup (\alpha\Box)$ as in 6.2. Then $\big(\neg(\forall\vec{x})B\big) \in \gamma$ and $\alpha R \gamma$, hence $\gamma \vdash (\exists\vec{x})\neg B$. Thus $\big(\neg B[\vec{x} := \vec{b}]\big) \in \gamma$, for some $\vec{b}$ in N_γ, therefore $B[\vec{x} := \vec{b}] \notin \gamma$. By the I.H. we have $\mathcal{I}_\gamma\big(B[\vec{x} := \vec{b}]\big) = \mathbf{f}$, hence (semantics of $\Box$) $\mathcal{I}_\alpha\big(\Box B\big) = \mathbf{f}$. ■

We can now prove (strong) completeness of M^3. Let $\mathcal{T} \models A$. Then

$$\mathcal{T} \models \forall A \tag{11}$$

Now, if $\mathcal{T} \nvdash \forall A$, then $\{\neg\forall A\} \cup \mathcal{T}$ is consistent. Let M be a Kripke model for $\{\neg\forall A\} \cup \mathcal{T}$. Then $\models_{\mathsf{M}} \mathcal{T}$ yet $\not\models_{\mathsf{M}} \forall A$, contradicting (11).

References

[1] Joseph R. Shoenfield, **Mathematical Logic**, Addison-Wesley, Reading, Massachusetts, 1967.

[3] A_i is $\forall C$ for some C, hence $\Box C \in \alpha$. Since α is deductively closed, we have that $\Box\forall C \in \alpha$ by axiom (M3).

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