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A MODAL LOGIC ϵ-CALCULUS

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1 *Introduction** First-order modal logics have been formulated in conventional axiom systems, Gentzen systems, natural deduction systems and tableau systems. In this paper we give a formulation based on the classical ϵ-calculus of Hilbert [4]. We deal only with S4 but a similar treatment of other modal logics is straightforward. Our proof of the analog of Hilbert's second ϵ-theorem is non-constructive and uses Kripke's model theory [3].

A straightforward attempt at producing an ϵ-calculus S4 by adding S4 axioms and rules to a classical logic ϵ-calculus does not work. A look at Kripke's model theory for S4 makes clear the reason for this failure. If X is a formula with one free variable, x, ϵxX classically is intended to be the name of a constant making $X(x)$ true, if any constant does (see [4] for a fuller classical discussion). However, in a Kripke S4 model [2, 3, 5] there are many possible worlds, and a constant making $X(x)$ true in one such world need not make it true in another. Thus in an ϵ-calculus S4, ϵxX would have to be a 'world-dependent' term, that is, possibly naming different constants in different worlds. Such things cannot be dealt with properly with the usual first-order S4 machinery. In [6, 7] Stalnaker and Thomason created an extension of ordinary first-order S4, by adding an abstraction operator, to handle similar 'world-dependent' terms (definite descriptions are things of this sort). We use this fundamental idea in an essential way in constructing our system. The syntactic purpose of the abstraction operator is to specify exactly the scope of a substitution for a free variable. Let us denote substitution of the term f for free x in X by $X(x/f)$. If f is a 'world-dependent' term, $[\Diamond X]\ (x/f)$ and $\Diamond[X(x/f)]$ could be taken in a natural way to have different semantic meanings. Let Γ be a possible world of a Kripke model and suppose f 'names' the object c in Γ. To say $[\Diamond X]\ (x/f)$ is true in Γ seems to say $[\Diamond X]\ (x/c)$ or $\Diamond X(x/c)$ is true in Γ. That is, for some world Δ possible relative to Γ, $X(x/c)$ is true in Δ.

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On the other hand, to say $\Diamond[X(x/f)]$ is true in Γ seems to say, for some world Δ' possible relative to Γ, $X(x/f)$ is true in Δ'. If f 'names' b in Δ' we would have that $X(x/b)$ is true in Δ'. Even if Δ and Δ' are the same there is no reason to suppose c and b are identical since f may 'name' different objects in Γ and Δ. See [6, 7] for a fuller discussion. By using the abstraction operator λ this apparent semantic distinction may be represented syntactically by $(\lambda x\ \Diamond X)(f)$ and $\Diamond(\lambda x\ X)(f)$, which are indeed two distinct formulas. Of course we must add axioms governing the use of this λ symbol.

We begin with the statement of a more usual axiomatic formulation of first-order S4, and its Kripke model theory, as found in [2, 3, 5]. Next we give our ϵ-calculus system and a Kripke type model theory suitable for it. Then we derive various formal results about the calculus to establish it as a convenient system of proof, and we show it is a conservative extension of the constant-free part of first-order S4. Finally we show the completeness of our ϵ-calculus relative to its model theory. Not all the axioms needed for model-theoretic completeness are needed to show we have an extension of constant-free first-order S4. In [1] we gave a system in which some axioms were missing and outlined some results.

2 A Fundamental First-Order S4 System (FS4) In this section we give a conventional axiomatic formulation of first-order S4, as found in [2] or [5], and its Kripke model theory, but in a slightly different notation than is usual.

In this and subsequent sections we take $\wedge$, $\sim$, $\exists$, $\Diamond$,), (as primitive, and consider $\vee$, $\supset$, $\equiv$, $\forall$, $\Box$ to be abbreviations in the usual way. We also use square and curly brackets informally. We assume we have a countable collection of n-place predicate letters for each natural number n, a countable collection of variables, and a separate countable collection of individual symbols, or constants. We use $x, y, z, w, v, \ldots$ to stand for arbitrary variables, and $a, b, c, d, \ldots$ to stand for arbitrary constants. We may also add subscripts or primes. The definition of formula is as usual, but we follow the terminology of [4] and reserve the word *formula* for the case where there are no free occurrences of variables; in the more general case we use the term *quasi-formula*. We use $X, Y, Z, \ldots$ to stand for arbitrary quasi-formulas. As indicated in section 1, by $X(t/u)$ we mean the result of replacing all occurrences of t in X (all free occurrences if t is a variable) by u.

The rules and axioms of **FS4** are as follows, wherein X and Y are any *formulas*.

Rules

FR1 $\dfrac{X \quad X \supset Y}{Y}$

FR2 $\dfrac{X}{\Box X}$.

FR3 $\dfrac{X \supset Y}{(\exists x)[X(c/x)] \supset Y}$ where c does not occur in Y.

Axiom schemas:

FA1 X, *where* X *is a classical tautology.*
FA2 $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$.
FA3 $\Box X \supset X$.
FA4 $\Box X \supset \Box\Box X$.
FA5 $X \supset (\exists x)[X(c/x)]$.

Remark: FA1 is, of course, an infinite collection of axioms, which can be replaced by a finite number of schemas, using any of the usual axiom systems for classical propositional logic.

Next we give a Kripke model theory for **FS4**. For convenience we assume the domain of any Kripke model consists of the set of constant symbols of **FS4**. This is sufficient, but not necessary.

By *an* **FS4** *model* we mean a quadruple, $\langle \mathscr{G}, \boldsymbol{R}, \models, \boldsymbol{P}\rangle$ where: $\mathscr{G}$ is a non-empty set; $\boldsymbol{R}$ is a transitive, reflexive relation on $\mathscr{G}$; $\boldsymbol{P}$ is a function on $\mathscr{G}$ ranging over non-empty sets of parameters; and $\models$ is a relation between elements of $\mathscr{G}$ and *formulas* of **FS4**. These are to satisfy the following, where $\Gamma \epsilon \mathscr{G}$.

1) If $\Delta \epsilon \mathscr{G}$ and $\Gamma \boldsymbol{R} \Delta$, then $\boldsymbol{P}(\Gamma) \subseteq \boldsymbol{P}(\Delta)$.
2) If $\Gamma \models X$, all constants of X are in $\boldsymbol{P}(\Gamma)$.
3) If all constants of X and Y are in $\boldsymbol{P}(\Gamma)$, then $\Gamma \models (X \wedge Y)$ if and only if $\Gamma \models X$ and $\Gamma \models Y$; $\Gamma \models \sim X$ if and only if not-$\Gamma \models X$ (often we write this as $\Gamma \not\models X$).
4) If X is a quasi-formula with at most one free variable, x, and all constants in $\boldsymbol{P}(\Gamma)$, then $\Gamma \models (\exists x)X$ if and only if $\Gamma \models X(x/c)$ for some $c \epsilon \boldsymbol{P}(\Gamma)$.
5) If all constants of X are in $\boldsymbol{P}(\Gamma)$, then $\Gamma \models \Diamond X$ if and only if for some $\Delta \epsilon \mathscr{G}$ such that $\Gamma \boldsymbol{R} \Delta$, $\Delta \models X$.

A formula, X, is called *valid* in the **FS4** model $\langle \mathscr{G}, \boldsymbol{R}, \models, \boldsymbol{P}\rangle$ if $\Gamma \models X$ for every $\Gamma \epsilon \mathscr{G}$ such that all constants of X belong to $\boldsymbol{P}(\Gamma)$. Proofs may be found in [2, 3, 5] (with a slightly different definition of model) that the set of formulas provable in **FS4** coincides with the set of formulas valid in all **FS4** models.

3 An ϵ-Calculus S4 (ϵS4) We take the same primitive symbols as **FS4** and use the same abbreviations. We no longer have constant symbols, but we add an abstraction symbol, λ, and a 'term forming' symbol, ϵ. We begin with a full definition of the notions of *quasi-formula, quasi-term,* and *free variable.*

1) Any variable is a quasi-term, and has itself as its only free variable.
2) If P is an n-place predicate letter and $t_1, \ldots, t_n$ are quasi-terms, $P(t_1, \ldots, t_n)$ is a quasi-formula. The free variables of $P(t_1, \ldots, t_n)$ are the free variables of $t_1, \ldots, t_n$.
3) If X and Y are quasi-formulas, so is $(X \wedge Y)$. The free variables of $(X \wedge Y)$ are those of X together with those of Y.
4) If X is a quasi-formula, so is $\sim X$. The free variables of $\sim X$ are those of X.

5) If X is a quasi-formula, so is $\Diamond X$. The free variables of $\Diamond X$ are those of X.
6) If X is a quasi-formula and x is a variable, $(\exists x)X$ is a quasi-formula. The free variables of $(\exists x)X$ are those of X other than x.
7) If X is a quasi-formula, x is a variable and t is a quasi-term, $(\lambda x\, X)(t)$ is a quasi-formula. The free variables of $(\lambda x\, X)(t)$ are those of X except for x, together with those of t.
8) If X is a quasi-formula and x is a variable, $\epsilon x X$ is a quasi-term. The free variables of $\epsilon x X$ are those of X other than x.

We use the word *formula* (respectively *term*) for quasi-formula (respectively quasi-term) having no free variables. We will use t, possibly primed or subscripted, to stand for an arbitrary quasi-term. We use $(\lambda x_1 \ldots x_n\, X)(t_1, \ldots, t_n)$ as an abbreviation for $(\lambda x_1\, (\lambda x_2 \ldots (\lambda x_n\, X)(t_n) \ldots)(t_2))(t_1)$. We also use $X(t_1/t_1', \ldots, t_n/t_n')$ as an abbreviation for $[\ldots[[X(t_n/t_n')]\,(t_{n-1}/t_{n-1}')] \ldots]\,(t_1/t_1')$. Moreover, we often use the following handy 'vector' notation: $\mathbf{x}$ for a sequence of variables, $\mathbf{t}$ for a sequence of quasi-terms, provided the full meaning is clear from context. Thus, we will use $(\lambda \mathbf{x}\; X)(\mathbf{t})$ for $(\lambda x_1 \ldots x_n\, X)(t_1, \ldots, t_n)$ and $X(\mathbf{t}/\mathbf{t}')$ for $X(t_1/t_1', \ldots, t_n/t_n')$.

Let X be a quasi-formula whose free variables are among $x_1, \ldots, x_n$. Let $t_1, \ldots, t_n$ be quasi-terms. If $(\lambda x_1 \ldots x_n\, X)(t_1, \ldots, t_n)$ is a *formula* we call it a λ-*closure* of X. If X has no free variables we consider it to be a λ-closure of itself. We use the phrase *t is free for x in X* in the standard way to mean that, on replacing all free occurrences of x in X by t, no free variable, y, of t becomes bound by a quantifier, $(\exists y)$, abstract symbol, λy, or ϵ-symbol, ϵy, of X.

The axioms and rules of ϵS4 are as follows.

Rules:

$\epsilon R1$ $\dfrac{X \quad X \supset Y}{Y}$ where X and Y are *formulas*.

$\epsilon R2$ $\dfrac{X}{\Box X}$ where X is a *formula*.

Axiom schemas: Let X and Y be quasi-formulas. We take as axioms *all λ-closures of* the following quasi-formulas.

First, structural axioms.

$\epsilon A1$ *If y is not free in X, but y is free for x in X,*

$$(\lambda x\, X)(t) \equiv [\lambda y\, X(x/y)]\,(t).$$

$\epsilon A2$ *If x is not free in X,*

$$(\lambda x\, X)(t) \equiv X.$$

$\epsilon A3$ *If $x \neq y$ and y is free for x in X,*

$$(\lambda yx\, X)(t, t) \equiv [\lambda y\, X(x/y)]\,(t).$$

$\epsilon A4$ *If* $x_1 \neq x_2$, x_1 *is not free in* t_2, x_2 *is not free in* t_1,

$$(\lambda x_1 x_2\ X)(t_1, t_2) \equiv (\lambda x_2 x_1\ X)(t_2, t_1).$$

$\epsilon A5$ $(\lambda x\ (\lambda x\ X)(x))(t) \equiv (\lambda x\ X)(t)$.

$\epsilon A6$ *If* A *is atomic*,

$$(\lambda \mathbf{x}\ \ A)(\mathbf{t}) \equiv A(\mathbf{x}/\mathbf{t}).$$

$\epsilon A7$ $[\lambda \mathbf{x}\ (X \wedge Y)]\ (\mathbf{t}) \equiv [(\lambda \mathbf{x}\ \ X)(\mathbf{t}) \wedge (\lambda \mathbf{x}\ Y)(\mathbf{t})]$.

$\epsilon A8$ $(\lambda \mathbf{x}\ {\sim}X)(\mathbf{t}) \equiv {\sim}(\lambda \mathbf{x}\ \ X)(\mathbf{t})$.

$\epsilon A9$ *If* y *is not free in any quasi-term of* $\mathbf{t}$ *and* y *is not in the sequence* $\mathbf{x}$,

$$[\lambda \mathbf{x}\ (\exists y)X]\ (\mathbf{t}) \equiv (\exists y)\ [(\lambda \mathbf{x}\ \ X)(\mathbf{t})].$$

Next, propositional axioms.

$\epsilon A10$ X, *where* X *is a classical tautology*.

$\epsilon A11$ $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$.

$\epsilon A12$ $\Box X \supset X$.

$\epsilon A13$ $\Box X \supset \Box\Box X$.

Finally, quantification.

$\epsilon A14$ $(\lambda x\ X)(t) \supset (\lambda x\ X)(\epsilon x X)$.

$\epsilon A15$ $(\lambda x\ \Diamond X)(t) \supset \Diamond(\lambda x\ X)(\epsilon x X)$.

$\epsilon A16$ $(\exists x)X \equiv (\lambda x\ X)(\epsilon x X)$.

This completes the system ϵS4. A model theory may be constructed as a natural extension of that for **FS4** in section **2**. To do this we need to give some model-theoretic meaning to quasi-terms and the abstraction operator. This may be done in a natural way.

The system ϵS4 as given above has no constant symbols, but for purposes of model theory we extend the language to allow them. We treat them as quasi-terms having no free variables, and allow them to enter into the formation of other quasi-terms. For the rest of this section, quasi-formulas and quasi-terms may contain constant symbols.

By an ϵS4 *model* we mean a quintuple, $\langle \mathscr{G}, \boldsymbol{R}, \models, \boldsymbol{P}, \boldsymbol{F} \rangle$ where: $\langle \mathscr{G}, \boldsymbol{R}, \models, \boldsymbol{P} \rangle$ is an **FS4** model (save that $\models$ is now a relation between elements of $\mathscr{G}$ and formulas of ϵS4), and $\boldsymbol{F}$ is a collection of functions defined on subsets of $\mathscr{G}$. These are to satisfy:

1) If $\epsilon x X$ is a *term*, there is an element $f_{\epsilon x X}$ in $\boldsymbol{F}$ such that: $f_{\epsilon x X}$ is a function with domain the set of all Γ in $\mathscr{G}$ such that $\boldsymbol{P}(\Gamma)$ contains all constants of X; if $\Gamma \epsilon$ domain $f_{\epsilon x X}$ then $f_{\epsilon x X}(\Gamma) \epsilon \Gamma$; if $\Gamma \models (\exists x)X$ then $\Gamma \models X(x/f_{\epsilon x X}(\Gamma))$.

(For simplicity in stating the next two items; if c is a constant, let f_c be the function with domain the set of Γ in $\mathscr{G}$ such that $c \epsilon \boldsymbol{P}(\Gamma)$, with values, $f_c(\Gamma) = c$.)

2) If $(\lambda x\ \ X)(t)$ is a formula,

$$\Gamma \models (\lambda x\ \ X)(t) \text{ if and only if } \Gamma \models X(x/f_t(\Gamma)).$$

3) If P is an n-place predicate letter and $t_1, \ldots, t_n$ are terms,

$$\Gamma \vDash P(t_1, \ldots, t_n) \text{ if and only if } \Gamma \vDash P(f_{t_1}(\Gamma), \ldots, f_{t_n}(\Gamma)).$$

Again, an ϵS4 formula X is called *valid* in the ϵS4 model $\langle \mathcal{G}, \boldsymbol{R}, \vDash, \boldsymbol{P}, \boldsymbol{F}\rangle$ if $\Gamma \vDash X$ for all $\Gamma \epsilon \mathcal{G}$ such that all constants of X are in $\boldsymbol{P}(\Gamma)$.

We leave it to the reader to verify that all axioms of ϵS4 are valid in any ϵS4 model and that the two rules preserve validity. Thus we have

Theorem 3.1 *All theorems of* ϵS4 *are valid in all* ϵS4 *models.*

Any **FS4** model $\langle \mathcal{G}, \boldsymbol{R}, \vDash, \boldsymbol{P}\rangle$ can be extended to an ϵS4 model $\langle \mathcal{G}, \boldsymbol{R}, \vDash, \boldsymbol{P}, \boldsymbol{F}\rangle$. We may extend $\vDash$ and define $\boldsymbol{F}$ by an induction on the degree of formulas. Then by the above theorem and the completeness of **FS4** we have

Theorem 3.2 *Let X be a formula of* **FS4** *with no constants. If X is not a theorem of* **FS4**, *X is not a theorem of* ϵS4.

4 Development of ϵS4 Since the system ϵS4 is rather unfamiliar we prove some metatheorems about it to show how one may work in the system, and to simplify such work. For instance, our axiom schemas are of the form: all λ-closures of X are provable; we begin with a generalization of our two rules to a corresponding form. This then has the appearance of working with universal closures; in fact, we show it is the same. Finally we show that ϵS4 is an extension of the constant-free part of **FS4**.

We use the notation $\vdash X$ to mean *all λ-closures of X are provable.* We begin by showing rule $\epsilon R1$ can be generalized.

Theorem 4.1 *Let X and Y be quasi-formulas. Then*

$$\frac{\vdash X \quad \vdash X \supset Y}{\vdash Y}$$

Proof: Suppose $\vdash X$ and $\vdash X \supset Y$. Let $(\lambda \mathbf{y}\ Y)(\mathbf{t}_2)$ be a λ-closure of Y we wish to prove. Let $\mathbf{x}$ be a sequence consisting of all the free variables of X other than those already in $\mathbf{y}$, and let $\mathbf{t}_1$ be a sequence of terms of the same length as $\mathbf{x}$. $\vdash X$ so $(\lambda \mathbf{xy} X)(\mathbf{t}_1, \mathbf{t}_2)$ is a theorem. $\vdash X \supset Y$, so similarly, $(\lambda \mathbf{xy}\ (X \supset Y))(\mathbf{t}_1, \mathbf{t}_2)$ is a theorem. Using $\epsilon A7$, $\epsilon A8$, and $\epsilon R1$ we get that $(\lambda \mathbf{xy}\ X)(\mathbf{t}_1, \mathbf{t}_2) \supset (\lambda \mathbf{xy}\ Y)(\mathbf{t}_1, \mathbf{t}_2)$ is a theorem. Then by $\epsilon R1$, $(\lambda \mathbf{xy}\ Y)(\mathbf{t}_1, \mathbf{t}_2)$ is a theorem. Since the variables in $\mathbf{x}$ are not free in Y, use of $\epsilon A2$ an appropriate number of times produces $(\lambda \mathbf{y}\ Y)(\mathbf{t}_2)$.

Remark: From now on we will use this result without specific mention; similarly for axiom $\epsilon A10$.

Theorem 4.2 *Let X be a quasi-formula. Then*

$$\frac{\vdash X}{\vdash \Box X}.$$

Proof: We show a representative special case. Suppose $\vdash X$, and $(\lambda x_1 x_2 \Box X)(t_1, t_2)$ is a λ-closure of $\Box X$ we wish to prove.

$\vdash \sim\sim X$, so $\vdash (\lambda x_1 x_2 \sim\sim X)(t_1', t_2')$ for *any* quasi-terms t_1' and t_2'. Then by

$\epsilon A8$, $\vdash \sim (\lambda x_1 x_2 \sim X)(t_1', t_2')$. Let us suppose t_1' and t_2' are chosen (as they in fact will be) so that $\sim(\lambda x_1 x_2 \sim X)(t_1', t_2')$ is a *formula* and hence provable. Then by $\epsilon R2$, $\Box \sim (\lambda x_1 x_2 \sim X)(t_1', t_2')$, i.e., $\sim \Diamond \sim \sim (\lambda x_1 x_2 \sim X)(t_1', t_2')$, or by standard S4 arguments,

(*) $\sim \Diamond (\lambda x_1 x_2 \sim X)(t_1', t_2')$.

But $\vdash (\lambda x_2 \Diamond \sim X)(t_2) \supset \Diamond (\lambda x_2 \sim X)(\epsilon x_2 \sim X)$ by $\epsilon A15$, so if we take t_2' to be $\epsilon x_2 \sim X$, $\vdash [\lambda x_1 ((\lambda x_2 \Diamond \sim X)(t_2) \supset \Diamond (\lambda x_2 \sim X)(t_2'))] (t_1)$, or

(**) $\vdash (\lambda x_1 (\lambda x_2 \Diamond \sim X)(t_2))(t_1) \supset (\lambda x_1 \Diamond (\lambda x_2 \sim X)(t_2'))(t_1)$.

Similarly, using $\epsilon A15$,

(***) $\vdash (\lambda x_1 \Diamond (\lambda x_2 \sim X)(t_2'))(t_1) \supset \Diamond (\lambda x_1 (\lambda x_2 \sim X)(t_2'))(t_1')$,

where t_1' is $\epsilon x_1(\lambda x_2 \sim X)(t_2')$. Now, from (*), (**), and (***), $\sim (\lambda x_1 (\lambda x_2 \Diamond \sim X)(t_2))(t_1)$, so $(\lambda x_1 x_2 \sim \Diamond \sim X)(t_1, t_2)$.

Lemma 4.1 $\dfrac{\vdash X \supset Y}{\vdash (\lambda \mathbf{x}\ X)(\mathbf{t}_0) \supset (\lambda \mathbf{x}\ Y)(\mathbf{t}_0)}$.

Proof: Suppose $\vdash X \supset Y$. Let $[\lambda \mathbf{y}\ ((\lambda \mathbf{x}\ X)(\mathbf{t}_0) \supset (\lambda \mathbf{x}\ Y)(\mathbf{t}_0))] (\mathbf{t}_1)$ be a λ-closure we wish to prove. Since $\vdash X \supset Y$, $(\lambda \mathbf{yx}\ (X \supset Y))(\mathbf{t}_1, \mathbf{t}_0)$ is a theorem. Using $\epsilon A7$ and $\epsilon A8$, $(\lambda \mathbf{yx}\ X)(\mathbf{t}_1, \mathbf{t}_0) \supset (\lambda \mathbf{yx}\ Y)(\mathbf{t}_1, \mathbf{t}_0)$. But this is the same as $(\lambda \mathbf{y}\ (\lambda \mathbf{x}\ X)(\mathbf{t}_0))(\mathbf{t}_1) \supset (\lambda \mathbf{y}\ (\lambda \mathbf{x}\ Y)(\mathbf{t}_0))(\mathbf{t}_1)$. Now using $\epsilon A7$, $\epsilon A8$ again, $[\lambda \mathbf{y}\ ((\lambda \mathbf{x}\ X)(\mathbf{t}_0) \supset (\lambda \mathbf{x}\ Y)(\mathbf{t}_0))] (\mathbf{t}_1)$.

Next we show an analog of *FR3*.

Theorem 4.3 *Let X and Y be quasi-formulas and suppose x is not free in Y. Then*

$$\frac{\vdash X \supset Y}{\vdash (\exists x)X \supset Y}.$$

Proof: $\vdash X \supset Y$ so by Lemma 4.1 and $\epsilon A2$, $\vdash (\lambda x\ X)(\epsilon x\ X) \supset Y$. Hence $\vdash (\exists x)X \supset Y$ by $\epsilon A16$.

Lemma 4.2 $\dfrac{\vdash X \equiv Y}{\vdash (\exists x)X \equiv (\exists x)Y}$.

Proof: Suppose $\vdash X \supset Y$. By Lemma 4.1, $\epsilon A7$, and $\epsilon A8$, $\vdash (\lambda x\ X)(t) \supset (\lambda x\ Y)(t)$ for any t. By $\epsilon A14$ and $\epsilon A16$, $\vdash (\lambda x\ X)(t) \supset (\exists x)Y$. Take t to be $\epsilon x X$ and then by $\epsilon A16$, $\vdash (\exists x)X \supset (\exists x)Y$. The converse implication is similar.

Lemma 4.3 a) $\dfrac{\vdash X \equiv Y}{\vdash \sim X \equiv \sim Y}$.

b) $\dfrac{\vdash X \equiv Y}{\vdash \Diamond X \equiv \Diamond Y}$.

c) $\dfrac{\vdash X_1 \equiv Y_1 \quad \vdash X_2 \equiv Y_2}{\vdash (X_1 \wedge X_2) \equiv (Y_1 \wedge Y_2)}$.

Now using a modification of standard proofs by induction on degree, and using Lemmas 4.1, 4.2, and 4.3, we may show

Theorem 4.4 (Replacement Theorem) *Let A, B, X, and Y be quasi-formulas. Let Y be the result of replacing, in X, the quasi-formula A at some or all of its occurrences (except within quasi-terms) by B. Then*

$$\frac{\vdash A \equiv B}{\vdash X \equiv Y}.$$

We use the following lemma on relabeling bound variables in a later section.

Lemma 4.4 *Suppose y is free for x in X. Then*

$$\vdash (\exists x)X \equiv (\exists y)X(x/y).$$

Proof: Let $X' = X(x/y)$. Then:

$$\begin{aligned} \vdash (\exists x)X &\supset (\lambda x\, X)(\epsilon x X) && \epsilon A16 \\ &\supset (\lambda y\, X')(\epsilon x X) && \epsilon A1 \\ &\supset (\lambda y\, X')(\epsilon y X') && \epsilon A14 \\ &\supset (\exists y)X'. && \epsilon A16 \end{aligned}$$

The converse implication is similar.

We next wish to establish the relationship between λ-closures and universal closures. All the expected properties of universal quantifiers can be proved in more-or-less the usual ways. In particular, these items (we indicate the chief axioms used).

1) $\vdash (\forall x)(\forall y)X \equiv (\forall y)(\forall x)X$. ($\epsilon A9$)
2) *If x is not free in X,* $\vdash (\forall x)X \equiv X$. ($\epsilon A2$)
3) $\vdash (\forall x)X \supset (\lambda x\, X)(t)$. ($\epsilon A14$, $\epsilon A16$)
4) $\vdash (\forall x)[X \supset Y] \supset [(\forall x)X \supset (\forall x)Y]$.
5) $\dfrac{\vdash X}{\vdash (\forall x)X}$. (similar to Theorem 4.3)

Now, let X be a quasi-formula whose free variables are $x_1, \ldots, x_n$. By a *universal closure* of X we mean $(\forall x_1) \ldots (\forall x_n)X$. Using item 1) above, any two universal closures of X are equivalent. We use $\boldsymbol{\forall} X$ to denote any one of them.

Theorem 4.5 (Closure Theorem) *Let X be a quasi-formula. Then $\vdash X$ if and only if $\boldsymbol{\forall} X$ is provable.*

Proof: If $\vdash X$, $\boldsymbol{\forall} X$ follows by repeated use of item 5) above. Conversely, if $\boldsymbol{\forall} X$ is provable, any λ-closure follows, using items 2)-5) above.

Finally we show that ϵS4 is an extension of the constant-free part of **FS4**. Let $X_1, X_2, \ldots, X_n$ be a proof of X_n in **FS4**. Let $a_1, a_2, \ldots, a_k$ be all the constants occurring in this proof, and let $x_1, x_2, \ldots, x_k$ be variables not occurring in the proof. For each $i = 1, 2, \ldots, n$, let $X_i^0 = X_i(\mathbf{a}/\mathbf{x})$. We claim $\vdash X_i^0$ in ϵS4. If X_i is an axiom this is straightforward; four of the **FS4** axioms are immediate and *FA5* follows primarily from $\epsilon A14$ and $\epsilon A16$. If X_i follows from X_j and $X_j \supset X_i$ by *FR1* we may use Theorem 4.1 to conclude $\vdash X_i^0$ from $\vdash X_j^0$ and $\vdash (X_j \supset X_i)^0$. Similarly, instances of rules *FR2* and *FR3* become Theorems 4.2 and 4.3, respectively. Thus $\vdash X_n^0$. Now if X_n has no constants, $X_n^0 = X_n$. Thus we have

Theorem 4.6 *If X has no constants and is a theorem of* **FS4**, *then X is a theorem of* ϵS4.

This together with Theorem 3.2 gives us

Theorem 4.7 ϵS4 *is a conservative extension of the constant-free part of* **FS4**.

We note that so far we have not used axioms $\epsilon A3$, $\epsilon A5$, or $\epsilon A6$. They are needed to show completeness of ϵS4 relative to its model theory, which we do in the next section. The system of [1] was ϵS4 without these three axioms.

5 Completeness of ϵS4 In the last section we showed completeness of ϵS4 relative to **FS4**. In this section we establish the completeness of ϵS4 with respect to the model theory of section **3**. Again, as in the model theory discussion of that section, we add constant symbols to the language, and we treat them as *terms*. This produces an extension of ϵS4, call it ϵS4*, but all the ϵS4 results of section **4** still hold for ϵS4*. All the work of this section is in ϵS4*. We will call terms and quasi-terms of ϵS4 *ϵ-terms* and *quasi-ϵ-terms* if it is necessary to distinguish them from terms which are constants. That ϵS4* is a conservative extension of ϵS4 is an easy corollary of the following lemma, which may be proved by induction on n.

Lemma 5.1 *Suppose* $X_1, \ldots, X_n$ *is a proof of* X_n *in* ϵS4*. *Let c be any constant and t be any term. Then* $X_1(c/t), \ldots, X_n(c/t)$ *is a proof of* $X_n(c/t)$ *in* ϵS4*.

Another corollary of this lemma which we will need is the following.

Lemma 5.2 *Suppose* $(\lambda \mathbf{x}\ X)(\mathbf{c})$ *is a theorem of* ϵS4*, *where* $\mathbf{c}$ *is a sequence of distinct constants, none of which occur in X, and* $\mathbf{x}$ *is a sequence of distinct variables. Then* $\vdash X$.

Proof: We consider a representative special case. Suppose $(\lambda x_1 x_2\ X)(c_1, c_2)$ is a theorem. Since $(\lambda x_1 (\lambda x_2\ X)(c_2))(c_1)$ is a theorem, by Lemma 5.1, if we substitute the term $\epsilon x_1 \sim (\lambda x_2\ X)(c_2)$ for c_1 we still have a theorem. It follows that $(\forall x_1)[(\lambda x_2\ X)(c_2)]$ is a theorem. By $\epsilon A8$ and $\epsilon A9$, $(\lambda x_2 (\forall x_1) X)(c_2)$ is a theorem, so again, we may substitute $\epsilon x_2 \sim (\forall x_1) X$ for c_2 and get $(\forall x_2)(\forall x_1) X$. Thus $\forall X$. Now by Theorem 4.5 we are done.

Our completeness proof is Henkin style. We begin with a series of dull preliminary definitions and results.

We call X a *key formula* if X is of the form $(\lambda \mathbf{x}\ Y)(\mathbf{c})$ where

1) the members of the sequence $\mathbf{c}$ are distinct constants (or else $\mathbf{x}$ and $\mathbf{c}$ are empty sequences).
2) the members of $\mathbf{x}$ are distinct variables.

3) Y contains no constants.
4) no variable of $\mathbf{x}$ occurs bound in Y.

We call Y the *kernal* of X and $\langle \mathbf{x}, \mathbf{c} \rangle$ the *shell*.

Our primary interest below will be in key formulas. Next we define the notion of *trivial variant* of a key formula. Let X be the key formula $(\lambda x_1 \ldots x_n\, Z)(c_1, \ldots, c_n)$.

1) $(\lambda x_{i_1} \ldots x_{i_n}\, Z)(c_{i_1}, \ldots, c_{i_n})$ is a trivial variant of X, where $\langle i_1, \ldots, i_n \rangle$ is a permutation of $\langle 1, \ldots, n \rangle$.
2) $(\lambda x_1 \ldots y \ldots x_n\, Z(x_i/y))(c_1, \ldots, c_n)$ is a trivial variant of X, where y does not occur in Z and y is not one of the x_j.
3) $(\lambda y\, (\lambda x_1 \ldots x_n\, Z)(c_1, \ldots, c_n))(d)$ is a trivial variant of X, where y is not one of the x_j, y is not in Z (free or bound) and d is not any c_j.
4) $(\lambda x_2 \ldots x_n\, Z)(c_2, \ldots, c_n)$ is a trivial variant of X, where x_1 is not free in Z.
5) Any trivial variant of a trivial variant of X is a trivial variant of X.

Thus, to get from X to a trivial variant we are allowed to permute the shell, relabel variables of the shell, and add or remove vacuous abstracts in the shell. By use of $\epsilon A4$, $\epsilon A1$, and $\epsilon A2$, any trivial variant of X is again (a key formula) equivalent to X.

Let P be a set of constants and M be a set of key formulas. We call *M maximal consistent with respect to P* if

1) all constants in M are from P.
2) M is consistent.
3) if $(\lambda \mathbf{x}\, X)(\mathbf{c})$ is a key formula with kernal X and all constants in P, either $(\lambda \mathbf{x}\, X)(\mathbf{c}) \in M$ or $(\lambda \mathbf{x}\, {\sim}X)(\mathbf{c}) \in M$.

A minor variant of the usual argument shows

Lemma 5.3 *Let C be a consistent set of key formulas all of whose constants are in P. Then C can be extended to a set M, maximal consistent with respect to P.*

Key formulas divide naturally into six classes depending on the type of formula the kernal is. We are interested now in those key formulas of the form $(\lambda \mathbf{x}\, (\lambda y\, X)(t))(\mathbf{c})$ where t is a quasi-ϵ-term and y is not bound in X. Call such formulas *pseudo-abstracts*. Note that since this is a key formula no variable in $\mathbf{x}$ can occur bound in $(\lambda y\, X)(t)$, and thus y is not in $\mathbf{x}$.

Let us say two pseudo-abstracts $(\lambda \mathbf{x}\, (\lambda y\, X)(t))(\mathbf{c})$ and $(\lambda \mathbf{z}\, (\lambda w\, Y)(t'))(\mathbf{d})$ are *congruent* if $t(\mathbf{x}/\mathbf{c})$ and $t'(\mathbf{z}/\mathbf{d})$ are the same ϵ-terms. By use of $\epsilon A1$ and $\epsilon A4$ we may show the following

Lemma 5.4 *Let X and Y be congruent pseudo-abstracts. Then there are pseudo-abstracts X' and Y' congruent and equivalent to X and Y respectively, such that X' is of the form $(\lambda \mathbf{x}\, (\lambda y\, Z)(t))(\mathbf{c})$ and Y' is of the form $(\lambda \mathbf{x}\, (\lambda y\, W)(t))(\mathbf{c})$.*

Let S be a set of key formulas. The above congruence relation is an equivalence relation on the set of pseudo-abstracts of S, and so partitions this part of S into disjoint sets which we call *congruence classes of* S. If $\mathcal{C}$ is one of the congruence classes of S and $(\lambda \mathbf{x}\ (\lambda y\ Y)(t))(\mathbf{c}) \in \mathcal{C}$, by *the* ϵ*-term corresponding to* $\mathcal{C}$ we mean the ϵ-term $t(\mathbf{x}/\mathbf{c})$.

Let $\mathcal{C}$ be one of the congruence classes of S. We say S is *term complete with respect to* $\mathcal{C}$ if there is some associated constant $b_{\mathcal{C}}$ such that

1) $b_{\mathcal{C}}$ does not occur in the ϵ-term corresponding to $\mathcal{C}$.
2) if $(\lambda \mathbf{x}\ (\lambda y\ Y)(t))(\mathbf{c}) \in \mathcal{C}$ where $b_{\mathcal{C}}$ is not in $\mathbf{c}$, then $(\lambda \mathbf{x}\ (\lambda y\ Y)(b_{\mathcal{C}}))(\mathbf{c}) \in S$.
3) if $(\lambda \mathbf{x}\ (\lambda y\ Y)(t))(\mathbf{c}) \in \mathcal{C}$ where $b_{\mathcal{C}}$ is c_i in $\mathbf{c}$, then $(\lambda \mathbf{x}\ Y(y/x_i))(\mathbf{c}) \in S$.

We say S is *term complete* if it is term complete with respect to each of its congruence classes.

Call Γ a *model element with respect to* P if Γ is maximal consistent with respect to P, and term complete. Call Γ a *model element* if, for some P, Γ is a model element with respect to P.

The principal result we need is the following.

Theorem 5.1 *Let M be maximal consistent with respect to P. Let $b_1, b_2, b_3, \ldots$ be a countable sequence of constants not in P. Then M can be extended to a set Γ which is a model element with respect to $P \cup \{b_1, b_2, b_3, \ldots\}$.*

Proof: Let S be the collection of all key formulas with constants from $P \cup \{b_1, b_2, b_3, \ldots\}$. Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots$ be the congruence classes of S. We define a sequence of extensions of M as follows.

Let $M_0 = M$. Suppose we have defined an extension M_n of M. Let $\mathcal{C}_\alpha$ be the first congruence class of S having an element in M_n but such that M_n is not term complete with respect to $\mathcal{C}_\alpha$. Let b_α be the corresponding member of the list of constants (which by construction will not occur in M_n). Let M_{n+1} consist of all formulas of M_n together with all *trivial variants* of formulas of the form $(\lambda \mathbf{x}\ (\lambda y\ Y)(b_\alpha))(\mathbf{c})$ such that for some quasi-ϵ-term t, $(\lambda \mathbf{x}\ (\lambda y\ Y)(t))(\mathbf{c}) \in M_n \cap \mathcal{C}_\alpha$. Finally, let $\Gamma = \bigcup_{n=0}^{\infty} M_n$.

We show four facts, from which the theorem follows.

Fact 1 *If M_n is consistent, so is M_{n+1}.*

Proof: Suppose M_{n+1} is not consistent. Then for some $W_1, \ldots, W_j \in M_n$ and $Z_1, \ldots, Z_k \in M_{n+1} - M_n$,

$$(W_1 \wedge \ldots \wedge W_j \wedge Z_1 \wedge \ldots \wedge Z_k) \supset f.$$

(f is $A \wedge \sim A$ for some constant-free formula A.)

Since $Z_1, \ldots, Z_k$ all arise from congruent formulas, as in Lemma 5.4 we may find equivalent formulas whose shells are the same; that is, each

Z_i is equivalent to a key formula of the form $(\lambda \mathbf{x}\ (\lambda y\ Y_i)(b_\alpha))(\mathbf{c})$. Then, using $\epsilon A7$ and $\epsilon A8$.

$$\{W_1 \wedge \ldots \wedge W_j \wedge (\lambda \mathbf{x}\ (\lambda y\ [Y_1 \wedge \ldots \wedge Y_k])(b_\alpha))(\mathbf{c})\} \supset f.$$

For convenience, let $W = W_1 \wedge \ldots \wedge W_j$ and $Y = Y_1 \wedge \ldots \wedge Y_k$. Thus

$$[W \wedge (\lambda \mathbf{x}\ (\lambda y\ Y)(b_\alpha))(\mathbf{c})] \supset f.$$

By $\epsilon A4$

$$[W \wedge (\lambda y\ (\lambda \mathbf{x}\ Y)(\mathbf{c}))(b_\alpha)] \supset f.$$

Now the only occurrence of b_α in this formula is the one indicated. By Lemma 5.1 we may replace it by the term $\epsilon y(\lambda \mathbf{x}\ Y)(\mathbf{c})$ and still have a theorem. Thus

$$[W \wedge (\exists y)(\lambda \mathbf{x}\ Y)(\mathbf{c})] \supset f.$$

By $\epsilon A9$

$$[W \wedge (\lambda \mathbf{x}\ (\exists y) Y)(\mathbf{c})] \supset f.$$

But then, using primarily $\epsilon A14$ and $\epsilon A16$,

$$[W \wedge (\lambda \mathbf{x}\ (\lambda y\ Y)(t))(\mathbf{c})] \supset f,$$

or

$$\{W_1 \wedge \ldots \wedge W_j \wedge (\lambda \mathbf{x}\ (\lambda y\ [Y_1 \wedge \ldots \wedge Y_k])(t))(\mathbf{c})\} \supset f.$$

But each of $(\lambda \mathbf{x}\ (\lambda y\ Y_i)(t))(\mathbf{c})$ is equivalent to a member of M_n. Thus M_n is inconsistent, a contradiction.

Fact 2 *If M_n is maximal consistent with respect to Q, M_{n+1} is maximal consistent with respect to $Q \cup \{b_\alpha\}$.*

Proof: Suppose $(\lambda \mathbf{x}\ X)(\mathbf{c})$ is a key formula (with kernal X), all the constants of $\mathbf{c}$ are in $Q \cup \{b_\alpha\}$ and $(\lambda \mathbf{x}\ X)(\mathbf{c}) \notin M_{n+1}$. We show $(\lambda \mathbf{x}\ \sim X)(\mathbf{c}) \in M_{n+1}$. If all the constants of $\mathbf{c}$ are in Q, the result follows since M_n is maximal consistent with respect to Q.

Suppose b_α occurs in $\mathbf{c}$ (of course only once). Let t_α be the ϵ-term corresponding to $\mathcal{O}_\alpha$. Let $\mathbf{a}$ be a sequence made up of the constants of t_α. Let $\mathbf{b}$ be those constants of $\mathbf{c}$ other than those of $\mathbf{a}$ or b_α. Let $\mathbf{y}$ and $\mathbf{z}$ be sequences of variables, all distinct, not in X, corresponding in length to $\mathbf{a}$ and $\mathbf{b}$, and let w be yet another 'new' variable. Let $t_\alpha^0 = t_\alpha(\mathbf{a}/\mathbf{y})$ and let $X^0 = [X(\mathbf{x}/\mathbf{c})](\mathbf{a}/\mathbf{y}, \mathbf{b}/\mathbf{z}, b_\alpha/w)$. Then

$$(\lambda \mathbf{yz}\ (\lambda w\ X^0)(t_\alpha^0))(\mathbf{a}, \mathbf{b}) \in \mathcal{O}_\alpha.$$

It must be that this formula is not in M_n, since $(\lambda \mathbf{x}\ X)(\mathbf{c})$ is a trivial variant of $(\lambda \mathbf{yz}\ (\lambda w\ X^0)(b_\alpha))(\mathbf{a}, \mathbf{b})$ and would then be in M_{n+1}. Since M_n is maximal consistent with respect to Q.

$$(\lambda \mathbf{yz} \sim (\lambda w\ X^0)(t_\alpha^0))(\mathbf{a}, \mathbf{b}) \in M_n.$$

It follows by $\epsilon A8$ that

$$(\lambda \mathbf{yz}\ (\lambda w \sim X^0)(t_\alpha^0))(\mathbf{a}, \mathbf{b}) \in M_n.$$

But this is also in $\mathcal{C}_\alpha$. Then all trivial variants of $(\lambda \mathbf{yz}\ (\lambda w \sim X^0)(b_\alpha))(\mathbf{a}, \mathbf{b})$ are in M_{n+1}, in particular, $(\lambda \mathbf{x} \sim X)(\mathbf{c})$.

Fact 3 *M_{n+1} is term complete with respect to $\mathcal{C}_\alpha$.*

Proof: Suppose $(\lambda \mathbf{x}\ (\lambda y\ Y)(t))(\mathbf{c}) \in M_{n+1} \cap \mathcal{C}_\alpha$. If this formula is in M_n we are done by construction. Otherwise, b_α occurs in $\mathbf{c}$. Without loss of generality let us suppose the formula is of the form $(\lambda \mathbf{z}\ (\lambda w\ (\lambda y\ Y)(t))(b_\alpha))(\mathbf{d})$.

We note that w can not be free in t. If it were, b_α would appear in the ϵ-term corresponding to $\mathcal{C}_\alpha$ and hence would occur in each formula in $\mathcal{C}_\alpha$. But $M_n \cap \mathcal{C}_\alpha \neq \emptyset$ and b_α is not a constant of M_n.

We claim $(\lambda \mathbf{z}\ (\lambda w\ (\lambda y\ Y)(t))(t))(\mathbf{d}) \in M_n \cap \mathcal{C}_\alpha$. It clearly belongs to $\mathcal{C}_\alpha$. If it did not belong to M_n, $(\lambda \mathbf{z}\ (\lambda w \sim (\lambda y\ Y)(t))(t))(\mathbf{d})$ would, but this too is in $\mathcal{C}_\alpha$, so

$$(\lambda \mathbf{z}\ (\lambda w \sim (\lambda y\ Y)(t))(b_\alpha))(\mathbf{c}) \in M_{n+1},$$

contradicting consistency.

Since $(\lambda \mathbf{z}\ (\lambda w\ (\lambda y\ Y)(t))(t))(\mathbf{d}) \in M_n$, by $\epsilon A3$,

$$(\lambda \mathbf{z}\ (\lambda w\ Y(y/w))(t))(\mathbf{d}) \in M_n.$$

But this is in $\mathcal{C}_\alpha$. Hence

$$(\lambda \mathbf{z}\ (\lambda w\ Y(y/w))(b_\alpha))(\mathbf{d}) \in M_{n+1},$$

and we are done.

Fact 4 *Suppose $M_n \cap \mathcal{C}_\beta \neq \emptyset$ and M_n is term complete with respect to $\mathcal{C}_\beta$. Then M_{n+1} is term complete with respect to $\mathcal{C}_\beta$.*

Proof: Suppose $(\lambda \mathbf{x}\ (\lambda y\ Y)(t))(\mathbf{c}) \in M_{n+1} \cap \mathcal{C}_\beta$. We treat only the case that b_β is not in $\mathbf{c}$. If this formula is in M_n we use the fact that M_n is term complete with respect to $\mathcal{C}_\beta$. Otherwise it must be that b_α is in the sequence $\mathbf{c}$. Again, we may suppose without loss of generality that the formula is of the form

(*) $(\lambda \mathbf{z}\ (\lambda w\ (\lambda y\ Y)(t))(b_\alpha))(\mathbf{d}) \in M_{n+1} \cap \mathcal{C}_\beta$.

Since this is in M_{n+1}, as above, for some quasi-ϵ-term t',

(**) $(\lambda \mathbf{z}\ (\lambda w\ (\lambda y\ Y)(t))(t'))(\mathbf{d}) \in M_n \cap \mathcal{C}_\alpha$.

Since (**) is a key formula, y is not in the sequence $\mathbf{z}$, thus y is not free in t'. Also w cannot be free in t, since (*) is in $\mathcal{C}_\beta$ and this would imply b_α is in every formula of $\mathcal{C}_\beta$, but $M_n \cap \mathcal{C}_\beta \neq \emptyset$, and b_α is not in M_n.

Now, by $\epsilon A4$,

$$(\lambda \mathbf{z}\ (\lambda y\ (\lambda w\ Y)(t'))(t))(\mathbf{d}) \in M_n.$$

This formula is in $\mathcal{C}_\beta$ since (*) is and w is not free in t. Since M_n is term complete with respect to $\mathcal{C}_\beta$,

$$(\lambda \mathbf{z}\ (\lambda y\ (\lambda w\ Y)(t'))(b_\beta))(\mathbf{d}) \in M_n.$$

But this formula is in $\mathcal{C}_\alpha$ since (**) is, and y is not free in t'. Thus $(\lambda \mathbf{z}\ (\lambda y\ (\lambda w\ Y)(b_\alpha))(b_\beta))(\mathbf{d}) \in M_{n+1}$ together with all trivial variants, in particular,

$$(\lambda \mathbf{z}\ (\lambda\, w\, (\lambda y\ Y)(b_\beta))(b_\alpha))(\mathbf{d}) \in M_{n+1}.$$

This completes the proof.

Now we proceed to construct a model from model elements. Let $\mathcal{G}$ be the collection of all model elements. If $\Gamma \in \mathcal{G}$, let $\boldsymbol{P}(\Gamma)$ be the set of constants occurring in formulas of Γ. For $\Gamma, \Delta \in \mathcal{G}$, let $\Gamma \boldsymbol{R} \Delta$ hold provided that whenever a key formula of the form $(\lambda \mathbf{x} \cdot \Box X)(\mathbf{c}) \in \Gamma$, then $(\lambda \mathbf{x}\ X)(\mathbf{c}) \in \Delta$. Let X be a formula with constants $\mathbf{a}$, and let $\mathbf{x}$ be some sequence of distinct variables, not in X, of the same length as $\mathbf{a}$. We say $\Gamma \models X$ if $(\lambda \mathbf{x}\ X(\mathbf{a}/\mathbf{x}))(\mathbf{a}) \in \Gamma$. Note that since Γ is maximal consistent with respect to $\boldsymbol{P}(\Gamma)$, this definition is independent of the ordering of $\mathbf{a}$ and of the choice of variables in $\mathbf{x}$. Finally, let $\epsilon x X$ be an ϵ-term with constants $\mathbf{a}$, all in $\boldsymbol{P}(\Gamma)$. Again, let $\mathbf{x}$ be some sequence of distinct variables, not in X, of the same length as $\mathbf{a}$. One and only one of

(*) $(\lambda \mathbf{x}\ [(\lambda x\ X)(\epsilon x X)]\,(\mathbf{a}/\mathbf{x}))(\mathbf{a})$
(**) $(\lambda \mathbf{x}\ [(\lambda x\ {\sim}X)(\epsilon x X)]\,(\mathbf{a}/\mathbf{x}))(\mathbf{a})$

belongs to Γ. Let $f_{\epsilon x X}(\Gamma)$ be the constant associated with the congruence class $\mathcal{C}$ of Γ such that (*) $\in \mathcal{C}$ or (**) $\in \mathcal{C}$. Thus we define a function $f_{\epsilon x X}$ on $\{\Gamma \in \mathcal{G} \mid$ *all constants of* X *belong to* $\boldsymbol{P}(\Gamma)\}$. Let $\boldsymbol{F}$ be the collection of all such functions.

We claim the structure $\langle \mathcal{G}, \boldsymbol{R}, \models, \boldsymbol{P}, \boldsymbol{F} \rangle$ so defined is an ϵS4 model. The proof is a straightforward adaptation of those usual in modal logic [2, 5] so we only exhibit a few of the more interesting parts.

Suppose $\Gamma \in \mathcal{G}$ and $\Gamma \models (\exists x)X$. We wish to show $\Gamma \models X(x/b)$ for some $b \in \boldsymbol{P}(\Gamma)$. Let the constants of X be $\mathbf{a}$, let $\mathbf{x}$ be a sequence of distinct variables not in $(\exists x)X$, of the same length as $\mathbf{a}$. Then $(\lambda \mathbf{x}\ (\exists x)\, X(\mathbf{a}/\mathbf{x}))(\mathbf{a}) \in \Gamma$. Let y be a variable not in this formula. By Lemma 4.4, if $X' = X(\mathbf{a}/\mathbf{x}, x/y)$, $(\lambda \mathbf{x}\ (\exists y) X')(\mathbf{a}) \in \Gamma$. By $\epsilon A16$, $(\lambda \mathbf{x}\ (\lambda y\ X')(\epsilon y X'))(\mathbf{a}) \in \Gamma$. Since Γ is term complete, for some constant, b, $(\lambda \mathbf{x}\ (\lambda y\ X')(b))(\mathbf{a}) \in \Gamma$. Thus $\Gamma \models X(x/b)$.

Suppose $\Gamma \models (\lambda x\ X)(a)$ where a is a constant. We wish to show $\Gamma \models X(x/a)$. Let us suppose, to simplify things, that x has no bound occurrences in X and that a does not occur in X. Let $\mathbf{c}$ be the constants of X and let $\mathbf{x}$ be a corresponding sequence of 'new' distinct variables; let y be 'new' to X, and not in $\mathbf{x}$. If $X' = X(\mathbf{c}/\mathbf{x})$, since $\Gamma \models (\lambda x\ X)(a)$, $(\lambda \mathbf{x} y\ (\lambda x\ X')(y))(\mathbf{c}, a) \in \Gamma$. By $\epsilon A1$ and $\epsilon A5$, $(\lambda \mathbf{x} y\ X'(x/y))(\mathbf{c}, a) \in \Gamma$. Thus $\Gamma \models X(x/a)$.

Suppose $\Gamma \models \Diamond X$. We show for some $\Delta \in \mathcal{G}$ such that $\Gamma \boldsymbol{R} \Delta$, $\Delta \models X$. Let the constants of X be $\mathbf{a}$ and let $\mathbf{x}$ be a corresponding sequence of distinct variables not in X. Then

$$(\lambda \mathbf{x}\ \Diamond X(\mathbf{a}/\mathbf{x}))(\mathbf{a}) \in \Gamma.$$

For convenience, let $X' = X(\mathbf{a}/\mathbf{x})$ so that

$$(\lambda \mathbf{x}\ \Diamond X')(\mathbf{a}) \in \Gamma.$$

Let E consist of the formula $(\lambda \mathbf{x}\ X')(\mathbf{a})$ together with all key formulas of the form $(\lambda \mathbf{y}\ Y)(\mathbf{c})$ such that $(\lambda \mathbf{y} \Box Y)(\mathbf{c}) \in \Gamma$. We claim E is consistent. If not, then for some $Z_1, \ldots, Z_n \in E$.

$$(Z_1 \wedge \ldots \wedge Z_n) \supset \sim(\lambda \mathbf{x}\ X')(\mathbf{a}).$$

For simplicity we only consider the $n = 1$ case. Thus we have

$$(\lambda \mathbf{y}\ Y)(\mathbf{c}) \supset \sim(\lambda \mathbf{x}\ X')(\mathbf{a}).$$

Let $\mathbf{b}$ be the constants common to $\mathbf{c}$ and $\mathbf{a}$, let $\mathbf{c}'$ be the constants of $\mathbf{c}$ not in $\mathbf{a}$, and $\mathbf{a}'$ the constants of $\mathbf{a}$ not in $\mathbf{c}$. Let $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ be corresponding appropriate sequences of variables. Let $Y^0 = Y(\mathbf{b}/\mathbf{v}, \mathbf{c}'/\mathbf{w})$ and $X^0 = X'(\mathbf{b}/\mathbf{v}, \mathbf{a}'/\mathbf{z})$. Then we have

$$(\lambda \mathbf{vwz}\ (Y^0 \supset \sim X^0))(\mathbf{b}, \mathbf{c}', \mathbf{a}').$$

Now, by Lemma 5.2, $\vdash Y^0 \supset \sim X^0$. Thus by Theorem 4.2, $\vdash \Box(Y^0 \supset \sim X^0)$, so $\vdash \Box Y^0 \supset \Box \sim X^0$. So in particular,

$$(\lambda \mathbf{vwz}\ (\Box Y^0 \supset \Box \sim X^0))(\mathbf{b}, \mathbf{c}', \mathbf{a}').$$

Then

$$(\lambda \mathbf{vw} \Box Y^0)(\mathbf{b}, \mathbf{c}') \supset (\lambda \mathbf{vz} \Box \sim X^0)(\mathbf{b}, \mathbf{a}').$$

Thus

$$\{(\lambda \mathbf{vw} \Box Y^0)(\mathbf{b}, \mathbf{c}'), \sim(\lambda \mathbf{vz} \Box \sim X^0)(\mathbf{b}, \mathbf{a}')\}$$

is inconsistent. It follows that

$$\{(\lambda \mathbf{y}\ \Box Y)(\mathbf{c}), (\lambda \mathbf{x} \sim\Box\sim X)(\mathbf{a})\}$$

is inconsistent, a contradiction.

Now that we have E consistent, we may extend it to a model element, Δ, with respect to Q, where $\boldsymbol{P}(\Gamma) \subseteq Q$. Then $\Delta \in \mathscr{G}$ and $\Gamma \boldsymbol{R} \Delta$. Moreover, $(\lambda \mathbf{x}\ X(\mathbf{a}/\mathbf{x}))(\mathbf{a}) \in \Delta$, so $\Delta \models X$.

Suppose $\Gamma \models (\lambda x\ X)(\epsilon y Y)$. We wish to show $\Gamma \models X(x/f_{\epsilon y Y}(\Gamma))$. Without loss of generality, let us assume x is not bound in X and does not occur in Y. Let $\mathbf{a}$ be the constants of Y and let $\mathbf{b}$ be the constants of X other than those already in $\mathbf{a}$. Then for suitable sequences of variables $\mathbf{v}$ and $\mathbf{w}$, if we let $Y^0 = Y(\mathbf{a}/\mathbf{v})$ and $X^0 = X(\mathbf{a}/\mathbf{v}, \mathbf{b}/\mathbf{w})$, we have

(*) $(\lambda \mathbf{vw}\ (\lambda x\ X^0)(\epsilon y Y^0))(\mathbf{a}, \mathbf{b}) \in \Gamma$.

Let $\mathscr{C}$ be the congruence class of Γ containing (*) and let $b_{\mathscr{C}}$ be the associated constant. It follows that $(\lambda \mathbf{vw}\ (\lambda x\ X^0)(b_{\mathscr{C}}))(\mathbf{a}, \mathbf{b}) \in \Gamma$, so $\Gamma \models X(x/b_{\mathscr{C}})$. Now it is easy to see that (*), $(\lambda \mathbf{v}\ (\lambda y\ Y^0)(\epsilon y Y^0))(\mathbf{a})$ and $(\lambda \mathbf{v}\ (\lambda y \sim Y^0)(\epsilon y Y^0))(\mathbf{a})$ are all congruent. It follows that $b_{\mathscr{C}} = f_{\epsilon y Y}(\Gamma)$.

We leave the other cases to the reader. Thus we have an ϵS4 model. We note that if X has no constants, $\Gamma \models X$ if and only if $X \in \Gamma$. Now we may finish simply. If X is a formula of ϵS4 with no constants, which is not a theorem, $\{\sim X\}$ is consistent. We may extend this set to a model element, Γ. $\Gamma \epsilon \not{b}$, and $\sim X \epsilon \Gamma$ so $\Gamma \not\models X$. We thus have

Theorem 5.2 *If X is valid in all* ϵS4 *models (where X is a formula of* ϵS4 *and hence contains no constants), X is a theorem of* ϵS4.

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