Logic and Logical Philosophy
Volume 19 (2010), 341–351
DOI: 10.12775/LLP.2010.013

Gemma Robles
José M. Méndez
Francisco Salto

A MODAL RESTRICTION OF R-MINGLE WITH THE VARIABLE-SHARING PROPERTY

Abstract. A restriction of R-Mingle with the variable-sharing property and the Ackermann properties is defined. From an intuitive semantical point of view, this restriction is an alternative to Anderson and Belnap's logic of entailment E.

Keywords: R-Mingle; relevance and entailment logics; variable-sharing property; Ackermann property.

1. Introduction

The axiom mingle and the restricted axiom mingle are, respectively, the following theses:

(M) $\quad A \to (A \to A)$

and

(M′) $\quad (A \to B) \to [(A \to B) \to (A \to B)]$

The axioms (M) and (M′) are equivalent in the context of Anderson and Belnap's relevance logic R (see Proposition 1 in Section 2 below). The logic R-Mingle RM is the result of adding (M) (or (M′)) to R.

On the other hand, and as it is well-known, according to Anderson and Belnap, a necessary property of any relevance logic S is the following (see [1]):

Received December 3, 2010

ISSN: 1425-3305

Definition 1 (Variable-Sharing Property – vsp). A logic S has the vsp iff in any theorem of S of the form $A \to B$, A and B share at least one propositional variable.

Not less well known is the fact that RM lacks the vsp (see [1], §29.5). The aim of this paper is to define a modal restriction of RM with the vsp. The idea is essentially to restrict the axiom of assertion

(a) $\quad A \to [(A \to C) \to C]$

and (M) to the case where A is an implicative formula; that is, to restrict (a) and (M) to:

(a′) $\quad (A \to B) \to [[(A \to B) \to C)] \to C]$

and (M′), respectively. The system resulting from this restriction is labelled $\mathrm{RM}^{\Box}$. It will be proved that not only $\mathrm{RM}^{\Box}$ has the vsp, but also that $\mathrm{RM}^{\Box}$ has exactly the same properties that are predicable of the logic of entailment E and of the logic of relevance R (see [1], §22.1.3).

On the other hand, $\mathrm{RM}^{\Box}$ is characterized as "modal" in the sense that it has the "Ackermann Property", to wit:

Definition 2 (Ackermann Property – AP). A logic S has the AP iff in any theorem of S of the form $A \to (B \to C)$, A contains at least any implicative formula (A is implicative iff A is of the form $X \to Y$).

According to Anderson and Belnap, the AP is a necessary property of any logic of entailment. Consequently, the AP, which is predicable of E but not of R, is the property that makes of E the logic of relevance and necessity, that is, the logic of entailment, R solely being the logic of relevance (see [1]). Therefore, and on the one hand, $\mathrm{RM}^{\Box}$ is dubbed "modal" in Anderson and Belnap's sense. But, on the other hand, given that $\mathrm{RM}^{\Box}$ has the vsp and the AP, from an intuitive semantical point of view, it can be considered to be an alternative to Anderson and Belnap's E.

2. The logic $\mathrm{RM}^{\Box}$

The logic RM can be axiomatized as follows (cf. [1]):

Axioms:

(A1)	$A \to A$
(A2)	$(A \to B) \to [(B \to C) \to (A \to C)]$
(A3)	$[A \to (A \to B)] \to (A \to B)$
(A4)	$A \to [(A \to B) \to B]$
(A5)	$A \to (A \to A)$
(A6)	$(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$
(A7)	$[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
(A8)	$A \to (A \vee B) \quad / \quad B \to (A \vee B)$
(A9)	$[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
(A10)	$[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
(A11)	$(A \to \neg B) \to (B \to \neg A)$
(A12)	$(\neg A \to B) \to (\neg B \to A)$
(A13)	$(A \to \neg A) \to \neg A$

Rules:

$$\textit{Modus ponens}\ (\mathrm{MP}){:}\ (\vdash A \to B\ \&\vdash A) \Rightarrow\ \vdash B$$
$$\textit{Adjunction}\ (\mathrm{Adj}){:}\ (\vdash A\ \&\vdash B) \Rightarrow\ \vdash A \wedge B$$

Then, the logic RM$^{\square}$ is the result of substituting (A4) and (A5) by

(A4′)	$(A \to B) \to [[(A \to B) \to C] \to C]$
(A5′)	$(A \to B) \to [(A \to B) \to (A \to B)]$

respectively.

Consider now the following axiom:

(A14) $\quad A \to [(A \to A) \to A]$

labelled "demodalizer" by Anderson and Belnap. It is proved:

PROPOSITION 1. RM *and* RM$^{\square}$ *plus* (A14) *are deductively equivalent logics.*

Proof. (a) It is obvious that RM$^{\square}$ is included in RM, (b) We have to prove that (A4) and (A5) are derivable in RM$^{\square}$ plus (A14), as we show below. Firstly, note that the axiom

(A15) $[(A \to A) \to A] \to A$

is immediate in RM$^{\square}$ by (A1) and (A4′). Next, we prove that (A4) and (A5) are derivable.

A4. $A \to [(A \to B) \to B]$

(A4′)	1. $[(A \to A) \to A] \to \{[[(A \to A) \to A] \to B] \to B\}$
(A14), 1	2. $A \to \{[[(A \to A) \to A] \to B] \to B\}$
(A15), 2	3. $A \to [(A \to B) \to B]$

A5. $A \to (A \to A)$

(A5′)	1. $[(A \to A) \to A] \to \{[(A \to A) \to A] \to [(A \to A) \to A]\}$
(A14), (A15), 1	2. $A \to (A \to A)$ ⊣

Note, finally, that (A13) is not independent in RM.

On the other hand, Anderson and Belnap's Logic of Entailment E can be axiomatized (cf. [2]) in respect of RM$^{\square}$, by dropping (A5′) and changing (A4′) for

(A16) $\{[(A \to A) \wedge (B \to B)] \to C\} \to C$

Then, it is proved:

Proposition 2. E *and* RM$^{\square}$ *are different logics.*

Proof. (a) Consider the following set of matrices where designated values are starred:

Matrix set I (MSI)

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
*1	0	1	2	3	2
*2	0	0	1	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
*1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
*1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

This set verifies E (that is, it satisfies the axioms and rules of E), but falsifies (A5′) ($v(A) = 1$ and $v(B) = 2$).

(b) Consider the following set of matrices where designated values are starred:

Matrix set II (MSII)

$\to$	0	1	2	3	4	5	$\neg$
0	4	4	4	4	4	4	5
*1	0	4	4	4	4	4	4
*2	0	0	2	0	4	4	2
*3	0	0	0	3	4	4	3
*4	0	0	0	0	4	4	1
*5	0	0	0	0	0	4	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
*1	0	1	1	1	1	1
*2	0	1	2	1	2	2
*3	0	1	1	3	3	3
*4	0	1	2	3	4	4
*5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
*1	1	1	2	3	4	5
*2	2	2	2	4	4	5
*3	3	3	4	3	4	5
*4	4	4	4	4	4	5
*5	5	5	5	5	5	5

This set verifies $\mathrm{RM}^{\Box}$ but falsifies (A16) ($v(A) = 3$, $v(B) = 2$, and $v(C) = 1$). ⊣

3. $\mathbf{RM}^{\Box}$ has the variable-sharing property

In fact, and as was pointed out in the introduction, we shall prove a stronger result: $\mathrm{RM}^{\Box}$ has exactly the same properties that are predicable of E and R (cf. [1], §22.1.3).

In order to prove that this is the case, we define (see [1], p. 240) *antecedent part* (ap) and *consequent part* (cp) of wff inductively as follows:

Definition 3 (ap and cp of wff). 1. A is a cp of A.

2. If $B \wedge C$ is a cp (ap) of A, then both B and C are cps (aps) of A.

3. If $B \vee C$ is a cp (ap) of A, then both B and C are cps (aps) of A.

4. If $B \to C$ is a cp (ap) of A, then B is an ap (cp) of A and C is a cp (ap) of A.
5. If $\neg B$ is a cp (ap) of A, then B is an ap (cp) of A.

Then, the properties we refer to are expressed in the following theorems (cf. [1], §22.1.3).

Theorem 1. *If $A \to B$ is provable (in* $\mathrm{RM}^{\Box}$*), then some variable occurs as an ap of both A and B, or else as a cp of both A and B.*

Theorem 2. *If A is provable (in* $\mathrm{RM}^{\Box}$*) and A contains no conjunctions as aps and no disjunctions as cps, then every variable in A occurs at least once as ap and at least once as cp.*

The proofs of Theorems 1 and 2 are based upon the ten-elements set of matrices from Fig. 1, where all values but 0 are designated.

Firstly, it is proved:

Proposition 3. *MSIII verifies* $\mathrm{RM}^{\Box}$.

Proof. It is left to the reader. $\dashv$

Next, we proceed into proving Theorem 1. We follow Anderson and Belnap's strategy in [1], §22.1.3.

Proof of Theorem 1. Suppose that $A \to B$ is a wff in which no variable occurs as an ap of both A and B or as a cp of both A and B. Then, the following six situations exhaust the possibilities in which each variable p occurring in $A \to B$ can appear in A and/or B:

	A	B
$p:$	cp	–
	ap	–
	–	ap
	–	cp
	cp	ap
	ap	cp

Read first row: p occurs as cp in A but does not occur in B. The rest of the rows are read similarly.

According to these six possibilities and given MSIII, the following assignation is defined for each variable p in $A \to B$:

$\to$	0	1	2	3	4	5	6	7	8	9	$\neg$
0	9	9	9	9	9	9	9	9	9	9	9
1	0	7	7	7	7	7	7	7	7	9	8
2	0	0	7	7	7	7	7	7	7	9	7
3	0	0	0	6	0	0	6	7	7	9	6
4	0	0	0	0	5	5	0	7	7	9	5
5	0	0	0	0	4	5	0	7	7	9	4
6	0	0	0	3	0	0	6	7	7	9	3
7	0	0	0	0	0	0	0	7	7	9	2
8	0	0	0	0	0	0	0	0	7	9	1
9	0	0	0	0	0	0	0	0	0	9	0

$\wedge$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1
2	0	1	2	2	2	2	2	2	2	2
3	0	1	2	3	2	2	3	3	3	3
4	0	1	2	2	4	4	2	4	4	4
5	0	1	2	2	4	5	2	5	5	5
6	0	1	2	3	2	2	6	6	6	6
7	0	1	2	3	4	5	6	7	7	7
8	0	1	2	3	4	5	6	7	8	8
9	0	1	2	3	4	5	6	7	8	9

$\vee$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	1	2	3	4	5	6	7	8	9
2	2	2	2	3	4	5	6	7	8	9
3	3	3	3	3	7	7	3	7	8	9
4	4	4	4	7	4	4	7	7	8	9
5	5	5	5	7	5	5	7	7	8	9
6	6	6	6	6	7	7	6	7	8	9
7	7	7	7	7	7	7	7	7	8	9
8	8	8	8	8	8	8	8	8	8	9
9	9	9	9	9	9	9	9	9	9	9

Figure 1. Matrix set III (MSIII)

	A	B	
$p:$	cp	–	3
	ap	–	6
	–	ap	4
	–	cp	5
	cp	ap	9
	ap	cp	0

Then, Theorem 1 is immediate from the following two lemmas.

Lemma 1. *For every ap C of A, $v(C) \in \{0, 3, 6\}$; and for every cp C of A, $v(C) \in \{3, 6, 9\}$.*

Lemma 2. *For every ap C of B, $v(C) \in \{4, 5, 9\}$; and for every cp C of B, $v(C) \in \{0, 4, 5\}$.*

The proofs of Lemma 1 and Lemma 2 are by induction on the length of C and they are easy by inspection of MSIII. In this sense, notice that fragments $\{0, 3, 6, 9\}$ and $\{0, 4, 5, 9\}$ are closed under $\rightarrow$, $\wedge$, $\vee$ and $\neg$.

Then, the proof of Theorem 1 is immediate. Given that each formula is a cp of itself, by Lemma 1 and Lemma 2, $v(A) \in \{3, 6, 9\}$ and $v(B) \in \{0, 4, 5\}$, whence by MSIII, $v(A \rightarrow B) = 0$, that is, $A \rightarrow B$ is not a theorem of $\mathrm{RM}^{\Box}$. Consequently, if $A \rightarrow B$ is a theorem of $\mathrm{RM}^{\Box}$, then some variable occurs either as an ap or else as a cp of both A and B. ⊣

The following is an immediate corollary of Theorem 1.

Corollary 1. $\mathrm{RM}^{\Box}$ *has the vsp.*

Next, we proceed to the proof of Theorem 2.

Proof of Theorem 2. Suppose that A is a wff in which some variable, say p, occurs only as a cp. Assign, then, the value 0 to p and the value 4 to the rest of the variables (distinct from p) appearing in A. Then according to MSIII, it is proved:

Lemma 3. *If B is any part of A in which p does not occur, then $v(B) \in \{4, 5\}$.*

PROOF. Induction on the length of B. As it was the case with lemmas 1 and 2, the proof is easy by inspection of MSIII. In this sense, remark that the $\{4, 5\}$ fragment of MSIII is closed under $\rightarrow$, $\wedge$, $\vee$ and $\neg$. $\dashv$

LEMMA 4. *If B is any part of A in which p does occur, then (a) if B is an ap of A, $v(B) = 9$ and (b) if B is a cp of A, $v(B) = 0$.*

PROOF. Induction on the length of B. If B is a propositional variable, then B is p and it occurs only as a cp. So, $v(B) = 0$.

Regarding the rest of the cases, we prove that of the conditional and leave the others to the reader.

B is of the form $C \rightarrow D$:

(a) *B is a cp*: Then C is an ap and D is a cp.

a1. *p occurs in C and in D*:

By hypothesis of induction (HI), $v(C) = 9$ and $v(D) = 0$. so, $v(C \rightarrow D) = 0$ by MSIII.

a2. *p occurs in C but not in D*:

By HI and Lemma 3, $v(C) = 9$ and $v(D) \in \{4, 5\}$. By MSIII, $v(C \rightarrow D) = 0$.

a3. *p occurs in D but not in C*:

By HI and Lemma 3, $v(C) \in \{4, 5\}$ and $v(D) = 0$. So, $v(C \rightarrow D) = 0$ by MSIII.

(b) *B is an ap*: Then C is a cp and D is an ap.

b1. *p occurs in C and in D*:

By HI, $v(C) = 0$ and $v(D) = 9$. so, $v(C \rightarrow D) = 9$ by MSIII.

b2. *p occurs in C but not in D*:

By HI and Lemma 3, $v(C) = 0$ and $v(D) \in \{4, 5\}$. By MSIII, $v(C \rightarrow D) = 9$.

b3. *p occurs in D but not in C*:

By HI and Lemma 3, $v(C) \in \{4, 5\}$ and $v(D) = 9$. So, $v(C \rightarrow D) = 9$ by MSIII.

With b3 ends the proof of the conditional case. The proof of the conjunction, disjunction and negation cases is similar. (Recall that conjunctions can only appear as cps and disjunctions only as aps). $\dashv$

The proof of Theorem 2 is now immediate. As each formula is a cp of itself, it follows from Lemma 4 that $v(A) = 0$. That is, A is not a theorem of $\mathrm{RM}^{\Box}$. Consequently, if A is a theorem of $\mathrm{RM}^{\Box}$ without conjunctions as aps and disjunctions as cps, then every variable in A occurs at least once as ap and at least once as cp. ⊣

4. RM$^{\Box}$ has the Ackermann Property

We prove that RM$^{\Box}$ has the Ackermann Property (cf. Definition 2).

Consider the following set of matrices where all values but 0 are designated:

Matrix set IV(MSIV)

$\to$	0	1	2	3	4	$\neg$
0	3	3	3	3	3	4
1	0	3	0	0	3	3
2	0	3	3	0	3	2
3	0	3	3	3	3	1
4	0	0	0	0	3	0

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	1
2	0	2	2	3	2
3	0	3	3	3	3
4	0	1	2	3	4

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	1	1	4
2	2	1	2	2	4
3	3	1	2	3	4
4	4	4	4	4	4

Firstly we have (the proof is left to the reader):

Proposition 4. *MSIV verifies* RM$^{\Box}$.

And then:

Proposition 5. RM$^{\Box}$ *has the AP.*

Proof. Let $A \to (B \to C)$ be any wff in which $\to$ does not appear in A. Then, assign all the variables in A the value 2. According to MSIV, $v(A) = 2$, and so, $v(A \to (B \to C)) = 0$. Therefore, RM$^{\Box}$ has the AP. ⊣

RM$^\Box$ has then both the vsp and the AP. Consequently, RM$^\Box$ is an alternative to E from an intuitive semantical point of view, as was pointed out in the introduction of this paper.

We end the paper with a problem: what is the semantics for RM$^\Box$? If we think in terms of the Routley-Meyer ternary relational semantics, corresponding postulates for each one of the axioms of RM$^\Box$, except for (A4′) and (A5′), are well-known for at least thirty years (see [2]). So, the problem can be reformulated as follows: which are the corresponding semantical postulates for (A4′) and (A5′) or, equivalently, for (A4″) $[(A \to A) \to B] \to B$ and (A5′)?

Acknowledgments. Work supported by the research projects FFI 2008-05859/FISO and FFI2008-01205/FISO, financed by the Spanish Ministry of Science and Innovation. G. Robles is supported by Program Ramón y Cajal of the Spanish Ministry of Science and Innovation.

References

[1] Anderson, A. R., and N. D. Belnap, Jr., *Entailment. The Logic of Relevance and Necessity*, vol. I, Princeton University Press, 1975.

[2] Routley, R., R. K. Meyer, V. Plumwood, and R. T. Brady, *Relevant Logics and their Rivals*, vol. 1, Atascadero, CA: Ridgeview Publishing Co., 1982.

G. Robles and F. Salto
Dpto. de Psicología, Sociología y Filosofía
Universidad de León
Campus de Vegazana, s/n, 24071, León, Spain
gemmarobles@gmail.com
francisco.salto@unileon.es

J. M. Méndez
Universidad de Salamanca
Campus Unamuno
Edificio FES, 37007, Salamanca, Spain
sefus@usal.es