A MODEL COMPLETE THEORY OF VALUED D-FIELDS

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Abstract. The notion of a D-ring, generalizing that of a differential or a difference ring, is introduced. Quantifier elimination and a version of the Ax-Kochen-Ershov principle is proven for a theory of valued D-fields of residual characteristic zero.

The model theory of differential and difference fields has been extensively studied (see for example [7, 3]) and valued fields have proven to be amenable to model theoretic analysis (see for example [1, 2]). In this paper we subject a theory of valued fields possessing either a derivation or an automorphism interacting strongly with the valuation to such an analysis. Our theory differs from C. Michaux's theory of henselian differential fields [8] on this last point: in his theory, the valuation and derivation have a very weak interaction.

In Section 1 we introduce the notion of a D-field and show that a differential ring may be regarded as a specialization of a difference ring. This formal connection supports the view that differential and difference algebra are instances of the same theory. We introduce our axioms in Section 5 and prove quantifier elimination in Section 7. This provides an example of a non-trivial difference ring admitting elimination of quantifiers in a natural language. Differential fields possessing a valuation compatible with the derivation in some way have appeared in many guises. In model theory, these fields have arisen as Hardy fields associated to O-minimal structures. However, in contrast to Hardy fields, the fields considered in this paper have the property that for each value in the value group there is some differential constant with that valuation. This restriction is intrinsic to the methods used here.

This paper derives from a chapter of my doctoral thesis [11] written under the direction of E. Hrushovski whom I now thank for his advice and careful reading of preliminary versions of this paper. I thank the referee for a very thorough reading of this paper, for detailed and constructive suggestions for improvements and for supplying a correct proof of Proposition 1.1.

1. Algebraic Preliminaries

In this paper, a valued field is a field K given together with a function $v : K \to \Gamma \cup \{\infty\}$ where Γ is an ordered abelian group and ∞ is a formal symbol defined by $(\forall \gamma \in \Gamma)\ (\infty > \gamma) \wedge (\infty + \gamma = \infty = \infty + \infty = \gamma + \infty)$. v must satisfy $v(x) = \infty \iff x = 0$, $v(x \cdot y) = v(x) + v(y)$, and $v(x + y) \geq \min\{v(x), v(y)\}$. The ring of integers of K is $\mathcal{O}_K := \{x \in K : v(x) \geq 0\}$. $\mathcal{O}_K$ is a local ring with maximal ideal $\mathfrak{m}_K := \{x \in K : v(x) > 0\}$. The residue field of K is $k_K := \mathcal{O}_K/\mathfrak{m}_K$. The quotient map $\pi : \mathcal{O}_K \to k_K$ is called the residue map. If R is any (unital) ring then we denote the units of R by $R^\times := \{x \in R : (\exists y \in R)\ xy = 1\}$. We may

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recover Γ as $K^\times/\mathcal{O}_K^\times$. We call an extension $(L,w)/(K,v)$ of valued fields immediate if $w(L^\times) = v(K^\times)$ and $\pi(\mathcal{O}_L) = \pi(\mathcal{O}_K)$. As described in Section 4, we think of a valued field as a three-sorted structure: (K, k_K, Γ).

Let R be a commutative local ring with maximal ideal $\mathfrak{m}$. Let $\pi : R \to R/\mathfrak{m}$ denote the reduction map. R is said to be *henselian* if Hensel's Lemma is valid in R. That is, whenever $P(X) \in R[X]$ and $a \in R$ such that $\pi(P'(a)) \neq 0 = \pi(P(a))$, then there is some $b \in R$ such that $P(b) = 0$ and $\pi(a) = \pi(b)$. A *henselization* of R is a local homomorphism $\varphi : R \to R^h$ such that R^h is henselian and whenever $R \to S$ is a local homomorphism from R to a henselian ring S, then there is a unique local homomorphism $R^h \to S$ such that the following diagram commutes.

$$\begin{array}{ccc} R & \longrightarrow & R^h \\ \downarrow & \swarrow \exists! & \\ S & & \end{array}$$

For a proof that henselizations exist, see [10].

By all rights the next proposition should be well known, but we could find no published proof. A proof of this proposition follows from Lemma 4.5 of the preprint [6]. The proof given below was supplied by the referee replacing my original erroneous proof.

Proposition 1.1. *Let (K,v) be a valued field of equicharacteristic zero. Let (L,w) be a finite immediate extension of K. There is some $\alpha \in \mathcal{O}_L$ integral over $\mathcal{O}_K$ such that $L = K(\alpha)$ and if $P(X) \in \mathcal{O}_K[X]$ is the minimal monic polynomial of α over $\mathcal{O}_K$, then $w(P'(\alpha)) = 0$.*

■ Let M be a normal closure of L. Let w' be an extension of w to M. Let R_L be the integral closure of $\mathcal{O}_K$ in L and R_M be the integral closure of $\mathcal{O}_K$ in M. Let $\mathfrak{p} := \{x \in R_M : w'(x) > 0\}$ be the prime ideal of w' in R_M.

Claim 1.2. *If $\sigma \in \mathrm{Gal}(M/K) \setminus \mathrm{Gal}(M/L)$, then $\sigma(\mathfrak{p}) \cap R_L \neq \mathfrak{p} \cap R_L$.*

✠ If $\sigma(\mathfrak{p}) \cap R_L = \mathfrak{p} \cap R_L$, then the valuations w' and $w' \circ \sigma^{-1}$ agree on L. Since the extension M/L is Galois, $\mathrm{Gal}(M/L)$ acts transitively on the set of extensions of w to M (see 14.1 of [4]). Thus, $w' \circ \sigma^{-1} = w' \circ \tau$ for some $\tau \in \mathrm{Gal}(M/L)$, but then $w' = w' \circ (\tau\sigma)$. Since the extension w of v to L is immediate, the residual and ramification degrees of the conjugates of w' over w are the same as they are over v. Call these common degrees f and e. As (K,v) and (L,w) have residual characteristic zero, they are each defectless in M (see Corollary 20.23 of [4]). Thus there are exactly $\frac{[M:K]}{ef}$ conjugates of w' under the action of $\mathrm{Gal}(M/K)$ and $\frac{[M:L]}{ef}$ conjugates of w' under the action of $\mathrm{Gal}(M/L)$. We conclude that the decomposition group of w' over K, the isotropy group of w' for the action of $\mathrm{Gal}(M/K)$, is a subgroup of $\mathrm{Gal}(M/L)$. So we must have $\tau\sigma \in \mathrm{Gal}(M/L)$ which implies that $\sigma \in \mathrm{Gal}(M/L)$ contrary to our hypothesis. ✠

By the Chinese remainder theorem we can find $\alpha \in R_L$ such that $\alpha \in \sigma(\mathfrak{p}) \cap R_L \Leftrightarrow \sigma \in \mathrm{Gal}(M/L)$. Such an α works. Since $\sigma(\alpha) = \alpha$ is only possible if $\alpha \in \sigma(\mathfrak{p})$ in which case $\sigma \in \mathrm{Gal}(M/L)$ we have the equality of fields $L = K(\alpha)$. The element α reduces to zero with respect to w' but every other conjugate of α over K is a w'-unit. Thus, 0 is a simple root of the reduction of the minimal monic polynomial of α over $\mathcal{O}_K$. ■

2. D-RINGS

To keep with the notation of the following sections, we use "e" rather than, say, "X", to denote an indeterminate.

Let R be a commutative $\mathbb{Z}[e]$-algebra. $\mathcal{D}_e(R)$ is the ring which as an abelian group is R^2 with multiplication defined by $(x_1, x_2) * (y_1, y_2) := (x_1 y_1, x_1 y_2 + y_1 x_2 + e x_2 y_2)$. $\mathcal{D}_e$ defines a functor from the category of commutative $\mathbb{Z}[e]$-algebras to the category of commutative rings.

The projection onto the first co-ordinate defines a ring homomorphism $\pi_0 : \mathcal{D}_e(R) \to R$. A $\mathcal{D}_e$-structure on R is given by a section of this projection map. Concretely, such a structure is given by an additive function $D : R \to R$ satisfying the twisted Leibniz rule $D(x \cdot y) = xDy + yDx + e(Dx)(Dy)$ and $D(1) = 0$ defining a section $\varphi : R \to \mathcal{D}_e(R)$ by $\varphi(x) = (x, Dx)$.

Remark 2.1. From the standpoint of logic, the restriction to $\mathbb{Z}[e]$-algebras corresponds to adding a constant symbol e to the language of rings.

Remark 2.2. Note that when $e = 0$ in R, a $\mathcal{D}_e$-structure on R is simply given by a derivation.

Definition 2.3. A *D-ring* is a $\mathbb{Z}[e]$-algebra R given with a function $D : R \to R$ defining a $\mathcal{D}_e$-structure on R.

Proposition 2.4. *Let (R, D) be a D-ring. The function $\sigma : R \to R$ defined by $x \mapsto eDx + x$ is a ring endomorphism of R.*

■ Additivity is clear as is the fact that $\sigma(1) = 1$. For multiplication:

$$\begin{aligned} \sigma(xy) &= eD(xy) + xy \\ &= e(xDy + yDx + eDxDy) + xy \\ &= e^2 DxDy + exDy + eyDx + xy \\ &= (eDx + x)(eDy + y) \end{aligned}$$

■

Remark 2.5. If e is a non-zero divisor, then σ determines D. So when e is a unit, a D-ring is just a difference ring in disguise. That is, if e is a non-zero divisor, then D and σ are inter-definable. If e is a unit and one includes e^{-1} as a constant, then D is term definable from σ as $Dx = \frac{\sigma(x) - x}{e}$.

Of course, for fields e being zero or a unit exhaust the possibilities, but for more general rings there is an intermediate case.

Remark 2.6. The Leibniz rule may also be written as $D(x \cdot y) = xDy + \sigma(y)Dx$.

Proposition 2.7. *If $x \in R^\times$, then $D(\frac{1}{x}) = \frac{-Dx}{x\sigma(x)}$.*

■

$$\begin{aligned} 0 &= D(1) \\ &= D(x^{-1}x) \\ &= x^{-1}D(x) + \sigma(x)D(x^{-1}) \end{aligned}$$

Subtracting, we find that $\sigma(x)D(x^{-1}) = -(x^{-1})Dx$. As $x \in R^\times$, we also have $\sigma(x) \in R^\times$. Therefore, $D(\frac{1}{x}) = \frac{-Dx}{\sigma(x)x}$. ■

Proposition 2.8. *If R is a D-ring and $S \subseteq R$ is a multiplicative subset of R containing 1 and is closed under σ, then there is a unique structure of a D-ring on the localization $S^{-1}R$.*

■ Proposition 2.7 shows how D must be defined. The original $\mathcal{D}_e$-structure on R corresponds to a map $R \xrightarrow{\varphi} \mathcal{D}_e(R)$ given by $x \mapsto (x, Dx)$. By functoriality of $\mathcal{D}_e$, there is a map $\mathcal{D}_e(R) \xrightarrow{\mathcal{D}_e(i)} \mathcal{D}_e(S^{-1}R)$. For any $s \in S$ there is an inverse to $i \circ \varphi(s) = (s, Ds)$ in $\mathcal{D}_e(R)$, namely, $(\frac{1}{s}, \frac{-Ds}{\sigma(s)s})$. By the universal property of $S^{-1}R$, there is a unique ring homomorphism $S^{-1}R \longrightarrow \mathcal{D}_e(S^{-1}R)$ making the following diagram commute.

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[d, "i"'] & \mathcal{D}_e(R) \arrow[d, "\mathcal{D}_e(i)"] \\
S^{-1}R \arrow[r, "\exists!"] & \mathcal{D}_e(S^{-1}R)
\end{tikzcd}$$

■

Let us also calculate Dx^n.

Proposition 2.9. *If R is a D-ring, $x \in R$, and n is a positive integer, then*

$$Dx^n = \sum_{i=1}^{n} \binom{n}{i} e^{i-1} x^{n-i} (Dx)^i$$

■ We check this by induction on n. For $n = 1$ the assertion is obvious. Let us now try the case of $n + 1$.

$$\begin{aligned}
Dx^{n+1} &= D(x^n x) \\
&= x^n Dx + x(Dx^n) + eDx(Dx^n) \\
&= x^n Dx + (x + eDx) \sum_{i=1}^{n} \binom{n}{i} e^{i-1} x^{n-i} (Dx)^i \\
&= x^n Dx + \sum_{j=1}^{n} \binom{n}{j} e^{j-1} x^{(n+1)-j} (Dx)^j + \sum_{\ell=1}^{n} \binom{n}{\ell} e^{\ell} x^{n-\ell} (Dx)^{\ell+1} \\
&= x^n Dx + \sum_{j=1}^{n} \binom{n}{j} e^{j-1} x^{(n+1)-j} (Dx)^j + \sum_{t=2}^{n+1} \binom{n}{t-1} e^{t-1} x^{(n+1)-t} (Dx)^t \\
&= (n+1)x^n Dx + \sum_{j=2}^{n+1} [\binom{n}{j} + \binom{n}{j-1}] e^{j-1} x^{(n+1)-j} (Dx)^j \\
&= \sum_{j=1}^{n+1} \binom{n+1}{j} e^{j-1} x^{(n+1)-j} (Dx)^j
\end{aligned}$$

■

Lemma 2.10. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume that $e \in \mathfrak{m}$. Then $\mathcal{D}_e(R)$ is also a local ring with maximal ideal $\pi_0^{-1}\mathfrak{m}$.*

■ Let $(x, y) \in \mathcal{D}_e(R) \setminus \pi_0^{-1}\mathfrak{m}$. That is, $x \in R^\times$. Since $e \in \mathfrak{m}$, $x + ey \in R^\times$ as well. The inverse to (x, y) is then $(\frac{1}{x}, \frac{-y}{x(x+ey)})$. ■

Proposition 2.11. *If R is a henselian local ring with maximal ideal $\mathfrak{m}$ and $e \in \mathfrak{m}$, then $\mathcal{D}_e(R)$ is also henselian.*

■ Denote the reduction map $R \to R/\mathfrak{m}$ by $x \mapsto \overline{x}$. Denote the induced map $R[X] \to (R/\mathfrak{m})[X]$ by $P(X) \mapsto \overline{P}(X)$ as well. Consider R as a subring of $\mathcal{D}_e(R)$ via $r \mapsto (r, 0)$. Let $P(X) \in \mathcal{D}_e(R)[X]$ and let $(x, y) \in \mathcal{D}_e(R)$ such that $\overline{x}$ is a simple root of $\overline{\pi_0(P)}(X)$. Since R is henselian, there is a unique $a \in R$ such that $\pi_0(P)(a) = 0$ and $\overline{a} = \overline{x}$. Let $\epsilon := (0, 1)$. Since $\pi_0(P)(a) = 0$, there is some $Q(Y) \in R[Y]$ with

$$P(a + \epsilon Y) \quad = \quad \epsilon Q(Y)$$

Taylor expand $P(a + \epsilon Y)$ to compute that the linear term of $Q(Y)$ is $\pi_0(P'(a))Y$ and that all the higher order terms involve e as a factor. Hence, $\overline{Q}$ is a linear polynomial and therefore has a unique solution in $R/\mathfrak{m}$. As R is henselian, there is a unique lifting of this solution to some $b \in R$. The pair (a, b) is then the unique solution to $P(X) = 0$ with $\overline{a} = \overline{x}$. ■

The free algebra in the variety of D-rings has a particularly simple description.

Proposition 2.12. *Let R be a D-ring. There is an extension of D-rings $R \to R\langle X\rangle$ universal with respect to simple extensions of R.*
As a ring, $R\langle X\rangle = R[\{D^n X\}_{n=0}^\infty]$, the polynomial ring in countably many indeterminates.

■ Let $R' := R[\{D^n X\}_{n=0}^\infty]$. Let $\varphi : R \to \mathcal{D}_e(R)$ be the ring homomorphism making R into a D-ring. The inclusion $R \to R'$ induces a map $\mathcal{D}_e(R) \to \mathcal{D}_e(R')$. Let $\varphi' : R \to \mathcal{D}_e(R')$ be the composite of this inclusion with φ. By the universality property of the polynomial ring, there is a unique map of rings $\tilde{\varphi} : R' \to \mathcal{D}_e(R')$ which agrees with φ' on R and which sends $D^n X \mapsto (D^n X, D^{n+1} X)$. Thus, R' is a D-ring extending R.

Let us check now universality. Let $\psi : R \to S$ be any map of D-rings. Let $s \in S$. By the universality property of the polynomial ring, there is a unique map of rings $\tilde{\psi} : R\langle X\rangle \to S$ which agrees with ψ on R and sends $D^n X \mapsto D^n s$. That this is a map of D-rings is equivalent to the commutativity of the following diagram.

$$\begin{tikzcd}
S \arrow[r] & \mathcal{D}_e(S) \\
R\langle X\rangle \arrow[u, "\tilde{\psi}"] \arrow[r] & \mathcal{D}_e(R\langle X\rangle) \arrow[u, "\mathcal{D}_e(\tilde{\psi})"']
\end{tikzcd}$$

Since $\psi : R \to S$ is a map of D-rings, this is commutative when restricted to R. By construction, $D\psi(D^n X) = D(D^n s) = D^{n+1}s = \psi(D^{n+1}X) = \psi(D(D^n X))$. Thus, this diagram is commutative on all the generators of $R\langle X\rangle$, and hence everywhere. ■

3. Notation and General Definitions

We refer to $R\langle X\rangle$ as the ring of D-polynomials over R.

In general, if R is a D-ring and $I \subseteq R$ is an ideal, then I is a *D-ideal* if $D(I) \subseteq I$. When $I \subseteq R$ is a D-ideal, then the $\mathcal{D}_e$-structure on R induces such a structure on R/I.

If R is a D-ring and $\Sigma \subseteq R$ is a subset, then $\langle \Sigma \rangle$ is the *D-ideal generated by* Σ. Since the intersection of a set of D-ideals is a D-ideal this notion is well-defined. Concretely, $\langle \Sigma \rangle = (\{D^n x\}_{x \in \Sigma, n \in \omega})$.

Define the *order* and *degree* of a D-polynomial by: The zero D-polynomial has order and degree ∞. A nonzero constant polynomial is considered to have order -1 and degree $-\infty$. Otherwise, $\text{ord}P := \min\{n : P \in R[X, \ldots, D^n X]\}$. If n is the order of P, then the degree is the degree of P as a polynomial in $D^n X$.

The D-polynomial P is *simpler than* the D-polynomial Q, written $P \ll Q$, if in the lexico-graphic order, the order-degree of P is less than that of Q.

If P is a D-polynomial of the form $P(X) = F(X, DX, \ldots, D^m X)$, then define $\frac{\partial}{\partial X_i} P$ to be the D-polynomial $(\frac{\partial}{\partial X_i} F)(X, DX, \ldots, D^m X)$. There are at least two natural ways to extend D to $R\langle X\rangle$. In the first case, we simply treat the new variables as constants. That is, for $P(X) = \sum p_\alpha X^{\alpha_0} DX^{\alpha_1} \cdots D^N X^{\alpha_N}$ we set $P^D(X) := \sum D(p_\alpha) X^{\alpha_1} \cdots D^N X^{\alpha_N}$. The other extension of D is the one coming from $D(D^i X) = D^{i+1} X$ used to make $R\langle X\rangle$ into the universal D-extension of R.

We may define a more refined degree: the *total degree*. $\text{T.deg}P := (\deg_{X_i} P)_{i=0}^\infty$ for nonzero P and $\text{T.deg}0 := \infty$. Notice that the image of T.deg on nonzero D-polynomials comprises the set $\mathbb{N}^{(\omega)} := \{(n_j)_{j=0}^\infty : n_j \in \mathbb{N}, n_j = 0 \text{ for } j \gg 0\}$. Define an ordering on $\mathbb{N}^{(\omega)}$ by $(n_j)_{j=0}^\infty < (m_j)_{j=0}^\infty$ iff there is some N such that $n_N < m_N$ and $n_j \leq m_j$ for $j > N$. Observe that this ordering is a well-ordering of $\mathbb{N}^{(\omega)}$. We define $P \prec Q$ if $\text{T.deg}P < \text{T.deg}Q$. The fact that this ordering on $\mathbb{N}^{(\omega)}$ is a well-ordering means that we can (and will) argue by induction with respect to $\prec$.

$\prec$ has the properties

- $\frac{\partial}{\partial X_i} P \preceq P$ and
- If $P \prec Q$ and $\tilde{P}$ and $\tilde{Q}$ differ from P and Q respectively by linear changes of variables – that is, if $P(X) = F(\{D^n X\})$ and $Q(X) = G(\{D^n X\})$ while $\tilde{P}(X) = F(\{a_n D^n X + b_n\})$ and $\tilde{Q}(X) = G(\{c_n D^n X + d_n\})$ with the parameters a_n, b_n, c_n, d_n taken from R with $a_n, c_n \neq 0$ – then $\tilde{P} \prec \tilde{Q}$.

If R is a D-ring then $R^D := \ker(D : R \to R)$. Observe that R^D is a ring, D is R^D-linear, and that $(R^D)^\times = R^D \cap R^\times$.

If L/K is an extension of D-fields and $a \in L$, then $K\langle\langle a\rangle\rangle$ denotes the D-subfield of L generated by K and a.

4. The Language

The models under consideration have three sorts (K, k, Γ).

- K is the valued field given with the signature of a $\mathcal{D}_e$-ring: $(+, \cdot, -, 0, 1, e, D)$.
- k is the residue field also given with the signature of a $\mathcal{D}_e$-ring and possibly with some extra predicates needed to ensure quantifier elimination for k.

- Γ is the value group given with the signature of an ordered abelian group with divisibility predicates $(+, -, \leq, 0, \{n|\cdot\}_{n\in\mathbb{Z}+})$ and possibly with some extra predicates needed for quantifier elimination.

For convenience, an extra symbol ∞ is added to the language. For instance, one defines $0^{-1} = \infty$ and $(\forall\gamma \in \Gamma)\gamma < \infty$. The sorts are connected by the functions $\pi : K \to k \cup \infty$ (the residue map) and $v : K \to \Gamma \cup \infty$ (the valuation). Denote the first-order language described above by $\mathcal{L}$.

If M is an $\mathcal{L}$-structure and P is a predicate, then we denote the realization of P in M by P_M. If P is a particular sort, then $S_{m,P}(A)$ denotes the space of m-types over A in the sort P. That is, each $p(x_1, \dots, x_m) \in S_{m,P}(A)$ must contain the formula $\bigwedge_{1\leq i\leq m} P(x_i)$. It will be proven that in the cases of $P = \Gamma$ or k that A may be replaced by P_A when A is an $\mathcal{L}$-substructure of a model of the theory described in Section 5.

5. Axioms

We restrict the models considered to those with a differential field of characteristic zero as residue field. The more general cases of positive residual characteristic or a difference field as residue field present technical problems. After the present paper was written the author found a way to treat some of these additional cases and will describe this argument in a forthcoming paper.

Let $\mathbf{k}$ be a differential field and $\mathbf{G}$ an ordered abelian group. We assume that $\mathbf{k}$ satisfies

1. $\operatorname{char}\mathbf{k} = 0$,
2. $(\mathbf{k}^\times)^n = \mathbf{k}^\times$ for each $n \in \mathbb{Z}_+$, and
3. any non-zero linear differential operator $L \in \mathbf{k}[D]$ is surjective as a map $L : \mathbf{k} \to \mathbf{k}$. We call a differential field satisfying this condition *linearly differentially closed*.

We also assume that enough predicates have been added to $\mathcal{L}$ on the sort k so that $\mathrm{Th}(\mathbf{k})$ admits elimination of quantifiers. Of course, one should take the language to be as simple as possible.

We also assume that the language for Γ is sufficiently rich so that $\mathrm{Th}(\mathbf{G})$ admits elimination of quantifiers. In many cases of interest (for example, $\Gamma = \mathbb{Z}$) one may achieve this with divisibility predicates defined by $n|x \iff (\exists y)\overbrace{y + \cdots + y}^{n\text{ times}} = x$ [13]. In general, more complicated predicates may be needed.

One can avoid cluttering $\mathcal{L}$ by taking $\mathbf{k} \models \mathrm{DCF}_0$ and $\mathbf{G} = \mathbb{Q}$.

The first axioms describe general valued $(\mathbf{k}, \mathbf{G})$-D-fields.

Axiom 1. *K and k are D-fields of characteristic zero and $k \models \mathrm{Th}_\forall(\mathbf{k})$.*

Axiom 2. *K is a valued field whose value group is a subgroup of Γ via the valuation v and whose residue field is a subfield of k via the residue map π and $v(e) > 0$.*

Axiom 3. $(\forall x \in K)$ $v(Dx) \geq v(x)$ *and* $\pi(Dx) = D\pi(x)$.

Axiom 4. $\Gamma \models \mathrm{Th}_\forall(\mathbf{G})$.

The next six axioms together with the first four describe $(\mathbf{k}, \mathbf{G})$-D-henselian fields.

Axiom 5. $(\forall x \in K)[([\exists y \in K]\ y^n = x) \iff n|v(x)]$.

Axiom 6. $\Gamma = v((K^D)^\times)$.

Axiom 7. $k = \pi(\mathcal{O}_K)$.

Axiom 8 (D-Hensel's Lemma)**.** *If $P \in \mathcal{O}_K\langle X\rangle$ is a D-polynomial, $a \in \mathcal{O}_K$, and $v(P(a)) > 0 = v(\frac{\partial}{\partial X_i}P(a))$ for some i, then there is some $b \in K$ with $P(b) = 0$ and $v(a-b) \geq v(P(a))$.*

If the hypotheses of the last axiom apply to P and a, then one says that DHL applies to P at a.

Axiom 9. $\Gamma \equiv \mathbf{G}$.

Axiom 10. $k \equiv \mathbf{k}$.

Remark 5.1. Axioms 1 and 2 imply that D is a derivation on k.

Remark 5.2. Axiom 8 may be strengthened to $v(b-a) = v(P(a))$.

We assumed that $\mathbf{k}$ is linearly differentially closed and is closed under roots in order to guarantee consistency of the theory of $(\mathbf{k}, \mathbf{G})$-D-henselian fields.

Proposition 5.3. *Axioms 1 - 8 together with $\mathbf{G} \neq 0$ imply that k is linearly differentially closed.*

■ Let K be a model of the first eight axioms. Let $L(X) = \sum_{i=1}^n a_i D^i X$ be a non-zero linear D-polynomial over k. Let $y \in k$ be given. By Axiom 7 there are $b_i \in \mathcal{O}_K$ such that $\pi(b_i) = a_i$ and $z \in \mathcal{O}_K$ such that $\pi(z) = y$. Since $\mathbf{G} \neq 0$, by Axiom 6 there is $\epsilon \in \mathcal{O}_K$ with $D\epsilon = 0$ and $v(\epsilon) > 0$. Let $P(X) = -\epsilon \cdot z + \sum_{i=1}^n b_i D^i X$.

Evaluating at zero we see that $v(P(0)) > 0$ and for some i we have $\pi(\frac{\partial}{\partial X_i}P(0)) = \pi(b_i) = a_i \neq 0$. So by DHL there is some $x \in \mathcal{O}_K$ such that $P(x) = 0$ and $v(x) \geq v(P(0)) = v(\epsilon)$. Let $x' = \frac{x}{\epsilon}$. We have

$$\begin{aligned} 0 &= P(x) \\ &= -\epsilon z + \sum_{i=1}^n b_i D^i x \\ &= \epsilon(-z + \sum_{i=1}^n b_i D^i x') \end{aligned}$$

Hence, $z = \sum_{i=1}^n b_i D^i x'$. Applying π, we find that $y = L(\pi(x'))$. ■

Proposition 5.4. *Axioms 1 - 8 imply that $(k^\times)^n = k^\times$ for each positive integer n.*

■ Let K be a model of the first eight axioms. Let $x \in k^\times$. By Axiom 7 there exists $y \in \mathcal{O}_K$ such that $\pi(y) = x$. The valuation of y is zero so $n | v(y)$ which implies by Axiom 5 that y has an n-th root z. Thus, $\pi(z)$ is an n-th root of x. ■

6. Consistency and the Standard D-Henselian Fields

$\mathbf{k}$ and $\mathbf{G}$ continue to have the same meaning as in the previous section.

The generalized power series fields $\mathbf{k}((\epsilon^{\mathbf{G}}))$ provide canonical models for the theory of D-henselian fields. For the reader's convenience, we recall the definition of these fields.

As a set, $\mathbf{k}((\epsilon^{\mathbf{G}})) = \{f : \mathbf{G} \to \mathbf{k} : \mathrm{supp}(f) := \{x \in \mathbf{G} : f(x) \neq 0\}$ is well-ordered in the ordering induced by $\mathbf{G}\}$.

We think of an element $f \in \mathbf{k}((\epsilon^{\mathbf{G}}))$ as a formal power series.

$$\begin{aligned} f &\leftrightarrow \sum_{\gamma \in \mathbf{G}} f(\gamma)\epsilon^{\gamma} \\ v(f) &:= \min \operatorname{supp}(f) \\ (f+h)(\gamma) &:= f(\gamma) + h(\gamma) \\ (fh)(\gamma) &:= \sum_{\alpha+\beta=\gamma} f(\alpha)h(\beta) \end{aligned}$$

For the time being we denote the derivation on $\mathbf{k}$ by ∂.

If we wish to have $e = 0$, then define

$$(Df)(\gamma) = \partial(f(\gamma))$$

Otherwise, take for e any non-zero element of positive valuation and define an endomorphism σ from which we recover D by the formula $Dx := \frac{\sigma(x)-x}{e}$. On $\mathbf{k}$ define σ by

$$\sigma(x) := \sum_{n=0}^{\infty} \frac{\partial^n x}{n!} e^n$$

Extend to all of $\mathbf{k}((\epsilon^{\mathbf{G}}))$ by the rule

$$\sigma(f) = \sum \sigma(f(\gamma))\epsilon^{\gamma}$$

$\mathbf{k}((\epsilon^{\mathbf{G}}))$ is a maximally complete valued field [9, 12].

That this field is a D-henselian field is clear except perhaps for DHL. We prove DHL for $K := \mathbf{k}((\epsilon^{\mathbf{G}}))$ in a *prima facie* stronger form.

Proposition 6.1. *If $P \in \mathcal{O}_K\langle X\rangle$ is a D-polynomial, $a \in \mathcal{O}_K$, and $v(P(a)) > 2v(\frac{\partial}{\partial X_i}P(a))$ for some i, then there is $b \in K$ such that $P(b) = 0$ and $v(a-b) \geq v(P(a)) - v(\frac{\partial}{\partial X_i}P(a))$.*

■ Let i be a non-negative integer such that $v(\frac{\partial}{\partial X_i}P(a)) \leq v(\frac{\partial}{\partial X_j}P(a))$ for all j. Let $\gamma_0 := v(\frac{\partial}{\partial X_i}P(a))$.

Inductively build an ordinal indexed Cauchy sequence of approximate solutions $\{x_\alpha\}$ from K. If at some point $P(x_\alpha) = 0$, stop. At each point in the construction ensure that $(\forall \beta < \alpha)\ v(P(x_\alpha)) > v(x_\alpha - x_\beta) + \gamma_0 > 2v(\frac{\partial}{\partial X_i}P(a))$ and that $v(x_{\alpha+1} - x_\alpha) \geq v(P(x_\alpha)) - \gamma_0$. By starting with $x_0 = a$, provided that one may construct the sequence so as to be cofinal in $\mathbf{G}$, the result is proven.

For each j, choose $c_j \in \mathbf{k}$ such that $v(c_j\epsilon^{\gamma_0} - \frac{\partial}{\partial X_j}P(a)) > \gamma_0$.

At successor stages, $\alpha+1$, try to modify x_α slightly so as to increase the valuation of P. Let $\gamma := v(P(x_\alpha)) - \gamma_0$ which by the inductive hypothesis in the case of $\alpha > 0$ or by the hypothesis of the theorem in case $\alpha = 0$ is greater than γ_0. Consider the expansion:

$$P(X\epsilon^{\gamma} + x_\alpha) = \sum_{m \geq 0} \sum_{|I|=m} \partial_I P(x_\alpha)\epsilon^{m\gamma} X^I \tag{1}$$

Every coefficient on the right hand side has value $\geq \gamma + \gamma_0$. For the constant term, this is because $\gamma + \gamma_0 = v(P(x_\alpha))$ by definition. For the linear terms, one knows that each of $v(\frac{\partial}{\partial X_j}P(x_\alpha))$ is at least γ_0. For the higher order terms, note that $m\gamma \geq 2\gamma = 2v(P(x_\alpha)) - 2\gamma_0 > v(P(x_\alpha)) + 2\gamma_0 - 2\gamma_0 = \gamma + \gamma_0$. (The strict inequality follows from the fact that $v(P(x_\alpha)) > 2\gamma_0$.) Thus, we may divide the right hand side by $\epsilon^{\gamma+\gamma_0}$ and still have a D- polynomial with integral coefficients. In the residue field, the equation is

$$\pi(\frac{P(X\epsilon^\gamma + x_\alpha)}{\epsilon^{\gamma+\gamma_0}}) = \pi(\frac{P(x_\alpha)}{\epsilon^{\gamma+\gamma_0}}) + \sum c_j D^j X \tag{2}$$

As $c_i \neq 0$, Equation 2 is a non-trivial inhomogeneous linear D-equation over $\mathbf{k}$. As $\mathbf{k}$ is linearly differentially closed we may find x which is a solution to this equation and set $x_{\alpha+1} = x\epsilon^\gamma + x_\alpha$. The inductive hypothesis is still true at $x_{\alpha+1}$ as $v(P(x_{\alpha+1})) > v(P(x_\alpha))$.

At limits, simply find any x_λ such that $v(x_\lambda - x_\alpha) < v(x_\lambda - x_\beta)$ for $\alpha < \beta < \lambda$. Such exists by completeness of K. Let $\delta < \alpha < \lambda$ and consider Equation 1. We compute by induction that $v(P(x_\lambda)) \geq \min\{v(P(x_\alpha)), v(x_\lambda - x_\alpha) + \gamma_0\} > v(x_\alpha - x_\delta) + \gamma_0 = v(x_\lambda - x_\delta) + \gamma_0 > 2\gamma_0$.

Let $b = \lim x_\alpha$. ■

7. Quantifier Elimination

Theorem 7.1. *With the restrictions imposed on $\mathbf{k}$ and $\mathbf{G}$ in Section 5, the theory of henselian $(\mathbf{k}, \mathbf{G})$-D-fields eliminates quantifiers and is the model completion of the theory of valued $(\mathbf{k}, \mathbf{G})$-D-fields. Its completions are determined by the isomorphism type of the valued D-field $\mathbb{Q}(\{D^n e\}_{n\in\omega})$.*

■ We prove this by a standard back-and-forth test.

Claim 7.2. *Let M_1 and M_2 be two $\aleph_1$-saturated $(\mathbf{k}, \mathbf{G})$-D-henselian fields. Let $A_1 \subset M_1$ and $A_2 \subset M_2$ be countable substructures. Let $f : A_1 \to A_2$ be an isomorphism of $\mathcal{L}$-structures. Let $b \in M_1$. Then f extends to a partial isomorphism from M_1 to M_2 having b in its domain.*

See Theorem 8.4.1 of [5] for a proof that Claim 7.2 implies that the theory of $(\mathbf{k}, \mathbf{G})$-D-henselian fields admits elimination of quantifiers.

We prove that the theory of D-henselian fields is the model completion of the theory of valued D-fields by showing that each of the constructions used in the proof of Claim 7.2 may actually be used to extend any valued D-field to a D-henselian field.

The description of the completions of the theory of $(\mathbf{k}, \mathbf{G})$-D-henselian fields follows from quantifier elimination.

Our strategy for the proof of Claim 7.2 is to extend f a little bit at a time so that the type of b over $\mathrm{dom}(f)$ becomes transparent. We work mostly within the sort of the valued field so that by "$x \in A_1$" we mean that x is an element of the valued field. We treat the extension of f as an inductive process in which at each stage the current domain of f is equal to A_1.

Before extending f we fix a countable elementary submodel $N_1 \prec M_1$ of M_1 containing A_1 and b. We actually extend f a little beyond N_1.

We start by enlarging A_1 freely so that Γ_{N_1} is contained in the divisible hull of Γ_{A_1}. We then extend f so that one has $n|\gamma \iff \gamma = nv(x)$ for some $x \in A_1$ for each $\gamma \in \Gamma_{A_1}$. Once this is done we ensure that A_1 has enough constants in the sense of Definition 7.3 below. At this point we extend A_1 so that the residue map is surjective onto k_{N_1}. The model N_1 will be an immediate extension of A_1 once this step has been achieved. Finally, we extend f to a particular immediate extension of N_1 by an inductive procedure. This step requires most of the work. ■

7.1. **Extensions of k and of Γ.**

Definition 7.3. The valued D-field (K, Γ) *has enough constants* if it satisfies Axiom 6.

Lemma 7.4. *If K is a valued D-field with enough constants, then for any value $\gamma \in \Gamma_K$ and any finite set of polynomials $Q_1(X), \ldots, Q_n(X) \in K[X]$ with $Q_i(X) = \sum_{j=0}^{m} q_{i,j} X^j$ there is some $\epsilon \in K$ with $v(\epsilon) = \gamma$, $D\epsilon = 0$, and $v(Q_i(\epsilon)) = \min\{v(q_{i,j}) + j\gamma\}$ for each Q_i from the above set.*

■ As K has enough constants there is some $\eta \in K$ with $D\eta = 0$ and $v(\eta) = \gamma$. For each i, let $\delta_i \in K$ with $v(\delta_i) = \min\{v(q_{i,j}) + j\gamma\}$. If for each i it were the case that

$$\pi\left(\frac{Q_i(\eta)}{\delta_i}\right) \neq 0 \tag{3}$$

then the desired result would be true with $\epsilon = \eta$ for Inequality 3 means simply that $v(Q_i(\eta)) = v(\delta_i) = \min\{v(q_{i,j}) + j\gamma\}$. Alas, it may happen that with our first choice of η, some instance of Inequality 3 fails. We replace η with $c\eta$ where $v(c) = 0$ and $Dc = 0$. We need only ensure $\pi(c)$ is not a solution to any of $\sum \pi(\frac{q_{i,j}\eta^j}{\delta_i})Y^j = 0$. With finitely many exceptions, any choice from $\mathbb{Q}$ will work. ■

Corollary 7.5. *Let K be an $\aleph_1$-saturated valued D-field with enough constants. Let $L \subset K$ be a countable subfield of K. Let $\gamma \in \Gamma_K$. There is some $\epsilon \in K$ such that $v(\epsilon) = \gamma$, $D\epsilon = 0$ and for any polynomial $Q(X) = \sum_{j=0}^{n} q_j X^j \in L[X]$ one has $v(Q(\epsilon)) = \min\{v(q_j) + j\gamma\}$.*

Remark 7.6. In the case that $\mathbf{k}$ already admits elimination of quantifiers in the natural language of differential rings, Corollary 7.5 may be used to give a quick proof that the isomorphism may be extended so that A_i has enough constants. Unfortunately, in general the adjunction of constants may enlarge the residue field so that there may be some ambiguity as to the extension unless the types of the new elements of the residue field are controlled.

Lemma 7.7. *If K is a valued field which is also a $\mathcal{D}_e$-ring in which $v(e) > 0$ and $a, b \in K^\times$ satisfy $v(Da) \geq v(a)$ and $v(Db) \geq v(b)$, then $v(D(\frac{a}{b})) \geq v(\frac{a}{b})$.*

■

$$\begin{aligned}
v(D(\frac{a}{b})) &= v(\frac{D(a)b - D(b)a}{b\sigma(b)}) \\
&\geq \min\{v(Da) - v(\sigma(b)), v(a) + v(Db) - (v(b) + v(\sigma(b)))\} \\
&= \min\{v(Da) - v(b), v(a) + v(Db) - 2v(b)\} \\
&\geq \min\{v(a) - v(b), v(a) + v(b) - 2v(b)\} \\
&= v(a) - v(b) \\
&= v(\frac{a}{b})
\end{aligned}$$

■

Lemma 7.8. *Let K be a valued D-field. Let $p \in S_{1,\Gamma}(\Gamma_K)$ be a 1-type in the value group sort extending $\{nx \neq \gamma : n \in \mathbb{Z}_+, \gamma \in v(K^\times)\}$. There is a unique (up to $\mathcal{L}_K$-isomorphism) structure of a valued D-field on $K(x)$, the field of rational functions over K in the indeterminate x, such that $v(x) \models p$, $Dx = 0$, and such that for any polynomial $P(x) = \sum_{i=0}^n p_i x^i \in K[x]$ one has $v(\sum p_i x^i) = \min\{v(p_i) + iv(x)\}$.*

■ The hypotheses completely describe the $\mathcal{D}_e$-structure and the valuation structure. Since there is no extension of the residue field, we need not consider the extra structure on it. Let us check now that this prescription actually gives a valued D-field. We need to check that $v(Dy) \geq v(y)$ for $y \in K(x)$. By Lemma 7.7, it suffices to consider $y = P(x) \in K[x]$. Write $P(x) = \sum p_i x^i$. Then

$$\begin{aligned} DP(x) &= \sum D(p_i x^i) \\ &= \sum D(p_i)x^i + \sigma(p_i)D(x^i) \\ &= \sum D(p_i)x^i \end{aligned}$$

Since K is a valued D-field, $v(D(p_i)) \geq v(p_i)$. Therefore,

$$\begin{aligned} v(DP(x)) &= \min\{v(D(p_i)) + iv(x)\} \\ &\geq \min\{v(p_i) + iv(x)\} \\ &= v(P(x)) \end{aligned}$$

■

Lemma 7.9. *If K is a valued D-field and L/K is an unramified valued field extension of K given with an extension of the $\mathcal{D}_e$-structure with $D(\mathcal{O}_L) \subseteq \mathcal{O}_L$, then L is also a valued D-field.*

■ Let $x \in L^\times$. Let $y \in K$ such that $v(x) = v(y)$. Let $\alpha = \frac{y}{x}$. The hypothesis that $D(\mathcal{O}_L) \subseteq \mathcal{O}_L$ means that $v(D\alpha) \geq v(\alpha) = 0$. Since $y \in K$, $v(Dy) \geq v(y)$. As $x = \frac{y}{\alpha}$, Lemma 7.7 shows $v(Dx) \geq v(x)$. ■

Remark 7.10. If in Lemma 7.9 one drops the requirement that the extension is unramified, then the result is not true. For an example take $K = \mathbb{Q}$ with the trivial valuation and derivation. Let $L = \mathbb{Q}((x))$ with the order of vanishing at the origin valuation and the derivation $\partial = \frac{d}{dx}$. $\frac{d}{dx}(\mathbb{Q}[[x]]) \subseteq \mathbb{Q}[[x]]$, but $\mathrm{ord}_x(\frac{d}{dx}x) = \mathrm{ord}_x(1) = 0 < 1 = \mathrm{ord}_x(x)$.

Lemma 7.11. *If K is a valued D-field, R is the henselization of $\mathcal{O}_K$, and L is the quotient field of R, then L has a unique structure of a valued D-field extending K.*

■ Let $\varphi : \mathcal{O}_K \to \mathcal{D}_e(\mathcal{O}_K)$ be the map $x \mapsto (x, Dx)$ expressing the D-structure on $\mathcal{O}_K$. Let $\tilde{\varphi} : \mathcal{O}_K \to \mathcal{D}_e(R)$ be the composition of φ with the inclusion $\mathcal{D}_e(\mathcal{O}_K) \hookrightarrow \mathcal{D}_e(R)$. By the universal property of the henselization, there is a unique map of local rings $\psi : R \to \mathcal{D}_e(R)$ compatible with the inclusion $\mathcal{O}_K \hookrightarrow R$ and the map $\tilde{\varphi}$. Define $D : R \to R$ as the function for which $\psi(x) = (x, Dx)$. Then R is a D-ring with respect to this function. By Proposition 2.8, L has a unique D-structure extending that on R. By Lemma 7.9, L is a valued D-field with respect to this structure. ■

Lemma 7.12. *Let K be a valued D-field. Given a type $p \in S_{1,\mathbf{k}}(k_K)$ and a D-polynomial $P \in \mathcal{O}_K\langle X\rangle$ such that*

- *if $x \models p$, then $\pi(P)$ is of minimal total degree among nonzero $Q(X) \in \pi(\mathcal{O}_K)\langle X\rangle$ with $Q(x) = 0$ and*
- $\mathrm{T.deg}P = \mathrm{T.deg}\pi(P)$,

there is a unique (up to $\mathcal{L}_K$-isomorphism) D-field $L = K\langle\langle a\rangle\rangle$ such that $P(a) = 0$ and $\pi(a) \models p$.

Remark 7.13. Lemma 7.12 applies equally well in the case that $P = 0$. Recall that in this case, $\mathrm{T.deg}P = \infty$.

■ We begin by proving uniqueness and then prove existence. We analyze $L = K\langle\langle a\rangle\rangle$ as a direct limit of valued field extensions $K_n := K(a, Da, \ldots, D^n a)$. Let $m := \mathrm{ord}P$ and $b := \pi(a)$.

- ($n \le m$) Every element of K_m is of the form $\frac{Q_1(a)}{Q_2(a)}$ with $Q_i \ll P$ so in order to pin down the valuation on K_m it suffices to compute $v(Q(a))$ for $Q(X) \in K\langle X\rangle$ with $Q \ll P$. Let $\alpha \in K$ such that $\alpha Q \in \mathcal{O}_K\langle X\rangle$ and $\pi(\alpha Q) \ne 0$. Since $\pi(\alpha Q)(b) \ne 0$, we must have $v(Q(a)) = -v(\alpha)$.
- ($n > m$) If $e = 0$, then $L = K_m$ so there is nothing more to do.

  We assume now that $e \ne 0$. Let $P(X) = F(X, \ldots, D^m X)$. Let $G(X) := F(a, \ldots, D^{m-1}a, X)$. In the case $e \ne 0$, σy and Dy are inter-definable, so it suffices to consider the extension

  $$K(a, Da, \ldots, D^{n-1}a, \sigma^{n-m}D^m a)/K(a, Da, \ldots, D^{n-1}a)$$

  $\sigma^{n-m}D^m a$ satisfies $\sigma^{n-m}G$. So the minimal polynomial Q_n of $\sigma^{n-m}D^m a$ over $K(a, \ldots, D^{n-1}a)$ divides $\sigma^{n-m}G$. Since σ reduces to the identity automorphism, $\pi(\sigma^{n-m}D^m a) = D^m b$ which is a simple root of $\pi(G) = \pi(\sigma^{n-m}G)$ (as $\frac{\partial}{\partial X_m}P \ll P$). Thus, $\pi(G')(D^m b) = \pi(\frac{\partial}{\partial X_m}(P))(b) \ne 0$. Thus, b is a simple root of $\pi(Q_n)$ so the extension is completely determined as an extension of valued fields.

We check now that the process used above to analyze such extensions may be used to produce them. Let $b \models p$ be a realization of p.

Let K' be the field of fractions of $K[X, DX, \ldots, D^m X]/(F(X, \ldots, D^m X))$. Let a denote the image of X in K'. K' is given a valuation structure by setting $v(Q(a)) := \max\{-v(\alpha) : \alpha Q \in \mathcal{O}_K\langle X\rangle\}$ for $Q(X) \in K\langle X\rangle$ with $Q \ll P$. In the case that $e = 0$, K' is already a differential field. In the case that $e \ne 0$, let Q_1 be the unique (up to multiplication by a unit) factor of $\sigma(G)$ (over $\mathcal{O}_{K'}[Y]$) for which $\pi(Q_1)(D^m b) = 0$ and $\pi(Q_1) \ne 0$. This was proven to exist in the course of the uniqueness proof. Since $D^m b$ is a simple root of $\pi(Q_1)(Y) = 0$, the equation $Q_1(Y) = 0$ has a unique solution y in R a henselization of $\mathcal{O}_{K'}$ such that $\pi(y) = D^m b$. Define σ on K as usual by $\sigma(x) := eDx + x$ and extend to $\sigma : K' \to K'(y)$ via $\sigma(D^i a) := eD^{i+1}a + D^i a$ for $i < m$ and $\sigma(D^m a) := y$. We formally define $Dz := \frac{\sigma(z)-z}{e}$ as a function $K' \to K'(y)$. At this point, we could continue to incrementally define D as the analysis in the uniqueness part of the proof might suggest. This works and the reader is invited to finish the argument this way. We take a different tack. First we check that the required inequalities continue to hold at least for $z \in K'$.

Claim 7.14. $v(D^{m+1}a) \ge 0$

✠ Since σ reduces to the identity on $\mathcal{O}_K[a,\dots,D^{m-1}a]/(e)$, $D^m a$ is a root to $\sigma(G)$ modulo e. Thus, $v(y - D^m a) \geq v(e)$. Since $D^{m+1}a = \frac{y-D^m a}{e}$, the result is now clear. ✠

The next claim is valid in general. That is, there is no restriction on e.

Claim 7.15. *If $Q(X) \ll P(X)$, then $v(Q(a)) \leq v(DQ(a))$.*

✠ By Lemma 7.7 we may assume that $Q(X) \in \mathcal{O}_K\langle X\rangle$ and $\pi(Q) \neq 0$. This implies $v(Q(a)) = 0$. Write $Q(X) = \sum q_i(D^m X)^i$ where $q_i \in \mathcal{O}_K[a,\dots,D^{m-1}a]$. Then $DQ(X) = \sum D(q_i)(D^m X)^i + \sigma(q_i)D(D^m X)^i \in \mathcal{O}_K\langle X\rangle$. For $j \leq m$, it is clear that $v(D^j a) \geq 0$. Claim 7.14 shows that $v(D^{m+1}a) \geq 0$ in the case that $e \neq 0$. In the case that $e = 0$, we observe that $D^{m+1}a = \frac{-P^D}{\frac{\partial}{\partial X_m}P}$ and $\frac{\partial}{\partial X_m}P \ll P$ so that its valuation is zero.

Since each of the $D^j a$ have non-negative valuation, $v(DQ(a)) \geq 0 = v(Q(a))$. ✠

Let R be the henselization of $\mathcal{O}_{K'}$ and let L be the field of fractions of R. Let $\varphi : K' \to \mathcal{D}_e(L)$ be the map $x \mapsto (x, Dx)$. By Claim 7.15, $\varphi(\mathcal{O}_{K'}) \subseteq \mathcal{D}_e(R)$. By Lemma 2.10, $\mathcal{D}_e(R)$ is a local ring with maximal ideal $\pi_0^{-1}\mathfrak{m}_R$. Since $\pi_0 \circ \varphi = \mathrm{id}_R$, φ is a local homomorphism. By Lemma 2.11, $\mathcal{D}_e(R)$ is henselian. Thus, there is a unique extension of φ to a local homomorphism $\tilde{\varphi} : R \to \mathcal{D}_e(R)$. By Lemma 2.7, $\tilde{\varphi}$ extends uniquely to a ring homomorphism $\psi : L \to \mathcal{D}_e(L)$. Let D denote the function $D : L \to L$ for which $\psi(x) = (x, Dx)$. Since $D(R) \subseteq R$, by Lemma 7.9, L is a valued D-field. ■

Proposition 7.16. *Let K be a valued D-field. Let $a \in K^\times$. There is an unramified extension L/K of valued D-fields of the form $L = K(x)$ for which $v(x) = v(a)$ and $Dx = 0$. Moreover, the $\mathcal{L}_K$-isomorphism type of L is determined by* $\mathrm{tp}(\pi(\frac{a}{x})/k_K)$.

■ We wish to find x so that $Dx = 0$ and $v(x) = v(a)$. This is equivalent to finding $y =$ "$\frac{a}{x}$". Such a y would have to satisfy $Dy = \frac{Da}{x} = \frac{Da}{a}\frac{a}{x} = \frac{Da}{a}y$. Conversely, if y satisfies $Dy = \frac{Da}{a}y$ and we define x by $x := \frac{a}{y}$, then $Dy = \frac{Da}{x} + \sigma(a)D(\frac{1}{x})$ as well so that $0 = D(\frac{1}{x})$ which implies that $Dx = 0$.

One would also need $v(y) = 0$ in order for $v(\frac{a}{y}) = v(a)$. That is, we need y to be a solution to $DY = \frac{Da}{a}Y$ with $v(y) = 0$. By Lemma 7.12, such extensions exist and they are determined by $\mathrm{tp}(y/k_K)$. ■

Proposition 7.17. *Let K be a valued D-field. Assume that $v(K^\times) = v((K^D)^\times)$. Let $\eta \in K^D$ such that $n|v(\eta)$. Assume that $v(\eta) \notin m \cdot v(K^\times)$ for each positive integer m dividing n. There exists a unique (up to $\mathcal{L}_K$-isomorphism) extension of valued D-fields of the form $K(\sqrt[n]{\eta})$.*

■ Let $\epsilon = \sqrt[n]{\eta}$. Since the extension is totally ramified, the valuation structure is completely determined. I claim that the $\mathcal{D}_e$-structure is determined by $D\epsilon = 0$. This fact would certainly fully specify the $\mathcal{D}_e$-structure, the content of my claim is that one must have $D\epsilon = 0$. When $e = 0$, this follows from the fact that $0 = D(\eta) = D(\epsilon^n) = n\epsilon^{n-1}D\epsilon$. When $e \neq 0$, the assertion $D\epsilon \neq 0$ is equivalent to $\sigma(\epsilon) = \omega\epsilon$ for some nontrivial n-th root of unity ω. But then $D\epsilon = \frac{\sigma(\epsilon)-\epsilon}{e} = \epsilon\frac{\omega-1}{e}$. Since $\omega \neq 1$, $v(\frac{\omega-1}{e}) = -v(e) < 0$. So that $v(D\epsilon) < v(\epsilon)$ which violates Axiom 3.

We check that this prescription correctly defines a valued D-field. Let $x = \sum_{i=0}^{n-1} x_i \epsilon^i \in L$. Then $v(x) = \min\{v(x_i) + \frac{i}{n} v(\eta)\}$ and $Dx = \sum_{i=0}^{n-1} Dx_i \epsilon^i$ so that visibly, the inequality $v(Dx) \geq v(x)$ holds. ■

Proposition 7.18. *The isomorphism may be extended so that $v(A_1^\times) = v((A_1^D)^\times)$.*

■ Let $a \in A_1$ such that there is no constant in A_1 having the same value as that of a. Let $p \in S_{1,\mathbf{k}}(k_{A_1})$ be some type containing the formula $Dx = \pi(\frac{Da}{a})x$ as well as the formulas $x \neq b$ for each $b \in k_{A_1}$. By the saturation hypotheses, p is realized in k_{M_1} by some b_1 and $f(p)$ is realized in k_{M_2} by some b_2. By the surjectivity of the residue map and DHL there is some $c_1 \in M_1$ and some $c_2 \in M_2$ such that $\pi(c_i) = b_i$, $Dc_1 = \frac{Da}{a} c_1$ and $Dc_2 = f(\frac{Da}{a})c_2$. By Proposition 7.16, the extension of f given by $c_1 \mapsto c_2$ (and therefore $\frac{a}{c_1} \mapsto \frac{f(a)}{c_2}$) is an isomorphism of $\mathcal{L}$-structures and the element $\frac{a}{c_1}$ is a constant with value equal to that of a. ■

Proposition 7.19. *The isomorphism extends so that A_1 has enough constants.*

■ By Proposition 7.18 we may now assume that $v(A_1^\times) = v((A_1^D)^\times)$. Let $\gamma \in \Gamma_{A_1} \setminus v(A_1^\times)$. In the case that $\mathrm{tp}(\gamma/v(A_1^\times)) \vdash \{n\gamma \neq v(a) : n \in \mathbb{Z}_+, a \in A_1^\times\}$, we find $\epsilon_1 \in N_1$ with $v(\epsilon_1) = \gamma$ and $D\epsilon_1 = 0$ and $\epsilon_2 \in M_2$ with $v(\epsilon_2) \models f(\mathrm{tp}(\gamma/v(A_1^\times))$ and $D\epsilon_1 = D\epsilon_2 = 0$ by Axiom 6. Lemma 7.8 shows that the extension given by $\epsilon_1 \mapsto \epsilon_2$ is an isomorphism of $\mathcal{L}$-structures.

In the case that $n\gamma \in v(A_1^\times)$ for some $n \in \mathbb{Z}_+$, take n minimal with this property and find some $\eta \in A_1^D$ with $v(\eta) = n\gamma$. By Axiom 5, we may find $\epsilon_i \in M_i$ such that $\epsilon_1^n = \eta$ and $\epsilon_2^n = f(\eta)$. By Proposition 7.17, this gives an isomorphism of $\mathcal{L}$-structures. ■

Proposition 7.20. *f extends so that $\Gamma_{N_1} \subseteq v((A_1^D)^\times)$.*

■ As we have assumed quantifier elimination in the value group sort, we may fix some elementary embedding $\overline{f} : \Gamma_{N_1} \to \Gamma_{M_2}$ extending f. By Proposition 7.19, we may extend f over $\overline{f}$. ■

Proposition 7.21. *The map extends so that $k_{N_1} \subseteq \pi(\mathcal{O}_{A_1})$.*

■ As the theory of the residue field of M_1 has quantifier elimination by assumption we may fix some isomorphism $\overline{f}$ between k_{N_1} and some countable elementary substructure of k_{M_2} extending f restricted to k_{A_1}.

Take $a \in \mathcal{O}_{N_1}$ so that $\pi(a)$ is a new element of $k_{N_1} \setminus \pi(\mathcal{O}_{A_1})$. If $\pi(a)$ is differentially transcendental over $\pi(\mathcal{O}_{A_1})$, then let $b_1 = a$. Let $b_2 \in \mathcal{O}_{M_2}$ such that $\pi(b_2) = \overline{f}(a)$. Necessarily, b_1 is D-transcendental over A_1 while b_2 is D-transcendental over A_2. Then Lemma 7.12 shows that we may extend the isomorphism by setting $f(b_1) = b_2$.

Otherwise, Let $P \in \mathcal{O}_{A_1}\langle X \rangle$ be such that $\pi(P)(a) = 0$, $\mathrm{T.deg} P = \mathrm{T.deg} \pi(P)$, $\pi(P) \neq 0$ and $\mathrm{T.deg} P$ is minimal with these properties. The minimality condition on P implies that $\pi(P)$ is a minimal D-polynomial for $\pi(a)$ over the residue field of A_1 so that for some i one has $v(\frac{\partial}{\partial X_i} P(a)) = 0$. By DHL in both N_1 and M_2 there is some $b_1 \in N_1$ and $b_2 \in M_2$ such that $P(b_1) = 0$, $f(P)(b_2) = 0$, $\pi(b_1) = \pi(a)$ and $\pi(b_2) = \overline{f}(\pi(a))$.

By Lemma 7.12, $A_1\langle\langle b_1\rangle\rangle \cong A_2\langle\langle b_2\rangle\rangle$ via an isomorphism extending f. ■

7.2. **Immediate Extensions.** The rest of this section is devoted to proving that the isomorphism may be extended to immediate extensions.

Definition 7.22.

(1) A *pseudo-convergent sequence* is a limit ordinal indexed sequence $\{x_\alpha\}_{\alpha<\kappa}$ of elements of K such that $(\forall \alpha < \beta < \gamma < \kappa)\ v(x_\alpha - x_\beta) < v(x_\beta - x_\gamma)$.
(2) If L/K is an extension of valued D-fields and $\{x_\alpha\}$ is a pseudo-convergent sequence from K, then the set of *pseudo-limits* of $\{x_\alpha\}$ in L is the set of $c \in L$ such that $(\forall \alpha < \beta < \kappa)\ v(x_\alpha - c) < v(x_\beta - c)$. In this case, one writes $x_\alpha \Rightarrow c$ and says that $\{x_\alpha\}$ *pseudo-converges* to c.
(3) The pseudo-convergent sequence $\{x_\alpha\}$ *pseudo-solves* the D-polynomial P if either $P(x_\alpha) \Rightarrow 0$ or $P(x_\alpha) = 0$ for $\alpha \gg 0$.
(4) A pseudo-convergent sequence from K is *strict* if it has no pseudo-limits in K.

Remark 7.23. If $\{x_\alpha\}_{\alpha<\kappa}$ is a pseudo-convergent sequence and one restricts to $\{x_\alpha\}_{\alpha\in I}$ where I is cofinal in κ, then the new sequence is also pseudo-convergent with the same pseudo-limits and D-polynomials which it pseudo-solves as the original sequence. As the value groups considered in this proof are countable, one could always assume that $\kappa = \omega$ by making such a restriction.

We will repeatedly use this process of restricting to cofinal subsequences. The next lemma shows that we may pass from one strict pseudo-convergent sequence to a subsequence of another sequence known to well-approximate the first without changing the set of pseudo-limits.

Lemma 7.24. *Let $\{x_\alpha\}_{\alpha<\kappa}$ be a strict pseudo-convergent sequence from K. Suppose that $\{y_\alpha\}_{\alpha<\kappa}$ is another sequence from K and that $v(y_\alpha - x_\alpha) \geq v(x_{\alpha+1} - x_\alpha)$ for all $\alpha < \kappa$. Then there is a cofinal $S \subseteq \kappa$ such that $\{y_\alpha\}_{\alpha\in S}$ is a strict pseudo-convergent sequence with the same pseudo-limits as those of $\{x_\alpha\}_{\alpha<\kappa}$.*

■ We define S by transfinite recursion. We start with $0 \in S$. At a stage α where we have decided that $\alpha \in S$, look for the least $\beta > \alpha$ such that $v(x_{\beta+1} - x_\beta) > v(y_\alpha - x_\beta)$ and let β be the next element of S.

Such a β must exist for if it did not, then y_α would be a pseudo-limit of $\{x_\gamma\}_{\gamma<\kappa}$ which is impossible as we assumed this sequence to be strict. In detail, if such a β failed to exist, then for each $\gamma > \beta > \alpha$, we would have $v(y_\alpha - x_\gamma) \geq v(x_{\gamma+1} - x_\gamma) > v(x_{\beta+1} - x_\beta) = v(x_\gamma - x_\beta) = \min\{v(y_\alpha - x_\gamma), v(x_\gamma - x_\beta)\} = v(y_\alpha - x_\beta)$ so that y_α would be a pseudolimit of this sequence.

At limit stages, we simply take the least $\lambda < \kappa$ not yet in S.

Let us check that this choice of S works. First we show that $\{y_\alpha\}_{\alpha\in S}$ is a pseudo-convergent sequence. Let $\alpha < \beta < \gamma$ be in S. Then $v(y_\gamma - y_\beta) \geq v(y_\gamma - x_\gamma + x_\gamma - x_\beta + x_\beta - y_\beta) \geq \min\{v(x_{\gamma+1} - x_\gamma), v(x_{\beta+1} - x_\beta)\} = v(x_{\beta+1} - x_\beta) > v(y_\alpha - x_\beta) = \min\{v(x_\beta - y_\alpha), v(y_\beta - x_\beta)\} = v(y_\beta - y_\alpha)$.

Let now a be a pseudo-limit of $\{x_\alpha\}_{\alpha<\kappa}$ in some extension of K. Let $\alpha < \beta \in S$. Let α' be the successor of α in S and β' the successor of β in S. Then $v(a - y_\beta) = v(a - x_{\beta'} + x_{\beta'} - y_\beta) = v(x_{\beta'+1} - x_{\beta'}) > v(x_{\alpha'+1} - x_{\alpha'}) = v(a - y_\alpha)$. So every pseudo-limit of $\{x_\alpha\}_{\alpha<\kappa}$ is a pseudo-limit of $\{y_\alpha\}_{\alpha\in S}$.

Let now a be a pseudo-limit of $\{y_\alpha\}_{\alpha\in S}$. Let $\alpha < \kappa$ be given. Let $\beta \in S$ be minimal with $\beta > \alpha$. Then $v(a - x_\alpha) = v(a - y_\beta + y_\beta - x_\beta + x_\beta - x_\alpha) = v(x_\beta - x_\alpha) =$

$v(x_{\alpha+1}-x_\alpha)$ as $v(a-y_\beta)=v(y_{\beta'}-y_\beta)\geq v(x_\beta-y_\beta)\geq v(x_{\beta+1}-x_\beta)>v(x_\beta-x_\alpha)$. So, a is also a pseudo-limit of $\{x_\alpha\}_{\alpha<\kappa}$.

Since $\{x_\alpha\}_{\alpha<\kappa}$ and $\{y_\alpha\}_{\alpha\in S}$ have the same pseudo-limits and $\{x_\alpha\}_{\alpha<\kappa}$ has no pseudo-limits in K, $\{y_\alpha\}_{\alpha\in S}$ is also strict. ■

Lemma 7.25. *Assume that the residue field of K is linearly differentially closed and that K has enough constants. If DHL applies to $P\in\mathcal{O}_K\langle X\rangle$ at $a\in\mathcal{O}_K$, then either there is some $b\in\mathcal{O}_K$ with $P(b)=0$ and $v(a-b)=v(P(a))$ or there is a strict pseudo-convergent sequence $\{x_\alpha\}_{\alpha<\lambda}$ from K pseudo-solving P with $v(x_\alpha-a)=v(P(a))$.*

■ This is proven exactly as in the proof of DHL for complete fields so we give only a sketch here referring the reader to the proof of Proposition 6.1 for the detailed computations.

We produce either a solution to $P(b)=0$ and $v(a-b)=v(P(a))$ or a sequence with the properties:

- $x_0=a$,
- $v(x_{\alpha+1}-x_\alpha)=v(P(x_\alpha))$, and
- $v(P(x_\alpha))$ is strictly increasing.

Start with $x_0=a$.

At a limit stage, look for x_λ such that $x_\alpha\Rightarrow x_\lambda$. If no x_λ exists, then stop – the sequence $\{x_\alpha\}_{\alpha<\lambda}$ is a strict pseudo-solution of P.

At a successor stage, since K has enough constants, there is some $\epsilon\in K^D$ with $v(\epsilon)=v(P(x_\alpha))$. We look to solve

$$P(x_\alpha+\epsilon Y) \quad = \quad 0$$

As in the proof of Proposition 6.1, we let $y\in\mathcal{O}_K$ be a lifting of a solution to

$$0 \quad = \quad \pi(\frac{P(x_\alpha)}{\epsilon})+\sum_{i=0}^{\mathrm{ord}P}\pi(\frac{\partial}{\partial X_i}P(a))D^iY$$

Set $x_{\alpha+1}=x_\alpha+\epsilon y$. ■

Definition 7.26. Let $\mathbf{N}\in\mathbb{N}^{(\omega)}\cup\{\infty\}$. The valued D-field K is $\mathbf{N}$- *full* if whenever $\{x_\alpha\}$ is a pseudo-convergent sequence from K and there is an immediate extension $K\langle\langle a\rangle\rangle$ with

- $x_\alpha\Rightarrow a$,
- $Q(a)=0$ for some $Q\in K\langle X\rangle$ with $\mathrm{T.deg}Q<\mathbf{N}$, and
- $\{x_\alpha\}$ pseudo-satisfies Q,

then $\{x_\alpha\}$ has a pseudo-limit in K.

Remark 7.27. Another way of stating the definition of $\mathbf{N}$-fullness is to say that no strict pseudo-convergent sequences satisfying the three listed hypotheses exist.

Definition 7.28. Let $A\in K\langle X\rangle$ be a non-zero D-polynomial. A *refinement* of A at a is a D-polynomial $G(Y)=\frac{A(\epsilon Y+a)-A(a)}{c}$ where $c,\epsilon\in(K^D)^\times$ and $G\in\mathcal{O}_K\langle X\rangle$ but $\pi(G)\neq 0$. ϵ is called the *internal scale* and c is the *external scale*. The D-polynomial $H(Y)=\frac{1}{c}A(a+\epsilon Y)=G(Y)+\frac{A(a)}{c}$ is called the *rescaling* of A at a with scales ϵ and c. Note that H need not have integral co-efficients.

Remark 7.29. Notice that G may be expressed as

$$G(Y) \quad = \quad \sum_{|I|>0} \frac{\epsilon^{|I|}}{c} \partial_I A(a) Y^I$$

Observe that the valuation of the external scale is entirely determined by the conditions $G(Y) \in \mathcal{O}_K\langle Y \rangle$ and $\pi(G) \neq 0$.

Definition 7.30. Let $\{x_\alpha\}$ be a pseudo-convergent sequence. Let A be a D-polynomial. A *refinement of A along $\{x_\alpha\}$* is a sequence of refinements of A at x_α having internal scale ϵ_α where $v(\epsilon_\alpha) = v(x_{\alpha+1} - x_\alpha)$. Likewise, a *rescaling of A along $\{x_\alpha\}$* is a sequence of rescalings of A at x_α having internal scale ϵ_α as above.

Definition 7.31. The non-zero D-polynomial $A(X) \in \mathcal{O}_K\langle X \rangle$ is *residually linear* if $\pi(A) \in k\langle X \rangle$ is a non-constant linear D-polynomial. A is *potentially residually linear* if some refinement of A is residually linear. We will stipulate that the zero polynomial is residually linear.

Proposition 7.32. *Let K be a valued D-field. Let $P(X) \in K\langle X \rangle$ be an irreducible D-polynomial. Assume that K has enough constants, has a linearly differentially closed residue field, and is* $\mathrm{T.deg}P$*-full. Let $\{x_\alpha\}_{\alpha<\kappa}$ be a strict pseudo-convergent sequence from $\mathcal{O}_K$.*

(1) *If $\{x_\alpha\}$ is a pseudo-solution of P, then there is an immediate extension of valued D-fields of the form $K\langle\langle a\rangle\rangle$ in which $P(a) = 0$ and $x_\alpha \Rightarrow a$.*
(2) *If $K\langle\langle a\rangle\rangle$ is an extension in which $x_\alpha \Rightarrow a$ and $P(a) = 0$, then $K\langle\langle a\rangle\rangle$ is unique up to $\mathcal{L}_K$-isomorphism.*
(3) *P is potentially residually linear. In fact, for $\alpha \gg 0$ any refinement of P along $\{x_\alpha\}$ is residually linear.*

Remark 7.33. Our proof of Proposition 7.32 is more complicated than one might expect it needs to be. The idea behind the proof is fairly simple, but technical problems arose for us. For the existence proof, one would like to take some sort of limit. Of course, $\{x_\alpha\}$ may be merely pseudo-convergent rather than convergent, so that there will not be a good notion of a completion. One might try to find the limit by working in some saturated extension and then specializing so as to eliminate excess infinitesimals. Instead, we employ an algebraic construction. For the uniqueness proof, one might like to argue that for $Q \prec P$ the sequence $v(Q(x_\alpha))$ is non-decreasing so that either $\{x_\alpha\}$ pseudo-solves Q and hence pseudo-converges to some $a \in K$ by the inductive hypothesis and fullness (contradicting the strictness of the sequence) or the value settles down. Again some technical problems arise, notably with controlling the qualitative behavior of $\{v(Q(x_\alpha))\}$, so that our actual proof is a bit more involved.

Remark 7.34. Proposition 7.32 allows us to finish extending f. We will arrange that the hypotheses are true of A_1 by an inductive argument. We use potential residual linearity to see that after a linear change of variables, any solution to a D-polynomial may be analyzed as an instance of DHL so that we can find the relevant solutions on both sides.

Definition 7.35. If $\{\gamma_\alpha\}_{\alpha<\kappa}$ is a sequence of elements of Γ, then we say that the limit of the sequence exists iff the sequence is eventually constant. In that case, we write $\lim \gamma_\alpha = \gamma$ where $\gamma_\alpha = \gamma$ for $\alpha \gg 0$.

Most of the lemmata proved in what follows will be employed to prove Proposition 7.32 and they depend inductively on Proposition 7.32. We indicate this by the condition

†: The hypotheses of Proposition 7.32 are assumed to hold and we assume inductively on $\mathrm{T.deg}P$ that Proposition 7.32 is true.

Lemma 7.36 (†). *Let $Q \in \mathcal{O}_K\langle X\rangle$. Suppose that $Q \prec P$ and DHL applies to Q at $a \in \mathcal{O}_K$. There is some $b \in K$ such that $Q(b) = 0$ and $v(b - a) = v(Q(a))$.*

■ By Lemma 7.25 either the lemma is true or there is a strict pseudo-convergent sequence $\{y_\beta\}_{\beta<\kappa}$ from K pseudo-solving Q with $y_0 = a$ and $v(y_\beta - a) = v(Q(a))$. By the inductive hypothesis (via †), there is an immediate extension $K\langle\!\langle c\rangle\!\rangle$ in which $Q(c) = 0$ and $y_\beta \Rightarrow c$. By $\mathrm{T.deg}P$-fullness of K, $\{y_\beta\}$ is not strict. This is a contradiction. ■

Lemma 7.37 (†). *Let $Q \in \mathcal{O}_K\langle X\rangle$. If $Q \prec P$, then $\lim v(Q(x_\alpha))$ exists.*

■ We prove this lemma by $\prec$-induction on Q. When Q is a constant D-polynomial, the result is obvious. We may now assume that Q is not constant. Thus, there is some i for which $\frac{\partial}{\partial X_i}Q$ is not the zero D-polynomial. By the inductive hypothesis, $\lim v(\frac{\partial}{\partial X_i}Q(x_\alpha))$ exists. We finish this proof by a series of lemmata. In each of these lemmata, we assume inductively † as well as

‡: Lemma 7.37 is true for $\tilde{Q} \prec Q$. ■

Lemma 7.38 (†, ‡). *Let $Q \prec P$. There is a pseudo-convergent sequence $\{y_\alpha\}$ having the same pseudo-limits as $\{x_\alpha\}$ such that if H_α is a refinement of Q at y_α with internal scale η_α with $v(\eta_\alpha) = v(y_{\alpha+1} - y_\alpha)$, then $v(H_\alpha(\frac{y_{\alpha+1}-y_\alpha}{\eta_\alpha})) = 0$.*

Remark 7.39. While the lemma and certainly the proof are stated in terms of the refinements of Q, the point of this lemma is to understand Q itself. In terms of Q, we find a pseudo-convergent sequence $\{y_\alpha\}$ having the same pseudo-limits as $\{x_\alpha\}$ such that for any α one has $v(Q(y_{\alpha+1})-Q(y_\alpha)) = \min\{v(\frac{\partial}{\partial X_i}Q(y_\alpha))+v(y_{\alpha+1}-y_\alpha) : 0 \le i \le \mathrm{ord}(Q)\}$.

■ We may assume that no cofinal sequence in $\{x_\alpha\}$ already works. For each α let G_α be a refinement of Q at x_α with internal scale ϵ_α having $v(\epsilon_\alpha) = v(x_{\alpha+1} - x_\alpha)$. So for $\alpha \gg 0$ we have that $v(G_\alpha(\frac{x_{\alpha+1}-x_\alpha}{\epsilon_\alpha})) > 0$.

We construct the sequence $\{y_\alpha\}$ allowing repetition and later thin using Lemma 7.24 to get an actual pseudo-convergent sequence.

Using the inductive hypotheses we may assume that the valuations of the partials of Q have stabilized (via ‡) and any refinement of Q along x_α is residually linear (via †).

Claim 7.40. *For each α, there is some $\beta > \alpha$ minimal with the properties that $v(G_\alpha(\frac{x_\beta-x_\alpha}{\epsilon_\alpha})) < v(\epsilon_\beta) - v(\epsilon_\alpha)$ and $v(G_\alpha(\frac{x_\gamma-x_\alpha}{\epsilon_\alpha})) \le v(G_\alpha(\frac{x_\beta-x_\alpha}{\epsilon_\alpha}))$ for $\gamma \ge \beta$.*

✠ There cannot be a cofinal sequence of γ's on which $v(G_\alpha(\frac{x_\gamma-x_\alpha}{\epsilon_\alpha}))$ is increasing for if this were to occur, by the inductive hypothesis for existence and fullness the sequence $\{x_\alpha\}$ would not be strict.

If the first condition were to fail, then since G_α is residually linear, DHL would apply at $\frac{x_\beta-x_\alpha}{\epsilon_\alpha}$ so that Lemma 7.36 would produce w_β with $G_\alpha(w_\beta) = 0$ and

$v(w_\beta - \frac{x_\beta - x_\alpha}{\epsilon_\alpha}) \geq v(\epsilon_\beta) - v(\epsilon_\alpha)$. Set $z_\beta := x_\alpha + \epsilon_\alpha w_\beta$. We may restrict to a subsequence of $\{z_\beta\}$ which is pseudo-convergent and has the same pseudo-limits as $\{x_\beta\}$ by Lemma 7.24.

For each β, one has $\tilde{Q}(z_\beta) := Q(z_\beta) - Q(x_\alpha) = 0$ so that by the inductive hypothesis, there is some extension $K(\zeta)$ of K in which $z_\beta \Rightarrow \zeta$ and $\tilde{Q}(\zeta) = 0$. By fullness, $\{z_\beta\}$ is not strict. Consequently $\{x_\alpha\}$ is not strict. This is a contradiction. ✠

We construct the sequence $\{y_\alpha\}$ now.

By Claim 7.40, there is a cofinal $S \subseteq \kappa$ such that for any $\alpha \in S$ if α' denotes the successor of α in S, then $v(G_\alpha(\frac{x_{\alpha'} - x_\alpha}{\epsilon_\alpha})) < v(\epsilon_{\alpha'}) - v(\epsilon_\alpha)$ and for any $\beta \geq \alpha'$ one has $v(G_\alpha(\frac{x_\beta - x_{\alpha'}}{\epsilon_\alpha})) \leq v(G_\alpha(\frac{x_{\alpha'} - x_\alpha}{\epsilon_\alpha})$. We restrict to S and omit it from the notation.

As in the proof of Claim 7.40, find z_α such that $G_\alpha(z_\alpha) = 0$ and $v(z_\alpha - \frac{x_{\alpha+1} - x_\alpha}{\epsilon_\alpha}) = v(G_\alpha(\frac{x_{\alpha+1} - x_\alpha}{\epsilon_\alpha})$. Define $y_\alpha := x_\alpha + z_\alpha \epsilon_\alpha$. Since $v(y_\alpha - x_\alpha) = v(z_\alpha \epsilon_\alpha) \geq v(\epsilon_\alpha) = v(x_{\alpha+1} - x_\alpha)$, Lemma 7.24 applies and we can find a cofinal pseudo-convergent subsequence of $\{y_\alpha\}_{\alpha \in S}$ having the same pseudo-limits as those $\{x_\alpha\}$.

Let us compute the valuation of $y_\beta - y_\alpha$ for $\beta > \alpha$ both from S.

$$\begin{aligned} v(y_\beta - y_\alpha) &= v(x_\beta + \epsilon_\beta z_\beta - x_\alpha - \epsilon_\alpha z_\alpha) \\ &= v(\epsilon_\beta z_\beta + \epsilon_\alpha(\frac{x_\beta - x_\alpha}{\epsilon_\alpha} - z_\alpha)) \\ &= v(\epsilon_\alpha) + v(G_\alpha(\frac{x_{\alpha+1} - x_\alpha}{\epsilon_\alpha})) \end{aligned}$$

The last equality uses the definition of z_α and the fact that $v(\epsilon_\beta) - v(\epsilon_\alpha) > v(G_\alpha(\frac{x_{\alpha+1} - x_\alpha}{\epsilon_\alpha})) = v(G_\alpha(\frac{x_\beta - x_\alpha}{\epsilon_\alpha}))$.

To finish the proof, let $\theta \in \Gamma$ be $\min_i\{v(\frac{\partial}{\partial X_i} Q(x_\alpha))\}$. (Recall that these values do not depend on α.) In what follows, H_β denotes a refinement of Q at y_β with internal scale η_β having $v(\eta_\beta) = v(y_{\beta+1} - y_\beta)$. Since H_β is residually linear, the minimal valuation of a coefficient of $Q(y_\alpha + \eta_\alpha Y) - Q(y_\alpha)$ is $\theta + v(\eta_\alpha)$.

Let now $\alpha < \beta \in S$

$$\begin{aligned} v(H_\alpha(\frac{y_\beta - y_\alpha}{\eta_\alpha})) &= v(Q(y_\beta) - Q(y_\alpha)) - \theta - v(\eta_\alpha) \\ &= v(Q(x_\beta)) - Q(x_\alpha)) - \theta - v(\epsilon_\alpha) - v(G_\alpha(\frac{x_{\alpha+1} - x_\alpha}{\epsilon_\alpha})) \\ &= v(G_\alpha(\frac{x_\beta - x_\alpha}{\epsilon_\alpha})) - v(G_\alpha(\frac{x_{\alpha+1} - x_\alpha}{\epsilon_\alpha})) \\ &= 0 \end{aligned}$$

The last equality is justified by the observations that H_α and its arguments are integral so that the final expression can be no less than zero, but it can be no more then zero because of the second condition of Claim 7.40. ■

Lemma 7.41 (†, ‡)**.** *If $Q \prec P$, then there is some $\beta < \kappa$ such that if L/K is any extension of valued D-fields and $c \in L$ with $v(x_{\beta+1} - c) > v(x_{\beta+1} - x_\beta)$, then $v(Q(c)) = v(Q(x_\beta))$. Moreover, $Q(x_\alpha) \Rightarrow Q(c)$.*

■ Let L be an extension of K (as a valued D-field) and in L take c with $x_\alpha \Rightarrow c$. Let $\{y_\alpha\}$, H_α and θ be as in Lemma 7.38. Since $v(H_\beta(\frac{y_{\beta+1}-y_\beta}{\eta_\beta})) = 0$ and $y_\beta \Rightarrow c$ (so that $\pi(\frac{c-y_\beta}{\eta_\beta}) = \pi(\frac{y_{\beta+1}-y_\beta}{\eta_\beta})$), we have

$$\begin{aligned} 0 &= v((H_\beta(\frac{c-y_\beta}{\eta_\beta})) \\ &= v(Q(c)-Q(y_\beta)) - \theta - v(\eta_\beta) \end{aligned}$$

So $v(Q(c)-Q(y_\beta)) = \theta + v(\eta_\beta)$. The right-hand side is growing with β so $Q(x_\beta) \Rightarrow Q(c)$. This implies that either $v(Q(c)) = v(Q(y_\beta))$ eventually (and we're done) or $v(Q(y_\beta))$ is increasing cofinally. In this second case, by † and fullness of K, $\{y_\beta\}$ is not strict. This contradicts the strictness of $\{x_\alpha\}$.

By the compactness theorem, there is some $\beta < \kappa$ for which we have $v(c-x_{\beta+1}) > v(x_\beta - x_{\beta+1}) \Rightarrow v(Q(c)) = v(Q(x_\beta))$. ■

Lemma 7.41 finishes the proof of Lemma 7.37.

Lemma 7.42 (†). *For $\alpha \gg 0$, any refinement of P along x_α is residually linear.*

■ By Lemma 7.37, $\lim v(\partial_I P(x_\alpha))$ exists for any non-zero multi-index I. From now on, work only with α large enough so that this common value has been attained.

For each α, let G_α be a refinement of P along x_α with scales ϵ_α and c_α. Write $G_\alpha(Y) = \sum_I g_{I,\alpha} Y^I$.

Claim 7.43. *The set of multi-indices $\{I : v(g_{I,\alpha}) = 0\}$ does not depend on α for α sufficiently large. In fact, all such I have the same length.*

✠ If it ever happens that for some I and J with $|I| < |J|$ that $v(g_{\alpha,I}) \leq v(g_{\alpha,J})$, then $v(g_{\beta,I}) < v(g_{\beta,J})$ for $\beta > \alpha$. To see this, observe that the hypothesis is that

$$\begin{aligned} v(\frac{\epsilon_\alpha^{|I|}}{c_\alpha}) + v(\partial_I P(x_\alpha)) &= v(g_{I,\alpha}) \\ &\leq v(g_{J,\alpha}) \\ &= v(\frac{\epsilon_\alpha^{|J|}}{c_\alpha}) + v(\partial_J P(x_\alpha)) \end{aligned}$$

That is,

$$v(\partial_I P(x_\alpha)) \quad \leq \quad (|J|-|I|)v(\epsilon_\alpha) + v(\partial_J P(x_\alpha))$$

Since the valuations of the partials are the same whether evaluated at x_α or x_β, $|J|-|I| > 0$, and $v(\epsilon_\beta) > v(\epsilon_\alpha)$, we conclude

$$v(\partial_I P(x_\beta)) \quad < \quad (|J|-|I|)v(\epsilon_\beta) + v(\partial_J P(x_\beta))$$

Reversing the above manipulations, the claim follows. ✠

If the lemma were not true, then for some non-zero multi-indices I and J and cofinal sequence of α's we would have

- $J = (i_0, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_n)$ where $I = (i_0, \ldots, i_n)$ and
- $v(g_{I,\alpha}) > 0 = v(g_{J,\alpha})$

Using the chain rule, we calculate

$$\partial_I G_\alpha(Y) = \frac{\epsilon_\alpha^{|I|}}{c_\alpha}\partial_I P(\epsilon_\alpha Y + x_\alpha) \tag{4}$$

By the expansion for G_α and the fact that $v(\partial_I P(x_\beta))$ is stable, we have

$$v(\partial_I G_\alpha(\frac{x_\beta - x_\alpha}{\epsilon_\alpha})) = v(g_{I,\alpha}) \tag{5}$$

for $\beta \geq \alpha$. So by Equation 4 with J playing the rôle of I and Equation 5, we have

$$\begin{aligned} v(\frac{\partial}{\partial X_j}\partial_I G(\frac{x_\beta - x_\alpha}{\epsilon_\alpha})) &= v(g_{J,\alpha}) \\ &= 0 \\ &< v(g_{I,\alpha}) \\ &= v(\partial_I G(\frac{x_\beta - x_\alpha}{\epsilon_\alpha})) \end{aligned}$$

for $\beta > \alpha$.

Claim 7.44. *There is another pseudo-convergent sequence $\{y_\delta\}$ having the same pseudo-limits as $\{x_\alpha\}_{\alpha<\kappa}$ but pseudo-satisfying $\partial_I P$.*

✠ We build $\{y_\delta\}$ while following along the sequence $\{x_\alpha\}$.

Start with $y_0 := x_0$. At stage α, if $y_\beta = y$ is constant for $\alpha > \beta \gg 0$ (N.B.: If α is a successor, this condition will always be true.) and $v(x_{\alpha+1} - y) \geq v(x_{\alpha+1} - x_\alpha)$, then set $y_\alpha := y$.

(N.B.: Since $\{x_\alpha\}$ is strict, cofinally we will not be in this case.)

Otherwise, since DHL applies to $\partial_I G_\alpha$ at $\frac{x_{\alpha+1}-x_\alpha}{\epsilon_\epsilon}$, by Lemma 7.36 there is some $w_\alpha \in K$ such that $\partial_I G_\alpha(w_\alpha) = 0$ and $v(w_\alpha - \frac{x_{\alpha+1}-x_\alpha}{\epsilon_\alpha}) \geq v(\partial_I G_\alpha(\frac{x_{\alpha+1}-x_\alpha}{\epsilon_\alpha})) > 0$. Define $y_\alpha := \epsilon_\alpha w_\alpha + x_\alpha$.

Let $S := \{\alpha < \kappa : y_\beta$ is **not** constant for $0 \ll \beta < \alpha\}$. S is cofinal in κ, so $\{x_\alpha\}_{\alpha\in S}$ is pseudo-convergent with the same pseudo limits as those of $\{x_\alpha\}_{\alpha<\kappa}$.

Equation 4 shows that if $\alpha \in S$, then $\partial_I P(y_\alpha) = 0$. By the construction, $v(y_\alpha - x_\alpha) \geq v(\epsilon_\alpha)$ so that by Lemma 7.24 there is a cofinal $J \subseteq S$ such that $\{y_\alpha\}_{\alpha\in J}$ is pseudo-convergent with the same pseudo-limits as those of $\{x_\alpha\}_{\alpha\in S}$ and hence of $\{x_\alpha\}_{\alpha<\kappa}$. ✠

By the inductive hypothesis since $\partial_I P \prec P$ and $\{y_\alpha\}$ is a pseudo-solution to $\partial_I P(X) = 0$, there is some b in some immediate valued D-field extension of K such that $y_\alpha \Rightarrow b$ and $\partial_I P(b) = 0$. By $\mathrm{T.deg}P$-fullness of K, $\{y_\alpha\}$ is not strict. This is a contradiction. ■

Lemma 7.45 (†)**.** *Suppose that $K\langle\langle a\rangle\rangle$ is a valued D-field extending K in which $P(a) = 0$ and $x_\alpha \Rightarrow a$. Then every rescaling of P along x_α has integral co-efficients for $\alpha \gg 0$.*

■ By Lemma 7.41 for any non-zero multi-index I we have $v(\partial_I P(a)) = v(\partial_I P(x_\alpha))$ for $\alpha \gg 0$. By Lemma 7.42 any refinement of P along x_α is residually linear. Work now only with α large enough so that the stable value of the partials has been achieved and so that the refinements are residually linear. A rescaling of P at x_α, with internal scale ϵ_α having $v(\epsilon_\alpha) = v(x_{\alpha+1} - x_\alpha)$ must have an external scale c with valuation $\min\{v(\frac{\partial}{\partial X_i}P(x_\alpha)) + v(\epsilon_\alpha)\} = \min\{v(\frac{\partial}{\partial X_i}P(a)) + v(\epsilon_\alpha)\}$. Thus, the

D-polynomial $H(Y) := \frac{1}{c}[P(a+\epsilon_\alpha Y) - P(a)] = \frac{1}{c}P(a+\epsilon_\alpha Y)$ is a refinement of P at a. In particular, $H(Y) \in \mathcal{O}_L\langle Y\rangle$. Since $v(a-x_\alpha) = v(x_{\alpha+1}-x_\alpha) = v(\epsilon_\alpha)$, the element $\frac{x_\alpha - a}{\epsilon_\alpha}$ is integral. Thus, $0 \leq v(H(\frac{x_\alpha - a}{\epsilon_\alpha})) = v(\frac{1}{c}P(x_\alpha))$. That is, the rescaling of P at x_α with scales c and ϵ_α is integral. ■

Lemma 7.46 (†). *The structure of a valued D-field on $K\langle\langle a\rangle\rangle$ is determined by $P(a) = 0$ and $x_\alpha \Rightarrow a$.*

■ When $e = 0$, this is an immediate consequence of Lemma 7.41 since every element of $K\langle\langle a\rangle\rangle$ is of the form $\frac{Q(a)}{R(a)}$ with $Q, R \ll P$.

Take now $e \neq 0$. Lemma 7.41 shows that $K(a, \ldots, D^m a)$ is an immediate extension of K. So the extension $K(a, \ldots, D^m a)/K(a, \ldots, D^{m-1}a)$ is generated by some single element b satisfying $F(b) = 0$, $\pi(b) = 0$ and $v(F'(b)) = 0$ for some irreducible $F \in \mathcal{O}_{K(a,\ldots,D^{m-1}a)}[X]$ by Proposition 1.1. Write $\sigma(F) = \prod_{i=1}^{\ell} G_i$ as a factorization into irreducibles of $\sigma(F)$ in $\mathcal{O}_{K(a,\ldots,D^m a)}[X]$. As zero is a simple root of the reduction of F, it is also a simple root of $\sigma(F)$. Hence, $\pi(G_i(0)) = 0$ for exactly one choice of i, say $i = 1$, and zero is a simple root of $\pi(G_1)$. The extension $K(a, \ldots, D^{m+1}a)/K(a, \ldots, D^m a, b)$ is then generated by $\sigma(b)$ which is the unique root of G_1 which reduces to zero. As in the proof of Lemma 7.12, the extension of D to the henselization of $\mathcal{O}_{K(a,\ldots,D^m a)}$ is now determined. ■

At this point we would like to show that this analysis may be used to produce a valued D-field extending K determined by the data $x_\alpha \Rightarrow a$ and $P(a) = 0$.

Lemma 7.47 (†). *Assume that $\{x_\alpha\}$ pseudo-satisfies P. There is an immediate extension of K of the form $K\langle\langle a\rangle\rangle$ in which $x_\alpha \Rightarrow a$ and $P(a) = 0$.*

■

We break the argument into cases depending on whether or not $m := \operatorname{ord} P$ is zero.

Consider first the case that $m = 0$. Let R be the henselization of $\mathcal{O}_K$. Let L be the field of fractions of R. By Proposition 7.32 and Lemma 7.45 all rescalings of P along $\{x_\alpha\}$ are residually linear for $\alpha \gg 0$. If $Q(Y)$ is one such rescaling at x_α with internal scale ϵ_α (having $v(\epsilon_\alpha) = v(x_{\alpha+1} - x_\alpha)$), we set $y_\beta := \frac{x_\beta - x_\alpha}{\epsilon_\alpha}$, and we find a solution $b \in R$ to $Q(b) = 0$ and $y_\beta \Rightarrow b$, then $a := x_\alpha + b\epsilon_\alpha$ is a solution to P and $x_\beta \Rightarrow a$. So, it suffices to find such a b.

Since Q is residually linear, there is a unique solution, call it b, to $Q(X) = 0$ in R with $\pi(y_\beta) = \pi(b)$. Taylor expand Q about b to see that $y_\beta \Rightarrow b$. By Lemma 7.11, the D-structure on K extends to a D-structure on L with respect to which L is a valued D-field.

Work now in the case that $m > 0$. Write $P(X) = F(X, DX, \ldots, D^m X)$ for some $F(X_0, \ldots, X_m) \in \mathcal{O}_K[X_0, \ldots, X_m]$.

We let K' be the field of fractions of $K[X_0, \ldots, X_m]/(F)$. Denote by $D^i a$ the image of X_i in K'. We define a valuation on K' by $v(G(a, Da, \ldots, D^m a)) := \lim v(G(x_\alpha, Dx_\alpha, \ldots, D^m x_\alpha))$ for $G \in K[X_0, \ldots, X_m]$ whose degree in X_m is less than that of F. By Lemma 7.41 these limits exist and one checks easily that this defines a valuation extending v on K having the same value group. In particular, if R denotes the henselization of $\mathcal{O}_{K(a,Da,\ldots,D^{m-1}a)}$ and L is its field of fractions, then K' embeds into L as a valued field.

We define $\varphi : \mathcal{O}_{K(a,Da,\dots,D^{m-1}a)} \to \mathcal{D}_e(R)$ by $\varphi(x) := (x, Dx)$ for $x \in K$ and $\varphi(D^i a) := (D^i a, D^{i+1} a)$ for $0 \leq i < m$. By the universal property of the henselization, this extends uniquely to $\psi : R \to \mathcal{D}_e(R)$. Let $D : R \to R$ be the function defined by $\psi(x) = (x, Dx)$. By Lemma 7.9, L is a valued D-field with respect to this valuation and D-structure. Moreover, $P(a) = 0$ in L and $x_\alpha \Rightarrow a$. ∎

We note now that the †-lemmata give a full proof of Proposition 7.32

∎ Via † we proceed by induction on $\mathrm{T.deg}P$. Lemma 7.46 proves uniqueness. Lemma 7.47 proves existence. Lemma 7.42 proves potential residual linearity. ∎

Proposition 7.32 is the last step needed to prove that any $(\mathbf{k}, \mathbf{G})$-D-field may be embedded into a $(\mathbf{k}, \mathbf{G})$-D-henselian field. We give the details of this in the next subsection.

To finish proving Theorem 7.1 it remains to show that for any particular $b \in M_1$ we can enlarge A_1 so that $A_1\langle\langle b\rangle\rangle$ is an immediate extension of A_1 and if $P \in A_1\langle X\rangle$ is of minimal $\mathrm{T.deg}$ such that $P(b) = 0$, then A_1 is $\mathrm{T.deg}P$-full. We show this using the following lemmata.

Lemma 7.48. *If $K \subseteq M_1$ is a sub-valued D-field whose value group is countable, then there is an unramified ∞-full extension L of K in M_1. If K has a linearly differentially closed residue field, one may take L to be an immediate extension.*

∎ Extend K to some K' first so that the residue field of K' is linearly differentially closed using Lemma 7.21. Let L be a maximal immediate extension of K' inside M_1. We show now that this L works. Take a minimal counter-example to ∞-fullness of L. That is, find some strict pseudo-convergent sequence $\{x_\alpha\}_{\alpha<\kappa}$ from L pseudo-solving some D-polynomial $P(X) \in K\langle X\rangle$ and some a in an extension valued D-field for which $P(a) = 0$, $x_\alpha \Rightarrow a$ and $\mathrm{T.deg}P$ is minimal.

Since $v(K^\times)$ is countable, we may assume that $\kappa = \omega$. Replacing a with $\frac{a}{\epsilon}$, x_α with $\frac{x_\alpha}{\epsilon}$, and P with $P(\epsilon X)$ for some $\epsilon \in K^D$ with $v(\epsilon) = v(a)$, we may assume that $v(a) = 0 = v(x_\alpha)$. By Proposition 7.32 and Lemma 7.45, by replacing P with a rescaling we may assume that P has integral coefficents and is residually linear. We note that DHL applies to P at each x_α ($\alpha > 0$): $\{x_\alpha\}$ pseudo-solves P, so $v(P(x_\alpha))$ is increasing. As everything in sight is integral, $v(P(x_\alpha)) \geq 0$. Hence, for $\alpha > 0$ we have $v(P(x_\alpha)) > v(P(x_0)) \geq 0$. As P is residually linear, there is some i such that $v(\frac{\partial}{\partial X_i} P(x_\alpha)) = 0$ or any α.

As M_1 is $\aleph_1$-saturated and satisfies DHL, the partial type $\{P(y) = 0\} \cup \{v(y - x_{n+1}) > v(y - x_n)\}_{n\in\omega}$ is realized by some element $b \in M_1$. By Proposition 7.32 again the extension $L\langle\langle b\rangle\rangle$ is immediate. As L is a maximal immediate extension of K' inside M_1, $b \in L$. This contradicts the strictness of $\{x_\alpha\}$. ∎

Lemma 7.49. *If L/K is an immediate extension of valued D-fields, L is ∞-full and has a linearly differentially closed residue field, and K has enough constants, then L may be realized as a direct limit $L = \cup_{i<|L\setminus K|} K_i$ where $K_0 = K$, $K_\lambda := \cup_{i<\lambda} K_i$ for λ a limit, and $K_{i+1} = K_i\langle\langle a_i\rangle\rangle$ where K_i is $\mathrm{T.deg}P$-full for $P(X) \in K_i\langle X\rangle$ a minimal D-polynomial for a_i over K_i.*

■ Well-order $L \setminus K$ in order type $|L \setminus K|$. Build K_i inductively. Set $K_0 := K$. For $i = \lambda$, a limit ordinal, let $K_\lambda := \cup_{i<\lambda} K_i$. At stage $i+1$, let $\mathbf{N}$ be $\min\{\mathrm{T.deg}(P) : P(X) \in K_i\langle X\rangle, P(a) = 0$ for some $a \in L \setminus K_i\}$. Let a_i be minimal in the well-ordering of $L \setminus K$ such that the $\mathrm{T.deg}P = \mathbf{N}$ for P a minimal D-polynomial for a_i over K_i.

Claim 7.50. *K_i is $\mathbf{N}$-full.*

✠ Consider a minimal counter-example to $\mathbf{N}$-fullness:

- a strict pseudo-convergent sequence $\{x_\alpha\}_{\alpha<\kappa}$ from K_i,
- a in some immediate extension of K_i and
- $Q(X) \in \mathcal{O}_{K_i}\langle X\rangle$

satisfying

- $\mathrm{T.deg}Q < \mathbf{N}$,
- $Q(a) = 0$,
- $\{x_\alpha\}$ pseudo-solves Q,
- $x_\alpha \Rightarrow a$, and
- $\mathrm{T.deg}Q$ is minimal subject to these conditions.

By Proposition 7.32 and ∞-fullness of L, $\{x_\alpha\}$ cannot continue to be strict in L. Let $b \in L$ such that $x_\alpha \Rightarrow b$. Since Q was chosen to have minimal $\mathrm{T.deg}$ among those D-polynomials witnessing the failure of $\mathbf{N}$-fullness, K_i is $\mathrm{T.deg}Q$-full. Thus, Proposition 7.32 shows that Q is potentially residually linear along $\{x_\alpha\}$. As $\{x_\alpha\}$ pseudo-solves Q, $v(Q(x_\alpha))$ is eventually increasing. Fix α large enough so that the valuations of the partials have stabilized, the refinements are residually linear, and for $\gamma > \beta > \alpha$ we have $v(Q(x_\gamma)) > v(Q(x_\beta))$ or $Q(x_\beta) = 0$. Let $A := \min\{v(\frac{\partial}{\partial X_i}Q(a))\}$.

Let $\beta > \alpha$. Taylor expand Q around x_β to conclude that $v(Q(b)) \geq \min\{v(Q(x_\beta), A + v(\epsilon_\beta)\}$ where as usual ϵ_β is a D-constant with $v(\epsilon_\beta) = v(x_{\beta+1} - x_\beta)$. If $v(Q(b)) < A + v(\epsilon_\beta)$, then we have $v(Q(b)) = v(Q(x_\beta))$. This can happen at most once since either $v(Q(x_\gamma)) > v(Q(x_\beta))$ or $Q(x_\gamma) = 0$ for $\gamma > \beta$ and our calculation used only the hypothesis that $\beta > \alpha$. Thus, $v(Q(b)) > A + v(\epsilon_\beta)$ for any $\beta > \alpha$. It follows that if $H(Y) = \frac{1}{c}Q(x_\alpha + \epsilon_\alpha Y)$ is a rescaling of Q at x_α, then $v(H(\frac{b-x_\alpha}{\epsilon_\alpha})) \geq v(\epsilon_\beta) - v(\epsilon_\alpha) > 0$ for every $\beta > \alpha$. By Lemma 7.36 there is some $b' \in L$ with $H(b') = 0$ and $v(b' - \frac{b-x_\alpha}{\epsilon_\alpha}) = v(H(\frac{b-x_\alpha}{\epsilon_\alpha}))$. Let $d := x_\alpha + \epsilon_\alpha b'$. Then $Q(d) = 0$ and $x_\beta \Rightarrow d$. The $\mathrm{T.deg}Q$-fullness of K_i contradicts the strictness of $\{x_\alpha\}$. ✠

Set $K_{i+1} := K_i\langle\langle a_i\rangle\rangle$. ■

We use these lemmata to finish the extension of f to b.

Proposition 7.51. *If A_1 has a countable value group, has a linearly differentially closed residue field and has enough constants and $A_1\langle\langle b\rangle\rangle$ is an immediate extension of A_1, then f extends so that b is in the domain of f.*

■ Let L be an ∞-full immediate extension of $A_1\langle\langle b\rangle\rangle$ produced by Lemma 7.48. We will extend f to all of L. Express $L = \bigcup_{i<|L\setminus A_1|} K_i$ as in Lemma 7.49. In extending f, the only issue comes at successor stages. At stage $i+1$, we need to define $f(a_i)$. Since K_i is $\mathrm{T.deg}P$-full where P is a minimal D-polynomial for a_i over K_i and K_i has enough constants and linearly differentially closed residue field, Proposition 7.32 implies that the extension $K_i\langle\!\langle a_i\rangle\!\rangle$ is entirely determined by a strict pseudo-convergent approximation $\{x_\alpha\}$ to a_i (which can be taken to be countable) and the equation $P(a_i) = 0$. By Proposition 7.32 and Lemma 7.45, any rescaling of P along x_α for $\alpha \gg 0$ is integral and residually linear, so that DHL applies to the rescaling. Thus, by DHL in M_2 and $\aleph_1$-saturation, there is some $b_i \in M_2$ satisfying $f(P)(b_i) = 0$ and $f(x_\alpha) \Rightarrow b_i$. Extend f by defining $f(a_i) = b_i$. ■

This completes the proof of completeness and quantifier elimination, though we recap the argument in the next subsection.

7.3. **Recap.** With the necessary lemmata now proven, let us go through the proof of Theorem 7.1 again.

Let us first prove quantifier elimination for the theory of $(\mathbf{k}, \mathbf{G})$-D-henselian fields.

We fix a countable elementary substructure $N_1 \prec M_1$ of M_1 containing A_1 and b. Using Proposition 7.20 we extend f so that its domain has enough constants and has value group equal to the value group of N_1. We then use Proposition 7.21 to extend f so that its domain has residue field containing k_{N_1}. Finally, we use Proposition 7.51 to extend f to an ∞-full immediate extension of N_1. In particular, the extension will be defined at b.

As to proving that the theory of $(\mathbf{k}, \mathbf{G})$-D-henselian fields is the model completion of the theory of $(\mathbf{k}, \mathbf{G})$-D-fields we use the existence parts of the lemmata cited above to show that any $(\mathbf{k}, \mathbf{G})$-D-field may be embedded into a $(\mathbf{k}, \mathbf{G})$-D-henselian field.

Let K be a $(\mathbf{k}, \mathbf{G})$-D-field. We use the existence part of Proposition 7.19 to prove that K may be enlarged so as to have enough constants. We use Lemma 7.17 to enlarge K to satisfy Axiom 5 and so that its value group is a model of $\mathrm{Th}(\mathbf{G})$. We use the existence part of Lemma 7.12 to enlarge K so that its residue field is a model of $\mathrm{Th}(\mathbf{k})$. Using Lemma 7.25 we produce the necessary pseudo-convergent sequence in K and then use Proposition 7.32 to actually find a solution in an immeditate valued D-field extension.

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