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A MODEL FOR THE FORMATION OF A SPHERICAL GALAXY

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SUMMARY

Numerical calculations have been made for a model representing the collapse of an initially gaseous proto-galaxy and the concurrent transformation of gas into stars. The assumed turbulent motions of the gas are represented by a simple model consisting of discrete colliding clouds, and the star formation rate is assumed to be given as a simple function of the density and turbulent velocity of the gas. The gas clouds and the stars are then treated separately by means of fluid-dynamical equations derived from the Boltzmann equation. It is found that, by assuming reasonable values for the various parameters of the model, it is possible in this way to reproduce reasonably well the observed properties of spherical and nearly spherical galaxies.

1. INTRODUCTION

It is generally believed that galaxies form as the result of the gravitational instability and collapse of condensations in an expanding universe (Field 1969). Evidence that a collapse occurred during the early history of our own galaxy has been provided by the work of Eggen, Lynden-Bell, & Sandage (1962). In the present investigation we adopt the basic picture of formation of galaxies by collapse, and we consider the problem of treating this collapse numerically. A number of calculations of the dynamical collapse of a spherical gas cloud have recently been made in connection with the problem of star formation (see for example Larson 1968, 1969); consequently it is of interest to see whether similar techniques might profitably be applied to the more difficult problem of galaxy formation.

In this investigation we shall make the fundamental assumption, following von Weizsaecker (1951), that the gaseous material in a proto-galaxy is in a highly turbulent, non-uniform state. This assumption is motivated by the idea that a chaotic and turbulent state for the pre-galactic material may in fact be a more ' natural ' one than the quiescent, nearly uniform state which has more commonly been assumed (as, for example, by Field 1969). This idea is supported by the recent work of Saslaw (1968), who concludes that, in any self-gravitating medium, those regions which are not already in a state of dynamical collapse will in general be characterized by large density fluctuations on approximately the Jeans scale, which are themselves just on the verge of gravitational collapse. If we assume a temperature of 10^4°K for the gas in a proto-galaxy (see Section 2), the Jeans mass at the relevant densities falls in the range of approximately $10^9 M_\odot$ to $10^7 M_\odot$; thus a proto-galaxy of $10^{11} M_\odot$ should contain a large number of these condensations or ' turbulent elements '.

Observationally, the existence of turbulence during the formative phases of our own galaxy appears to be indicated by the very large velocity dispersion of the extreme Population II stars, and by the fact that a significant fraction of these stars and some globular clusters are known to have retrograde orbits (Oort 1965; Woolley 1966). If we take the velocity dispersion of the Population II stars to be indicative of the velocity dispersion of the gas from which they formed, the turbulent velocities must have been of the order of 100 km s^{-1}, which for a gas temperature of 10^4 °K is highly supersonic (Mach number ~ 10).

A theory of star formation based on the idea of supersonic turbulence and bearing some points of similarity to the present picture of galaxy formation has been described by McCrea (1960).

2. A MODEL FOR THE TURBULENCE

We have represented the turbulent flow field by an idealized model in which the gas is assumed to be entirely concentrated into discrete spherical clouds of unifom internal density. We assume that in any locality the clouds all have the same properties, and that they occupy some fixed fraction f of the volume available; the cloud density ρ_c is then related to the local mean gas density ρ by $\rho_c = \rho/f$. We have for the most part adopted $f = 0{\cdot}1$; this is approximately the value appropriate for the interstellar clouds in the solar vicinity (Allen 1963).

In the dynamical calculations it will be important to specify the rate at which the turbulent motions are dissipated in collisions between the clouds. In calculating the dissipation rate we have assumed that the clouds have an isotropic Maxwellian velocity distribution, and that the collisions are completely inelastic, i.e. that whenever two clouds collide they are brought completely to rest in the centre of mass frame. In the following we shall denote by N the number density of clouds, by σ the collision cross section, and by α the variance of the velocity distribution (i.e., the mean squared random velocity) in any one coordinate direction. It can be shown, by integrating over the velocity distributions of the two clouds involved in a collision, that the number of collisions per unit time experienced by any cloud is given by

$$\text{collision rate} = 4\pi^{-1/2} N\sigma\alpha^{1/2} \tag{1}$$

and that the rate of decrease of α due to collisions is given by

$$\frac{d\alpha}{dt} = -\frac{8}{3}\pi^{-1/2} N\sigma\alpha^{3/2}. \tag{2}$$

If we equate σ to the geometrical cross section πR^2 of a cloud, then $N\sigma$ is related to the cloud mass M and the mean gas density ρ by

$$N\sigma = \pi\left(\frac{3f}{4\pi}\right)^{2/3}\left(\frac{\rho}{M}\right)^{1/3}. \tag{3}$$

In order to estimate the cloud mass M, we have assumed that the size of the clouds is given approximately by the Jeans length corresponding to the cloud density ρ_c and temperature T. More specifically, we have used the formula adopted by Larson (1969) for the radius of a cloud about to begin gravitational collapse:

$$R = 0{\cdot}41\,\frac{GM}{\mathcal{R}T} \tag{4}$$

where G is the gravitational constant and $\mathcal{R}$ is the gas constant. It then remains to estimate the cloud temperature T.

The cloud temperature will be determined through the effects of heating of the clouds in collisions, and cooling by the emission of radiation. We assume for simplicity that the proto-galactic material consists of pure hydrogen; then, as was argued by Hoyle (1953), the temperature is expected to be maintained most of the time near 10^4 °K, since at higher temperatures the material is ionized and cools rapidly by the emission of radiation, whereas at lower temperatures the material recombines and ceases to radiate. If heavier elements are present and contribute significantly to the cooling, the temperature would of course become lower than 10^4 °K, but a similar type of argument may still be applicable, since each cooling mechanism tends to produce a characteristic temperature. We have made some allowance for this possibility, as well as for other uncertainties in the model, by trying different values for some of the important parameters.

The assumption of 10^4 °K is of course valid only as long as the ionized hydrogen remains optically thin. Examination of the results shows that this is, in fact, always the case in the present calculations.

Making use of equations (3) and (4), we can now write equations (1) and (2) as follows:

$$\text{collision rate} = C\rho^{1/2}\alpha^{1/2} \tag{5}$$

$$\frac{d\alpha}{dt} = -\tfrac{2}{3}C\rho^{1/2}\alpha^{3/2} \tag{6}$$

where

$$C = 2\left(3f\frac{0\cdot 41G}{\mathscr{R}T}\right)^{1/2}. \tag{7}$$

Taking $f = 0\cdot 1$ and $T = 10^4$ °K, and assuming a composition of pure ionized hydrogen, we obtain $C = 1\cdot 3 \times 10^{-10}$ (cgs).

3. ASSUMPTIONS CONCERNING STAR FORMATION

In the dynamical calculations it will be necessary to specify both the rate at which the gas is transformed into stars and the velocity distribution of the stars at formation. We have assumed the velocity distribution of the stars at formation to be the same as that of the gas clouds discussed in the previous section. Unfortunately, the physics of fragmentation and star formation in a turbulent medium is insufficiently understood to provide any quantitative estimate of the star formation rate; therefore, we shall consider only the possible form of the star formation rate, leaving the free parameters to be determined, if possible, by fitting the results to observational data.

In a study of the evolution of the stars and gas in the solar neighbourhood, Schmidt (1959) assumed the star formation rate per unit volume to be proportional to some power n of the gas density, and he found that various observational data appeared to favour a value for n of about 2. In a later paper (Schmidt 1963) allowance was made for a star formation rate varying with mass, and it was concluded that the bulk of the stars (i.e., those of small or moderate mass) appear to have formed at a rate characterized by a somewhat smaller value of n than 2.

Field & Saslaw (1965) and Field & Hutchins (1968) have discussed a model for star formation in which interstellar clouds are assumed to collide and coalesce until a massive gravitationally unstable cloud is formed, which then collapses and forms stars. With the assumption that the cloud properties are independent of the mean gas density, this model predicts a star formation rate proportional to the square of the gas density. However, the details of this model are rather different from what we have imagined to be the case in a protogalaxy (Section 2), so this result may not be directly relevant to the present problem.

In seeking a physical basis for the star formation rate which is consistent with the turbulence model described in Section 2, one possibility that may be imagined is that star formation is connected with the collisions between the clouds, and may in fact be triggered by the compression produced when two clouds collide. If we assume that in each collision a certain fraction of the total mass involved goes into stars, the star formation rate per unit mass of gas will be proportional to the collision rate as given by equation (5), and the star formation rate per unit volume will be given by

$$\frac{d\rho_s}{dt} \propto C\rho_g{}^{3/2}\alpha^{1/2}, \tag{8}$$

where ρ_s is the density of stars and ρ_g is the density of the gas.

Equation (8) was in fact the first form tried for the star formation rate; however, for reasons which will be explained later, it did not lead to a satisfactory model for a spherical galaxy. We have, therefore, allowed the star formation rate to be a more general function of ρ_g and α, having the form

$$\frac{d\rho_s}{dt} = A\rho_g{}^p\alpha^q, \tag{9}$$

where the values of most interest for the parameters p and q are given approximately by $1.5 \lesssim p \lesssim 2.0$ and $0 \lesssim q \lesssim 0.5$, and A is left as a completely free parameter.

We shall neglect any effects of ejection of material from evolving stars, except insofar as this may be accounted for by a decrease in the effective star formation rate. Also, we neglect any feedback of energy from the stars to the gas, caused by such effects as expansion of H II regions, supernova explosions, etc.

4. THE FLUID-DYNAMICAL EQUATIONS

In deriving the equations describing the dynamical evolution of the system, we have adopted a fluid-dynamical approach for both the gas clouds and the stars; that is, the gas clouds and the stars are considered as the basic particles of a two-component fluid, and fluid-dynamical equations are derived from the Boltzmann equation for the gaseous and stellar components individually. Since we are considering a system with spherical symmetry, we take as the coordinates in the six-dimensional phase space the spherical polar coordinates r, θ, and ϕ, and the corresponding velocity components u, v, and w. The density of particles in phase space will then be described by functions of the form $f(r, u, v, w, t)$ for both the gas and the stars. The Boltzmann equation, which is essentially just an equation of continuity for f, may then be written

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial r} + \dot{u}\frac{\partial f}{\partial u} + \dot{v}\frac{\partial f}{\partial v} + \dot{w}\frac{\partial f}{\partial w} = \left(\frac{\partial f}{\partial t}\right)^* \tag{10}$$

where

$$\left.\begin{aligned} \dot{u} &= -\frac{\partial \Phi}{\partial r} + \frac{v^2+w^2}{r} \\ \dot{v} &= -\frac{uv}{r} + \frac{w^2}{r\tan\theta} \\ \dot{w} &= -\frac{uw}{r} - \frac{vw}{r\tan\theta} \end{aligned}\right\}. \tag{11}$$

In these equations Φ is the gravitational potential of the proto-galaxy, and $(\partial f/\partial t)^*$ denotes the rate of change of f due to star formation.

We now derive the equations for the moments of the velocity distribution, considering moments up to the second order. We consider first the gas component, which we assume to have an isotropic velocity distribution with

$$\langle (u-\langle u\rangle)^2\rangle = \langle v^2\rangle = \langle w^2\rangle = \alpha,$$

where the mean value of any quantity Q is defined by

$$\rho\langle Q\rangle = \iiint fQ \, du \, dv \, dw.$$

For an isotropic velocity distribution, the third order moments vanish, so that the moment equations may be terminated at the second order. We assume that the velocity distribution of the gas which is transformed into stars is the same as for the rest of the gas; then

$$\iiint Q\left(\frac{\partial f}{\partial t}\right)^* du \, dv \, dw = \langle Q\rangle\left(\frac{\partial \rho}{\partial t}\right)^*,$$

where

$$\left(\frac{\partial \rho}{\partial t}\right)^* = -A\rho^p\alpha^q$$

is the rate of change of the gas density due to star formation. Multiplying equation (10) successively by 1, u, and $(u^2+v^2+w^2)$, and integrating over all velocities, we now obtain, after some reductions, the following equations:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho\langle u\rangle) = \left(\frac{\partial \rho}{\partial t}\right)^* \quad (12)$$

$$\frac{\partial\langle u\rangle}{\partial t}+\langle u\rangle\frac{\partial\langle u\rangle}{\partial r}+\frac{\partial\Phi}{\partial r}+\frac{1}{\rho}\frac{\partial}{\partial r}(\rho\alpha) = 0 \quad (13)$$

$$\frac{\partial\alpha}{\partial t}+\langle u\rangle\frac{\partial\alpha}{\partial r}+\frac{2}{3}\frac{\alpha}{r^2}\frac{\partial}{\partial r}(r^2\langle u\rangle) = \left(\frac{\partial\alpha}{\partial t}\right)_{\text{coll}}. \quad (14)$$

In equation (14) the term

$$\left(\frac{\partial\alpha}{\partial t}\right)_{\text{coll}} = -\tfrac{2}{3}C\rho^{1/2}\alpha^{3/2}$$

has been inserted on the right-hand side to provide for the collisional dissipation of the turbulent motions as given by equation (6). We note that equations (12), (13), and (14) are essentially just the fluid-dynamical equations for the conservation of mass, momentum, and energy, respectively.

We now consider the moment equations for the stellar component of the proto-galaxy. In the case of the stars, we have allowed for different velocity dispersions in the radial and transverse directions, and we write

$$\langle (u-\langle u\rangle)^2\rangle = \alpha, \quad \langle v^2\rangle = \langle w^2\rangle = \beta.$$

We assume that the stars are formed with the same velocity distribution as the gas, and we also assume that the distribution of the random velocities of the stars is symmetric with respect to the coordinate directions, so that the third order moments vanish. Multiplying equation (10) successively by 1, u, u^2, and v^2 (or

w^2), and integrating over all velocities, we then obtain

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho\langle u\rangle) = \left(\frac{\partial \rho}{\partial t}\right)^* \tag{15}$$

$$\frac{\partial\langle u\rangle}{\partial t}+\langle u\rangle\frac{\partial\langle u\rangle}{\partial r}+\frac{\partial\Phi}{\partial r}+\frac{1}{\rho}\frac{\partial}{\partial r}(\rho\alpha)+\frac{2}{r}(\alpha-\beta) = \frac{1}{\rho}(\langle u_g\rangle-\langle u\rangle)\left(\frac{\partial \rho}{\partial t}\right)^* \tag{16}$$

$$\frac{\partial\alpha}{\partial t}+\langle u\rangle\frac{\partial\alpha}{\partial r}+2\alpha\frac{\partial\langle u\rangle}{\partial r} = \frac{1}{\rho}[(\langle u_g\rangle-\langle u\rangle)^2+(\alpha_g-\alpha)]\left(\frac{\partial \rho}{\partial t}\right)^* \tag{17}$$

$$\frac{\partial\beta}{\partial t}+\langle u\rangle\frac{\partial\beta}{\partial r}+2\frac{\langle u\rangle\beta}{r} = \frac{1}{\rho}(\beta_g-\beta)\left(\frac{\partial \rho}{\partial t}\right)^* \tag{18}$$

where

$$\left(\frac{\partial \rho}{\partial t}\right)^* = A\rho_g{}^p\alpha_g{}^q$$

is the rate of increase of the star density due to star formation. In these equations the subscript g denotes the properties of the gas.

Equations (12)–(18) now provide all the equations required to compute the dynamical evolution of a spherical proto-galaxy, given suitable initial and boundary conditions. The numerical techniques which we have employed for solving these equations are similar to those which were used by Larson (1968, 1969) in calculating the collapse of a proto-star, and they are outlined in the Appendix. The calculations have also been made with a similar grid structure and a similar level of accuracy, i.e. of the order of 20 per cent or so.

The effects of neglecting the third order moments of the stellar velocity distribution have been alleviated somewhat in a calculation to be described in Section 8, in which the stars have been divided into two ‘ populations ’ with independent density and velocity distributions, each described by its own set of moment equations.

5. INITIAL AND BOUNDARY CONDITIONS

In the present calculations we have found it convenient to take the fundamental units of mass, length, and time to be 1 $M_\odot$, 1 pc, and 10^6 yr respectively. In this system the unit of velocity is 1 pc/10^6 yr = 0·978 km s^{-1}, and the unit of density is 1 $M_\odot$/pc^3 = $6{\cdot}77\times10^{-23}$ g cm^{-3}. The value of the gravitational constant G becomes $4{\cdot}50\times10^{-3}$, and the value of the constant C derived in section 2 becomes $3{\cdot}3\times10^{-3}$.

In most cases we have adopted the simplest possible assumptions for the initial and boundary conditions. The collapsing proto-galaxy is assumed to start from rest with a uniform density distribution, and the outer boundary is assumed to remain fixed in space. A mass of 10^{11} $M_\odot$ has been assumed, and the radius of the fixed outer boundary has in most cases been taken equal to 5×10^4 pc; the initial density is then $1{\cdot}9\times10^{-4}$ $M_\odot$/pc^3 ($1{\cdot}3\times10^{-26}$ g cm^{-3}), and the corresponding free-fall time is $5{\cdot}9\times10^8$ yr. Calculations have also been made assuming a radius of 10^5 pc, corresponding to an initial density of $2{\cdot}4\times10^{-5}$ $M_\odot$/pc^3. The initial value of α for the gas has in most cases been obtained from equation (4) with $\mathcal{R}T$ replaced by α; the resulting value for α is approximately the maximum for which the proto-galaxy is initially gravitationally bound. With a radius of 5×10^4 pc this gives $\alpha = 3{\cdot}7\times10^3$ in the adopted units, corresponding to an initial velocity dispersion of 59 km s^{-1}. A calculation has also been made with the initial value of α reduced by one order of magnitude to $3 \cdot 7 \times 10^2$; as will be seen (Section 8), this makes no large difference to the final result.

The assumption of a fixed boundary may be a reasonable one if we are considering galaxy formation in a cluster of galaxies; however, for field galaxies, it would presumably be more realistic to assume a boundary expanding with the expansion of the universe. We have, therefore, made some calculations in which the boundary is assumed to expand with time according to $R = R_0 t^{2/3}$. If the mean density within radius R is just equal to the density required to close the universe, we have $R_0 = (4 \cdot 5\, GM)^{1/3}$, which for a mass $M = 10^{11}\, M_\odot$ gives $R_0 = 1 \cdot 27 \times 10^3$ in the adopted units. In order to get the material to stop expanding and condense into a galaxy within a reasonable length of time, however, it is necessary to use a somewhat smaller value for R_0; for purposes of illustration, we have adopted $R_0 = 9 \times 10^2$, corresponding to a mean density of about 2·8 times the density required to close the universe. The proto-galaxy has again been assumed to start with a uniform density distribution, and also to be initially in a state of uniform expansion.

6. RESULTS

We describe first the results obtained assuming a radius of 5×10^4 pc and an initial value for α of $3 \cdot 7 \times 10^3$. After a number of trial calculations, it was found that a reasonable model for a spherical galaxy could be produced with the following simple choice for the parameters A, p, and q specifying the star formation rate (equation 9): $A = 1 \cdot 0$, $p = 2$, and $q = 0$. We shall describe first the results obtained with this choice of parameters, and we shall leave to Section 8 a consideration of the effects of varying the parameters.

During the initial stages of the collapse, the density and the star formation rate are so low that only a small amount of material is transformed into stars. As is generally the case in spherical collapse problems, the density distribution, although initially uniform, soon becomes sharply peaked at the centre, and the central density peak continues to develop at an ever accelerating rate as the density increases and the collapse time decreases. With the present choice of parameters, it happens that the rate of dissipation of the turbulent motions is nearly balanced by the rate at which energy is fed into the turbulence by the collapse; consequently the 'temperature' parameter α for the turbulent gas remains nearly constant. The density distribution, therefore, assumes approximately the form characteristic for an isothermal collapse (Bodenheimer & Sweigart 1968; Larson 1969).

As the central density rises, the star formation rate increases rapidly, as does the central density of stars. The star density overtakes the gas density at the centre after about 8×10^8 yrs, at which time the central density for both stars and gas is about $2 \times 10^{-2}\, M_\odot/\text{pc}^3$. After this point the gravitational field in the central part of the proto-galaxy is dominated by the stars, and becomes considerably greater than the gravitational field produced by the gas alone. As a result the gravitational energy input to the turbulent motions begins to exceed the dissipation rate, and the velocity dispersion of the gas increases substantially.

As a result of the increase in the 'temperature' of the turbulent gas, the pressure due to the random turbulent motions becomes increasingly important in relation to the gravitational forces; the collapse in the central part of the proto-galaxy then becomes substantially slowed down from a free fall, and the gas in this region begins to approach hydrostatic equilibrium. An approximate ' steady state ' is then achieved, in which the gas is converted into stars almost as fast as it flows into the central region, and the gas density increases only slowly in comparison with the dynamical time scale. Meanwhile, the density distribution of the stars continues to become more sharply peaked at the centre, since star formation takes place preferentially at the centre, where the gas density is highest. The majority of the stars are formed during this phase of the collapse.

Throughout the collapse, the velocity dispersion of the stars remains nearly the same as for the gas, particularly near the centre where the star formation rate is highest and the bulk of the stars have only recently formed from the gas. However, since there is no dissipation of the random motions of the stars and no ' sink ' of stars at the centre of the proto-galaxy, the mean inward motion of the stars is much smaller than for the gas, and negligible in comparison with the free-fall velocity. In fact, except for the outermost regions, the stellar component of the protogalaxy is quite close to being in hydrostatic equilibrium throughout the major phases of star formation.

After about $1 \cdot 7 \times 10^9$ yrs, the gas density at the centre reaches a maximum value of $0 \cdot 9\ M_\odot/\text{pc}^3$, and then begins to decrease. At this time about 97 per cent of the original proto-galactic material has been converted into stars. The calculations were continued until about $2 \cdot 5 \times 10^9$ yrs after the start of the collapse, at which time 99·6 per cent of the material had been transformed into stars. The calculations were stopped at this point because of numerical difficulties, but other similar calculations have shown that no very large changes can be expected to occur after this time. (The central density of stars, for example, may increase by a further 30–40 per cent, but the density distribution is otherwise hardly altered.)

The density distributions for the stars and gas after $2 \cdot 5 \times 10^9$ yrs are illustrated in Fig. 1. The central stellar density at this time is about $1 \cdot 3 \times 10^3\ M_\odot/\text{pc}^3$, and the radius at which the density drops to half its central value is about 65 pc; approximately 1 per cent of the total mass is enclosed within this radius. The radius of a

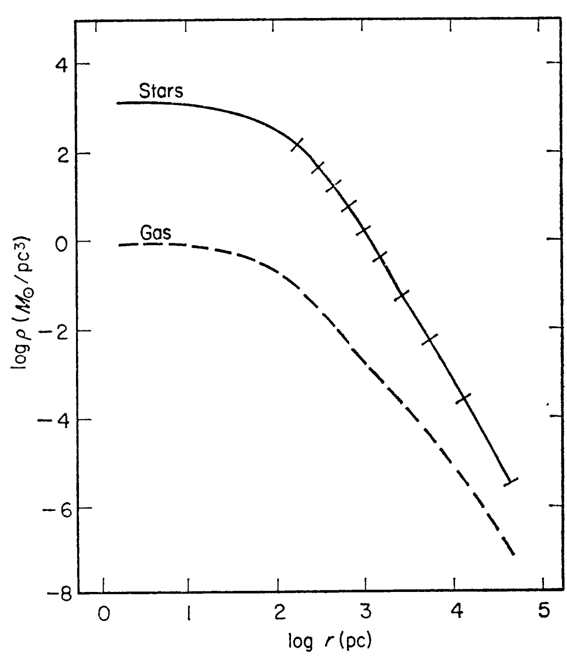

FIG. 1. *The density distributions for the stars and gas after* $2 \cdot 5 \times 10^9$ yrs. *The ticks along the curve divide the galaxy into ten zones of equal mass.*

sphere enclosing half of the total mass is about $1{\cdot}1 \times 10^3$ pc. We note in Fig. 1 that the plot of $\log \rho$ vs $\log r$ for the stars is nearly linear over a wide range in r; the average slope of the nearly linear section is about $-3{\cdot}4$, corresponding to a density distribution approximately of the form $\rho \propto r^{-3\cdot 4}$.

The radial and transverse components of the stellar velocity dispersion (i.e., $\alpha^{1/2}$ and $\beta^{1/2}$ respectively) have been plotted vs $\log r$ in Fig. 2. We see from this figure that $\alpha^{1/2}$ is equal to about 290 km s^{-1} at the centre and reaches a maximum value of about 330 km s^{-1} at $r \simeq 170$ pc, decreasing to much smaller values in the outermost part of the galaxy. It is interesting to note also that the velocity distribution is nearly isotropic over a large part of the galaxy. This result may be understood from the fact that the bulk of the stellar component of the proto-galaxy has been in a near-equilibrium configuration from the time of its formation, and has evolved not by any general collapse but just by the addition of more stars at the centre; the nearly isotropic velocity distribution of the stars is then a simple consequence of the assumed isotropic velocity distribution of the gas.

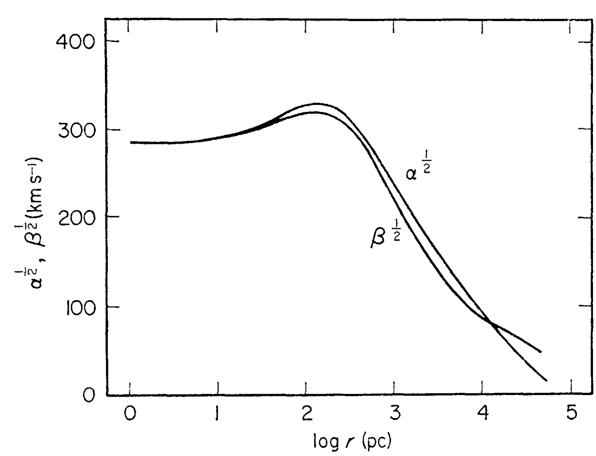

FIG. 2. *The radial component $\alpha^{1/2}$ and the transverse component $\beta^{1/2}$ of the stellar velocity dispersion after $2{\cdot}5 \times 10^9$ yrs.*

We note that our result that a spherical galaxy is formed nearly in equilibrium is inconsistent with a commonly-made assumption (see e.g. Lynden-Bell 1967), namely that a stellar system is far out of equilibrium at the time of its formation, and subsequently undergoes a violent collapse and ‘ relaxation ’ process. The present results show that, at least in the model considered here, the ‘ relaxation ’ associated with the formation of a spherical galaxy occurs almost entirely during the gaseous phase of its evolution.

We note parenthetically that during the final stages of the formation of the galaxy, the amount of gas in the dense nucleus of the galaxy becomes comparable to or somewhat less than the mass of one of the clouds of the turbulence model discussed in Section 2. Under these circumstances, the adopted turbulence model clearly begins to break down, and equation (6) for the turbulent dissipation rate begins to lose any physical basis that it may have had.

7. COMPARISON WITH OBSERVATIONS

It is of interest to compare the final density distribution found in the present calculations with the observed light intensity distributions in spherical and nearly spherical galaxies. We shall be interested mainly in the region well away from the centre, where the calculated density distribution has approximately the form $\rho \propto r^{-n}$. Assuming a constant mass-to-light ratio, the corresponding predicted intensity distribution is of the form $I \propto r^{-n+1}$. With $n \simeq 3{\cdot}4$, as obtained from the results of Section 6, the predicted intensity distribution would then have approximately the form $I \propto r^{-2{\cdot}4}$.

The empirical formulas given by Hubble (1930) and Baum (1955) for I(r) in spherical and elliptical galaxies both approach the asymptotic form $\mathrm{I} \propto r^{-2}$ at large radii. However, this formula implies a logarithmically divergent mass for the galaxy, whereas, the bulk of more recent data (e.g. Liller 1960) seem to favour a finite mass and a steeper slope for I(r) at large radii. Abell & Mihalas (1966) have adopted $\mathrm{I} \propto r^{-3}$ as a better representation of the data at large r. The detailed photometry of NGC 3379 by Miller & Prendergast (1962) gives a logarithmic intensity gradient varying between about $-1{\cdot}6$ and $-2{\cdot}5$, the magnitude of the slope increasing outwards from the centre. Rood & Baum (1968) give logarithmic intensity gradients for some elliptical galaxies in the Coma cluster; these gradients vary considerably from case to case, but for the most part they fall in the range between about $-2{\cdot}0$ and $-3{\cdot}3$, the slope again tending to increase outwards from the centre.

From the above we conclude that, while one can probably not expect accurate quantitative agreement between the theory and the observations in any particular case, in general the agreement may be considered qualitatively satisfactory if the predicted density distribution is of the form $\rho \propto r^{-n}$ with n somewhere between 3 and 4. Thus it appears that the value $n \simeq 3{\cdot}4$ obtained in Section 6 is in satisfactory agreement with the observations. In Section 8 we shall indicate how n depends on the various parameters of the model, and what other choices of parameters also give acceptable values of n.

It is also of interest to compare the central density found from the calculations with the values inferred from observations. Miller & Prendergast (1962) quote a value of $4{\cdot}4 \times 10^2\ M_\odot/\mathrm{pc}^3$ for the central density of NGC 3379. It is known, however (Spitzer & Saslaw 1966), that some galaxies have very much higher central densities —up to $\sim 10^6\ M_\odot/\mathrm{pc}^3$ and possibly higher. Consequently the central density of $1{\cdot}3 \times 10^3\ M_\odot/\mathrm{pc}^3$ obtained in Section 6 appears to be well within the range of observed central densities in elliptical galaxies.

Finally, we consider the comparison of predicted and observed velocity dispersions. Minkowski (1962) has measured velocity dispersions in the central regions of a number of elliptical galaxies, and has obtained values ranging from 100 km s^{-1} to 500 km s^{-1}. The predicted central velocity dispersion of about 300 km s^{-1} (Fig. 2) appears to be an entirely typical value for elliptical galaxies, and is, therefore, again in satisfactory agreement with the observations.

8. RESULTS OBTAINED WITH OTHER ASSUMPTIONS

8.1. *The star formation rate*

We consider first the effect of changing the star formation rate. It is found that when the constant coefficient A in equation (9) is varied (within reasonable limits), there is little change in the *form* of the resulting density distribution; i.e., the plot of log ρ *vs.* log r retains very nearly the same shape as in Fig. 1. The central density is, however, quite sensitive to the value of A, in the sense that a smaller value of A

results in a higher central density. This occurs because when the star formation rate is reduced, the central condensation of the gas is able to proceed to a more extreme degree before the gas becomes exhausted. As an example, we have made a calculation with the star formation rate reduced by a factor of 2·5 (i.e., with A equal to 0·4); the resulting central density is then increased by 3 orders of magnitude to $1{\cdot}4 \times 10^6\, M_\odot/\mathrm{pc}^3$. The radius at which the density drops to half its central value is in this case about 6 pc, and the velocity dispersion in the central part of the galaxy is about 900 km s^{-1}.

Thus, it appears that it may be possible to interpret the wide range of observed central densities in spherical and elliptical galaxies as being due to relatively small differences in the star formation rate. It is interesting to note in particular that, if star formation is inhibited for some reason, it may be possible to obtain extremely high central densities in stellar systems with only a relatively moderate decrease in the star formation rate (provided, of course, that spherical symmetry may always be assumed).

As might be expected, the central density is also rather sensitive to some of the other parameters of the model. In experimenting with these other parameters, our practice has usually been to readjust the value of A so as to bring the resulting central density back to a ' standard ' value of roughly $10^3\, M_\odot/\mathrm{pc}^3$.

According to the discussion of Section 3, the values of the parameters p and q in equation (9) are expected to be given approximately by $1{\cdot}5 \lesssim p \lesssim 2{\cdot}0$ and $0 \lesssim q \lesssim 0{\cdot}5$. We have made calculations for $p = 2{\cdot}0$, 1·8, and 1·7, with $q = 0$ and $q = 0{\cdot}5$ in each case. It is found that in all cases the resulting density distribution may again be represented approximately by $\rho \propto r^{-n}$ over a wide range in r. The values of n obtained for the various choices of p and q are listed in Table I. The fourth column in this table shows the approximate value of A required to produce a central density near $10^3\, M_\odot/\mathrm{pc}^3$. We note in Table I that decreasing the value of p has the effect of

TABLE I

p	q	n	(A)
2·0	0	3·4	1·0
2·0	0·5	3·8	0·004
1·8	0	3·0	0·5
1·8	0·5	3·4	0·003
1·7	0	2·7	0·3
1·7	0·5	3·2	0·003

decreasing n; with $q = 0$, the smallest value of p which can be considered acceptable in accordance with the discussion of Section 7 is about 1·8. With $q = 0{\cdot}5$, acceptable values of n may be achieved with somewhat smaller values of p. The density distributions obtained with $p = 1{\cdot}7$ and with $q = 0$ and $q = 0{\cdot}5$ are illustrated in Fig. 3.

We note that it is not possible to obtain a reasonable result with p equal to 1·5 or less, since in this case the star formation rate increases with density no faster than the free-fall compression rate; consequently star formation is never able to halt the increase in the gas density at the centre, and the central density of both gas and stars increases indefinitely.

For comparison with the present results, the value of A found by Schmidt (1959) for the case $p = 2$, $q = 0$ is about 3×10^{-3}, i.e. about 300 times smaller than the value used here. This may be an indication that star formation in the galactic disk is strongly inhibited relative to the conditions existing in a collapsing proto-galaxy; it is known, for example (Field & Saslaw 1965), that the time scale for star formation in the present interstellar medium is much longer than either the free-fall time or the cloud collision time. In any case, it is clear that a simple proportionality of the star formation rate to the square of the gas density cannot adequately represent the star formation rate in all circumstances, and that other factors such as (perhaps) the turbulent velocity must in general be taken into account.

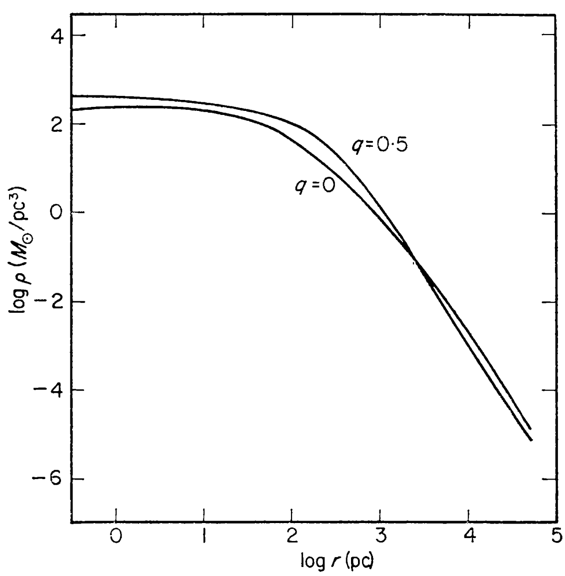

Fig. 3. *The stellar density distributions obtained with $p = 1\cdot7$ and $q = 0$ and $q = 0\cdot5$.*

8.2. *The turbulent dissipation rate*

We consider next the effect of changing the constant coefficient C in equation (6) for the turbulent dissipation rate. It is found that when C is changed by a factor of 2 (corresponding to a factor of 4 in the temperature or a factor of 8 in the mass of the clouds postulated in Section 2), the results described in Section 6 can be fairly closely recovered by changing A by a factor of about 2·4 in the same direction, all other parameters being left the same. For example, with $C = 6\cdot6 \times 10^{-3}$ and $A = 2\cdot4$, the resulting density distribution is almost the same as shown in Fig. 1, except that the logarithmic density gradient is slightly smaller ($n \simeq 3\cdot2$ instead of 3·4). With $C = 1\cdot65 \times 10^{-3}$ and $A = 0\cdot42$, we again obtain a similar result, this time with $n \simeq 3\cdot5$.

Since the cloud mass M appearing in equation (3) may not vary with density in the way that was assumed in Section 2, we have considered also the alternative assumption that the cloud mass simply remains constant as the density increases. In this case the exponent of ρ in equation (6) becomes $\frac{1}{3}$ instead of $\frac{1}{2}$. In Fig. 4 we show the density distribution resulting when this change is made, all the other parameters being left as in Section 6. The curve is again qualitatively similar to Fig. 1, with a mean slope of about $-3\cdot5$. However, it is less nearly linear, and appears to agree less well with observed density distributions.

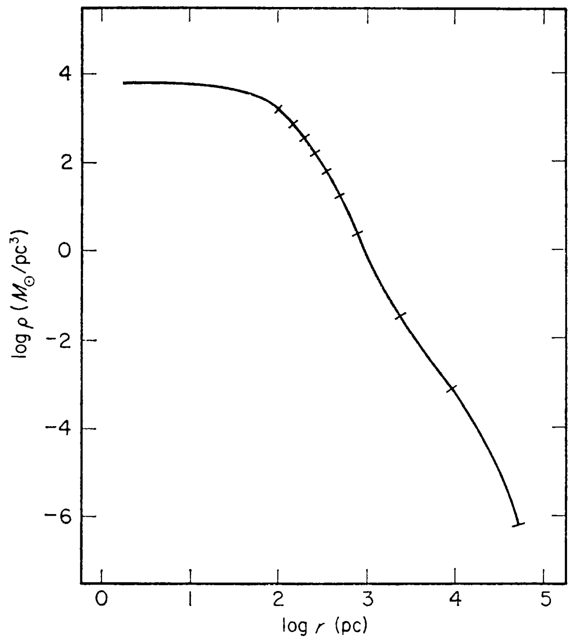

FIG. 4. *The density distribution obtained when the exponent of* ρ *in equation* (6) *is changed from* $\frac{1}{2}$ *to* $\frac{1}{3}$. *The ticks along the curve divide the galaxy into ten zones of equal mass.*

8.3. *Initial and boundary conditions*

To test the importance of the assumed initial velocity dispersion of the gas, a calculation was made with the initial value of α reduced by an order of magnitude to $3{\cdot}7 \times 10^2$, corresponding to a velocity dispersion of 19 km s^{-1}. The resulting density distribution is again similar to Fig. 1, except that the logarithmic density gradient is slightly steeper ($n \simeq 3{\cdot}7$). Since this value is still acceptable from the point of view of comparison with observations, it appears that the assumed initial velocity dispersion of the gas is not a matter of critical importance.

A calculation has been made with the fixed outer boundary of the proto-galaxy set at a radius of 10^5 pc instead of 5×10^4 pc. In this case it was found that if A is increased from 1·0 to 1·4, the density distribution of Fig. 1 is reproduced quite closely, the value of n being about 3·5 instead of 3·4.

Finally, some calculations have been made with an expanding boundary instead of a fixed boundary, as was described in Section 5. As an example, we describe the results of a calculation in which the boundary was assumed to expand with time according to $R = 900t^{2/3}$; the calculations were started at $t = 10^2$ (i.e., 10^8 yrs) with an initial density of $3{\cdot}3 \times 10^{-3}\, M_\odot/\mathrm{pc}^3$ and an initial value for α of 10^4, corresponding to a velocity dispersion of 98 km s^{-1}. The parameter A was set equal to 1·5. With these assumptions, the proto-galactic cloud continues to expand until $t \simeq 5 \times 10^8$ yrs, at which time the radius is about 5×10^4 pc, the central density about $5 \times 10^{-4}\, M_\odot/\mathrm{pc}^3$, and the central velocity dispersion about 40 km s^{-1}. Thereafter, the central part of the cloud begins to collapse, and the collapsing region gradually grows until it eventually includes almost the entire mass of the proto-galaxy. Because the proto-galaxy is expanding, the time scale for its evolution continually lengthens; after 10^{10} yrs, only about 96 per cent of the material has been converted into stars, and the central density is about $2 \times 10^3\, M_\odot/\mathrm{pc}^3$ and still slowly increasing. The calculations were continued until $1 \cdot 9 \times 10^{10}$ yrs, at which time 98 per cent of the material had been converted into stars and the central density had reached $4 \times 10^3 \; M_\odot/\mathrm{pc}^3$. The density distribution at this time is illustrated in Fig. 5. The value of n appropriate for this density distribution is about 3·2, still in acceptable agreement with observations.

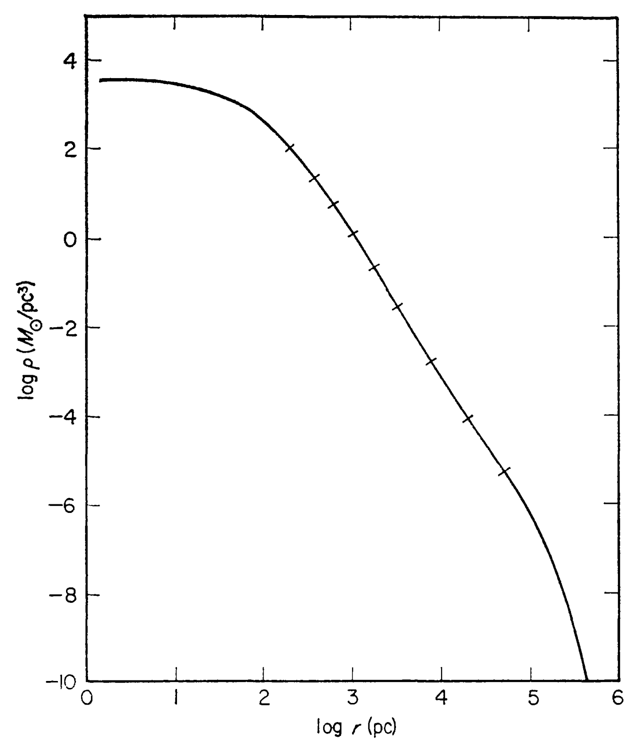

FIG. 5. *The density distribution obtained from the calculations with an expanding boundary.*

8.4. *Results with two star populations*

In order to alleviate to some extent the assumptions on the stellar velocity distribution, and to provide some possibility of keeping track of the stars formed at different stages during the collapse, a calculation was made with the stars divided into two independent ' populations ', each described by its own set of moment equations. These two populations have been chosen to consist approximately of the first 50 per cent and the second 50 per cent of the stars to be formed. The resulting density distribution obtained for all stars together is indicated by the solid curve in Fig. 6, and the density distributions of the first 50 per cent and the second 50 per cent of the stars to be formed are indicated by the dashed curve and the dot-dash curve respectively. The solid curve is almost identical to that in Fig. 1, showing that the form of the resulting density distribution is not significantly affected by the division into two populations.

In Fig. 7 are plotted the radial and transverse components of the velocity dispersions for the two stellar populations. Looking at Figs. 6 and 7, we see that the second population of stars to be formed has a more centrally concentrated density distribution than the first population, but a smaller central velocity dispersion. It is interesting to note that this situation bears a qualitative resemblance to the observed density and velocity distributions of Population I and Population II stars in the vicinity of the galactic plane. This suggests that the present approach may be

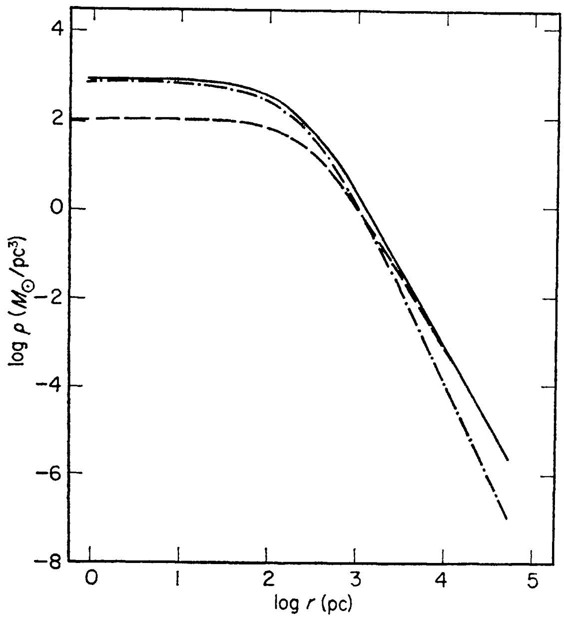

FIG. 6. *The density distribution obtained when the stars are divided into two populations. The solid curve is the density distribution for all stars, the dashed curve is the density distribution of the first* 50 *per cent of the stars to form, and the dot-dash curve is the density distribution of the second* 50 *per cent of the stars to form.*

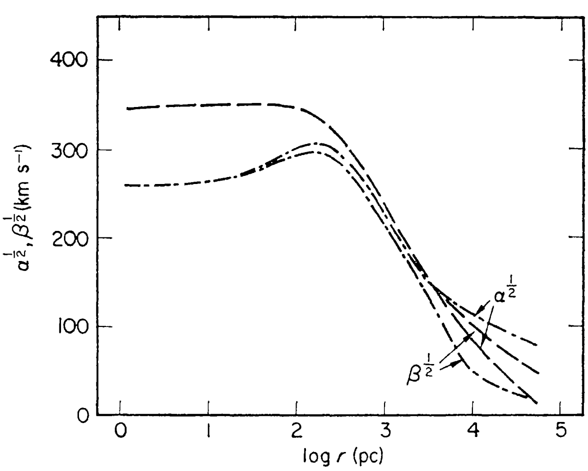

FIG. 7. *The radial component* $\alpha^{1/2}$ *and the transverse component* $\beta^{1/2}$ *of the velocity dispersions for the two stellar populations. The dashed curve refers to the first population to form, and the dot-dash curve to the second population. In the case of the first population, the radial and transverse components are so nearly equal that the two curves have been drawn as one for* $\log r < 3{\cdot}5$.

capable, with more refinements, of predicting the density and velocity distributions of the various population classes in a galaxy.

9. CONCLUSIONS

It appears that the type of model which we have described, based on the hypothesis of galaxy formation by collapse and fragmentation, may be capable of representing reasonably well the observed structure of spherical and nearly spherical galaxies. The nature of the results is not significantly affected by the details of the initial and boundary conditions, but is dependent mainly on the assumed star formation rate and the turbulent dissipation rate. Reasonable results were obtained with a star formation rate proportional to the square of the gas density, and it was found that the star formation rate cannot differ too much from this form, if the results are to remain reasonable. Likewise, it appears that the form of the turbulent dissipation rate cannot differ too much from that given in equation (6). The constant coefficient A in the star formation rate remains arbitrary, and it serves mainly to determine the central density at which the collapse is halted. Clearly it would be desirable to put the star formation rate and the turbulent dissipation rate on a firmer pnysical foundation, but pending this, models of the type we have described may nevertheless be of some use for understanding some of the more general aspects of galaxy formation.

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APPENDIX

THE DIFFERENCE EQUATIONS

In the present calculations we have used an implicit Eulerian difference scheme similar to that used by Larson (1968) for computing the collapse of a proto-star. The quantities r_i, u_i, m_i, and m_i' (defined below), $i = 1 \ldots N$, are evaluated at the boundaries of a set of fixed concentric shells, and the quantities $\rho_{i-1/2}$, $\alpha_{i-1/2}$, and $\beta_{i-1/2}$ are evaluated at the mid-points of these shells. The difference equations are written in implicit form with backward time differences, and are solved iteratively by the Newton–Raphson technique.

For numerical purposes it is convenient to rewrite the equation of continuity in terms of two variables m and m', where m is defined as the total mass of gas (or stars) within a sphere of radius r, and m' is defined as the mass within radius r of that material which at the beginning of the current time step was in the form of gas (or stars). In the following we shall use a superscript n to denote the values of quantities at the beginning of the time step Δt, and we shall use $r_{i-1/2}$ to denote $\frac{1}{2}(r_i + r_{i-1})$. We write the difference equations for the gas as follows:

$$\frac{m_i' - m_{i-1}'}{r_i^3 - r_{i-1}^3} = \frac{4\pi}{3}(\rho_{i-1/2} + A\rho_{i-1/2}{}^p \alpha_{i-1/2}{}^q \Delta t) \qquad (19)$$

$$\frac{u_i - u_i^n}{\Delta t} + u_i \frac{u_{i+1} - u_i}{r_{i+1} - r_i} + \frac{G(m_i' + m_{s,i}')}{r_i^2} + (\alpha_{i-1/2}\alpha_{i+1/2})^{1/2} \frac{\ln(\rho\alpha)_{i+1/2} - \ln(\rho\alpha)_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} = 0 \qquad (20)$$

$$\frac{\alpha_{i-1/2} - \alpha_{i-1/2}{}^n}{\Delta t} + u_i \frac{\alpha_{i+1/2} - \alpha_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} + 2\alpha_{i-1/2} \frac{r_i^2 u_i - r_{i-1}^2 u_{i-1}}{r_i^3 - r_{i-1}^3} + \tfrac{2}{3} C \rho_{i-1/2}{}^{1/2} \alpha_{i-1/2}{}^{3/2} = 0 \qquad (21)$$

where

$$m_i' = m_i^n - 4\pi r_i^2 (\rho_{i-1/2}\rho_{i+1/2})^{1/2} u_i \Delta t \qquad (22)$$

and $m_{s,i}'$ is the mass within radius r_i of that material which at the beginning of the time step Δt was in the form of stars.

For the stars the difference equations have been written as follows:

$$\frac{m_i' - m_{i-1}'}{r_i^3 - r_{i-1}^3} = \frac{4\pi}{3}(\rho_{i-1/2} - \zeta_{i-1/2}\Delta t) \qquad (23)$$

$$\frac{u_i - u_i^n}{\Delta t} + u_i \frac{u_{i+1} - u_i}{r_{i+1} - r_i} + \frac{G(m_i' + m_{g,i}')}{r_i^2} + \left(\frac{\alpha_{i-1/2} + \alpha_{i+1/2}}{2}\right) \frac{\ln(\rho\alpha)_{i+1/2} - \ln(\rho\alpha)_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} + \frac{(\alpha - \beta)_{i-1/2} + (\alpha - \beta)_{i+1/2}}{r_i} = (u_g - u)_i \left[\left(\frac{\zeta}{\rho}\right)_{i-1/2} \left(\frac{\zeta}{\rho}\right)_{i+1/2}\right]^{1/2} \qquad (24)$$

$$\frac{\alpha_{i-1/2} - \alpha_{i-1/2}{}^n}{\Delta t} + u_i \frac{\alpha_{i+1/2} - \alpha_{i-1/2}}{r_{i+1/2} - r_{i-1/2}} + 2\alpha_{i-1/2} \frac{u_i - u_{i-1}}{r_i - r_{i-1}} = [\tfrac{1}{4}(u_{g,i-1} - u_{i-1} + u_{g,i} - u_i)^2 + (\alpha_g - \alpha)_{i-1/2}] \left(\frac{\zeta}{\rho}\right)_{i-1/2} \qquad (25)$$

$$\frac{\beta_{i-1/2}-\beta_{i-1/2}{}^n}{\Delta t}+u_i\frac{\beta_{i+1/2}-\beta_{i-1/2}}{r_{i+1/2}-r_{i-1/2}}+2\beta_{i-1/2}\frac{u_{i-1}+u_i}{r_{i-1}+r_i} = (\alpha_g-\beta)_{i-1/2}\left(\frac{\zeta}{\rho}\right)_{i-1/2} \quad (26)$$

where

$$m_i' = m_i{}^n - 4\pi r_i{}^2(\rho_{i-1/2}\rho_{i+1/2})^{1/2}u_i\Delta t. \quad (27)$$

In the above equations the subscript g denotes properties of the gas, and

$$\zeta = A\rho_g{}^p\alpha_g{}^q$$

is the star formation rate.

The procedure for using these equations to advance all quantities by one time step Δt has been as follows: First equations (19)–(22) have been applied to compute new values for the variables describing the gas, assuming no change in the mass distribution of that material which at the beginning of the time step was in the form of stars, i.e., putting $m_{s,i}' = m_{s,i}{}^n$ in equation (20). This is expected to be a reasonable approximation, since the mean motion of the stars is usually much smaller than for the gas. A new mass distribution $m_{g,i}'$ for the gas and a new star formation rate ζ have then been computed, and these have been used in equations (23)–(27) to compute new values for the variables describing the stars.

The grid structure and the numerical accuracy of the present calculations are again similar to those of Larson (1968). The shell radii r_i have been distributed with a constant ratio of $\sqrt{2}$ between neighbouring radius values, and the corresponding level of numerical accuracy is of the order of 20 per cent, which appears to be adequate for the present purposes. The calculations made with an expanding boundary, described in Section 8.4, were done with a modified Eulerian scheme, in which the ratios between successive shell radii were held constant but the whole grid structure was made to expand by a time-dependent scale factor.