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A model for two coupled turbulent fluids Part III: Numerical approximation by Finite Elements

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Summary. This paper introduces a scheme for the numerical solution of a model for two turbulent flows with coupling at an interface. We consider a variational formulation of the coupled model, where the turbulent kinetic energy equation is formulated by transposition. We prove the convergence of an approximation to this formulation for 2D flows by piecewise affine triangular elements. Our main contribution is to prove that the standard Galerkin - Finite Element approximation of the Laplace equation approximates in L^2 norm its solution by transposition, for data with low smoothness. We include some numerical tests for simple geometries that exhibit the behaviour predicted by our analysis.

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1 Introduction

This paper deals with the numerical solution of two turbulent flows coupled through a common interface and, in particular, of air - atmosphere turbulence.

Numerical turbulence models are widely used in engineering to predict the behaviour of many kinds of flows of applied interest. A long development of physical turbulence models has taken place since the early 70's to model flows of increasing complexity (Cf. Wilcox [19]). Recently, an interest for the theoretical and numerical analysis of these models has taken place, motivated by the need of constructing numerical solvers on mathematical grounds.

The main issue in the mathematical analysis of turbulence models is to find a convenient weak formulation for the equations of the turbulence statistics. Lewandowski (Cf. [12]) analyses the so-called φ-θ model, a two-equations model with larger levels of turbulent diffusion than the well-known k-ε one. An existence result is derived, for given smooth velocity fields. In this case, the usual concept of weak solution of Lions (Cf. [14]) is enough to obtain an existence result. This essentially occurs because the source terms in the φ and θ equations are nonpositive. This result is improved in Gómez and Ortegón (Cf. [10]) by considering a more realistic φ-θ model, which includes coupling with the velocity field equations. Still, the usual weak formulation of the equations for φ and θ is used.

In Lewandowski (Cf. [13]) the mixing - length model is analyzed. This is a one-equation model, including velocity and turbulent kinetic energy (TKE in what follows), for which an existence result is proved. For this model, the production term for the TKE is nonnegative. As it has only L^1 regularity, this equation is formulated in the renormalized sense of Lions and Murat (Cf. [16]), and *a priori* estimates of Boccardo–Gallouët type [4] are used. However, no existence results of any kind of weak solutions for the full k-ε model has been reported, up to our knowledge.

In Mohammadi and Pironneau [17] an analysis of some aspects of the numerical approximation of the k-ε model is performed. Concretely, some Finite Element numerical schemes that use the method of characteristics for time discretization are introduced. Althoug these schemes are proved to be stable, no convergence results are reported.

Recently, Bernardi et al. have considered in [2] a model for the coupling of two turbulent flows separated by a fixed interface. The flow i $(i = 1, 2)$ is assumed to occupy a bounded domain Ω_i of $\mathbf{R}^2$, and to be represented by the (vector) velocity field $\mathbf{u}_i$, the pressure p_i and the turbulent kinetic energy (TKE in what follows) k_i. The interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is assumed to coincide with the intersection of both $\bar{\Omega}_1$ and $\bar{\Omega}_2$ with the hyperplane $x_2 = 0$, while Ω_1 and Ω_2 are contained in the half-spaces $x_2 > 0$ and $x_2 < 0$ respectively, where we denote by $\mathbf{x} = (x_1, x_2)$ a generic point of $\mathbf{R}^2$. The flows are assumed to be governed by the following boundary value problem:

$$
\begin{cases}
-\mathrm{div}\,\big(\alpha_i(k_i)\,\nabla\mathbf{u}_i\big) + \mathbf{grad}\,p_i = \mathbf{f}_i & \text{in } \Omega_i, \quad i=1,2,\\
\mathrm{div}\,\mathbf{u}_i = 0 & \text{in } \Omega_i, \quad i=1,2,\\
-\mathrm{div}\,\big(\gamma_i(k_i)\,\nabla k_i\big) = \alpha_i(k_i)\,|\nabla\mathbf{u}_i|^2 & \text{in } \Omega_i, \quad i=1,2,\\
\mathbf{u}_i = \mathbf{0} & \text{on } \Gamma_i, \quad i=1,2,\\
k_i = 0 & \text{on } \Gamma_i, \quad i=1,2,\\
\alpha_i(k_i)\,\partial_{n_i}\mathbf{u}_i = (u_{j_i} - u_i)\,|u_i - u_{j_i}| \quad, v_i = 0, & \text{on } \Gamma, \quad i=1,2,\\
 & \quad \text{with } j_1 = 2,\ j_2 = 1\\
k_i = |\mathbf{u}_1 - \mathbf{u}_2|^2, & \text{on } \Gamma, \quad i=1,2,
\end{cases}
\tag{1}
$$

where u_i and v_i respectively denote the horizontal and vertical components of the velocity field $\mathbf{u}_i$. Also, the quantities $\alpha_i(k)$ and $\gamma_i(k)$ are the eddy viscosities, assumed to be bounded and continuous functions of the argument k.

System (1) is motivated by the coupling of two turbulent fluids F_i, $i = 1$ and 2, such as in the framework ocean/atmosphere or in the case of two layers of a stratified fluid as is the ocean (see *e.g.* [12], Chap. 1 & 3). These fluids F_i are coupled through the interface condition in the sixth line of (1), on the part of the boundary Γ supposed to be fixed. Indeed, we assume that the so-called "rigid lid hypothesis" holds, which is standard in geophysics and oceanography. Basically, Γ is a mean interface and the values of $\mathbf{u}_i$, p_i and k_i on Γ are in fact mean values of the velocity, pressure and energy. So, the turbulent mixing layer of the two turbulent fluids is modelled by the sixth and seventh lines in (1) which contain the information related to a realistic interface ocean/atmosphere (see [12], Section 1.4 for more details about this modelling).

Model (1) is given in [2] a weak formulation as follows: Because of the boundedness of each function α_i, following the ideas of Lions (Cf. [14]) leads to a weak mixed formulation of the first two lines. However, because of low smoothness of the boundary conditions at the interface Γ, the renormalization in the sense of Lions and Murat [16] for elliptic equations does not seem the best way for the study of the TKE equations in the present problem. For this reason, the unknown k_i is at first transformed in an equivalent unknown, in order to replace the operator div $\big(\gamma_i(k_i)\,\nabla\big)$ by a simpler Laplace operator. The corresponding new equation is proved to have a sense by transposition, according to the ideas of Stampacchia [18] and of Lions and Magenes [15], Chap. 2, Section 6. Under this formulation, the existence of a solution of the global system (1) is proved if the domains Ω_1 and Ω_2 are either convex, or have $C^{1,1}$ boundaries. Uniqueness of solutions for small data is also proved.

Recently, Bernardi and al. have considered in [3] a spectral discretization of system (1) and perform its numerical analysis. The convergence of the method is proven in the two-dimensional case, together with optimal error estimates for smooth solutions.

The goal of this paper is to introduce a Finite Element numerical discretization of system (1). The equations for velocity and pressure are approximated by a standard mixed method. The main difficulty that we face is to derive efficient approximations of the solution by transposition of the TKE equation. Essentially, this equation reads as

$$\begin{cases} -\Delta k = \alpha & \text{in } \Omega, \\ k \quad\;\; = \delta & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where k (resp., Ω) stands for either k_1 or k_2 (resp., Ω_1 or Ω_2), α is a function of $L^1(\Omega)$ and δ a function of $L^4(\partial\Omega)$. Since either Ω is convex, or $\partial\Omega$ is $C^{1,1}$, the Laplace operator $\mathcal{L}$ which associates with data g in $H^{-1}(\Omega)$ the solution $\varphi = \mathcal{L}g \in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta \varphi = g & \text{in } \Omega, \\ \varphi \quad\;\; = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

is an isomorphism from $L^2(\Omega)$ into $H^2(\Omega) \cap H_0^1(\Omega)$ (See [11], Thm. 3.2.1.2). The solution by transposition of (2) is now defined as follows :

$$\begin{cases} \text{Find } k \in L^2(\Omega) \text{ such that } \forall g \in L^2(\Omega), \\ \displaystyle\int_\omega k\, g\, d\mathbf{x} = -\int_{\partial\Omega} \delta\, \partial_n(\mathcal{L}g)\, d\tau + \int_\Omega \alpha\, (\mathcal{L}g)\, d\mathbf{x}. \end{cases} \tag{4}$$

A direct numerical approximation of this formulation would need to explicitely know $\mathcal{L}$ acting on the elements of the discrete space, or some suitable approximation. The approximation introduced in this paper starts from the observation that the approximation by Finite Elements of problem (1) provides in particular a regularization of the data appearing in equation (2). Then, the standard Galerkin approximation of (2) makes sense, and only internal approximations of $H_0^1(\Omega)$ are needed. We prove that indeed the standard Finite Element Galerkin solution of the TKE equation (2) approximates the solution by transposition in L^2 norm. As a consequence, we prove the strong convergence of a subsequence of our global approximation to a solution of the continuous model (1), in H^1 and L^2 norms for velocity and kinetic energy, respectively. We consider piecewise linear Finite Elements because the low regularity of the solutions of problem (1) makes it unefficient the use of higher degree elements. However, our convergence analysis may be extended to general Finite Element approximations with some technical work.

Our paper is organized as follows. In Section 2 we introduce our first approximation and prove the existence of solutions. Section 3 is devoted to prove the convergence of this approximation, in the terms mentioned above. Finally, in Section 4 we report some numerical experiments for domains with simple geometries that exhibit the convergence in the norms predicted by our theory.

2 Numerical approximation

We shall at first describe the weak formulation of problem (1) introduced in [2]. This is based upon a reformulation of the TKE equation, as follows. We assume that α_i and γ_i are continuous bounded functions from the set of nonnegative real numbers $\mathbf{R}_+$ onto $\mathbf{R}$, which satisfies

$$\forall k \in \mathbf{R}_+, \quad M \geq \alpha_i(k) \geq \nu \quad \text{and} \quad M \geq \gamma_i(k) \geq \nu, \tag{5}$$

where M and ν are positive constant. Let us define the functions G_i, $i = 1$ and 2, by

$$G_i(k) = \int_0^k \gamma_i(\kappa)\, d\kappa. \tag{6}$$

The functions G_i are differentiable with bounded derivative, and also increasing and nonnegative on $\mathbf{R}_+$, so that they admit an inverse G_i^{-1} from $\mathbf{R}_+$ into $\mathbf{R}_+$. Moreover, the functions $\tilde{\alpha}_i$, $i = 1$ and 2, defined by

$$\tilde{\alpha}_i = \alpha_i \circ G_i^{-1}, \tag{7}$$

satisfy the same properties as the α_i, namely they are continuous, bounded and satisfy

$$\forall \ell \in \mathbf{R}_+, \quad \tilde{\alpha}_i(\ell) \geq \nu. \tag{8}$$

The unknowns k_i are replaced by the new unknowns $\ell_i = G_i(k_i)$. The equation for the TKE in (1) is formally replaced by

$$\begin{cases} -\Delta \ell_i = \tilde{\alpha}_i(\ell_i)\, |\nabla \mathbf{u}_i|^2 & \text{in} \quad \Omega_i, \\ \quad \ell_i = 0 & \text{in} \quad \Gamma_i, \\ \quad \ell_i = G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) & \text{in} \quad \Gamma. \end{cases} \tag{9}$$

Let us now introduce the spaces

$$X_i = \big\{\mathbf{u}_i = (u_i, v_i) \in H^1(\Omega_i) \times H_0^1(\Omega_i);\ u_i = 0 \text{ on } \Gamma_i\, \},$$

$$L_0^2(\Omega_i) = \{\, q_i \in L^2(\Omega_i);\ \int_{\Omega_i} q_i d\mathbf{x} = 0\, \}, \quad i = 1, 2;$$

and define, for simplicity of notation,

$$E_i = X_i \times L_0^2(\Omega_i) \times L^2(\Omega_i).$$

The formulation introduced in [2] is

Obtain $(\mathbf{u}_i,\ p_i,\ \ell_i) \in E_i,\ i = 1, 2$ such that $\forall(\mathbf{v}_i, q_i, g_i) \in E_i$,

$$\begin{aligned} &\tilde{a}_i(\ell_i; \mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) - b_i(\mathbf{u}_i, q_i) \\ &\quad + n(\mathbf{u}_i, \mathbf{u}_{j_i}, \mathbf{v}_i) = \langle \mathbf{f}_i, \mathbf{v}_i \rangle, \end{aligned} \tag{10}$$

$$\begin{aligned} \int_{\Omega_i} \ell_i\, g_i\, d\mathbf{x} = &- \int_\Gamma G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2)\, \partial_{n_i}(\mathcal{L}_i g_i)\, d\tau \\ &+ \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, |\nabla \mathbf{u}_i|^2\, \mathcal{L}_i g_i\, d\mathbf{x}; \end{aligned} \tag{11}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality between X_i and X_i', the forms $\tilde{a}_i(\cdot, \cdot, \cdot)$, $b_i(\cdot, \cdot)$ and $n(\cdot, \cdot, \cdot)$ are defined by

$$\begin{aligned} \tilde{a}_i(\ell_i; \mathbf{u}_i, \mathbf{v}_i) = \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i\, d\mathbf{x}, \quad b_i(\mathbf{v}_i, q_i) \\ = - \int_{\Omega_i} q_i\, (\operatorname{div} \mathbf{v}_i)\, d\mathbf{x}, \end{aligned} \tag{12}$$

$$n(\mathbf{u}_i, \mathbf{u}_{j_i}, \mathbf{v}_i) = \int_\Gamma (u_i - u_{j_i})\, |u_i - u_{j_i}|\, v_i\, d\tau, \tag{13}$$

and the operators $\mathcal{L}_i$ are defined by (3) with $\Omega = \Omega_i$. Equation (10) is a weak formulation of the two first equations in (1), plus the boundary conditions set by the fourth and sixth equations. Equation (11) is the formulation by transposition of the third equation in (1), plus the boundary conditions set by the fifth and seventh equations, via the change of unknowns $\ell_i = G_i(k_i)$.

In [2] it is proved by a fixed point argument that problem (10)-(11) admits at least one solution, that belongs to $X_i \times L^2_0(\Omega_i) \times H^s(\Omega_i)$, for all $s < 1/2$, $i = 1, 2$. This solution is proved to provide a solution in the sense of distributions of the original model (1).

To describe our discretization, we assume that the Ω_i are polygonal, and consider at first a family of triangular meshes $\mathcal{T}_{ih}$ of Ω_i, $i = 1, 2$. We assume the following hypothesis on the meshes :

Hypothesis 1

a) The meshes are regular, in the usual sense of the Finite Element Method (Cf. Ciarlet [6]).

b) The meshes are compatible on Γ*, in the sense that the sets*

$$\partial \mathcal{T}_{ih} = \{T \cap \Gamma,\ for\, T \in \mathcal{T}_{ih}\}, \quad i = 1, 2 \tag{14}$$

are equal.

We shall also assume the following hypothesis on the domains:

Hypothesis 2 *Both domains Ω_i are polygonal and have no fissures.*

As discrete velocity - pressure spaces we shall use for simplicity the Mini-Element ($\mathbf{P}1 \bigoplus$ Bubble, $\mathbf{P}1$) on both Ω_1 and Ω_2. These spaces are respectively defined by

$$X_{ih} = \big([V_{ih}]^2 \cap X_i\big) \oplus \mathcal{B}_{ih}, \quad M_{ih} = V_{ih} \cap L_0^2(\Omega_i), \tag{15}$$

where V_{ih} is the piecewise affine Finite Element space,

$$V_{ih} = \{v_{ih} \in C(\bar{\Omega}_i) \mid v_{ih|T} \text{ is affine } \forall T \in \mathcal{T}_{ih}\},$$

and $\mathcal{B}_{ih}$ is the “bubble” space,

$$\mathcal{B}_{ih} = \{b_{ih} \in C(\bar{\Omega}_i) \mid b_{ih|T} \text{ is proportional to } \quad b_{iT} \ \forall T \in \mathcal{T}_{ih}\},$$

b_{iT} denoting the local bubble on element T, defined as the product of all barycentric coordinates associated to the vertex of T.

The family $\{X_{ih}, M_{ih}\}_{h>0}$, for $i = 1, 2$, is well known to satisfy the discrete Babuška - Brezzi inf-sup condition on Ω_i : There exists a constant $\tilde{\beta}_i > 0$ such that

$$\forall q_{ih} \in M_{ih}, \quad \sup_{\mathbf{v}_{ih} \in X_{ih}} \frac{b_i(\mathbf{v}_{ih}, q_{ih})}{\|\mathbf{v}_{ih}\|_{H^1(\Omega_i)^2}} \geq \tilde{\beta}_i \, \|q_{ih}\|_{L^2(\Omega_i)}. \tag{16}$$

Notice that due to Hypothesis 1 b), the velocity spaces X_{1h} and X_{2h} are compatible on Γ, in the sense that the trace spaces

$$Z_{ih} = \{\mathbf{v}_{ih|\Gamma}, \text{ for } \mathbf{v}_{ih} \in X_{ih}, \}, \quad i = 1, 2$$

are equal.

We also consider the subspace K_{ih} of $H_0^1(\Omega_i)$ formed by piecewise affine Finite Elements on triangulation $\mathcal{T}_{ih}$,

$$K_{ih} = V_{ih} \cap H_0^1(\Omega_i).$$

We look for a discrete approximation ℓ_{ih} of ℓ_i as $\ell_{ih} = \ell_{0ih} + D_{ih}$, where $\ell_{0ih} \in K_{ih}$ and D_{ih} is a suitable extension of the boundary data $G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2)$ to Ω_i. Let us define the operator $P_{ih} : Z_{ih} \mapsto X_{ih}$, by

$$P_{ih}\mathbf{v}_h = \sum_{\alpha_j \in \Gamma} \mathbf{v}_h(\alpha_j)\, \varphi_j^i,$$

where we denote by α_j the nodes of interpolation of the X_{ih} on Γ (which by Hypothesis 1 b) do not depend on i), and by φ_j^i the canonical basis functions of space X_{ih} associated to each node α_j defined by $\varphi^i(\alpha_j) = \delta_{ij}$. We now define

$$D_{ih} = G_i(|P_{ih}(\mathbf{u}_{1h} - \mathbf{u}_{2h})|^2), \quad i = 1, 2.$$

Notice that by construction, $D_{ih|_\Gamma} = G_i(|\mathbf{u}_{1h} - \mathbf{u}_{2h}|^2)_{|\Gamma}$. Also, as each G_i is differentiable with bounded derivative, then D_{ih} belongs to $H^1(\Omega_i)$, $i = 1, 2$.

Let us define the space $E_{ih} = X_{ih} \times M_{ih} \times K_{ih}$. Our approximation of formulation (10) - (11) may now be stated as follows :

Obtain $(\mathbf{u}_{ih},\ p_{ih},\ \ell_{0ih}) \in E_{ih},\ i = 1, 2$ such that $\forall(\mathbf{v}_{ih}, q_{ih}, g_{ih}) \in E_{ih}$,

$$\begin{aligned}\tilde{a}_i(\ell_{ih}; \mathbf{u}_{ih}, \mathbf{v}_{ih}) + b_i(\mathbf{v}_{ih}, p_{ih})\\ -b_i(\mathbf{u}_{ih}, q_{ih}) + n(\mathbf{u}_{ih}, \mathbf{u}_{j_ih}, \mathbf{v}_{ih}) = \langle \mathbf{f}_i, \mathbf{v}_{ih}\rangle,\end{aligned} \tag{17}$$

$$\begin{aligned}&\int_{\Omega_i} \nabla \ell_{ih} \cdot \nabla g_{ih}\, d\mathbf{x}\\ &\quad = \int_{\Omega_i} \tilde{\alpha}_i(\ell_{ih})\, |\nabla \mathbf{u}_{ih}|^2\, g_{ih}\, d\mathbf{x};\end{aligned} \tag{18}$$

$$\ell_{ih} = \ell_{0ih} + D_{ih}, \text{ with } \quad D_{ih} = G_i(|P_{ih}(\mathbf{u}_{1h} - \mathbf{u}_{2h})|^2). \tag{19}$$

We shall prove in the next Section that this problem admits at least one solution. Our main result in the paper is the following

Theorem 1 *Assume Hypothesis 1 and 2, then there exists a subsequence of the solutions* $(\mathbf{u}_{1h},\ p_{1h},\ \ell_{1h})$, $(\mathbf{u}_{2h},\ p_{2h},\ \ell_{2h})$ *provided by method (17)-(19) that converge strongly in*

$$(H^1(\Omega_1)^2 \times L^2_0(\Omega_1) \times H^s(\Omega_1)) \times (H^1(\Omega_2)^2 \times L^2_0(\Omega_2) \times H^s(\Omega_2)),$$

for $0 \le s < 1/2$, *to a solution of problem (10)-(11).*

If this solution is unique, then the whole sequence converges to it.

Remark 1 In Dauge (Cf. [8]) it is proved that $\mathcal{L}_i$ is an isomorphism from $H^{-s}(\Omega_i)$ on to $H^{2-s}(\Omega_i) \cap H^1_0(\Omega_i)$, if $s > s_0 = 0$ when Ω_i is convex, or

$$s > s_0 = 1 - \frac{\pi}{\omega}, \tag{20}$$

where ω is the largest internal angle of $\partial\Omega_i$. In the last case, as we are assuming by Hypothesis 2 that the Ω_i have no fissures, then $w < 2\pi$ and $s_0 < 1/2$.

Remark 2 It is proved in [2] that for small data $\mathbf{f_1}$ and $\mathbf{f_2}$ problem (10)-(11) admits a unique solution. Thus, in this case Theorem 1 provides a complete convergence result.

Remark 3 We may obtain a maximum principle for the TKE under some additional restriction on the triangulations $\mathcal{T}_{ih}$. Indeed, if all angles of elements of $\mathcal{T}_{ih}$ are acute, then the matrix associated to the discrete Laplacian operator in (18) is an M-matrix. As the boundary data given by (19) and the r.h.s. of (18) are nonnegative, we deduce that ℓ_{ih}, and then k_{ih} also are nonnegative. Moreover, $\ell_{ih} = 0$ if and only if $\mathbf{u}_{ih}$ is constant and $\mathbf{u}_{1h} \equiv \mathbf{u}_{2h}(\equiv$ constant) on Γ.

3 Convergence analysis

We first state an existence result for discretization (17)-(18)-(19). We shall assume in this Section that Hypotheses 1 and 2 hold. For simplicity, we shall not state this in each result.

Our proof uses Brouwer's fixed point theorem. We shall consider the following partial problems :

Problem 1 Given $(\bar{\mathbf{u}}_{ih}, \bar{\ell}_{0ih}) \in X_{ih} \times K_{ih}$, $i = 1, 2$,

$$\text{Obtain } \left((\mathbf{u}_{1h},\ p_{1h}), (\mathbf{u}_{2h},\ p_{2h}) \right) \in \left((X_{1h} \times M_{1h}), (X_{2h} \times M_{2h}),\ \right),$$
$$\text{such that } \forall \left((\mathbf{v}_{1h}, q_{1h}),\ (\mathbf{v}_{2h},\ q_{2h}) \right) \in \left((X_{1h} \times M_{1h}), (X_{2h} \times M_{2h}),\ \right),$$

$$\begin{aligned} &\tilde{a}_i(\bar{\ell}_{ih}; \mathbf{u}_{ih}, \mathbf{v}_{ih}) + b_i(\mathbf{v}_{ih}, p_{ih}) - b_i(\mathbf{u}_{ih}, q_{ih}) + n(\mathbf{u}_{ih}, \mathbf{u}_{j_i h}, \mathbf{v}_{ih}) \\ &\quad = \langle \mathbf{f}_i, \mathbf{v}_{ih} \rangle, \end{aligned} \tag{21}$$

where

$$\bar{\ell}_{ih} = \bar{\ell}_{0ih} + \bar{D}_{ih}, \quad \text{with} \quad \bar{D}_{ih} = G_i(|P_{ih}(\bar{\mathbf{u}}_{1h} - \bar{\mathbf{u}}_{2h})|^2),$$

for $i = 1, 2$.

Problem 2 Given $(\mathbf{u}_{ih}, \widehat{\ell_i}) \in X_{ih} \times L^2(\Omega_i)$, $i = 1, 2$,

$$\text{Obtain} \quad \ell_{0ih} \in K_{ih},,\ i = 1, 2$$

such that

$$\int_{\Omega_i} \nabla \ell_{ih} \cdot \nabla g_{ih}\, d\mathbf{x} = \int_{\Omega_i} \tilde{\alpha}_i(\widehat{\ell_i})\, |\nabla \mathbf{u}_{ih}|^2\ g_{ih}\, d\mathbf{x}, \quad \forall g_{ih} \in K_{ih} \tag{22}$$

where

$$\ell_{ih} = \ell_{0ih} + D_{ih}, \quad D_{ih} = G_i(|P_{ih}(\mathbf{u}_{1h} - \mathbf{u}_{2h})|^2).$$

Notice that (21) is a coupled problem for $\mathbf{u}_{1h}$ and $\mathbf{u}_{2h}$, while (22) in reality represents two decoupled problems for ℓ_{01h} and ℓ_{02h}.

Lemma 1 *Problems 1 and 2 admit unique solutions, which satisfy the estimates*

$$\|\mathbf{u}_{1h}\|_{H^1(\Omega_1)^2} + \|\mathbf{u}_{2h}\|_{H^1(\Omega_2)^2} \le \frac{C}{\nu} \left[\|\mathbf{f}_1\|_{L^2(\Omega_1)^2} + \|\mathbf{f}_2\|_{L^2(\Omega_2)^2} \right], \tag{23}$$

$$\begin{aligned} \|p_{1h}\|_{L^2(\Omega_1)} + \|p_{2h}\|_{L^2(\Omega_2)} \le \frac{C}{\nu} \Big[&\|\mathbf{f}_1\|^2_{L^2(\Omega_1)^2} + \|\mathbf{f}_2\|^2_{L^2(\Omega_2)^2} \\ &+ \|\mathbf{f}_1\|_{L^2(\Omega_1)^2} + \|\mathbf{f}_2\|_{L^2(\Omega_2)^2} \Big], \end{aligned} \tag{24}$$

$$\begin{aligned} \|\ell_{ih}\|_{H^s(\Omega_i)} \le C_s\, (1 + h^{1/2-s}) \Big[&\|\tilde{\alpha}_i(\widehat{\ell_{ih}})\, |\nabla \mathbf{u}_{ih}|^2 \\ &\|_{L^1(\Omega_i)} + \|\mathbf{u}_{1h}|_\Gamma - \mathbf{u}_{2h}|_\Gamma\|^2_{L^4(\Gamma)} \Big], \end{aligned} \tag{25}$$

for $s_0 < s \le 1/2$, where C_s is a constant depending only on s, Ω_1, Ω_2, Γ, $\|\gamma_1\|_{L^\infty(\mathbf{R})}$, $\|\gamma_2\|_{L^\infty(\mathbf{R})}$ and the aspect ratio of triangulations $\mathcal{T}_{1h}$ and $\mathcal{T}_{2h}$. Notice that the estimates (23) and (24) are independent of the data.

Proof. **Problem 1** The existence of solution of the velocities $\mathbf{u}_{ih}$ in $H^1(\Omega_i)$ is deduced as in Section 3 of [1]: Define the spaces

$$V_{ih} = \{\mathbf{v}_{ih} \in X_{ih} \text{ such that } b_i(\mathbf{v}_{ih}, q_{ih}) = 0,\ \forall\, q_{ih} \in M_{ih}\}, \quad \text{for } i = 1, 2.$$

Then, the pair $(\mathbf{u}_{1h},\ \mathbf{u}_{2h}) \in V_{1h} \times V_{2h}$ satisfies, $\forall\, (\mathbf{v}_{1h},\ \mathbf{v}_{2h}) \in V_{1h} \times V_{2h}$,

$$\tilde{a}_i(\bar{\ell}_{ih}; \mathbf{u}_{ih}, \mathbf{v}_{ih}) + n(\mathbf{u}_{ih}, \mathbf{u}_{j_i h}, \mathbf{v}_{ih}) = \langle \mathbf{f}_i, \mathbf{v}_{ih} \rangle, \tag{26}$$

for $i = 1, 2$.

Following [1], this problem admits a unique solution, due to the monotonicity of the whole operator appearing in (26), acting on $V_{1h} \times V_{2h}$. This solution is readily proved to satisfy estimate (23).

To obtain the pressures, let us define $\Phi_{ih} \in X'_{ih}$ by

$$\langle \Phi_{ih}, \mathbf{v}_{ih} \rangle = \tilde{a}_i(\bar{\ell}_{ih}; \mathbf{u}_{ih}, \mathbf{v}_{ih}) + n(\mathbf{u}_{ih}, \mathbf{u}_{jh}, \mathbf{v}_{ih}) - \langle \mathbf{f}_i, \mathbf{v}_{ih} \rangle;$$

and the discrete gradient operator $\mathcal{G}_{ih} : M_{ih} \mapsto X'_{ih}$ by

$$\langle \mathcal{G}_{ih}(q_{ih}), \mathbf{v}_{ih} \rangle = b_i(\mathbf{v}_{ih}, q_{ih}), \quad \forall q_{ih} \in M_{ih}.$$

Notice that $V_{ih} = (Im(\mathcal{G}_{ih}))^{\perp}$. Due to (26),$\Phi_{ih} \in V_{ih}^{\perp} = Im(\mathcal{G}_{ih})$. Thus, there exists some $p_{ih} \in M_{ih}$ such that $\Phi_{ih} = \mathcal{G}_{ih}(-p_{ih})$. Or, in other words, $(\,(\mathbf{u}_{1h},\ p_{1h}), (\mathbf{u}_{2h},\ p_{2h})\,)$ is a solution of Problem (21). This solution is unique due to the inf-sup condition (16).

To obtain the $L^2(\Omega_i)$ estimate for the pressures, let us consider that, as Γ is C^∞ and bounded, then $H^{1/2}(\Gamma)$ is injected in $L^p(\Gamma)$, for $1 \le p \le \infty$, with compact injection. Also, as $\partial\Omega_i$ is Lipschitz-continuous, then $\mathbf{u}_{ih|\Gamma} \in H^{1/2}(\Gamma)$. Then,

$$\begin{aligned}
|n(\mathbf{u}_{ih}, \mathbf{u}_{j_i h}, \mathbf{v}_{ih})| &\le \| \mathbf{u}_{1h|\Gamma} - \mathbf{u}_{2h|\Gamma} \|^2_{L^3(\Gamma)^2} \| \mathbf{v}_{ih|\Gamma} \|_{L^3(\Gamma)^2} \\
&\le C \left[\|\mathbf{u}_{1h}\|^2_{H^1(\Omega_1)^2} + \|\mathbf{u}_{2h}\|^2_{H^1(\Omega_2)^2} \right] \|\mathbf{v}_{ih}\|_{H^1(\Omega_i)^2},
\end{aligned}$$

for some constant $C > 0$ independent of h. Estimate (24) follows from the discrete inf-sup condition (16). □

Proof. **Problem 2** As $D_{ih} \in H^1(\Omega_i)$ and $\tilde{\alpha}_i(\widehat{\ell_i})\, |\nabla \mathbf{u}_{ih}|^2 \in L^\infty(\Omega_i)$, it is clear that Problem 2 admits a unique solution.

To deduce estimate (25), observe that $H^s(\Omega_i) = H_0^s(\Omega_i)$ if $0 \le s \le \frac{1}{2}$ (Cf. Grisvard [11]). Then, using the reflexivity of $H_0^s(\Omega_i)$, $H^s(\Omega_i) = \big(H^{-s}(\Omega_i)\big)'$. From Hypothesis 2, for $s_0 < s \le 1/2$, we deduce (see Remark 1)

$$\begin{aligned}
\|\ell_{ih}\|_{H^s(\Omega_i)} &= \sup_{Q \in B - \{0\}} \frac{\langle \ell_{ih}, -\Delta Q \rangle}{\|\Delta Q\|_{H^{-s}(\Omega_i)}} \\
&\le C \sup_{Q \in B - \{0\}} \frac{\langle \ell_{ih}, -\Delta Q \rangle}{\|Q\|_{H^{2-s}(\Omega_i)}};
\end{aligned} \tag{27}$$

where $B = H^{2-s}(\Omega_i) \cap H_0^1(\Omega_i)$, and $\langle \cdot \rangle$ stands for the $H^s(\Omega_i) - H^{-s}(\Omega_i)$ duality product.

Notice that $H^{2-s}(\Omega_i)$ is continuously injected in $C^0(\Omega_i)$ if $s < 1$. Thus, for $Q \in B$, its standard interpolate on K_{ih} (defined by $Q_h(\alpha_j) = Q(\alpha_j)$, for all α_j node of $\mathcal{T}_{ih}$), is well defined. We write

$$
\begin{aligned}
\langle \ell_{ih}, -\Delta Q \rangle &= \int_{\Omega_i} \nabla \ell_{ih} \cdot \nabla Q \, d\mathbf{x} - \int_\Gamma \ell_{ih} \, \partial_{ni} Q \, d\tau \\
&= \int_{\Omega_i} \nabla \ell_{ih} \cdot \nabla Q_h \, d\mathbf{x} + \int_{\Omega_i} \nabla \ell_{ih} \cdot \nabla (Q - Q_h) \, d\mathbf{x} \\
&\quad - \int_\Gamma \delta_{ih} \, \partial_{ni} Q \, d\tau \\
&= \int_{\Omega_i} \tilde{\alpha}_i(\widehat{\ell_i}) \, |\nabla \mathbf{u}_{ih}|^2 \, Q_h \, d\mathbf{x} + \int_{\Omega_i} \nabla D_{ih} \cdot \nabla (Q - Q_h) \, d\mathbf{x} \\
&\quad + \int_{\Omega_i} \nabla \ell_{0ih} \cdot \nabla (Q - Q_h) \, d\mathbf{x} - \int_\Gamma \delta_{ih} \, \partial_{ni} Q \, d\tau \\
&:= \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}, \qquad (28)
\end{aligned}
$$

where $\delta_{ih} = G_i(|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}|^2)$.

We next estimate these four summands.

Estimates of I As $\|Q_h\|_{L^\infty(\Omega_i)} \le \|Q\|_{L^\infty(\Omega_i)}$,

$$
\begin{aligned}
|\mathbf{I}| &\le \|\tilde{\alpha}_i(\widehat{\ell_i}) \, |\nabla \mathbf{u}_{ih}|^2 \|_{L^1(\Omega_i)} \, \|Q_h\|_{L^\infty(\Omega_i)} \\
&\le C \, \|\tilde{\alpha}_i(\widehat{\ell_i}) \, |\nabla \mathbf{u}_{ih}|^2 \|_{L^1(\Omega_i)} \, \|Q\|_{H^{2-s}(\Omega_i)} \qquad (29)
\end{aligned}
$$

Estimates of II A slight modification of the arguments of Exercice 8.3 of Dupont and Scott (Cf. [9]) proves that there exists a constant $C > 0$ depending only on Ω_i such that

$$
\|\nabla (Q - Q_h)\|_{L^2(\Omega_i)} \le C \, h^{1-s} \, \|Q\|_{H^{2-s}(\Omega_i)}. \qquad (30)
$$

Also, denote $\mathbf{z}_h = P_{ih}(\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma})$. Then,

$$
\begin{aligned}
\|\nabla D_{ih}\|_{L^2(\Omega_i)} &= \|G_i'(|\mathbf{z}_h|^2) \, \nabla \left(|\mathbf{z}_h|^2 \right) \|_{L^2(\Omega_i)} \\
&\le \|\gamma_i\|_{L^\infty(\mathbf{R})} \, \|\nabla \left(|\mathbf{z}_h|^2 \right) \|_{L^2(\Omega_i)}.
\end{aligned}
$$

Denote by $\mathcal{C}_h$ the set of elements $K \in \mathcal{T}_{ih}$ such that $meas(\partial K \cap \Gamma) > 0$, and by $\{\alpha_1, \alpha_2, \cdots, \alpha_M\}$ the nodes of $\mathcal{T}_{ih}$ that lie on Γ. Then, there exists a constant $C > 0$ depending on the aspect ratio of the triangulations, such that

$$
\|\nabla \left(|\mathbf{z}_h|^2 \right) \|^2_{L^2(\Omega_i)} = 4 \sum_{T \in \mathcal{C}_h} |\nabla \mathbf{z}_{h|_T}|^2 \int_T |\mathbf{z}_h|^2 \, d\mathbf{x} \le C \sum_{i=1}^M |\mathbf{z}_h(\alpha_i)|^4.
$$

This occurs because the regularity of the triangulations $\mathcal{T}_{ih}$ ensures

$$|\nabla \mathbf{z}_{h\,|_T}| \le C \sum_{i,j=1}^{3} \left| \frac{\mathbf{z}_h(\alpha_{iT}) - \mathbf{z}_h(\alpha_{jT})}{h} \right|^2,$$

$$\int_T |\mathbf{z}_h|^2 \, d\mathbf{x} \le C\, h^2 \sum_{i=1}^{M} |\mathbf{z}_h(\alpha_{iT})|^2,$$

where α_{iT}, $i = 1, 2, 3$ are the nodes of element T.

By a scaling argument,

$$h \sum_{i=1}^{M} |\mathbf{z}_h(\alpha_i)|^4 \le C\, \|\mathbf{z}_{h\,|_\Gamma}\|^4_{L^4(\Gamma)}.$$

Then,

$$\|\nabla \left(|\mathbf{z}_h|^2\right)\|_{L^2(\Omega_i)} \le C\, h^{-1/2}\, \|\mathbf{z}_{h\,|_\Gamma}\|^2_{L^4(\Gamma)},$$

and

$$\|\nabla D_{ih}\|_{L^2(\Omega_i)} \le C\, h^{-1/2}\, \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)}. \tag{31}$$

Combining now estimates (30) and (31), we deduce

$$\begin{aligned} |\mathbf{II}| &\le \|\nabla D_{ih}\|_{L^2(\Omega_i)}\, \|\nabla (Q - Q_h)\|_{L^2(\Omega_i)} \\ &\le C\, h^{1/2-s}\, \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)}\, \|Q\|_{H^{2-s}(\Omega_i)}. \end{aligned} \tag{32}$$

Estimates of III From (30),

$$|\mathbf{III}| \le C\, h^{1-s}\, \|\nabla \ell_{0ih}\|_{L^2(\Omega_i)}\, \|Q\|_{H^{2-s}(\Omega_i)}. \tag{33}$$

To estimate the first factor above, let us consider that

$$\|\nabla \ell_{0ih}\|_{L^2(\Omega_i)} = \sup_{g_h \in K_{ih} - \{0\}} \frac{\displaystyle\int_{\Omega_i} \nabla \ell_{0ih} \cdot \nabla g_h \, d\mathbf{x}}{\|\nabla g_h\|_{L^2(\Omega_i)}}.$$

From here, using equation (22), we deduce

$$\begin{aligned} \|\nabla \ell_{0ih}\|_{L^2(\Omega_i)} \le C\Bigg[& \|\tilde{\alpha}_i(\widehat{\ell_i})\, |\nabla \mathbf{u}_{ih}|^2\|_{L^1(\Omega_i)} \sup_{g_h \in K_{ih} - \{0\}} \frac{\|g_h\|_{L^\infty(\Omega_i)}}{\|\nabla g_h\|_{L^2(\Omega_i)}} \\ & + \|\nabla D_{ih}\|_{L^2(\Omega_i)} \Bigg]. \end{aligned} \tag{34}$$

Given $g_h \in K_{ih}$, there exists a node α_0 of triangulation $\mathcal{T}_{ih}$ such that $|g_h(\alpha_0)| = \|g_h\|_{L^\infty(\Omega_i)}$, as g_h is a piecewise affine function. As the triangulations are regular, there exists a chain of nodes of $\mathcal{T}_{ih}$, $\alpha_0, \alpha_2, \cdots, \alpha_N$, that rely α_0 with $\alpha_N \in \partial\Omega_i$, such that $N = O(h^{-1})$. Using $g_h(\alpha_N) = 0$, we deduce

$$\|g_h\|_{L^\infty(\Omega_i)} \leq \sum_{i=1}^{N} |g_h(\alpha_i) - g_h(\alpha_{i-1})| \leq N^{1/2} \left(\sum_{i=1}^{N} |g_h(\alpha_i) - g_h(\alpha_{i-1})|^2 \right)^{1/2}$$

$$\leq C\, h^{-1/2} \left(\sum_{i=1}^{N} |g_h(\alpha_i) - g_h(\alpha_{i-1})|^2 \right)^{1/2}. \tag{35}$$

Consider now that

$$\begin{aligned} \|\nabla g_h\|^2_{L^2(\Omega_i)} &\geq C \sum_{T \in \mathcal{T}_{ih}} \sum_{i,j=1}^{3} \int_T \frac{|g_h(\alpha_{i,T}) - g_h(\alpha_{j,T})|^2}{h^2}\, d\mathbf{x} \\ &\geq C \sum_{T \in \mathcal{T}_{ih}} \sum_{i,j=1}^{3} |g_h(\alpha_{i,T}) - g_h(\alpha_{j,T})|^2 \\ &\geq C\, h\, \|g_h\|^2_{L^\infty(\Omega_i)}, \end{aligned}$$

the last inequality being due to estimate (35). Then,

$$\|g_h\|_{L^\infty(\Omega_i)} \leq C\, h^{-1/2}\, \|\nabla g_h\|_{L^2(\Omega_i)}. \tag{36}$$

Combining now estimates (36) and (31) with (34) and (33), we deduce

$$\begin{aligned} |\mathbf{III}| \leq C\, h^{1/2-s} \Big[& \|\tilde{\alpha}_i(\widehat{\ell}_i)\, |\nabla \mathbf{u}_{ih}|^2\|_{L^1(\Omega_i)} \\ & + \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)} \Big] \|Q\|_{H^{2-s}(\Omega_i)}. \end{aligned} \tag{37}$$

Estimates of IV As Γ is C^∞, then $H^{1/2-s}(\Gamma)$ is continuously imbedded in $L^q(\Gamma)$ with $q > 2$. Then,

$$|\mathbf{IV}| \leq \|\delta_{ih}\|_{L^4(\Gamma)} \|\partial_{ni} Q\|_{L^{4/3}(\Gamma)} \leq C \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)} \|Q\|_{H^{2-s}(\Omega_i)}. \tag{38}$$

Inserting estimates (29), (32), (37) and (38) in (28) and (27), we deduce (25). $\square$

This result allows to prove the existence of solutions of our numerical approximation:

Theorem 2 *The discrete problem (17) - (18) - (19) always admits a solution, which satisfies the estimates*

$$(39)\quad \|\mathbf{u}_{1h}\|_{H^1(\Omega_1)^2} + \|\mathbf{u}_{2h}\|_{H^1(\Omega_2)^2} \le \frac{C}{\nu}\,\Big[\,\|\mathbf{f}_1\|_{L^2(\Omega_1)^2} + \|\mathbf{f}_2\|_{L^2(\Omega_2)^2}\Big]\,,$$

$$(40)\quad \|p_{1h}\|_{L^2(\Omega_1)} + \|p_{2h}\|_{L^2(\Omega_2)} \le \frac{C}{\nu}\Big[\,\|\mathbf{f}_1\|^2_{L^2(\Omega_1)^2} + \|\mathbf{f}_2\|^2_{L^2(\Omega_2)^2} + \|\mathbf{f}_1\|_{L^2(\Omega_1)^2} + \|\mathbf{f}_2\|_{L^2(\Omega_2)^2}\Big],$$

$$(41)\quad \|\ell_{ih}\|_{H^s(\Omega_i)} \le C_s\,(1+h^{1/2-s})\Big[\,\|\tilde{\alpha}_i(\ell_{ih})\,|\nabla \mathbf{u}_{ih}|^2 \|_{L^1(\Omega_i)} + \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)}\,\Big],$$

for $s_0 < s \le 1/2$, where C_s is a constant depending only on s, Ω_1, Ω_2, Γ, $\|\gamma_1\|_{L^\infty(\mathbf{R})}$, $\|\gamma_2\|_{L^\infty(\mathbf{R})}$ and the aspect ratio of triangulations $\mathcal{T}_{1h}$ and $\mathcal{T}_{2h}$.

Proof We use Brouwer's fixed point Theorem. Let us consider the transformation $\mathcal{F}$ from space $E_h = (X_{1h} \times K_{2h}) \times (X_{2h} \times K_{2h})$ onto itself defined as follows : The image by $\mathcal{F}$ of an element $\big((\bar{\mathbf{u}}_{1h}, \bar{\ell}_{01h}),\ (\bar{\mathbf{u}}_{2h}, \bar{\ell}_{02h})\big) \in E_h$ is the element $(\,(\mathbf{u}_{1h}, \ell_{01h}),\ (\mathbf{u}_{2h}, \ell_{02h})\,) \in E_h$ defined in two steps :

Step 1: $\mathbf{u}_{1h}$ and $\mathbf{u}_{2h}$ are the velocity components of the solution of Problem 1 with data $\big((\bar{\mathbf{u}}_{1h}, \bar{\ell}_{01h}),\ (\bar{\mathbf{u}}_{2h}, \bar{\ell}_{02h})\big)$.

Step 2: ℓ_{01h} and ℓ_{02h} are the solution of Problem 2 with data $(\mathbf{u}_{1h}, \widehat{\ell}_{1h})$ and $(\mathbf{u}_{2h}, \widehat{\ell}_{2h})$, where $\widehat{\ell}_{ih} = \bar{\ell}_{0ih} + \bar{D}_{ih}$, $\bar{D}_{ih} = G_i(|P_{ih}(\bar{\mathbf{u}}_{1h} - \bar{\mathbf{u}}_{2h})|^2)$, $i = 1, 2$.

Transformation $\mathcal{F}$ is well defined, due to the uniqueness of solutions of Problems 1 and 2. It is a continuous mapping from space E_h onto itself, as all operator terms appearing in Problems 1 and 2 are continuous, when acting on spaces of finite dimension as it is the case.

Further, to estimate ℓ_{0ih}, let us recall that $\ell_{0ih} = \bar{\ell}_{ih} - D_{ih}$, with $\bar{\ell}_{ih}$ solution of (22) and $D_{ih} = G_i(|P_{ih}(\mathbf{u}_{1h} - \mathbf{u}_{2h})|^2)$. On one hand, estimate (25) reads

$$(42)\quad \|\bar{\ell}_{ih}\|_{L^2(\Omega_i)} \le C\,\Big[\|\tilde{\alpha}_i(\widehat{\ell}_{ih})\,|\nabla \mathbf{u}_{ih}|^2\|_{L^1(\Omega_i)} + \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)}\Big].$$

On another hand, denote $\mathbf{w}_h = \mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}$, $\mathbf{z}_h = P_{ih}(\mathbf{w}_h)$. Then, using the notation of the proof of Lemma 1, and the arguments employed in the estimate of **II**,

$$\|D_{ih}\|^2_{L^2(\Omega_i)} \le C\int_{\Omega_i}|\mathbf{z}_h|^4 = C\sum_{T\in\mathcal{C}_h}\int_T |\mathbf{z}_h|^4\,d\mathbf{x} \le C\,h^2\sum_{i=1}^M |\mathbf{w}_h(\alpha_i)|^4$$
$$\le C\,h\,\|\mathbf{w}_h\|^4_{L^4(\Gamma)} \le C\,h\,[\|\mathbf{u}_{1h}\|^2_{L^4(\Gamma)} + \|\mathbf{u}_{2h}\|^2_{L^4(\Gamma)}\,]^2.$$

Then, using the continuity of the injection of $H^{1/2}(\Gamma)$ into $L^4(\Gamma)$,

(43) $$\|D_{ih}\|_{L^2(\Omega_i)} \leq C\, h^{1/2}\, [\|\mathbf{u}_{1h}\|^2_{H^1(\Omega_1)^2} + \|\mathbf{u}_{2h}\|^2_{H^1(\Omega_2)^2}\,], \quad i = 1, 2.$$

Using estimates (42) and (43) and the boundedness of the $\tilde{\alpha}_i$,

(44) $$\|\ell_{0ih}\|_{L^2(\Omega_i)} \leq C\,(1 + h^{1/2})\,[\|\mathbf{u}_{1h}\|^2_{H^1(\Omega_1)^2} + \|\mathbf{u}_{2h}\|^2_{H^1(\Omega_2)^2}\,], \;\; i = 1, 2$$

Using now estimate (23), we deduce

$$\|\mathcal{F}\left((\bar{\mathbf{u}}_{1h}, \bar{\ell}_{01h}),\ (\bar{\mathbf{u}}_{2h}, \bar{\ell}_{02h})\right)\|_{H^1(\Omega_1)^2\times L^2(\Omega_1)\times H^1(\Omega_2)^2\times L^2(\Omega_2)} \leq M,$$

where

$$M = C\left[\,\|\mathbf{f}_1\|^2_{L^2(\Omega_1)^2} + \|\mathbf{f}_2\|^2_{L^2(\Omega_2)^2}\right].$$

By Brouwer's fixed point Theorem, we conclude that Problem (17) - (18) - (19) always admits a solution. Estimates (39), (40) and (41) are directly deduced from (23), (24) and (25). We are now ready to prove Theorem 1. □

Proof of Theorem 1 We perform this proof in three steps.

Step 1 : A priori estimates.

Let us start by finding a bound for the energies ℓ_{ih}. By Theorem 2, the sequence $\{\ell_{ih}\}_{h>0}$ is bounded in $H^s(\Omega_i)$, $i = 1, 2$, for any $s_0 < s \leq 1/2$.

Also, from estimates (39) and (40), the sequences $\{\mathbf{u}_{ih}\}_{h>0}$ and $\{p_{ih}\}_{h>0}$ are respectively bounded in $H^1(\Omega_i)^2$ and $L^2_0(\Omega_i)$, $i = 1, 2$.

Step 2 : Limit Momentum Equations.

Let us recall now that the embedding of $H^s(\Omega_i)$ in $L^2(\Omega_i)$ for any $s > 0$ is compact. Then, for $i = 1$ and $i = 2$ the sequence $\{\ell_{ih}\}_{h>0}$ contains a subsequence which is strongly convergent in $L^2(\Omega_i)$ to a function ℓ_i.

From the estimates for the velocities, we may find a subsequence of $\{\mathbf{u}_{ih}\}_{h>0}$ weakly convergent in $H^1(\Omega_i)^2$ to a function $\mathbf{u}_i$, $i = 1, 2$. Under this situation, it is first proved in [2] that $\{\mathbf{u}_{ih\,|_\Gamma}\}_{h>0}$ contains a subsequence strongly convergent in $L^3(\Gamma)$ to $\mathbf{u}_{i\,|_\Gamma}$. Next, that the corresponding subsequence $\{\tilde{\alpha}_i(\ell_{ih})\,|\nabla\mathbf{u}_{ih}|^2\}_{h>0}$ converges strongly to $\tilde{\alpha}_i(\ell)\,|\nabla\mathbf{u}_i|^2$ in $L^1(\Omega_i)$ or, equivalently, that $\{\mathbf{u}_{ih}\}_{h>0}$ converges to $\mathbf{u}_i$ in $H^1(\Omega_i)^2$ (We denote all subsequences in the same way).

Also, the corresponding subsequence of pressures $\{p_{ih}\}_{h>0}$ contains a subsequence weakly convergent in $L^2(\Omega_i)$ to p_i, $i = 1, 2$. Then, we may pass to the limit in (17) and deduce that $(\mathbf{u}_1, p_1, \ell_1)$ and $(\mathbf{u}_2, p_2, \ell_2)$ verify equation (10).

Step 3 : Limit KTE equation.

Consider $g_i \in \mathcal{D}(\Omega_i)$, $i = 1, 2$, and denote $Q_i = \mathcal{L}_i g_i$. As in the proof of Lemma 1, we have

$$\begin{aligned}\int_{\Omega_i} \ell_{ih}\, g_i\, d\mathbf{x} \;&=\; \int_{\Omega_i} \nabla\ell_{ih}\cdot\nabla Q\, d\mathbf{x} - \int_\Gamma \ell_{ih}\,\partial_{ni} Q\, d\tau \\ &=\; \int_{\Omega_i} \nabla\ell_{ih}\cdot\nabla Q_h\, d\mathbf{x} + \int_{\Omega_i} \nabla\ell_{ih}\cdot\nabla(Q - Q_h)\, d\mathbf{x}\end{aligned}$$

$$
\begin{aligned}
&- \int_\Gamma \delta_{ih}\, \partial_{ni} Q \, d\tau \\
&= \int_{\Omega_i} \tilde{\alpha}_i(\ell_{ih})\, |\nabla \mathbf{u}_{ih}|^2\, Q_h \, d\mathbf{x} + \int_{\Omega_i} \nabla D_{ih} \cdot \nabla (Q - Q_h)\, d\mathbf{x} \\
&\quad + \int_{\Omega_i} \nabla \ell_{0ih} \cdot \nabla (Q - Q_h)\, d\mathbf{x} - \int_\Gamma \delta_{ih}\, \partial_{ni} Q \, d\tau \\
&:= \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV},
\end{aligned}
\tag{45}
$$

where $\delta_{ih} = G_i(|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}|^2)$.

We next analyze the convergence of these four summands.

Limit of I As in Exercice 8.3 of Dupont and Scott (Cf. [9]), using the continuous embedding of $H^2(\Omega_i)$ into $W^{1/2,\infty}(\Omega_i)$,

$$
\|Q - Q_h\|_{L^\infty(\Omega_i)} \le C\, h^{1/2}\, \|Q\|_{W^{1/2,\infty}(\Omega_i)} \le C\, h^{1/2}\, \|Q\|_{H^2(\Omega_i)}.
$$

As $\{\tilde{\alpha}_i(\ell_{ih})\, |\nabla \mathbf{u}_{ih}|^2\}_{h>0}$ converges strongly to $\tilde{\alpha}_i(\ell)\, |\nabla \mathbf{u}_i|^2$ in $L^1(\Omega_i)$, then

$$
\lim_{h\to 0} \int_{\Omega_i} \tilde{\alpha}_i(\ell_{ih})\, |\nabla \mathbf{u}_{ih}|^2\, Q_h\, d\mathbf{x} = \int_{\Omega_i} \tilde{\alpha}_i(\ell_i)\, |\nabla \mathbf{u}_i|^2\, Q\, d\mathbf{x}.
$$

Limit of II Similarly to the estimation of **II** in the proof of Lemma 1, we have

$$
\begin{aligned}
|\mathbf{II}| &\le C\, h^{1/2}\, \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)}\, \|Q\|_{H^2(\Omega_i)} \\
&\le C\, h^{1/2}\, [\|\mathbf{u}_{1h}\|^2_{H^1(\Omega_1)^2} + \|\mathbf{u}_{2h}\|^2_{H^1(\Omega_2)^2}]\, \|Q\|_{H^2(\Omega_i)}.
\end{aligned}
$$

Then, the summand **II** vanishes in the limit $h \to 0$.

Limit of III Proceeding as in the estimate of **III** in the proof of Lemma 1, we obtain

$$
\begin{aligned}
|\mathbf{III}| \le C\, h^{1/2-s} &\left[\|\tilde{\alpha}_i(\bar{\ell}_i)\, |\nabla \mathbf{u}_{ih}|^2\|_{L^1(\Omega_i)} + \|\mathbf{u}_{1h\,|_\Gamma} - \mathbf{u}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)} \right] \\
&\times \|Q\|_{H^{2-s}(\Omega_i)}.
\end{aligned}
$$

Then, the summand **III** vanishes in the limit $h \to 0$.

Limit of IV Due to the continuity of the embedding of $H^{1/2}(\Gamma)$ in $L^4(\Gamma)$, $\mathbf{u}_{ih\,|_\Gamma}$ converges strongly to $\mathbf{u}_{|_\Gamma}$ in $L^4(\Gamma)$. Then, δ_{ih} converges strongly to $G_i(|\mathbf{u}_{1\,|_\Gamma} - \mathbf{u}_{2\,|_\Gamma}|^2)$ in $L^2(\Gamma)$. Then,

$$
\lim_{h\to 0} \int_\Gamma \delta_{ih}\, \partial_{ni} Q = \int_\Gamma G_i(|\mathbf{u}_{|_\Gamma} - \mathbf{u}_{|_\Gamma}|^2)\, \partial_{ni} Q.
$$

We have deduced that ℓ_i satisfies (11) for test functions $g_i \in \mathcal{D}(\Omega_i)$, $i = 1, 2$. Given $g_i \in L^2(\Omega_i)$, there exists a sequence $\{g_{in}\}_{n\ge 1} \subset \mathcal{D}(\Omega_i)$ convergent to g_i in $L^2(\Omega_i)$. Then, $\mathcal{L}_i g_{in}$ converges to $\mathcal{L}_i g_{in}$ in $H^2(\Omega_i)$ and, consequently,

$\partial_{n_i}(\mathcal{L}_i g_{in})$ converges to $\partial_{n_i}(\mathcal{L}_i g_i)$ in $H^{1/2}(\Gamma)$ and $\mathcal{L}_i g_{in}$ converges to $\mathcal{L}_i g_{in}$ in $C^0(\bar{\Omega}_i)$. Then, we may pass to the limit and conclude that (11) holds for any $g_i \in L^2(\Omega_i)$, $i = 1, 2$.

This completes the proof of Theorem 1. □

Remark 4 The extension of this result to 3D flows faces hard technical difficulties. Indeed, in 3D estimate (36) reads

$$\|g_h\|_{L^\infty(\Omega_i)} \le C\, h^{-1}\, \|\nabla g_h\|_{L^2(\Omega_i)}.$$

Then, estimates (25) becomes

$$\|\ell_{ih}\|_{H^s(\Omega_i)} \le C_s \Big[(1 + h^{-s})\, \|\tilde{\alpha}_i(\widehat{\ell_{ih}})\, |\nabla \bar{\mathbf{u}}_{ih}|^2\|_{L^1(\Omega_i)} \\ +(1 + h^{1/2-s})\, \|\bar{\mathbf{u}}_{1h\,|_\Gamma} - \bar{\mathbf{u}}_{2h\,|_\Gamma}\|^2_{L^4(\Gamma)} \Big].$$

This only yields that the sequence $\{\ell_{ih}\}_{h>0}$ is bounded in $L^2(\Omega_i)$, thus invalidating the compactness argument that proves Theorem 1.

This could be overcomed extending the analysis of Bramble [5], where the estimate

$$\|g_h\|_{L^\infty(\Omega_i)} \le C\, |\, log\, h\, |\, \|\nabla g_h\|_{L^2(\Omega_i)},$$

is proved. However, this only applies to 2D finite difference discretizations on uniform grids.

4 Numerical experiments

The goal of this section is to show some numerical results that simulate a realist physic case, validate the model with its boundaries conditions, and test the accuracy of the method with respect to the convexity of the domain.

Physically, we simulate the behaviour of a coupled model for ocean-atmosphere flow where the ocean is forced by the atmosphere. The flow in the atmosphere is generated by a imposed horizontal wind.

We propose two numerical test, where the computational domains have simple geometries. The difference between both tests is the convexity of the computational domains, which seemingly will affect the accuracy of the scheme. Indeed, the regularity of the continuous TKE ℓ_1 and ℓ_2 will depend on the smoothing properties of the inverse Laplacian operators $\mathcal{L}_i$. In their turn, these depend on the degree of convexity of the domain.

- **Test1: Convex Domain.** As computational domain for the atmosphere, we take a rectangular box, $\Omega_1 = (0, 5) \times (0, 1)$. Its boundary is decomposed into $\partial\Omega_1 = \Gamma_1^- \cup \Gamma_1^+ \cup \Gamma$, where
  - $\Gamma = [0, 5] \times \{0\}$ (interface);
  - $\Gamma_1^- = \{(x, y) \in \overline{\Omega}_1 \,/\, x = 0\}$ (inflow boundary); and

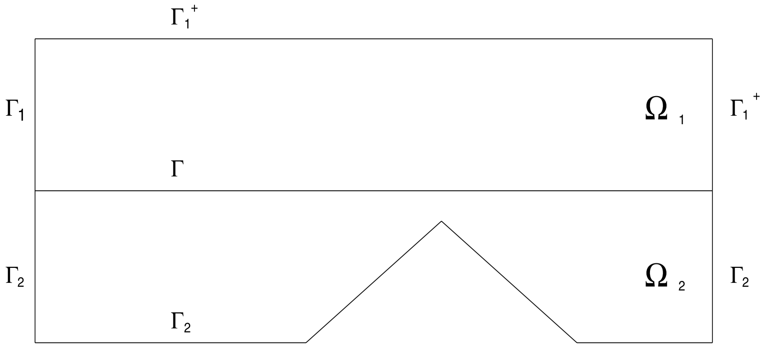

Fig. 1. Geometric configuration for Test 2

– $\Gamma_1^+ = \{(x, y) \in \overline{\Omega}_1 \,/\, y = 1\} \cup \{(x, y) \in \overline{\Omega}_1 \,/\, x = 5\}$ (outflow boundary).

The ocean is located in another rectangular domain $\Omega_2 = (0, 5) \times (-1, 0)$. Its boundary is decomposed into $\partial\Omega_2 = \Gamma_2 \cup \Gamma$, where $\Gamma_2 = \partial\Omega_2 \setminus \Gamma$. We shall impose no-slip boundary conditions on Γ_2.

- **Test2: Non-Convex Domain.** For this test the domain for the atmosphere is the same as in the Test1. For the ocean, we assume that there is a large submarine mountain. Concretely, we set

$$\Omega_2 = \{(x, y) \in \mathbf{R}^2 \,/\, 0 \le x \le 5,\ H(x) \le y \le 0\},$$

where

$$H(x) = \begin{cases} -1 & 0 \le x \le 2 \\ -2.6 + 0.8x & 2 < x \le 3 \\ 2.2 - 0.8x & 3 < x \le 4 \\ -1 & 4 < x \le 5 \end{cases}$$

The boundaries are decomposed as in Test1. The geometric configuration of Tests 1 and 2 is schematized in Figure 1.

The idea is to force the atmosphere-ocean system by a steady wind imposed in boundary Γ_1^- and look for a steady state of this forcing.

The system that we have solved is the following one:

$$(46)\qquad \begin{cases} \mathbf{u}_i \cdot \nabla\mathbf{u}_i - \operatorname{div}\big(\alpha_i(k_i)\,\nabla\mathbf{u}_i\big) + \nabla p_i = 0 & \text{in } \Omega_i,\ i = 1, 2, \\ \operatorname{div}\mathbf{u}_i = 0 & \text{in } \Omega_i,\ i = 1, 2, \\ \mathbf{u}_i \cdot \nabla k_i - \operatorname{div}\big(\gamma_i(k_i)\,\nabla k_i\big) = \alpha_i(k_i)\,|\nabla\mathbf{u}_i|^2 & \text{in } \Omega_i,\ i = 1, 2, \end{cases}$$

We have included here convection effects to solve a realistic flow.

We have imposed the following boundary conditions.

Boundary conditions To include an incoming wind into the atmosphere we consider the following modifications of the boundary conditions of the original system (1):

- Velocities and pressures:
  - $\mathbf{u}_1 = \mathbf{u}_1^-$ on Γ_1^-;
  - $\partial_n \mathbf{u}_1 + p_1 \cdot \mathbf{n} = 0$ on Γ_1^+;
  - $\mathbf{u}_2 = 0$ on Γ_2
  - $\tilde{\alpha}_i(\ell_i)\partial_{n_i}\mathbf{u}_i = c_{f_i}(u_{j_i} - u_i)|u_i - u_{j_i}|$ on Γ; $i = 1, 2$ with $j_1 = 2$ and $j_2 = 1$;

  Here, we take

$$\mathbf{u}_1^- = \begin{pmatrix} A_1 + A_2 y \\ 0 \end{pmatrix} \tag{47}$$

  where A_1 and A_2 are constant (linear velocity profile).
  The constant A_1 represents the jump of the horizontal velocity $u_1 - u_2$ across the interface Γ. It has been taken as a free parameter.
  Also, the constants c_{f_1} and c_{f_2} represent the relative effects of the roughness of the interface between atmosphere and ocean onto each flow. These are also free parameters of our model.
  To determine the constant A_2 we use the conditions

$$\tilde{\alpha}_1(\ell_1)\partial_{n_1}\mathbf{u}_1 = c_{f_1}(u_2 - u_1)|u_1 - u_2| \text{ on } \Gamma$$

  and

$$\partial_{n_1}\mathbf{u}_1 = A_2,$$

  issued from (47) using the expression for $\alpha_1(k)$ (See (49)). This yields

$$A_2 = \frac{c_{f_1}}{3.3 \times 10^{-4}} A_1^2.$$

  In our computations we have taken $A_1 = 0.5$, and $c_{f_1} = c_{f_2} = 10^{-3}$.
- Turbulent kinetic energy:
  - $k_2 = 0$, on Γ_2.
  - $k_1 = 0$, on Γ_1^-.
  - $\dfrac{\partial k_1}{\partial n} = 0$, on Γ_1^+.
  - $k_i = c_i|u_1 - u_2|^2$, on Γ.

To determine the constants c_1 and c_2, we consider that physically the TKE is of the order of a few percents of the kinetic energy of the mean flow. We have thus taken

$$c_1 = c_2 = 0.05.$$

Turbulent viscosities We have considered turbulent viscosities of the form

$$\alpha_i(k) = \gamma_i(k) = \nu_i + \nu_{i\tau}(k),$$

where

- ν_i is the cinematic viscosity, with values $\nu_1 = 1/3.000$ (air) and $\nu_2 = 1/300$ (water), and
- $\nu_{i\tau}$ is the eddy viscosity.

This expression for $\alpha_i(k)$ is proposed in [17], for a TKE + Mixing length one-equation model. The expression for the eddy viscosities is

$$\nu_{i\tau}(k) = c_\mu l_{i\mu} k^{1/2};$$

where $l_{i\mu}$ is a mixing length and c_μ is an empirical constant. In the previous expression $l_{i\mu}$ must contain the damping effects in the region close to the wall. In our case this wall is the interface Γ and we assume that the boundary layer is concentrated on Γ. Then, following [17] we set

$$l_{i\mu} = \chi c_\mu^{-3/4} y_i \left(1 - exp\left(\frac{-\,y_i^+}{100}\right)\right) \tag{48}$$

with $y_i^+ = \sqrt{k_i}\dfrac{1}{\nu_i} y_i$.

On the other hand, the experience shows that the turbulent boundary layer corresponds to $20 \le y_i^+ \le 100$. We have considered $y_i^+ = 100$, for $i = 1, 2$. This corresponds to

$$y_1 = \frac{54}{10^7} k^{-1/2}, \quad y_2 = \frac{36}{10^8} k^{-1/2}.$$

We have replaced these expressions in (48) to define the actual mixing length that we have used.

The constants $c_\mu = 0.09$ and $\chi = 0.41$ (Prandtl constant) are obtained via experimentation. Consequently, if denote $d_i = l_{i\mu}(y_i)$ then,

$$\begin{aligned} &\alpha_i(k) = \nu_i + d_i k^{-1/2}, \qquad i = 1, 2, \ \text{ with} \\ &d_1 = 0.277 \times 10^{-4} \ \text{ and } \ d_2 = 0.185 \times 10^{-5}. \end{aligned} \tag{49}$$

This expression does not satisfy (5). In practice, to compute $\alpha_i(k)$, we replace k by max $\{k,\ k_0\}$ for some very small $k_0 > 0$.

The functions G_i defined in (6) are now:

$$G_i(k) = \nu_i k + \frac{2}{3} d_i k^{3/2}.$$

For small k, the leading summand in this expression is largely the second one. In practice, we have taken

$$G_i(k) = \frac{2}{3} d_i k^{3/2}. \tag{50}$$

We stress that our purpose is to perform a qualitative analysis of our numerical results for a realist flow, rather than performing a highly accurate computation.

Discretization of the problem We replace k_{ih} by the new unknowns $\ell_i = G_i(k_i)$ where G_i are defined in (50), then we consider the problem:

$$\left\{\begin{array}{ll}
\mathbf{u}_i\,\nabla\mathbf{u}_i - \operatorname{div}\left(\tilde{\alpha}_i(\ell_i)\,\nabla\mathbf{u}_i\right) + \nabla p_i = 0 & \text{in } \Omega_i,\quad i = 1,2,\\
\operatorname{div}\mathbf{u}_i = 0 & \text{in } \Omega_i,\quad i = 1,2,\\
\mathbf{u}_i\,\nabla[G_i^{-1}(\ell_i)] - \Delta\ell_i = \tilde{\alpha}_i(\ell_i)\,|\nabla\mathbf{u}_i|^2 & \text{in } \Omega_i,\quad i = 1,2,\\
\text{with the boundary conditions} & \\
\mathbf{u}_1 = \mathbf{u}_1^- & \text{on } \Gamma_1^-;\\
\partial_n\mathbf{u}_1 + p_1\cdot\mathbf{n} = 0 & \text{on } \Gamma_1^+;\\
\mathbf{u}_2 = 0 & \text{on } \Gamma_2;\\
\ell_i = 0 & \text{on } \Gamma_i,\quad i = 1,2,\\
\tilde{\alpha}_i(\ell_i)\partial_{n_i}\mathbf{u}_i = c_{f_i}(u_{j_i} - u_i)|u_i - u_{j_i}| & \text{on } \Gamma;\quad i = 1,2,\ j_1 = 2\ j_2 = 1;\\
\ell_i = G_i^{-1}(c_i|\mathbf{u}_1 - \mathbf{u}_2|^2) & \text{on } \Gamma\quad i = 1,2.
\end{array}\right. \tag{51}$$

Due to the changement of boundary conditions, we now look for a solution of problem (51) such that

$$\mathbf{u}_1 \in \tilde{X}_1, \quad \text{with } \tilde{X}_1 = \{\mathbf{v} = (v_1, v_2) \in H^1(\Omega_1)\times H^1(\Omega_1);\ v_2 = 0 \ \text{ on } \Gamma\cup\Gamma_1^-\}$$

while the remaining variables are searched for in the same spaces as in problem (1).

Using the notations of section 2, we consider two triangulations $\mathcal{T}_{ih}$, one of each Ω_i, $i = 1, 2$, compatible in the sense of Hypothesis 1, and define the space

$$Y_{1h} = [V_{1h}]^2 \times \tilde{X}_1.$$

For simplicity of notation, we shall denote $Y_{2h} = X_{2h}$. We look for discrete velocities $\mathbf{u}_{1h}$ and $\mathbf{u}_{2h}$ respectively in Y_{1h} and Y_{2h}, and for pressures $p_{ih} \in M_{ih}$ (cf. 15). The pairs of spaces (Y_{ih}, M_{ih}) do not satisfy the discrete inf-suf condition (16). We shall overcome this difficulty by using a stabilized discretization technique.

To solve problem (51) we have used a time-stepping strategy which consists in looking at the solutions of (51) as steady states of the corresponding evolution problem. The time discretization is given by

$$\frac{d\mathbf{u}}{dt} \approx \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t^n}, \qquad \frac{dk}{dt} \approx \frac{k^{n+1} - k^n}{\Delta t^n},$$

where Δt^n is computed every step and we have used a strategy of local time step based on a local CFL condition.

To treat the incompressibility restriction we have used a Penalty Stabilized Method. The actual problem that we have solved is the following:

Step 1 Given $(\mathbf{u}^n_{ih}, \ell^n_{0ih}) \in Y_{ih} \times K_{i0h}$, $i = 1, 2$, obtain $(\mathbf{u}^{n+1}_{ih}, p^{n+1}_{ih}) \in Y_{ih} \times M_{ih}$, $i = 1, 2$, such that

$$\begin{cases} \left(\dfrac{\mathbf{u}^{n+1}_{ih} - \mathbf{u}^n_{ih}}{\Delta t^n}, \mathbf{v}_h\right) + a_{ih}(k^n_{ih}; \mathbf{u}^{n+1}_{ih}, \mathbf{v}_h) + b_i(\mathbf{v}_h, p^{n+1}_{ih}) - b_i(\mathbf{u}^{n+1}_{ih}, q_{ih}) \\ \quad + n_{ih}(\mathbf{u}^{n+1}_{ih}, \mathbf{u}^{n+1}_{jh}; \mathbf{u}^n_{ih}, \mathbf{u}^n_{jh}, \mathbf{v}_h) - T_{ES}(p^{n+1}_{ih}, q_{ih}) = 0 \\ \forall (\mathbf{v}_{ih}, q_{ih}) \in [Y_{ih} \cap H^1_0(\Omega)]^2 \times M_{ih}, \ i = 1, 2; \end{cases}$$
(52)

where

(53) $$k^n_{ih} = G_i^{-1}(l^n_{i0h} + D^n_{ih}), \ D^n_{ih} = G_i(|P_{ih}(u^n_{1h} - u^n_{2h})|^2),$$

and

- $a_{ih}(k; \mathbf{u}, \mathbf{v}) = \sum_{T \in \mathcal{T}_{ih}} \alpha_{iT}(k)(\nabla \mathbf{u}, \nabla \mathbf{v})_T$,
  $\alpha_{iT}(k)$ being a constant value on each element. This value is calculated as the arithmetic mean of the values $\alpha_i(k(a_{jT}))$, where a_{1T}, a_{2T}, a_{3T} are the vertex of triangle T;
- $n_{ih}(\mathbf{u}_i, \mathbf{u}_j; \mathbf{w}_i, \mathbf{w}_j, \mathbf{v}) = \sum_{I \in Z_h} ((u_i - u_j)|w_i - w_j|, v)_I$.
  Here we recall that, for instance, u_i denotes the first component of the velocity field $\mathbf{u}_i$.
  This form is a linearization of the form n defined in (13) and Z_h is the set defined by (14);
- $T_{ES}(p, q) = \sum_{T \in \mathcal{T}_{ih}} \tau_T (\nabla p, \nabla q)_T$,
  τ_T being a stabilizing coefficient of size $\mathcal{O}(h_T^2)$. This is the penalty stabilizing term of the pressure discretization. The actual expression for τ_T that we have taken is the one which is obtained by static condensation of the bubble in the Mini-Element (cf. [7]).

Step 2 Given $\mathbf{u}_{ih}^{n+1}, \ell_{i0h}^{n} \in Y_{ih} \times K_{i0h}, i = 1, 2$. Obtain $\ell_{i0h}^{n+1} \in K_{i0h}, i = 1, 2$ by

$$(54) \quad \begin{cases} \left(\dfrac{\ell_{i0h}^{n+1} - \ell_{i0h}^{n}}{\Delta t^n}, g_{ih} \right) + (\nabla \ell_{i0h}^{n+1}, \nabla g_{ih}) = C_{ih}(k_{ih}^{n}; \mathbf{u}_{ih}^{n+1}, g_{ih}) \\ \qquad -(\nabla D_{ih}^{n+1}, \nabla g_{ih}) - (\mathbf{u}_{ih}^{n+1} \cdot \nabla k_{ih}^{n}, g_{ih}), \ \forall g_{ih} \in K_{i0h}, \end{cases}$$

where $D_{ih}^{n+1} = G_i(|P_{ih}(\mathbf{u}_{1h}^{n+1} - \mathbf{u}_{2h}^{n+1})|^2)$, k_{ih}^{n} is given by (53), and

$$C_{ih}(k; \mathbf{u}, g) = \sum_{T \in \tau_{ih}} \alpha_{iT}(k)(|\nabla \mathbf{u}|^2, g)_T$$

where $\alpha_{iT}(k)$ is the value defined in Step 1.

Step 3 Once $\ell_{ih}^{n+1} = \ell_{i0h}^{n+1} + D_{ih}^{n+1}$ is known, we compute k_{ih}^{n+1} as

$$k_{ih}^{n+1} = \left(\frac{3}{2} \frac{1}{d_i} \ell_{ih}^{n+1} \right)^{2/3}.$$

The expression for a_{ih} and n_{ih} that we have considered maintain the coerciveness and monotonicity properties of the continuous linear Navier-Stokes operator.

Also, the expression for the source term $C_{ih}(k; \mathbf{u}, g)$ for ℓ_{ih} is a weak form of the expression $a_{ih}(k; \mathbf{u}, \mathbf{v})$. This ensures that this source term is bounded in $L^1(\Omega_i)$, similarly to the continuous case.

Finally, for simplicity we use an explicit discretization to the convection term in (54). Although we have not included the contribution of the term D_{ih}^{n+1} to the time derivative, this does not change the actual steady state of equation (54).

We stress that a special care must be put in treating the point where the boundaries $\partial\Omega_1$, $\partial\Omega_2$ and Γ meet. This should not be considered as an interface point to assign boundary conditions, but rather as a boundary point for both domains Ω_1 and Ω_2. Otherwise, an incompatibility between the boundary conditions on $\partial\Omega_1$, $\partial\Omega_2$ and the interface conditions on Γ may occur, yielding unphysical pressure results.

To start the above time-stepping procedure, we have set initial conditions which meet the prescribed boundary conditions, as simple as posible, starting from an ocean in rest.

The initial velocity is calculated in such a way that in the atmosphere it is the linear profile, and in the ocean it vanishes,

$$\mathbf{u}_{1h}^{0}(x, y) = \mathbf{u}_1^{-}, \ \text{ on } \Omega_1, \quad \mathbf{u}_{2h}^{0}(x, y) = \mathbf{0}, \ \text{ on } \Omega_2.$$

Also, we want the turbulent kinetic energy in the ocean at the initial instant to take the value cero. This is incompatible with the boundary condition $k_2 = c_2|u_1 - u_2|^2$, unless $c_2 = 0$, but this would yield $k_2(t) = 0$ at any $t > 0$. To solve this problem, we have set

$$k_{1h}^0(x, y) = c_1|u_{1h}^0(x, y) - u_{2h}^0(x, y)|^2 \quad \text{on } \Gamma,$$

$$k_{2h}^0(x, y) = c(t)|u_{1h}^0(x, y) - u_{2h}^0(x, y)|^2 \quad \text{on } \Gamma,$$

where $c(t)$ is a linear function which is zero at $t = 0$, and takes the value c_2 at $t = t_0$ (t_0 prefixed).

Numerical results Now we show the numerical results obtained for Tests 1 and 2, following the algorithm described above. For both tests, we use a reference triangulation $\mathcal{T}_{ih}$, $i = 1$ and 2, with 3200 nodes and 6004 triangular elements. These meshes satisfy Hypotheses 1 as we can see in Figure 2. We suppose that the system arrives to a stationary steady when

$$\frac{\|\mathbf{u}_i^{n+1} - \mathbf{u}_i^n\|_{L^2}}{\|\mathbf{u}_i^n\|_{L^2}} < 10^{-6}, \qquad \text{and} \qquad \frac{\|k_i^{n+1} - k_i^n\|_{L^2}}{\|k_i^n\|_{L^2}} < 10^{-6}.$$

To test the quality of our numerical results we have performed some qualitative and quantitative tests.

We may perform a quantitative test using that the flow in the atmosphere generated by our boundary condition is in fact a mixing layer flow. Then, the self-similarity is a good test for any numerical solver of steady states of this flow.

The basic parameter to define self-similarity profiles of mixing layers is the thickness δ of the layer. To define δ, we denote by x the longitudinal variable along the layer, and by y the cross-flow variable. For each x, we define y_1 such that

$$u_1(y_1) = u_1 + \sqrt{0.9}(u_2 - u_1)$$

where $u_1 = u(x, 1)$ and $u_2 = u(x, 0)$.

We then define the thickness $\delta = y_1$. Now, we may define the similarity profiles u_1^S and k^S for the velocity and energy, as

$$u_1^S(x, y^S) = \frac{u(x, y) - u_1}{u_2 - u_1}, \; k^S(x, y^S) = \frac{k(x, y)}{max_y k(x, y)}, \text{ whit } y^S = \frac{y}{\delta}$$

We say that a solution is self-similar if the corresponding similarity profiles are independent of x.

As we may observe in Figures 3 and 4, our results lead to good self-similar profiles for mean velocity and kinetic energy. These figures are obtained at

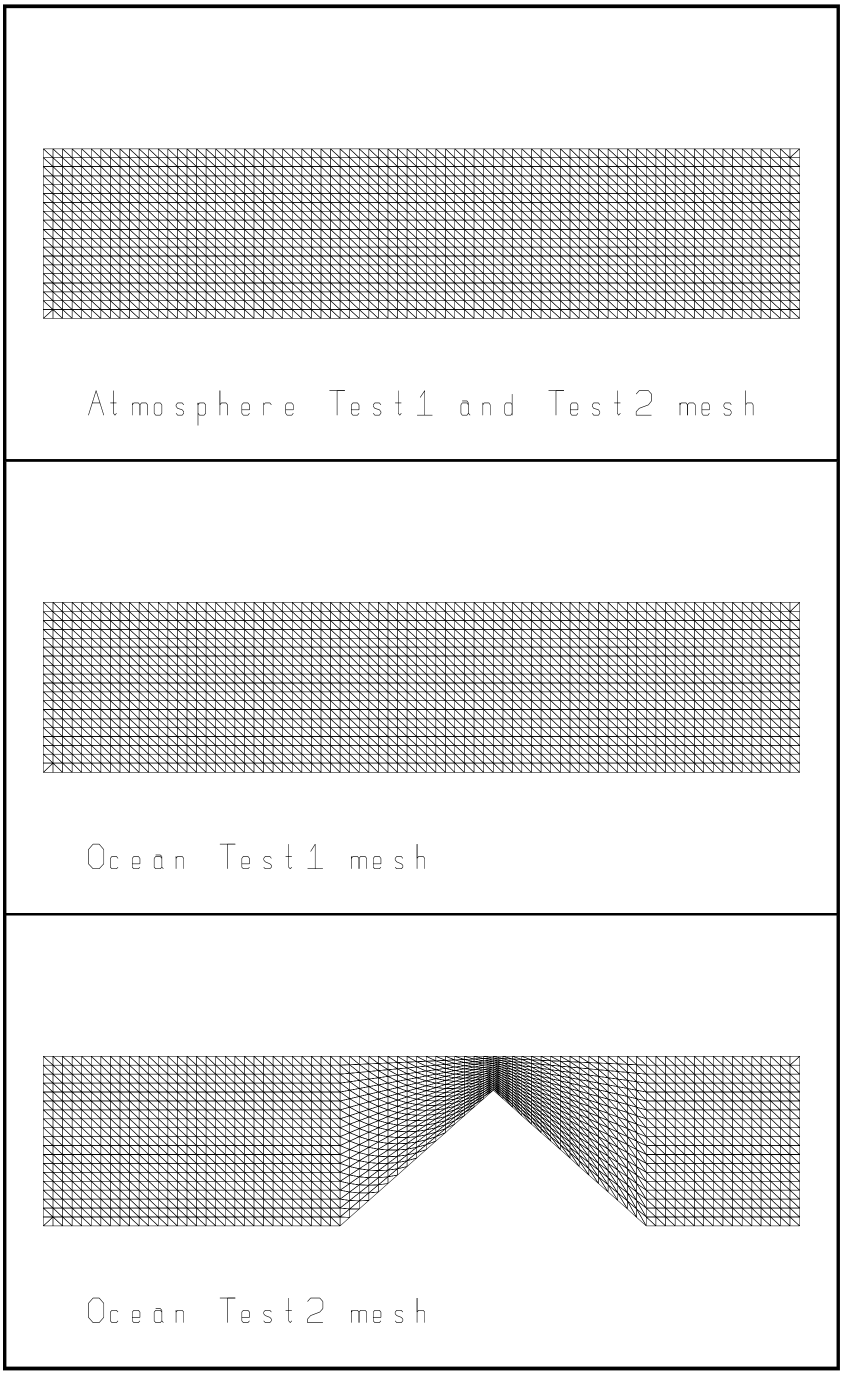
Atmosphere Test1 and Test2 mesh
Ocean Test1 mesh
Ocean Test2 mesh

Fig. 2. Meshes

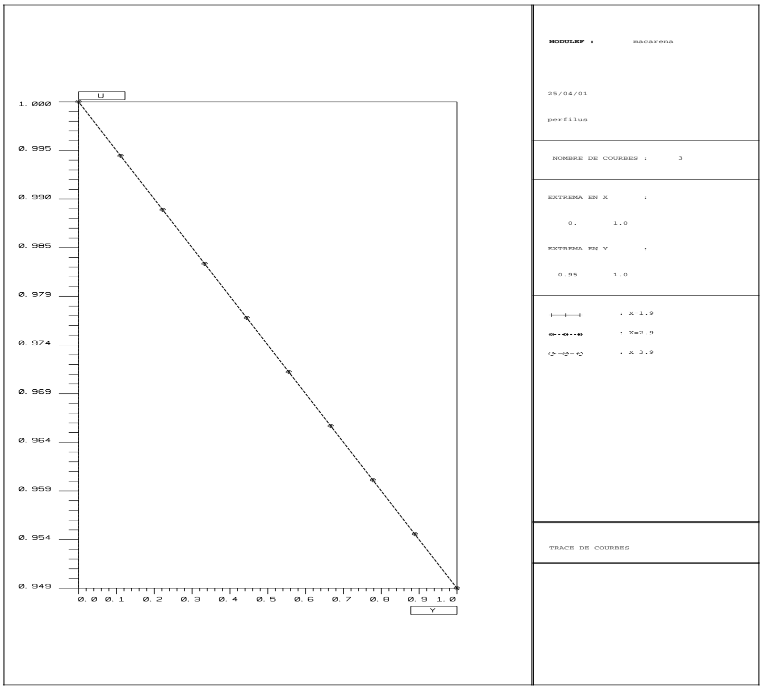

Fig. 3. Self-similar profile for velocity

different distances from the leading edge, concretely $x = 1.9$, $x = 2.9$ and $x = 3.9$.

To qualitatively test our solver, we look at the physical coherence of the results.

In Figures 5 and 6 we show the velocity fields for Test 1 and Test 2. Notice that the presence of the obstacle dramatically affects the oceanic flow, while the atmospheric flow remains practically unchanged. For Test 2, we may observe how the flow slows down a before arriving to the straitness and accelerates again after crossing it. On the other hand, the no-slip boundary conditions originate a recirculation in the right side of the mountain, and moreover we may see how the upper layer of the deep recirculating water mixes with the water near the interface while the bottom layer enters the left side of the mountain following a nearly parabolic profile (Figure 7).

Figure 8 displays the iso-pressure contour lines. The pressure in the atmosphere is nearly linear far from the leading edge, where an important decompression takes place. In the ocean, there is a respectively large decompression-compression in the initial and final points of the interface. The presence of the obstacle also produces a compression-decompression effect.

In Figure 9 we have represented the iso-TKE lines. A typical mixing layer appears for the atmospheric flow. Also, the TKE is created at the interface

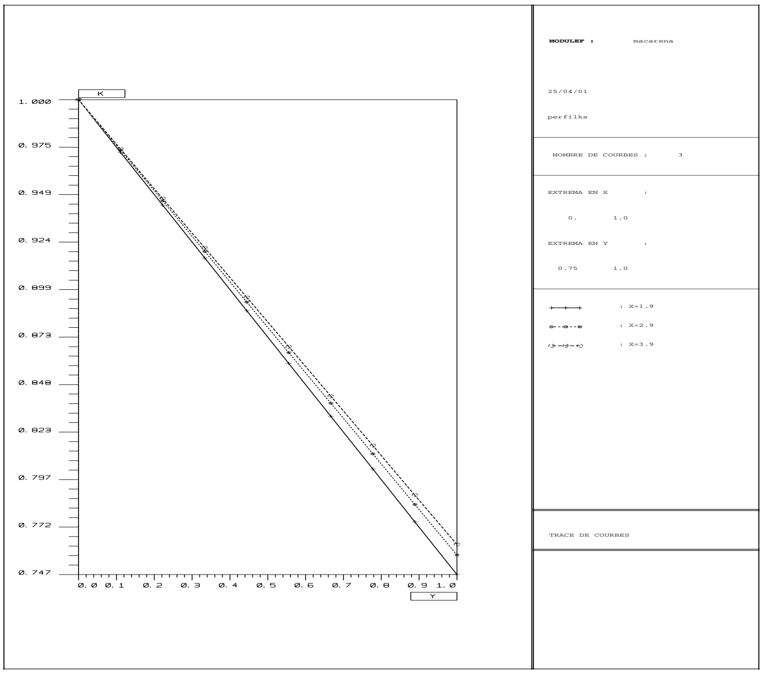

Fig. 4. Self-similar profile for kinetic energy

and is transported into the ocean by the recirculating flow, for both tests. Some instabilities appear due to convection dominance, which do not seem to be an obstacle for our purposes of estimating the effect of convexity on the accuracy of our scheme.

Finally, we have estimated the convergence order of the kinetic energy in both tests in the following way: let us call h the discretization parameter, and denote by $e(h) = k - k_h$, the discretization error in TKE, where k is the exact solution, and k_h is the approximated solution obtained by the previous algorithm.

Furthermore, we assume that the error $e(h)$ admits an asymptotic expansion of the form:

$$e(h) = \mu\, h^p + \mathcal{O}(h^{p+1}).$$

Then, taking three different values for h, we deduce that an approximation to p is a solution of the nonlinear equation

$$\frac{\|k_{h_1} - k_{h_2}\|_{L^2}}{\|k_{h_1} - k_{h_3}\|_{L^2}} = \frac{(h_2/h_1)^p - 1}{(h_3/h_1)^p - 1}.$$

The value h_1 corresponds to our reference grid, while $h_2 = 0.66 h_1$, $h_3 = 1.33 h_1$.

The result obtained for Test 1 (convex domain) is $p \approx 2.2$ and for Test 2 (non-convex domain) $p \approx 1.6$.

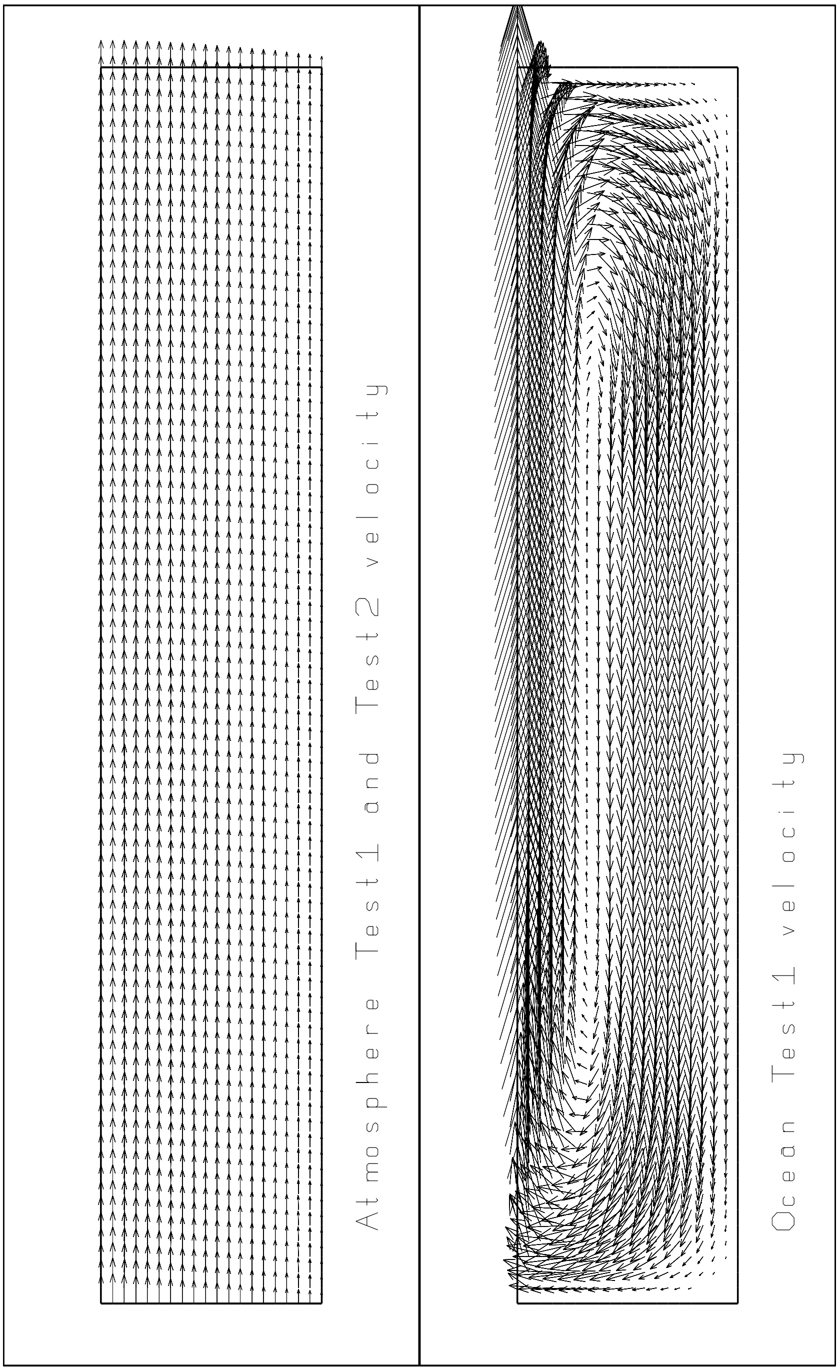

Fig. 5. Velocity

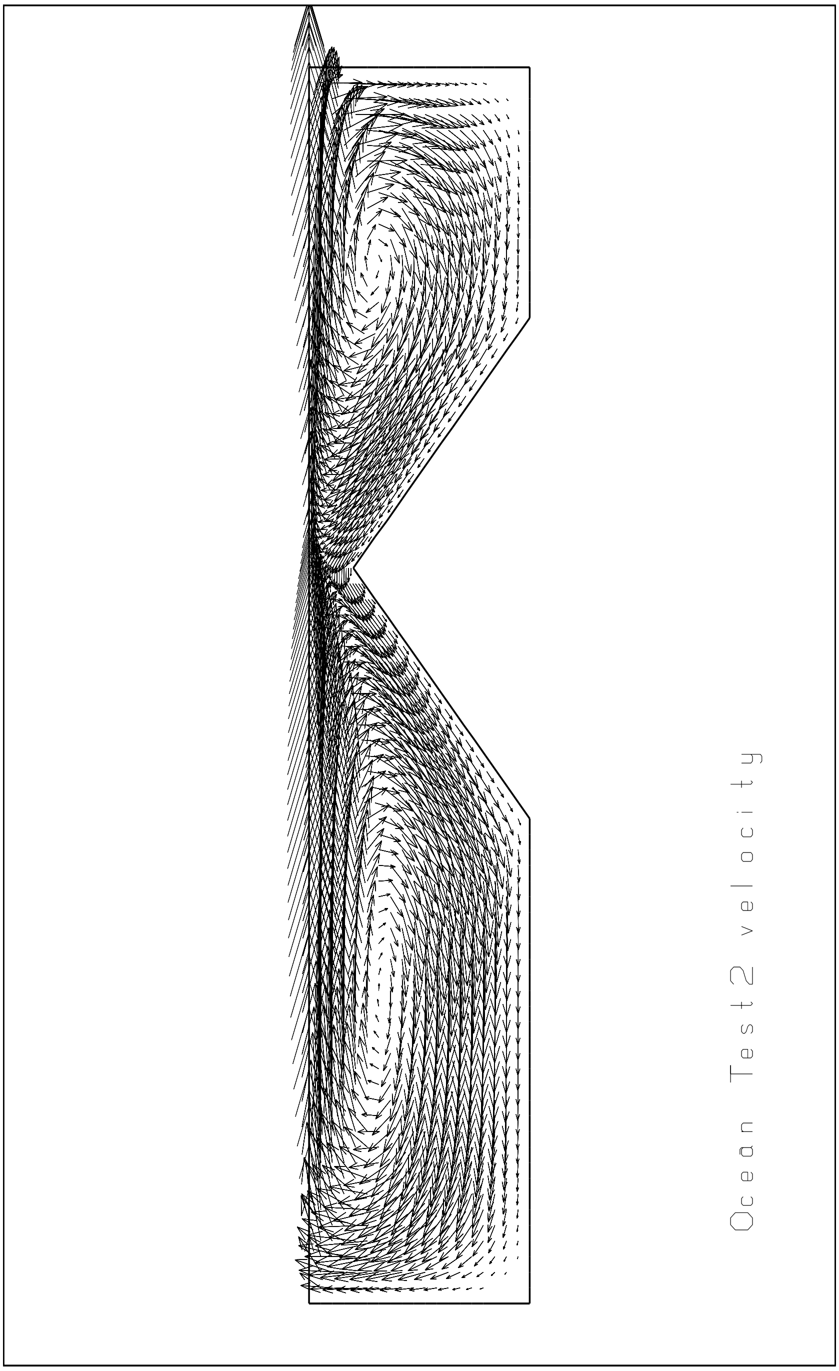

Fig. 6. Velocity for Test2

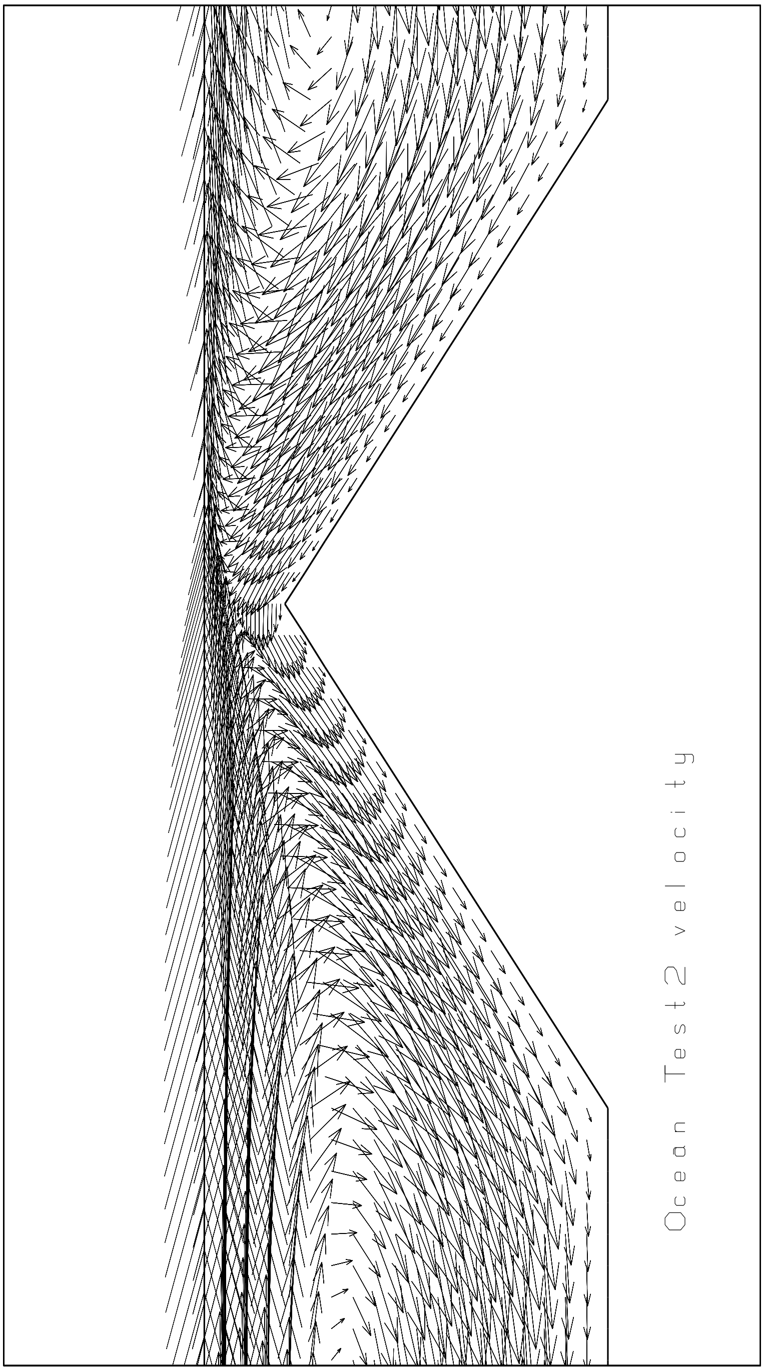

Fig. 7. Zoom of velocity for Test 2

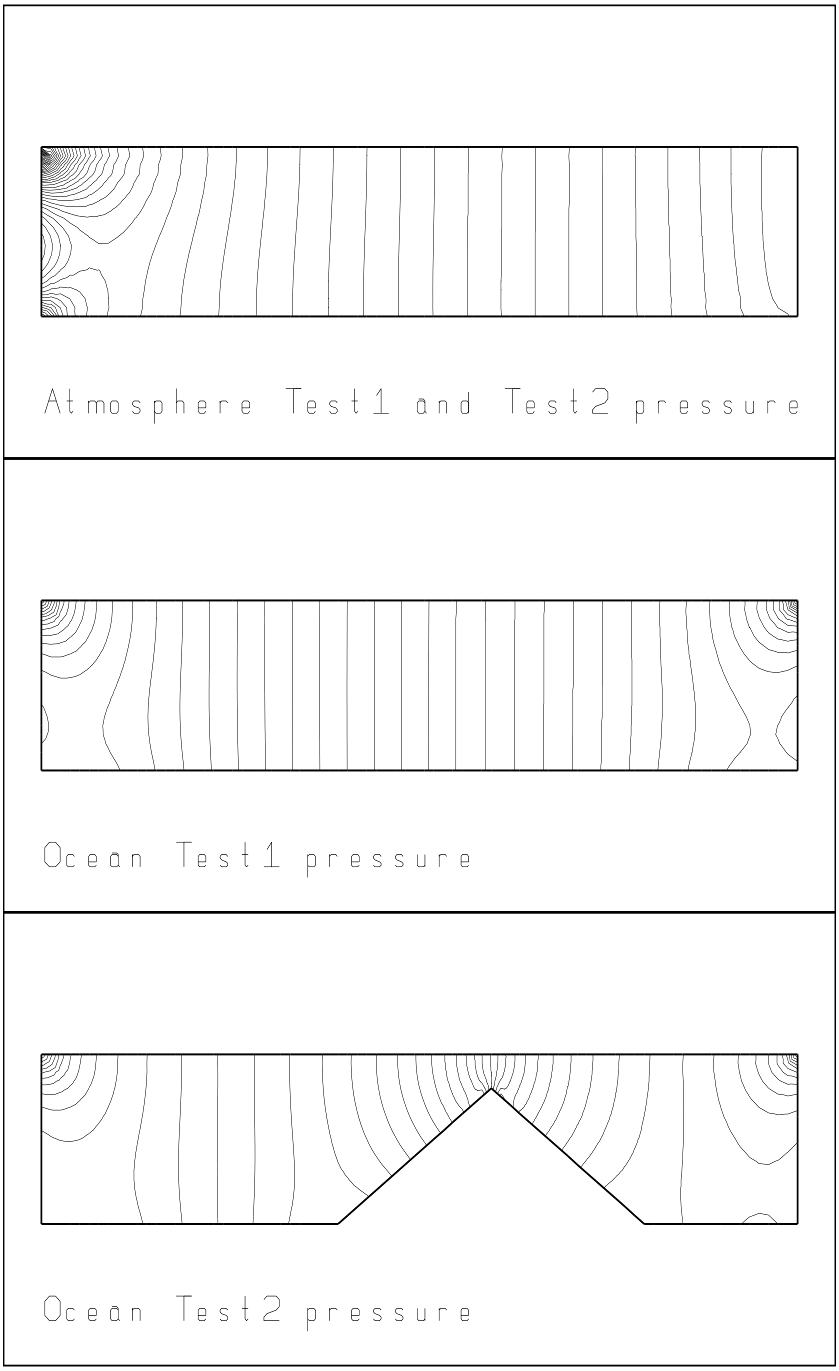

Fig. 8. Pressure

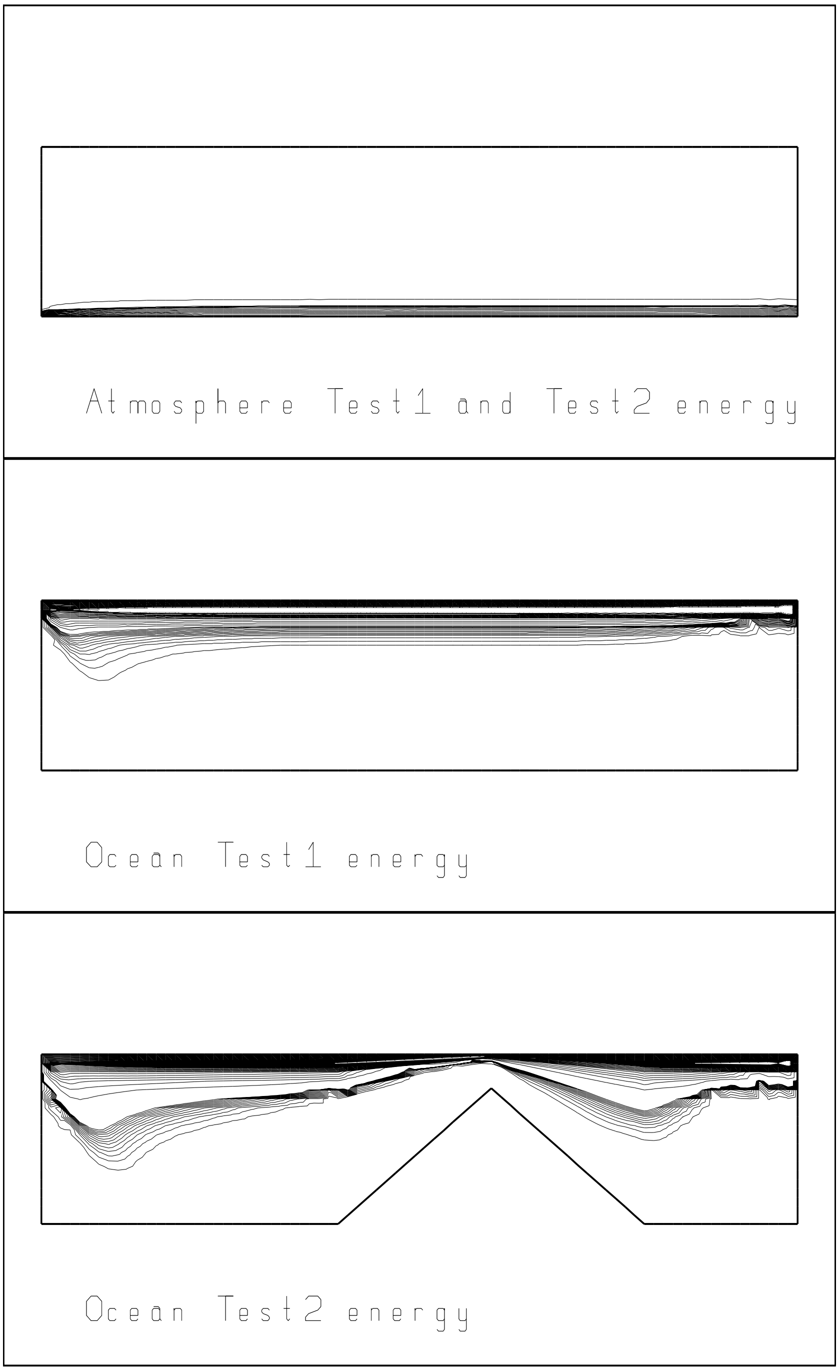

Fig. 9. Turbulent kinetic energy

As we expected, the convexity of the domain plays a relevant role in the accuracy of our numerical scheme.

Conclusion

As a conclusion, we may consider that our scheme satisfactorily reproduces the overall qualitative behaviour of a coupled turbulent system formed by two stratified flows. Also, that our convergence analysis holds in practice for realistic flows. These results encourage to deepen in the research developed in this paper, in two ways: By extending the convergence analysis to 3D flows an to more complex turbulent models, and also by performing more realistic simulations, in order to use our coupled numerical model with predictive purposes.

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