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A Model of Handwriting

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Abstract. The research reported here is concerned with hand trajectory planning for the class of movements involved in handwriting. Previous studies show that the kinematics of human two-joint arm movements in the horizontal plane can be described by a model which is based on dynamic minimization of the square of the third derivative of hand position (jerk), integrated over the entire movement. We extend this approach to both the analysis and the synthesis of the trajectories occurring in the generation of handwritten characters. Several basic strokes are identified and possible stroke concatenation rules are suggested. Given a concise symbolic representation of a stroke shape, a simple algorithm computes the complete kinematic specification of the corresponding trajectory. A handwriting generation model based on a *kinematics from shape* principle and on dynamic optimization is formulated and tested. Good qualitative and quantitative agreement was found between subject recordings and trajectories generated by the model. The simple symbolic representation of hand motion suggested here may permit the central nervous system to learn, store and modify motor action plans for writing in an efficient manner.

1 Introduction

Dynamic optimization has been suggested lately as an organizing principle for a broad class of motor tasks (Hogan 1982; Flash 1983; Nelson 1983; Hogan 1984). This paper describes an application of dynamic optimization to the study of movements involved in the production of cursive script. It proposes a model which relies on handwriting data and is constructive in that it not only explains the regularities apparent in the data but also produces cursive script which is qualitatively and quantitatively similar to a real one.

1.1 The Need for a Simple Description of Handwriting

Handwriting movements are an important subject of study in motor control theory for several reasons. Hollerbach (1981), e.g., mentions richness of trajectory shapes, ease of measurement and relatively small feedback influence, permitting insights into the central programming of the movement. There is another side to handwriting which differentiates it (together with speech articulation) from such motor behaviors as reaching or locomotion: it is a physical manifestation of complex cognitive processes.

The process of generation of handwritten characters involves translating a stream of symbols (which represent the characters on the cognitive level) into a stream of muscle activation commands. Several questions about the details of this translation procedure suggest themselves:

1. How do the parameters of a movement that produces a character shape relate to the geometry of that shape?

2. How can the central nervous system (CNS) represent the motor plans associated with writing movements in a simple and robust manner?

3. Are there planning principles which are common to handwriting and other, simpler multi-joint movements in the plane?

A simple, physiologically plausible mapping of character symbols into a set of primitive movements could provide constructive answers to all of the above questions. We propose that such a mapping exists and can be modelled by a relatively simple computer program. This model represents cursive characters as concatenations of *strokes*. A simple relation is proposed between the geometric shape of the strokes and the kinematics of their production. This relation is provided by an organizing principle recently shown to hold for a broad class of planar hand movements. Its existence permits to translate a symbolic description of a character into a specification of the hand trajectory which generates its written shape.

1.2 Existing Results on Dynamically Optimized Trajectory Planning

An organizing principle for hand movements in a plane proposed by Flash (1983) and Hogan (1984; Flash and Hogan 1985) is that of *dynamic optimization.* According to this principle, Cartesian hand trajectories are planned with the minimization of a cost function as an objective. The form of the cost function reflects the characteristics of the desired trajectory. For example, if the requirement is to produce smooth trajectories, then the cost function should penalize high values of the third derivative of position with respect to time.

A cost function penalizing the mean square of the third derivative of position *(jerk)* has the following form:

$$C=\int_0^{t_f}\left(\left(\frac{d^3x}{dt^3}\right)^2+\left(\frac{d^3y}{dt^3}\right)^2\right)dt, \tag{1}$$

where $x(t)$ and $y(t)$ are the hand coordinates. Note that if the boundary conditions demand complete rest (zero velocity and acceleration) at the trajectory endpoints

$$\begin{cases} \dot{x}(0)=0,\ \dot{y}(0)=0 \\ \ddot{x}(0)=0,\ \ddot{y}(0)=0 \\ \dot{x}(t_f)=0,\ \dot{y}(t_f)=0 \\ \ddot{x}(t_f)=0,\ \ddot{y}(t_f)=0 \end{cases} \tag{2}$$

then it is impossible to minimize mean square derivative of order lower than 3 [the results would be incompatible with (2) (Flash 1983)]. The cost function C and the boundary conditions (2) define a variational problem, which can be solved e.g. by integrating its associated Euler-Poisson equations (Flash and Hogan 1985). Jerk minimization, as expressed by (1), results in smooth movements, for which $x(t)$ and $y(t)$ are fifth-order polynomials in t: $x(t)=\sum_{n=0}^{5} a_n t^n$, $y(t)=\sum_{n=0}^{5} b_n t^n$. The coefficients a_n and b_n depend on the boundary conditions [including the initial and final positions $x(0)$, $y(0)$, $x(t_f)$, and $y(t_f)$].

Minimum jerk (MJ) point-to-point trajectories showed good agreement with experimental data (recordings of real movements). The MJ model was extended to simple curved movements by adding an intermediate point through which the trajectory was constrained to pass (a *via point*). The dependence of position on time in that case was found by using Pontryagin's Maximum principle and augmenting the cost function by the via point constraints (Bryson and Ho 1975). The position then turned out to be of the form

$$x(t)=\sum_{n=0}^{5} a_n t^n + p_x(t-t_1)_+^5, \tag{3}$$

where p_x depended on the boundary conditions and on the movement duration t_f, with $(t-t_1)_+$ defined as follows:

$$(t-t_1)_+=\begin{cases} t-t_1, & \text{if } t\geqq t_1; \\ 0, & \text{otherwise}. \end{cases}$$

Here t_1 is the time of passage through the via point, obtained together with a_n and p_x by solving the minimization problem. Equation (3) corresponds to a familiar result from spline theory, stating that a natural spline of degree $2m+1$ minimizes the L_2 norm of $d^m x/dt^m$ (de Boor and Lynch 1966).

Previous work (Flash and Hogan 1985) showed agreement between the MJ trajectories and measured hand movements for several kinds of motion.

Straight Movements. The requirement of point to point motion produced straight trajectories with bell-shaped velocity profiles, invariant under translation and rotation and scalable under changes in the movement amplitude A and duration t_f. The MJ model predicts the following relation between the maximum velocity v_{max} and the scale parameters: $v_{max}=1.875\,A/t_f$. This relation held for the recorded data within experimental error.

Curved Movements. The requirements of obstacle avoidance, passage through a via point or tracing a given curve all produced trajectories consisting of two low-curvature segments connected by a high-curvature one. The velocity profile was bimodal, with the minimum in the middle corresponding to the maximum in curvature. The size of the velocity peaks grew when the distance from the respective end points to the site of maximal curvature increased. The invariance under translation and rotation and the scaling property were present for curved movements too. Another feature of the recorded data was related to the isochrony principle (Viviani and Terzuolo 1982): the durations of the portions of movement before and after the via point were roughly equal.

A few fine details of the kinematics of real movements were not captured by the MJ model. These included the tendency of the first velocity peak to be higher than predicted in some movements, and the existence of small trajectory irregularities ("hooks") near the end point.

2 Relating the Kinematics of Handwriting to the Trajectory Shape

The model of Flash and Hogan (1985) was consistent with the data of Abend et al. (1982) on hand movements in the horizontal plane, and with the findings of Viviani and Terzuolo (1980, 1982) on handwriting. Specifically, the tangential velocity was found to be inversely related to the curvature of the path, and the trajectory characteristics were time- and amplitude-invariant. It was logical therefore to try and extend the MJ model to the generation of handwriting. The purpose of this extension was to find a method of computing the *kinematics* of handwritten trajectories from their *geometrical shape*. A simple relationship between shape and kinematics would add to our understanding of the way the CNS plans the details of a movement, given that the goal of the movement is to draw a character (i.e. to generate a certain geometrical shape).

2.1 Partitioning Handwriting into Strokes

Our basic hypothesis states that handwritten trajectories are planned in terms of simple segments, similar to the straight and curved movements whose characteristics were described above. The segmentation is manifest at the stage when people learn to draw their first characters: they do it stroke by stroke. As the writer's experience grows, the motor programs for various characters and perhaps for entire words may be compiled into a more efficient form. Developmentally, however, segmented writing is more basic.

We started therefore by postulating a representation of cursive characters in terms of simpler curve segments (strokes).

It appeared that the following stroke repertoire was sufficient: *hook*, *cup*, *gamma*, and *oval* (see Fig. 1). In addition, there were the straight, or almost straight strokes (as the vertical part in *t*). The hook and the cup were considered as separate strokes, despite their similarity. This distinction was due to the differences in the performance of the generation models for these two strokes. The breakdown of each standard script lowercase character into a sequence of strokes belonging to the chosen set is obvious. The number of strokes per character ranges from one (e.g. for an *o*) to three or four (for an *m*).

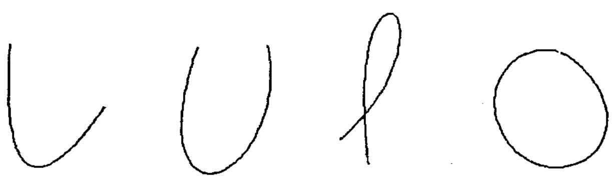

Fig. 1. The four basic stroke types – *hook*, *cup*, *gamma*, and *oval*. All cursive characters can be represented as combinations of rotated, scaled and translated strokes belonging to this set

Note that no claim is made as to the uniqueness of the suggested stroke set. Other partitions of characters into strokes are possible, provided that the basic set remains simple and compatible with the findings from human subjects (Wing 1980).

2.2 Definition of Extended Models

Of the four stroke types, only the hook can be adequately described by the MJ model with one via point per stroke (call it MJ_1). The number of parameters (degrees of freedom) required by MJ_1 per stroke per coordinate, which is equal to 7 (6 coefficients a_n plus p), is insufficient to produce more complex shapes like a cup or a gamma. The reason for this is as follows. The difference between a gamma and a cup on one hand, and a hook on the other, lies in the slope of the trajectory (dy/dx) near the start and end points. Under MJ_1 these slopes are determined by the boundary conditions and cannot be varied freely. A straightforward solution to this shortcoming of MJ_1 is to "upgrade" to MJ_2, i.e. use two via points instead of one. However, we find this solution lacking for two reasons:

1. The increase in the number of via points is bound to result in better and better matches between the model and the data simply because it is equivalent to approximating the data with splines of progressively finer knot resolution. Parsimony of representation requires therefore to restrict the number of points defining the curve to a minimum.

2. The placement of the additional via points is not dictated in any clear way by the shape of the curve. This is different from MJ_1, where the via point location is determined by the curvature maximum. Since the goal of modelling in the case of handwriting is to let the curve shape determine the kinematics of the motion, one must have a good reason for the choice of every via point locus.

An alternative to the increase of the number of via points is to improve the fit by imposing additional constraints at the end points and/or the "natural" via point of the curve (which is the curvature maximum for a hook, cup or gamma, or simply the point most distant from the endpoints for an oval). A kind of constraint which can turn a hook into something resembling a gamma or an oval is the specification of x- and y-components of the tangential velocity at the via point, v_{1z} and v_{1y} (the positive x and y directions are defined as left to right and bottom to top of the page, respectively). Thus, a $v_{1x} < 0$ condition should produce a gamma, and leaving v_{1x} positive while increasing its absolute value – a cup (making the path more rounded to accommodate the increased velocity at the via point without excessive jerk). An oval could be produced in a manner similar to a cup, but with the endpoints coinciding.

Let us call a minimum jerk model with one via point and a velocity constraint at the via point MJ_{1v}. The equations for $x(t)$ and $y(t)$ for this model can be obtained by solving the minimization problem adjoined by the appropriate constraints at the via point. The resulting expression for a coordinate is:

$$x(t) = \sum_{n=0}^{5} a_n t^n + p_{1x}(t-t_1)_+^4 + p_{2x}(t-t_1)_+^5 . \tag{4}$$

Note that the number of degrees of freedom of the model is increased from 7 to 8, to permit the addition of one more constraint per coordinate.

A somewhat different line of reasoning leads to another minimization model whose goal is to generate the postulated stroke types in a parsimonious manner. The form of this model becomes apparent after the following argument. Hook is well approximated by MJ_1, while cup is not. The discrepancy between the original and the simulation for a cup is at the beginning and the end of the trajectory. It makes sense therefore to try and improve the degree of fit there by dictating the initial and final *slopes* of the trajectory.

2.3 *Kinematics from Shape*

An important advantage of this formulation over MJ_{1v} lies in the fact that it translates the *shape* of the stroke (using slopes, as opposed to velocity) into the *kinematics* of the underlying trajectory. In other words, a relationship is established between the purpose of movements involved in handwriting (which is the production of certain shapes) and the details of their execution.

In order to specify the value of dy/dx at $t=0$ for a movement which obeys the initial conditions (2) one has to use L'Hôpital's rule and force the value of d^3y/dx^3 instead. Note that it is just the ratio of the y- and x-components of jerk, $j_y = \dddot{y}$ and $j_x = \dddot{x}$. Since in a minimum jerk model the initial and final jerk values are completely determined by the boundary conditions, minimization of a higher-order (fourth) derivative, *snap*, is necessary. On the other hand, constraining only the ratio and not the individual component values results in a discrepancy between the number of variables and the number of initial conditions that yield the equations for these variables. The j_x and j_y components must therefore be specified separately.

Snap minimization was considered before by Flash (1983). She found that a MS_1 model is at least as good as MJ_1 at describing the straight and simple hooked trajectories, with the boundary jerk values set to zero. The expression for $x(t)$ under MS_1 is

$$x(t) = \sum_{n=0}^{7} a_n t^n + p_x(t-t_1)_+^7 . \tag{5}$$

Analogously to (3), it may be seen as a spline of degree seven.

Taking the kinematics from shape principle to the limit, the magnitudes as well as the directions of the initial and final jerk in MS_1 can be computed from the relative positions of the start, via and end points. The relationships are as follows. The magnitudes can be be made proportional to the size of the stroke, expressed by the Euclidean distance from the start (end) point to the via point:

$$\begin{aligned} |j_0| &= J\sqrt{(x_1-x_0)^2+(y_1-y_0)^2} , \\ |j_f| &= J\sqrt{(x_f-x_1)^2+(y_f-y_1)^2} . \end{aligned} \tag{6}$$

The factor J in (6) depends on the subject, the stroke type and the writing conditions (more about that in the discussion).

The initial and final directions of jerk, θ_0 and θ_f (see Fig. 2) can be estimated from the planned shape in the following manner. Let α_0 and α_f be the directions from the start and the end point respectively to the via point. Then

$$\begin{aligned} \theta_0 &= \begin{cases} \alpha_0 + \delta(S), & \text{for} \quad S \in \{cup,\ oval,\ hook\}; \\ \alpha_0 - \delta(S), & \text{for} \quad S \in \{gamma\} \end{cases} \\ \theta_f &= \begin{cases} \alpha_f - \delta(S), & \text{for} \quad S \in \{cup,\ oval,\ hook\}; \\ \alpha_f + \delta(S), & \text{for} \quad S \in \{gamma\}, \end{cases} \end{aligned} \tag{7}$$

where $\delta(S)$ is an angle increment which depends on the stroke type, on J and on other parameters in a manner that assures the correct classification (but not necessarily the exact shape) of the resulting stroke.

The formulae (6) and (7) are independent of the desired orientation or absolute position of the planned shape. They also relate $|j_0|$ and $|j_f|$ to the size of the stroke, resulting in rotation, scale and translation invariant trajectory planning.

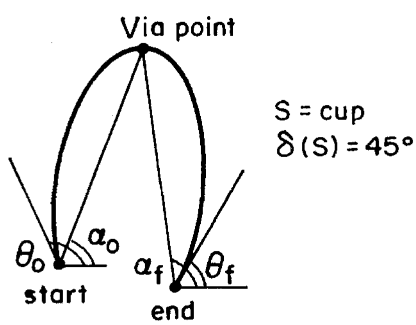

Fig. 2. The initial and final slopes of a stroke of any given type can be computed from the relative positions of the start, via and end points (see text)

The next section describes the experimental setup we used to collect handwriting data from subjects and the evaluation procedures applied to the two models, MJ_{1v} and MS_1, to test their compatibility with the data.

3 Experimental Measurements

3.1 Setup Description

We used a GTCO digitizing tablet to sample the position of the writing stylus at 100 Hz. The tablet was connected to a Symbolics 3670 Lisp Machine which recorded the data in real time and did the subsequent processing and analysis. The path described by the stylus and the time courses of velocity, acceleration and curvature were displayed on the screen within seconds. Several choices of preprocessing were available to the user.

Filters. The standard filter through which all data were passed was a digital FIR low-pass with a 5-Hz cutoff and Hamming window of width 11 (Chen 1979). The function of the filter was to reduce the high-frequency noise originating in writing surface friction and to average out the quantization error of the tablet. A moving average filter was available, to be used in cases when stronger smoothing was needed. This filter, normally of width 11, was compensated for window effects at the start and end positions.

Differentiation. Numeric differentiation is an ill-posed problem, requiring care when the precision of the results is important (Torre and Poggio 1986). Since we were going to use the values of jerk computed from the tablet data in model verification, we had to make sure those values made sense. Several differentiation methods were therefore compared. The first, standard one was the Lagrange polynomial method (see e.g. Atkinson and Harley 1983). It incorporates strong smoothing and works quite fast, but is far from precise. Atkinson and Harley suggest the *interpolating* cubic spline as a better alternative. For noisy data differentiation by computing the *approximating* cubic spline and taking its derivative is optimal (Torre and Poggio 1986). Both these methods were available to cross-check the results, but smoothing followed by spline interpolation was used in most cases, being less time-consuming than approximation.

3.2 Experimental Procedure

A set of 10 samples was gathered for each one of the 4 stroke types, for 3 subjects. The subjects were instructed to produce curves similar in shape to the standard stroke types. Two of the three subjects were naive as to the details of the models under test. Each time a curve was drawn, its complete kinematic description appeared on the computer screen (see e.g. Fig. 3) and the experimenter decided whether the recording was successful. To be accepted for further analysis a recording had to start and end at rest as stated by (2). The digitizing tablet operated in continuous mode, outputting every 10 ms a pair of coordinates and a z-axis bit signifying whether pressure was applied to the stylus. The data were recorded starting with the sample for which the z-axis bit came on until it went off. To enforce boundary conditions (2), the stylus had to be pressed before it was moved laterally and released after a complete stop. The regions of zero lateral motion were detected and discarded later by the software.

3.3 Model Evaluation Method

A good method to test whether a model fits the experimental data is to extract the parameters critical for the model from the data and use them in a simulation. Similarity between real and simulated results in this case indicates that the model and the chosen parameter set have captured the essential regularities of the data.

So far most models of handwriting production were evaluated by a subjective judgement of the degree of visual similarity between the real and the artificial trajectories (position and velocity time courses). This study used a more rigorous method of comparison. A numerical estimate of the degree of fit was obtained by computing the normalized correlation index for the following six data time sequence with their simulated counterparts: $x(t), \dot{x}(t), \ddot{x}(t), y(t), \dot{y}(t), \ddot{y}(t)$. The average of the six correlations served as a single figure of merit for each simulation. The correlation index for two sequences $a(t)=\{a_0, a_1, \ldots, a_n\}$ and $b(t)=\{b_0, b_1, \ldots, b_n\}$ was defined as

$$c(a,b)=\max_{0\leq r\leq R}\frac{\sum_{i=0}^{n-r}(a_i-\bar{a})(b_{i+r}-\bar{b})}{(n-r)\sqrt{\frac{1}{n}\sum_{i=0}^{n}(a_i-\bar{a})^2}\sqrt{\frac{1}{n}\sum_{i=0}^{n}(b_i-\bar{b})^2}}, \tag{8}$$

where $\bar{a}$ and $\bar{b}$ were the sequence means. R (the maximum permitted "time" shift between the two vectors) was equal to $0.1n$ (Bendat and Piersol 1966). This definition yields $c=1.0$ for $a_i=kb_i$ $(k>0)$; $c=-1.0$ for $a_i=kb_i$ $(k<0)$ and $-1.0<c<1.0$ otherwise. Thus, similar (but not necessarily identical) curves have high positive correlation values.

Each one of the two models, MJ_{1v} and MS_1, was tested on every one of the 120 recorded curves, yielding 240 correlation indices. The locations of the via points were set by the experimenter at the curvature maxima, while all the other parameters were extracted from the data. The precise description of the extraction procedure appears below.

3.4 Parameter Extraction

The determination of velocities at the via point, to be used by the MJ_{1v} model, presented no problems. The experimenter pointed at the desired location on the curve, and the values of v_x and v_y at the corresponding moment were extracted. Note that the time t_1 at the via point was computed by the model and not taken from the data.

The MS_1 model used the values of $\dddot{x}$ and $\dddot{y}$ at the locations where the derivative data were the most unreliable – at the trajectory endpoints. In order to

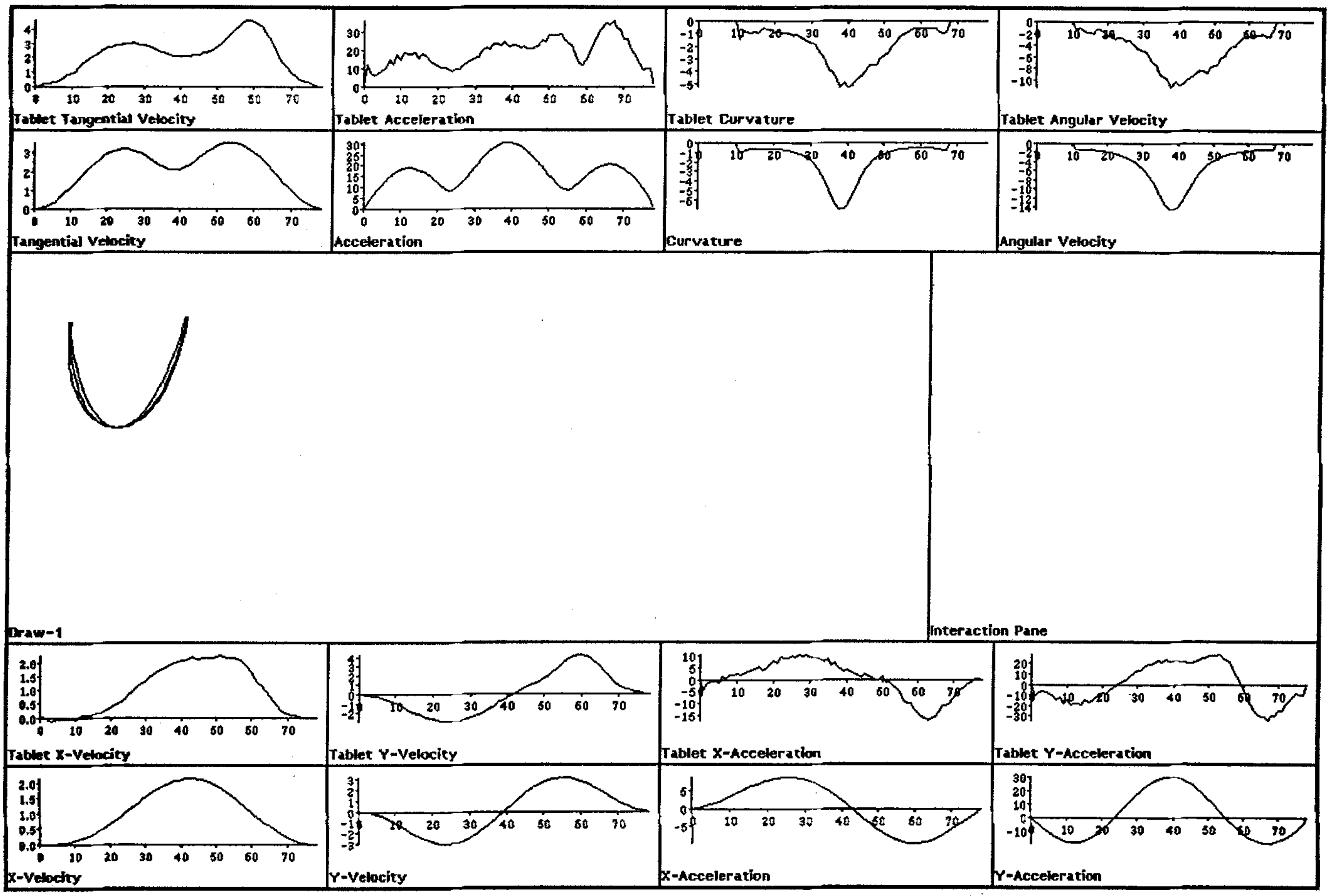

Fig. 3. A recording of a *hook* stroke and its simulations using the MJ_{1v} and the MS_1 models. In this and the next two figures the curves are barely distinguishable. The graphs marked "tablet" refer to the original recording. The others refer to the MS_1 simulation. The correlations are as follows:

Model	Avg	x	$\dot{x}$	$\ddot{x}$	y	$\dot{y}$	$\ddot{y}$
MJ_{1v}	0.966	0.985	0.955	0.923	0.982	1.000	0.954
MS_1	0.980	0.980	0.958	0.998	0.973	1.019	0.952

provide an estimate of the error in these values, the program drew short line segments tangent to the path at its ends (corresponding to the computed slopes). An option was available to specify the slopes manually, when there was a large discrepancy between the real and the computed values for a particular record. The magnitudes of jerk ($|j| = \sqrt{\dddot{x}^2 + \dddot{y}^2}$) were computed from the third derivatives of position [and also found using (6), with generally similar results].

4 Experimental Results

Two rounds of simulation were performed on the same data: the first time immediately after the records were made, the second – after all data have been gathered and the initial results were available. The purpose of the first round was to discard the bad recordings (these which didn't comply with conditions (2), and these in which the synchronization between the Lisp Machine and the tablet controller has slipped). During that round no indication was present of the precision of the j_0 and j_f values used by the snap minimization model MS_1. Also, the use of Lagrange polynomial differentiation method resulted in distorted derivative values near the beginning and the end of the movement. Nevertheless, the mean correlation indices for both models were high: 0.934 and 0.943 for MS_1 and MJ_{1v} respectively.

In the second round of simulations the time derivatives were computed using smoothing, followed by cubic spline interpolation (instead of the Lagrange polynomial method). The slopes were specified manually in the cases where the discrepancy between the real and the computed values was obvious. Jerk magni-

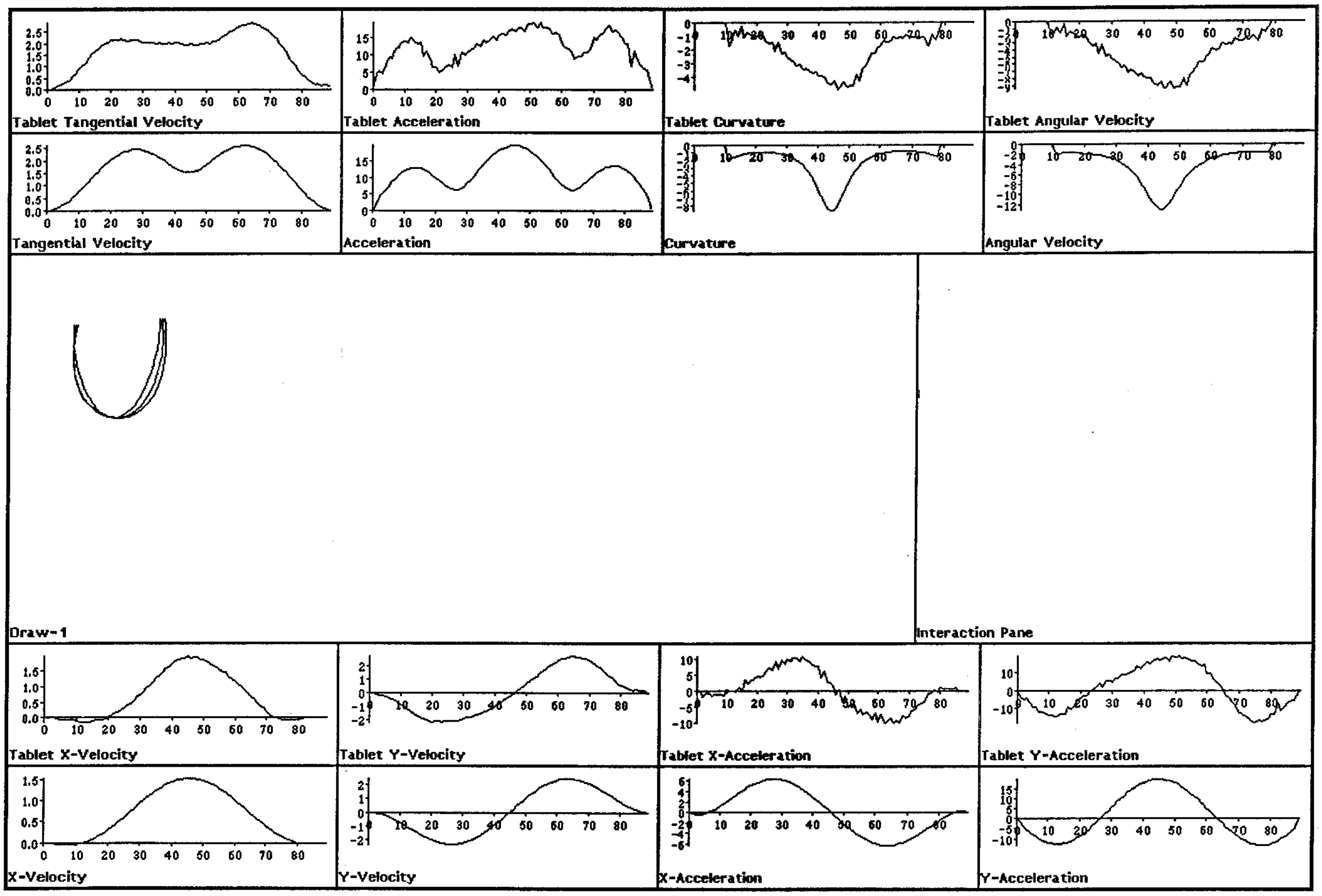

Fig. 4. A recording of a *cup* stroke and its simulations using the MJ_{1v} and the MS_1 models. The correlations are as follows:

Model	Avg	x	$\dot{x}$	$\ddot{x}$	y	$\dot{y}$	$\ddot{y}$
MJ_{1v}	0.989	0.987	0.993	0.993	0.984	0.999	0.976
MS_1	0.981	0.983	0.973	0.962	0.985	1.007	0.975

Table 1. Simulation statistics (means and standard deviations of the average correlation coefficients). For each subject, the upper and the lower lines correspond to the MJ_{1v} and the MS_1 models, respectively

Stroke *Subject*	*hook*	*cup*	*gamma*	*oval*
AVR	0.949 ± 0.020	0.981 ± 0.010	0.967 ± 0.024	0.951 ± 0.015
	0.962 ± 0.016	0.987 ± 0.014	0.967 ± 0.018	0.971 ± 0.015
DYM	0.980 ± 0.008	0.979 ± 0.015	0.938 ± 0.030	0.962 ± 0.017
	0.977 ± 0.007	0.969 ± 0.030	0.930 ± 0.070	0.975 ± 0.032
EDE	0.961 ± 0.021	0.969 ± 0.020	0.953 ± 0.024	0.948 ± 0.011
	0.964 ± 0.023	0.975 ± 0.018	0.969 ± 0.011	0.988 ± 0.010

tudes in all cases were computed using (6), putting to test the kinematics from geometry hypothesis.

Examples of recorded and simulated trajectories for each stroke type appear in Figs. 3–6. The statistical results of the second simulation are summarized in Table 1. The mean correlation measure was 0.962 ± 0.002 for MJ_{1v}, and 0.970 ± 0.003 for MS_1. The superiority of the minimum snap model was significant (Duncan's multiple-range test, critical range = 0.0065). The results were subjected to a three-way analysis of

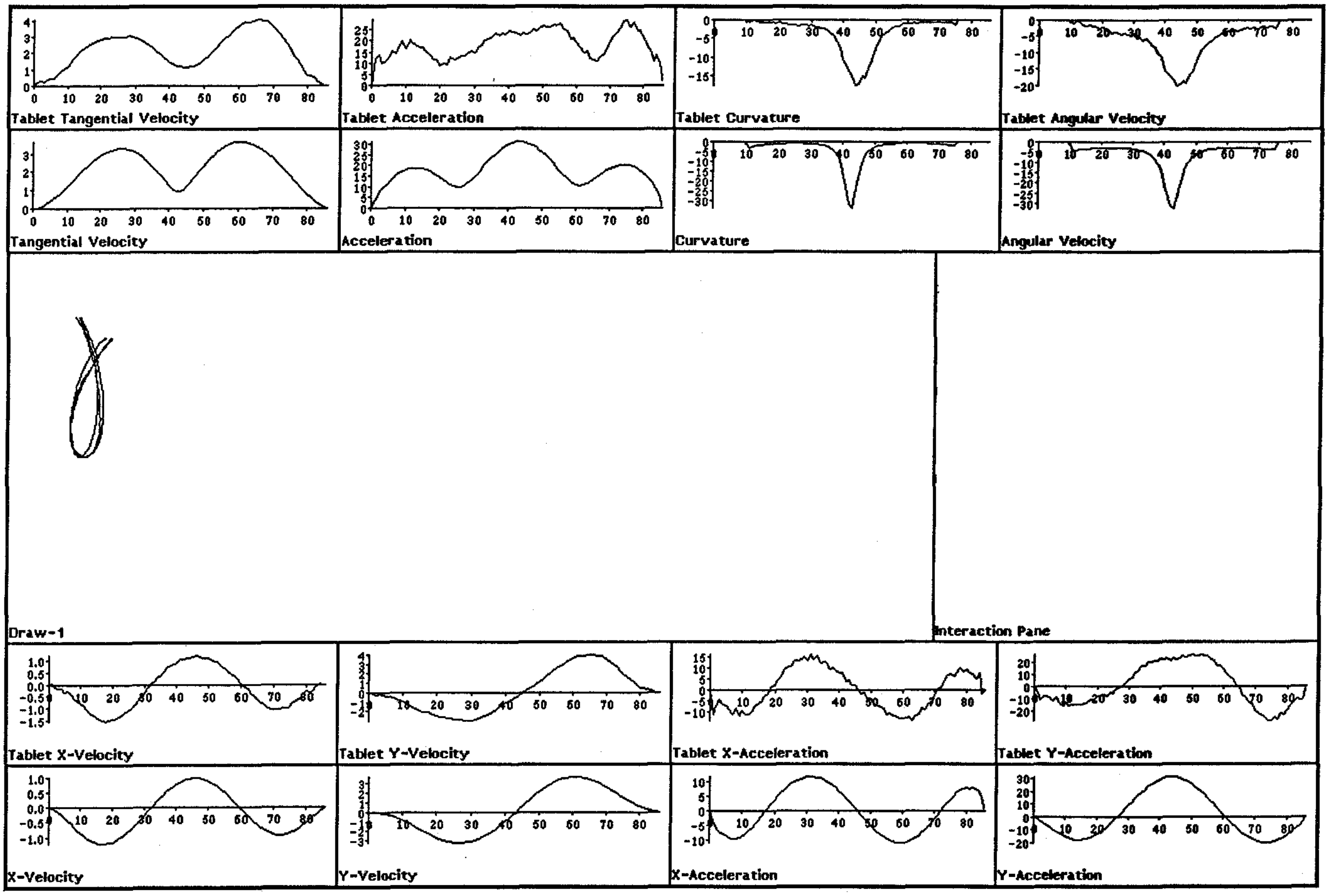

Fig. 5. A recording of a *gamma* stroke and its simulations using the MJ_{1v} and the MS_1 models. The correlations are as follows:

Model	Avg	x	$\dot{x}$	$\ddot{x}$	y	$\dot{y}$	$\ddot{y}$
MJ_{1v}	0.979	0.958	0.982	0.947	0.988	1.009	0.991
MS_1	0.988	0.962	0.999	0.976	0.982	1.020	0.988

variance (ANOVA). The three-way interaction, $F(12, 216) = 0.43 < 1$, and the main effect for subjects, $F(2, 216) = 0.35 < 1$, were not significant, as opposed to the main effects for strokes, $F(3, 216) = 9.10$, $p < 0.0001$, and models, $F(1, 216) = 7.07$, $p < 0.0084$. Duncan's test grouped all three subjects together, and distinguished between various stroke types [with the mean correlation for cup being the best (0.977), for oval and hook – intermediate (0.966), and for gamma – the worst (0.954)].

Of the six correlations averaged in the degree of fit estimation, the results for the x- and y-positions and velocities were usually better than for the accelerations. The simulated accelerations were smoother than the recorded ones, a possible indication that the minimization criterion incorporated in the models was too stringent, or that the optimization should include some other cost component in addition to snap or jerk.

The statistical data indicate that the MS_1 model performs better than MJ_{1v}. Visual judgement usually supported this conclusion. Because of that, and bearing in mind the arguments of Sect. 2 in favor of MS_1, we shall discuss below only the snap minimization model.

5 Discussion

5.1 How to Plan Trajectory Kinematics from Its Intended Shape

In this study we tried to come up with a plausible organization principle governing the production of handwritten trajectories as specified by a hypothesized simple representation of the character shapes. The MS_1 model provides such a principle. Its input parameters are as follows:

1. The initial and final "hand" (really, the writing implement) positions.

2. A via point, situated approximately in the middle of the planned trajectory. The role of this point is to set the amplitude of the movement (hence, the size of the resulting form) and to determine the gross direction.

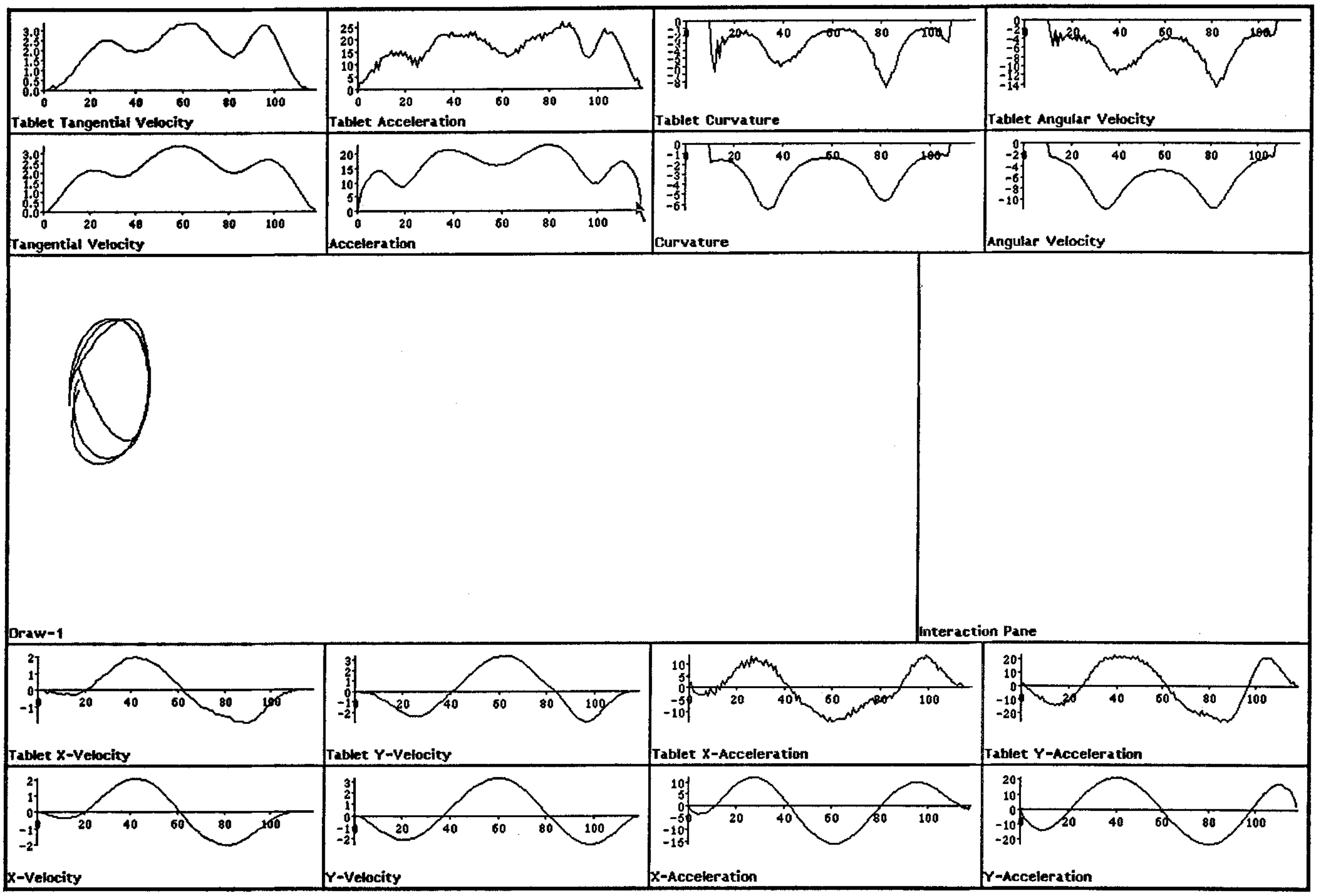

Fig. 6. A recording of an *oval* stroke and its simulations using the MJ_{1v} and the MS_1 models. The innermost curve was produced by MJ_{1v}. Note the clearly better performance of MS_1. The correlations are as follows:

Model	Avg	x	$\dot{x}$	$\ddot{x}$	y	$\dot{y}$	$\ddot{y}$
MJ_{1v}	0.957	0.969	0.956	0.865	1.005	0.993	0.952
MS_1	0.992	0.996	1.011	0.977	1.015	0.996	0.956

3. The initial and final directions of motion (corresponding to the directions of the jerk vectors there). These can be computed from the start, end and via positions.

4. The initial and final relative jerk magnitudes. These control the relative size of the first and second parts of the movement (before and after the via point). They too can be computed from position data.

5. The global jerk magnitude. This, together with the location of the via point, determines the global scale of the path form.

In the process of testing the model, all parameter values were extracted from real data. The degree of approximation of all kinematic characteristics of the trajectory was good, provided that the jerk estimation process succeeded (a problematic issue for noisy numerical data). More important, all the parameters except the global jerk magnitude could be reliably computed from the geometrical description of the resulting form, illustrating the kinematics from geometry principle. The MS_1 model incorporating this principle permitted the trajectory planner to distinguish between the important aspects of the intended movement and the irrelevant details. We argue for this distinction as follows.

The goal of the simulation as stated in the previous section has been to approximate the *exact shape* of the recorded trajectory. However, the goal of a person who is about to draw a letter shape is less well-defined. It can be stated as *the production of a curve which has a high probability of being recognized by a potential reader* as the intended letter. This goal is implicit in the writing process even if the writer is unaware of it, or fails to adhere to it because of haste or neglect (barring the cases of intentional bad penmanship). The realization of this fact helps one to distinguish between the parameters of model MS_1 that are important to subsequent recognition of the result and those that are not. Specifically, the size and roundness of the stroke and the small details of its shape are *not* important and can be excluded from the internal representation.

Such a representation can be purely symbolic and encode the geometrical and topological relationships (such as *above* and *connected*) between figure parts, without any metric constraints. It follows the spirit of Bernstein (1984, p. 114) in retaining only the parameters that are relevant to the purpose of the movement. The metric factors like the global jerk magnitude are free to vary under this scheme. This variation can account for the differences between handwriting samples taken under changing conditions (i.e. pen on paper vs. chalk on blackboard) or from different people.

5.2 A Simpler Model Doesn't Work

The success of MS_1 strengthens the arguments against using more via points in the simulations. While additional via points would be superfluous, the one used is necessary, as demonstrated by the totally inadequate performance of an MS_0 model (snap minimization without via point constraints). It was tested under the same conditions as MS_1 and failed to produce the oval shapes. For the other three stroke types there were large discrepancies from the real trajectories. The MS_1 model can therefore be considered as the simplest working model of handwriting production which uses the approach of minimization of a mean square derivative of position.

5.3 Extension to Cursive Script Production

Every handwriting generation model which describes characters in terms of strokes must incorporate a method for stroke concatenation. This comes quite naturally for MS_1. Obviously, connecting two strokes at a cusp is easy, since no slope continuity conditions are imposed. For a smooth join of two strokes one has to modify conditions (2) to allow nonzero final (initial) velocity for the first (second) stroke. This should result in a smooth (differentiable) join between the strokes, since the velocity is continuous. Such a splice requires from the motor system to plan for a coasting, rather than braking, final phase of the first stroke.

This concatenation scheme, together with the *kinematics from shape* principle outlined above, was incorporated into a computer program which "wrote down" character strings given by the user. The program first looked up the symbolic stroke description of the given string. For example, the printed representation of the Lisp structure describing the letter "a" is

```
# <Character "a"
  # <Stroke [OVAL, small, regular, middle-zone]>
  # <Splice [CUSP]>
  # <Stroke [HOOK, small, regular, middle-zone]>
  >
```

Note that all the slots in the Stroke structure are symbol-valued. Their meaning is as follows: small describes the size of the stroke (big is the other possibility); regular means no inversion of $\dot{x}$ at the via point (for a gamma it is retrograde). The zone slot indicates where should the stroke be placed with respect to the line. Its value can be upper, middle or lower.

The coordinates of the start, via and end points for each stroke were computed according to a set of rules from its symbolic description. The rest of the parameters of the MS_1 model were then computed using the relationships (6) and (7). The characters were drawn on the screen, and the detailed kinematics of the trajectories computed and displayed. Close resemblance to human handwriting was evident (see Fig. 7).

The program described above produced a standard idealized script, whose characteristics could be varied by changing the values of the default parameters, such as J and $\delta(S)$. If the writing is fast (as it is with proficient writers), the transitions between strokes may become less pronounced (e.g. a cusp may turn into a bend if there is no time for a full stop). A discrete model such as MS_1 can account for these changes by introducing random perturbations of the parameter values. In a forthcoming paper we will show how a continuous variation of the parameters defining a stroke can transform a gamma shape into a cusp, and vice versa.

5.4 Comparison with Previous Work

Of the more elaborate existing models of handwriting two chose to consider strokes as the basic building blocks (Mermelstein and Eden 1964; Morasso and Mussa Ivaldi 1982), while the third one used a continuous approach (Hollerbach 1981). *Mermelstein and Eden* represented a connected piece of script as a series of upstrokes and downstrokes, joined at the loci of zero $\dot{y}$ component. The strokes were characterized by sinusoidal velocity profiles. They were able to produce highly idealized-looking script; no comparisons with the kinematics of real writing movements were performed at the time and no entire word simulations were made.

As opposed to the piecemeal (stroke by stroke) trajectory generation, *Hollerbach* (1981) suggested a continuous model in which handwriting was produced by two orthogonal oscillatory movements (horizontal and vertical in the plane of the writing surface), superimposed on a constant-velocity rightward horizontal sweep. Letter shapes emerged as the oscillations were modulated in phase and amplitude. Hollerbach measured the constancy of the invariants of his model across subjects, but did not estimate the degree of fit by substituting parameters from actual handwriting samples into the model. The process of trajectory planning

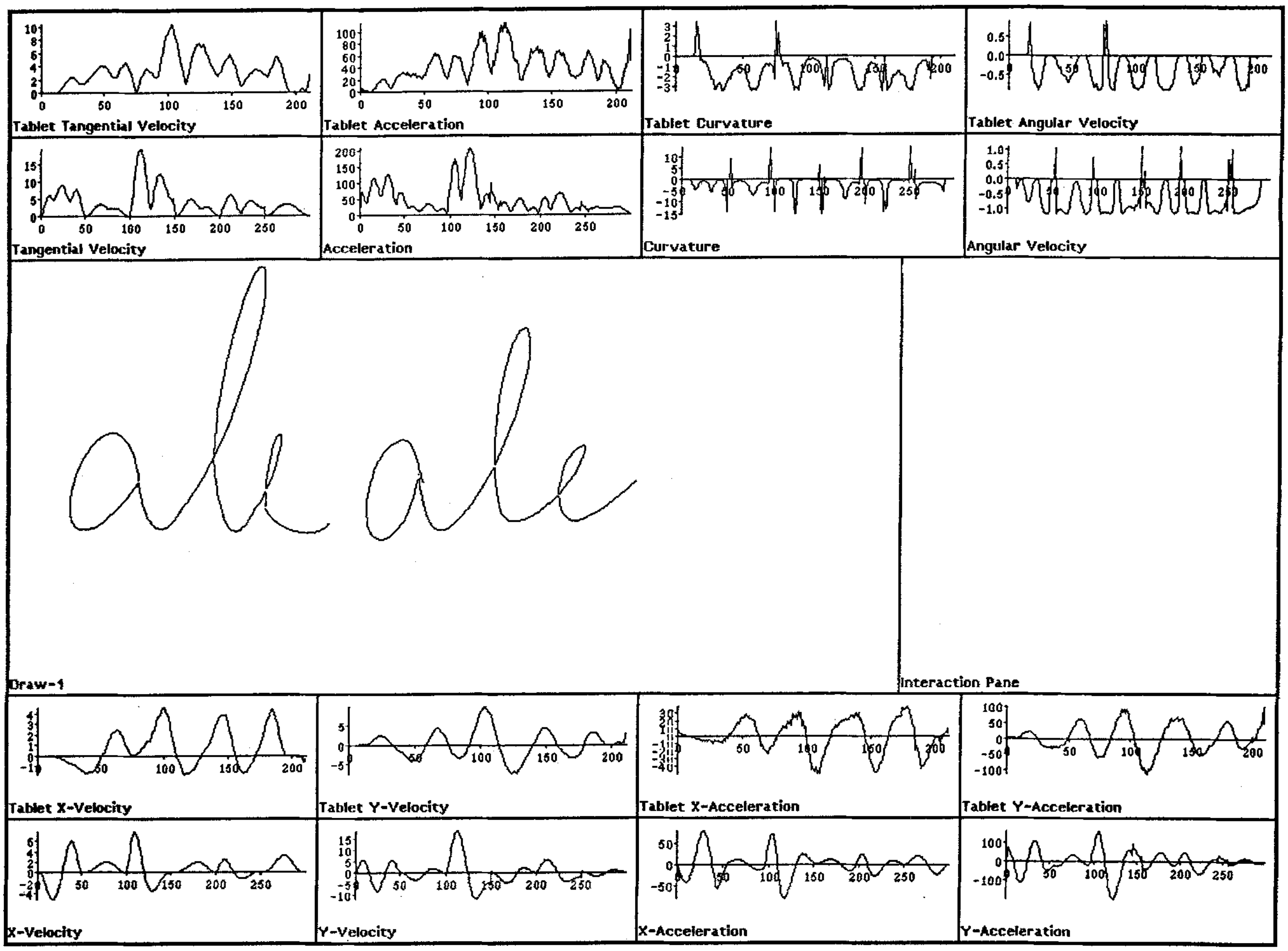

Fig. 7. An example of a recorded (right) and simulated (left) word, "ale". Similar features appear shifted on the graphs, because of the unequal durations. The vertical scale on the curvature and angular velocity graphs is compressed, to accomodate the high values at the cusps

using a symbolic description of a word was not fully specified by the oscillation model. On the other hand, Hollerbach's theory was compatible with a simple model of hand dynamics, under which the oscillations were produced by the equivalent of two coupled orthogonal spring systems.

Morasso and Mussa Ivaldi (1982) used a variant of the piecemeal generation approach, in which time-overlapped strokes combined linearly to produce the trajectory. Two different types of strokes were postulated: *(i)* rectilinear, with cubic time dependence and *(ii)* circular, with sinusoidal one. In the end, only the circular stroke version was retained, as having better agreement with the experimental data. The degree of fit was decided by visual comparison of the trajectory shapes and velocity and curvature time profiles. No explanation was attempted either for the overlapping-strokes approach, or for the origin of the stroke shapes. Morasso and Mussa Ivaldi made the suggestion that the stroke descriptions should ultimately be symbolic and not numeric, but did not implement it.

The main point of comparison between our model and those of Morasso and Mussa Ivaldi and of Hollerbach lies in the nature of the basic representation of the movement. In the model of Morasso and Mussa Ivaldi, the kinematics of motion are ultimately determined by the parameters of the constituent circular segments. These parameters, in turn, are found from the experimental data by an iterative procedure.

The oscillation model of Hollerbach uses *kinematic* parameters such as velocities and relative phases and amplitudes of motion to represent the movement. According to this model, motor programs for handwriting trajectories consist of sequences of phase and amplitude settings for the coupled oscillations underlying the handwriting.

The kinematics from shape principle, put forward in this paper, permits a simpler representation of handwriting movements, one that does not involve any kinematic parameters. Instead, our model uses a symbolic description of a stroke, determined by its shape. Using such descriptions, the CNS may plan handwriting movements in a high-level language, leaving the details of the motor programs to the implementation stage. The translation rules from the symbolic description into detailed trajectory plans may be common to other human movements, as indicated by the successful application of the dynamic optimization principle to a broad class of hand movements in the horizontal plane.

6 Summary

The main features of the proposed handwriting generation model are as follows:

1. It explicitly shows how *the trajectory form can determine the movement kinematics* [to paraphrase the title of Viviani and Terzuolo's paper (1982)].

2. It treats handwriting and other types of planar arm movement in a uniform manner, using the same general organizing principle: dynamic optimization.

3. It seems plausible from the developmental point of view. Children usually start by learning how to write bars, ovals and hooks (i.e. strokes) before graduating to letters and words. A study on the motor development in handwriting could provide the relevant comparative data.

The model is consistent with the available data on human arm movements in the horizontal plane in the following respects:

– The trajectory descriptions produced by the model are scalable with respect to the movement amplitude and duration.

– Substitution of real start, end and via points and slopes results in good agreement between the kinematics of the recorded and the simulated trajectories.

– Characters which cannot be interpreted as single strokes or as discontinuous concatenations of such, have places with near-zero acceleration, which may correspond to the inter-stroke splices.

– In characters consisting of a given stroke followed by one of a set of possible different strokes (e.g. *b*, *f*, *l*, *h*), the kinematics of the first stroke remain invariant.

We have presented a model of handwriting which postulates a hierarchical planning structure for the handwritten cursive script. According to the model, the movement is synthesized from strokes whose general topological and geometrical parameters (but not the exact shape) are determined by a simple computation from a concise symbolic description of the desired shape. The important parameters are the approximate relative locations of the start, via and end points, the roundness of the shape and its classification with respect to the direction of the via point velocity (inverted – like a gamma – or not). The movements that produce the strokes are guided by the same principle of dynamic optimization as other planar hand movements. An algorithm is suggested which permits the planning of trajectory kinematics from a symbolic description of the desired shape. The model is consistent with the available data and its implications are the subject of further investigation in our laboratory.

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