A Model of Random Industrial SAT

D. Barak-Pelleg¹ D. Berend^{2,3}

J.C. Saunders^{4,5,6}

Abstract

One of the most studied models of SAT is random SAT. In this model, instances are composed from clauses chosen uniformly randomly and independently of each other. This model may be unsatisfactory in that it fails to describe various features of SAT instances, arising in real-world applications. Various modifications have been suggested to define models of industrial SAT. Here, we focus mainly on the aspect of community structure. Namely, here the set of variables consists of a number of disjoint communities, and clauses tend to consist of variables from the same community. Thus, we suggest a model of random industrial SAT, in which the central generalization with respect to random SAT is the additional community structure.

There has been a lot of work on the satisfiability threshold of random k-SAT, starting with the calculation of the threshold of 2-SAT, up to the recent result that the threshold exists for sufficiently large k.

In this paper, we endeavor to study the satisfiability threshold for the proposed model of random industrial SAT. Our main result is that the threshold in this model tends to be smaller than its counterpart for random SAT. Moreover, under some conditions, this threshold even vanishes.

1 Introduction

For both historical and practical reasons, the Boolean satisfiability problem (SAT) is one of the most important problems in theoretical computer science. It was the first problem proven to be NP-complete [11]. Since its introduction, there has been growing interest in the problem, and many aspects of the problem have been researched.

In this problem, one is required to determine whether a certain Boolean formula is satisfiable. An instance of the problem consists of a Boolean formula in several variables v_1, \ldots, v_n . The formula is usually given in conjunctive normal form (CNF). The basic building block of the formula is a *literal*, which is either a variable v_j or its negation \overline{v}_j . A *clause* is a disjunction of the form $l_1 \lor \ldots \lor l_k$ of several distinct literals. Thus, altogether, the formula looks like $C_1 \land C_2 \land \ldots \land C_m$,

¹Department of Mathematics, Ben-Gurion University, Beer Sheva 84105, Israel. E-mail: dinabar@post.bgu.ac.il

²Departments of Mathematics and Computer Science, Ben-Gurion University, Beer Sheva 84105, Israel. E-mail: berend@math.bgu.ac.il

³Research supported in part by the Milken Families Foundation Chair in Mathematics.

⁴Department of Mathematics, Ben-Gurion University, Beer Sheva 84105, Israel. E-mail: saunders@post.bgu.ac.il ⁵Research supported by an Azrieli Fellowship from the Azrieli Foundation.

⁶Current address: Department of Mathematics & Statistics, University of Calgary, 2500 University Drive NW, Calgary, Alberta, Canada, T2N 1N4.

where each C_i is a clause, say $C_i = l_{i,1} \vee ... \vee l_{i,k_i}$. Given a formula, one may assign a TRUE/FALSE value to each of the variables $v_1, ..., v_n$. The formula is *satisfiable*, or SAT, if there exists an assignment under which the formula is TRUE, and is *unsatisfiable*, or UNSAT, otherwise.

The k-satisfiability (k-SAT) problem is a special case of SAT, in which each clause is a disjunction of up to k literals. Some authors restrict k-SAT to instances with exactly k literals in each clause, which terminology we will follow here. Given n and k, let $\Omega(n, k)$ denote the set of all $\binom{n}{k}2^k$ possible clauses of length k over n Boolean variables. A random k-SAT instance with m clauses is a uniformly random element of $(\Omega(n, k))^m$. Namely, it consists of m clauses, selected uniformly randomly and independently from $\Omega(n, k)$. Thus, clause repetitions are allowed, and two instances, differing in the order of the clauses only, are considered as distinct.

The ratio m/n is the *density* and denoted by α . This parameter turns out to be very important. If α is sufficiently small, then a large random instance with density α is SAT with high probability, whereas if it is sufficiently large then a large random instance is UNSAT with high probability. Despite its loose name, the notion of "with high probability" is well defined. Let $(E_j)_{j=1}^{\infty}$ be a sequence of events. The event E_j occurs with high probability (w.h.p.) if $P(E_j) \xrightarrow{j \to \infty} 1$. In our case, we take larger and larger random instances with some fixed density, and inquire whether they are SAT or UNSAT. For $k \ge 2$, denote [1]:

$$\begin{split} r_k &\equiv \sup\{\alpha: \text{A random density-}\alpha \text{ instance is SAT w.h.p.}\} \text{,} \\ r_k^* &\equiv \inf\{\alpha: \text{A random density-}\alpha \text{ instance is UNSAT w.h.p.}\} \text{.} \end{split}$$

For k = 2, it was proved long ago [10, 16, 22] that $r_2 = r_2^* = 1$. The Satisfiability Threshold Conjecture claims that, in fact, $r_k = r_k^*$ for all k [10]. This conjectured common value is the *satisfiability threshold*. It has been a subject of interest among researchers, theoretically and empirically, to prove the conjecture for $k \ge 3$ and find the threshold. Recently, the conjecture has been proved for large enough k [13].

As part of this research, lower and upper bounds were obtained on r_k and r_k^* for $k \ge 3$. In [17] it was proven that $r_k^* \le 2^k \ln 2$. This has been improved in [25] to $r_k^* \le 2^k \ln 2 - \frac{1}{2}(1 + \ln 2) + \varepsilon_k$. From the other side, a sequence of successive improvements led finally to the bound $r_k \ge 2^k \ln 2 - \frac{1}{2}(1 + \ln 2) + \varepsilon_k$ [12]. Thus, with the satisfiability conjecture settled in [13] for large k, it follows that $r_k = r_k^* = 2^k \ln 2 - \frac{1}{2}(1 + \ln 2) + \varepsilon_k$ for such k. For small values of k, more specific results were obtained. For k = 3, the best bounds seem to be $r_3 \ge 3.52$ [23, 24], and $r_3^* \le 4.4898$ [14]. Experiments and other results of heuristics, based on statistical physics considerations, indicate that $r_3 \approx 4.26$ [29, 30], $r_4 \approx 9.93$, $r_5 \approx 21.12$, $r_6 \approx 43.37$, $r_7 \approx 87.79$ [29].

Much more is known about 2-SAT. First, unlike k-SAT for $k \ge 3$, which is an NP-complete problem, 2-SAT instances may be solved by a linear time algorithm [10, 22]. Also, there is quite precise information about 2-SAT for density very close to the threshold $r_2 = 1$ [9, 38].

It has been argued that instances of random k-SAT do not in fact represent real-world, or industrial, instances [28, 33, 34]. One of the major differences between industrial and random SAT instances is that the set of variables in industrial instances often consists of a disjoint union of subsets, referred to as *communities*; clauses tend to comprise variables from the same community, with only a minority of clauses containing variables from distinct communities [7, 32]. There are several additional variations [4, 6]. For example, the variables may be selected non-uniformly (say, according to a power-law distribution [5, 20]), and/or the clauses may be of non-constant length.

In this paper we work with a (generalization of a) model introduced by [18]. Our model is similar to the random model, except for the partition of the variables into communities. These communities are of the same size. There are several clause types (defined precisely in the next section), differing in the number of variables from the same or distinct communities in each clause.

Our focus is on the satisfiability threshold in this model. The question has been studied in [18], mostly experimentally, for the model suggested there. We show that the findings in that paper, whereby the threshold tends to be smaller when there are many single-community clauses (i.e., clauses consisting of variables from the same random community) remain true in the general model. In fact, if the communities are small, the threshold may even be 0.

We present our model in Section 2. The main results are stated in Section 3, and the proofs follow in Section 4. In Section 5 we present some simulation results.

2 A Model of Random Industrial SAT

In industrial SAT, the strength of the community structure of an instance is usually measured by its modularity [7, 19, 35]. Roughly speaking, given a graph, its modularity gives an indication of the tendency of the vertices to be connected to other vertices, which are similar to them in some way. In our case, an instance defines the following undirected graph. The set of nodes is the set of variables $\{v_1, \ldots, v_n\}$. There is an edge (v_i, v_j) for $i \neq j$ if there exists a clause in the instance, containing both variables v_i (or its negation) and v_j (or its negation). More precisely, we view this object as a multi-graph; if both v_i and v_j (or their negations) appear in several clauses, there are several edges connecting them. Given an instance, high modularity indicates that there exists a partition of the set of variables into subsets, such that a large portion of the edges connect vertices of the same subset, compared to a random graph with the same number of vertices and same degrees [31, 35].

As in the regular model, we have n Boolean variables and m clauses in an instance. Each clause is chosen independently of the others. Each variable in each clause is negated with a probability of 1/2, independently of the other variables. The model differs from the regular model in several aspects: There is a community structure on the set of variables, and we also do not necessarily assume all clauses to be of the same length. Specifically, the set of variables $\{v_1, \ldots, v_n\}$ is partitioned into B disjoint (sets of variables referred to as) communities $C_1, C_2, ..., C_B$. For simplicity, we assume all communities to be of the same size h, so that $n = B \cdot h$. Without loss of generality, we will assume that $C_i = \{v_i : (i-1)h + 1 \le j \le ih\}, 1 \le i \le B$. As n grows, so do usually both B and h (although at times one of them may remain fixed), and we will write B(n) and h(n) when we want to relate to their dependence on n. For an ℓ -tuple $\mathbf{K} = (k_1, \ldots, k_\ell)$ with non-increasing, positive integer entries, denote by $\Omega_B(n, \mathbf{K})$ the set of all clauses of length $k_1 + \cdots + k_\ell$, formed of k_1 variables from some community C_{i_1} , k_2 from another community C_{i_2} , ..., k_ℓ from some ℓ -th community $C_{i_{\ell}}$, where the indices i_j are mutually distinct. We will refer to such a clause as a clause of type K. We will always implicitly assume that $k_i \leq h$ for each i, so that we can indeed choose the required number of variables from the various communities. Similarly, we implicitly assume that $\ell \leq B$. Let $P_{\mathbf{K}}$ be the uniform measure on $\Omega_B(n, \mathbf{K})$.

Example 2.1. (a) Let n = 1000, B = 10 and h = 100. The clauses

$$(\overline{v}_{237} \lor \overline{v}_{250} \lor v_{911} \lor \overline{v}_{917} \lor v_{939}),$$

 $(v_{401} \vee \overline{v}_{423} \vee v_{427} \vee \overline{v}_{450} \vee v_{500})$

are of types (3, 2) and (5) (single-community clause), respectively. (In general, the type of single-community clauses of length k will be written as (k).) The clauses above belong to the spaces Ω_{10} (1000, (3, 2)) and Ω_{10} (1000, (5)), respectively.

- (b) The space $\Omega_{10}(1000, (3, 2))$ consists of $a = 10 \cdot 9\binom{100}{3}\binom{100}{2} \cdot 2^5$ clauses, and $\Omega_{10}(1000, (5))$ consists of $b = 10\binom{100}{5} \cdot 2^5$ clauses.
- (c) Under the measure $P_{(3,2)}$, each clause in Ω_{10} (1000, (3,2)) is chosen with probability 1/a; under $P_{(5)}$, each clause in Ω_{10} (1000, (5)) is chosen with probability 1/b.

The random instances we will be dealing with are of the following structure. There is some number $T \ge 1$ of clause types $\mathbf{K}_1, \ldots, \mathbf{K}_T$. Each $\mathbf{K}_t, 1 \le t \le T$, is a vector $\mathbf{K}_t = (k_{1t}, \ldots, k_{\ell t})$. These vectors are mutually distinct. Each clause in the instance is of one of these types. The probability of each clause to be of type \mathbf{K}_t is p_t , where $p_t, 1 \le t \le T$, are arbitrary fixed real numbers, with $\sum_{t=1}^T p_t = 1$. More formally, we select independently m clauses from the space $\bigcup_{t=1}^T \Omega_B(n, \mathbf{K}_t)$ according to the measure $P = \sum_{t=1}^T p_t \cdot P_{\mathbf{K}_t}$. Using similar notations to [18], denote by F(n, m, B, P) the probability space of instances. Namely, the sample space of F(n, m, B, P) is the m-fold Cartesian product $\left(\bigcup_{t=1}^T \Omega_B(n, \mathbf{K}_t)\right)^m$ of the space corresponding to the selection of a single clause, endowed with the product measure P^m . (For more on the notions of a product of measure spaces and of the product measure, see, for example, [2, Sec. 2.5].) For concreteness, in Algorithm 1 we present the exact mechanism for selecting an instance of F(n, m, B, P).

Note that, when employing Algorithm 1, we care about the order of choices, so that each clause may be obtained in several ways. This is easier to implement and has no bearing on the probability of obtaining each possible clause.

Thus, the regular model of random k-SAT is, with the notations above, $F(n, m, 1, P_{(k)})$. Instances in the model presented in [18] include clauses of length k of two types: (i) singlecommunity clauses – all k variables belong to the same community $(B\binom{h}{k} \cdot 2^k \text{ possible choices})$, and (ii) the k variables belong to k distinct communities $\binom{B}{k} \cdot (2h)^k$ possible choices). For some 0 , each clause is of type (i) with probability p and of type (ii) with probability <math>1 - p. With the above notations, their probability space is

$$F\left(n,m,B,p\cdot P_{(k)}+(1-p)\cdot P_{\left(\underbrace{1,\ldots,1}_{k}\right)}\right)$$

for some k.

Example 2.2. With n, B and h as in Example 2.1, and k = 3, the instance

$$(\overline{v}_{423} \lor v_{459} \lor v_{496}) \land (v_{156} \lor \overline{v}_{437} \land v_{626}),$$

is an instance in

$$F\left(1000, 2, 10, 0.2 \cdot P_{(3)} + 0.8 \cdot P_{(1,1,1)}
ight)$$
 .

Algorithm 1: Choosing an instance in F(n, m, B, P)

Input: $n, m, B, K_1, ..., K_T, p_1, ..., p_T$. **Output:** An instance \mathcal{I} $\mathcal{I} \leftarrow \emptyset;$ for $i \leftarrow 1$ to m do $C \leftarrow \emptyset$: Choose a clause type – each \mathbf{K}_t has probability p_t ; Suppose **K**_{*t*} = $(k_1, ..., k_\ell)$; Select ℓ distinct integers i_1, \ldots, i_{ℓ} in the range [1, B] (with the same probability $\frac{1}{B(B-1)\cdots(B-\ell+1)}$ for each possible choice); for $j \leftarrow 1$ to ℓ do Choose k_j distinct integers a_1, \ldots, a_{k_j} in the range [1, h] (with the same probability $\frac{1}{h(h-1)\cdots(h-k_j+1)}$ for each possible choice); for $d \leftarrow 1$ to k_j do $\begin{array}{|c|c|c|c|} x \leftarrow (i_j - 1) \cdot h + a_d; \\ \text{Negate } v_x \text{ with probability } 1/2; \\ C \leftarrow C \lor v_x; \end{array}$ $\mathcal{I} \leftarrow \mathcal{I} \wedge C;$ return \mathcal{I}

The first clause is of type (3) as all three variables $v_{423}, v_{459}, v_{496}$ belong to the same community $C_5 = \{v_i : 401 \le i \le 500\}$, while the second clause is of type (1,1,1), as the variables $v_{156}, v_{437}, v_{626}$ belong to three distinct communities: C_2 , C_5 and C_7 , respectively.

As our interest in this paper is in instances constructed as above, from this point on we will use the term "community-structured" instead of the more general "industrial".

3 The Main Results

As explained above, the clauses in an community-structured instance tend to include variables from the same community. In this paper, moreover, we usually deal with the case where one (or more) of the clause types is a single-community type, namely $\mathbf{K}_t = (k)$ for some $1 \le t \le T$ and $k \le h$. In some results, we will further restrict ourselves to the case T = 1, where the only clause type is a single-community type (equivalently, $P = P_{(k)}$ for some k).

In [18] it was observed empirically that, when the modularity of the variable incidence graph of the instance increases, the threshold decreases. Now, the modularity in our case is larger when more clauses consist of variables from the same community and when the communities are small. Our first result is quite straightforward, but it already hints that instances in the model suggested in Section 2 tend to be no more satisfiable than random k-SAT instances. Note that the first part of the proposition is one of the initial results for random SAT [17].

Proposition 3.1. Let \mathcal{I} be a random instance in $F(n, \alpha n, B, P)$.

- (a) Suppose that for each \mathbf{K}_t , $1 \le t \le T$, the clause length is at most k. If $\alpha > 2^k \ln 2$, then \mathcal{I} is UNSAT w.h.p.
- (b) Let T = 1 and $P = P_{(k)}$ for some $k \ge 2$.
 - (i) If $\alpha > r_k^*$, then \mathcal{I} is UNSAT w.h.p. (ii) If $h(n) = \Theta(n)$ and $\alpha < r_k$, then \mathcal{I} is SAT w.h.p.

Our next result points out a significant difference between random instances and communitystructured ones. One might expect the threshold to be different for community-structured instances, but it turns out that this difference may be not just quantitative. The following result shows that, surprisingly, under certain conditions the satisfiability threshold is 0. To this end, we will consider mas some function of n, not necessarily $m = \alpha n$, and write m(n) instead of m.

For real functions f and g, we write $f = \Omega(g)$ if g = O(f), and $f = \omega(g)$ if g = o(f). We also write f = polylog(g) if $f = O(\ln^{\theta} g)$ for some θ .

Theorem 3.2. Let \mathcal{I} be a random instance in F(n, m(n), B, P), where $\mathbf{K}_t = (k)$ for some $1 \leq t \leq T$.

(b) If
$$h(n) = o\left(\frac{\ln n}{\ln \ln n}\right)$$
 and $m(n) = \Omega\left(\frac{n}{\operatorname{polylog}(n)}\right)$, then \mathcal{I} is UNSAT w.h.p.

- (c) If $h(n) = o(\ln n)$ and $m(n) = \Omega(n \cdot e^{-\beta \cdot \ln n/h(n)})$ for some $\beta < 1/r_k^*$, then \mathcal{I} is UNSAT w.h.p.
- (d) Let $h(n) = O(\ln n)$ and T = 1. Then there exists some $\varepsilon_0 > 0$ such that, if $m(n) = \alpha n$ with $\alpha > r_k^* - \varepsilon_0$, then \mathcal{I} is UNSAT w.h.p.
- **Remark 3.3.** (a) The ε_0 in part (d) is effective. Namely, as will follow from the proof, one can present such an ε_0 explicitly (in terms of the implicit constant in the equality $h(n) = O(\ln n)$).
- (b) Still in case (d), one can deal with the general case of arbitrary T as long as the weight of $P_{(k)}$ in P, namely the probability that a clause is of type $P_{(k)}$, is sufficiently large.

In Theorem 3.2 there are four types of results for the asymptotic satisfiability of a random community-structured instance with n variables, m(n) clauses, B communities of size h(n) = n/B, and probability measure P. Namely, either the probability of satisfiability (i) tends to 0 as $n \to \infty$, or (ii) it tends to 1, or (iii) it is bounded away from 1, or (iv) it is bounded away from both 0 and 1. These results are summarized in Table 1. In general, we assume that $\mathbf{K}_t = (k)$ for some $1 \le t \le T$ and $k \ge 1$. In the third column we place a '1' or a '*', depending on whether

Parameters			Result
h(n)	m(n)	T	
O(1)	$o\left(n^{1-1/2^k}\right)$	1	SAT w.h.p.
O(1)	$\Theta\left(n^{1-1/2^k}\right)$	1	$\in (\delta, 1 - \delta)$
O(1)	$\Theta\left(n^{1-1/2^k}\right)$	*	$\in (0, 1 - \delta)$
O(1)	$\omega\left(n^{1-1/2^k}\right)$	*	UNSAT w.h.p.
$O\left(\frac{\ln n}{\ln\ln n}\right)$	$\Omega\left(n/polylog(n)\right)$	*	UNSAT w.h.p.
$o\left(\ln n\right)$	$\Omega(n^{1-1/(r_k^*+\varepsilon)h(n)})$	*	UNSAT w.h.p.
$O\left(\ln n\right)$	$> (r_k^* - \varepsilon_0) n$	1	UNSAT w.h.p.

Table 1: Asymptotic satisfiability of a random instance with small communities in F(n, m(n), B, P).

T is required to be 1 or is arbitrary, respectively. The notation $\in (0, 1 - \delta)$ indicates a probability bounded away from 1, and the notation $\in (\delta, 1 - \delta)$ indicates a probability bounded away both from 0 and 1.

The proof of Theorem 3.2 will use the following lemma.

Lemma 3.4. Consider the spaces F(n, m(n), B, P) and F(n, m'(n), B, P), where $m'(n) = \omega(m(n))$. If a random instance in F(n, m(n), B, P) is UNSAT with probability bounded away from 0, then a random instance in F(n, m'(n), B, P) is UNSAT w.h.p.

In the proof of Theorem 3.2 (and that of Theorem 3.6), we use some results regarding the classical "balls and bins" problem. In this problem, there are M balls and B bins. Each ball is placed uniformly randomly in one of the bins, independently of the other balls. One quantity of interest is the *maximum load*, which is the maximum number of balls in any bin. There are several papers studying the size of the maximum load, as well as generalizations of this problem. It seems that [37] contains all previous results. Our next result seem not to be covered by previous results regarding the balls and bins problem. It will be employed in the proof of Theorem 3.2, and is of independent interest.

Given a sequence $(X_n)_{n=1}^{\infty}$ of random variables and a probability law \mathcal{L} , we let $X_n \xrightarrow[n \to \infty]{} \mathcal{L}$ denote the fact that X_n converges to \mathcal{L} in distribution as $n \to \infty$. Denote by $Po(\lambda)$ the Poisson distribution with parameter λ .

Theorem 3.5. Consider the balls and bins problem with B bins and M = M(B) balls, where $B \to \infty$. Let $s \ge 2$ be an arbitrarily fixed integer.

(a) If $M(B) = o(B^{1-1/s})$, then the maximum load is at most s - 1 w.h.p.

(b) If $M(B) = \omega(B^{1-1/s})$, then the maximum load is at least s w.h.p.

(c) If $M(B) = \Theta(B^{1-1/s})$, then the maximum load is either s - 1 or s w.h.p. Moreover, suppose

$$M(B) = C \cdot B^{1-1/s} \left(1 + o(1) \right), \tag{1}$$

and let X_B be the number of bins that contain exactly s balls. Then $X_B \xrightarrow{\mathcal{D}} \operatorname{Po}(C^s/s!)$.

Theorem 3.5 will be proven in Appendix A.

As noted earlier, random 2-SAT is much better understood than random k-SAT for general k. This enables us to obtain a stronger result than Theorem 3.2 in the case $P = P_{(2)}$.

Theorem 3.6. Let \mathcal{I} be a random instance in $F(n, \alpha n, B, P_{(2)})$.

- (a) There exists an $0 < \varepsilon_0 < 1$ such that, if $h(n) = o(\sqrt{n})$ and $\alpha > 1 \varepsilon_0$, then \mathcal{I} is UNSAT w.h.p.
- (b) For $h(n) = \Theta(\sqrt{n})$:
 - (i) If $1 \varepsilon_0 < \alpha < 1$, where ε_0 is as in (a), then \mathcal{I} is SAT with probability bounded away from both 0 and 1.

(ii) If
$$\alpha = 1$$
 then \mathcal{I} is UNSAT w.h.p.

(c) For
$$h(n) = \omega(\sqrt{n})$$
 with $h(n) = o(n)$.

(i) If $\alpha < 1$ then \mathcal{I} is SAT w.h.p.

(ii) If $\alpha = 1$ then \mathcal{I} is UNSAT w.h.p.

(d) For $h(n) = \Theta(n)$:

(i) If $\alpha < 1$ then \mathcal{I} is SAT w.h.p.

(ii) If $\alpha = 1$ then \mathcal{I} is SAT with probability bounded away from both 0 and 1.

Remark 3.7. As in Remark 3.3.(b), one can deal with the more general case of arbitrary T, as long as one of the clause types \mathbf{K}_t is of the form (2) and is of sufficiently large weight.

Similarly to Table 1, we summarize the results of Theorem 3.6 in Table 2. Here, we always assume k = 2, $m(n) = \alpha n$ and T = 1. The notations are as in Table 1.

As we have seen, when clauses tend to be formed of variables in the same community, the instance tends to become unsatisfiable. One may wonder what happens in the opposite case, namely when variables tend to belong to distinct communities. Intuitively, this constraint should usually make little difference, as anyway few clauses may be expected to contain variables from the same community. However, if there are very few communities, this constraint is more significant. Specifically, consider the extreme case of B = 2, $P = P_{(1,1)}$. In this case, we disallow about half of the possible clauses of the classical model B = 1, $P = P_{(2)}$. Does it affect the satisfiability threshold? Namely, if when moving from the classical case to a case with most clauses from the same community, we tend to make the instance unsatisfiable, will the constraint of having in each clause variables from distinct communities tend to make it "more satisfiable"? The following theorem shows that it makes a very small difference if at all.

$h(n)$ α	$\in (1 - \varepsilon_0, 1)$	= 1
$= o\left(\sqrt{n}\right)$	UNSAT w.h.p	UNSAT w.h.p.
$= \Theta\left(\sqrt{n}\right)$	$\in (\delta, 1-\delta)$	UNSAT w.h.p.
$\in \omega\left(\sqrt{n}\right)\cap o\left(n\right)$	SAT w.h.p.	UNSAT w.h.p
$=\Theta\left(n\right)$	SAT w.h.p	$\in (\delta, 1-\delta)$

Table 2: Asymptotic satisfiability of a random instance in $F(n, \alpha n, B, P_{(2)})$.

Theorem 3.8. Let $0 \le p \le 1$ and $B \ge 2$ arbitrary and fixed. The satisfiability threshold in the model

$$F(n, m, B, pP_{(1,1)} + (1-p)P_{(2)})$$

is 1.

One may still ask whether the regular random model $F(n, m, 1, P_{(2)})$ and the model $F(n, m, 2, P_{(1,1)})$ display some difference in behaviour near the threshold, namely for $m = n \cdot (1 + o(1))$. More precisely, recall that, by [9], for m in some range of size $\Theta(n^{2/3})$ around n, the satisfiability probability for the random model is bounded away from both 0 and 1. (See (7) below for a more accurate formulation.) Do the two models behave in the same way for $m = n + \theta n^{2/3}$ for fixed θ ?

We studied this question by a large simulation. We detail the experiment in Section 5. The results seem to indicate strongly that the two models behave in the same way also in the window $m = n \pm \Theta(n^{2/3})$.

4 **Proofs**

Proof of Proposition 3.1:

(a) We follow the proof in the random model [17]. Fix a truth assignment and consider *I*. Each clause has at most k literals. The variables are negated with probability 1/2 independently of each other, and hence each clause is satisfied with probability of at most 1 − 2^{-k}, independently of the other clauses. The expected number of satisfying truth assignments is therefore at most 2ⁿ · (1 − 2^{-k})^{αn}. As α > 2^k ln 2, we have

$$2^n \cdot \left(1 - 2^{-k}\right)^{\alpha n} \xrightarrow[n \to \infty]{} 0.$$

(Note that we have not used in this part the specific mechanism by which clauses are selected. The variables in each clause may be selected arbitrarily. As long as all clauses are of length at most k, and the sign of each variable in each clause is selected uniformly randomly, and independently of all other variables in this clause and all the others, the conclusion holds.) Thus, \mathcal{I} is UNSAT w.h.p.

(b) A random instance \mathcal{I} in $F(n, \alpha n, B, P_{(k)})$ decomposes into B sub-instances \mathcal{I}_i , $1 \leq i \leq B$, where each \mathcal{I}_i is formed of those clauses consisting of variables solely from \mathcal{C}_i . Obviously, \mathcal{I} is SAT if and only if all \mathcal{I}_i -s are such. For $1 \leq i \leq B$, let $U_i = 1$ if \mathcal{I}_i is satisfiable, and $U_i = 0$ otherwise. The variable $U = \prod_{i=1}^{B} U_i$ indicates whether \mathcal{I} is satisfiable. Let W_i denote the number of clauses in \mathcal{I}_i . Since each of the αn clauses consists of variables from \mathcal{C}_i with probability 1/B, independently of all other clauses, W_i is binomially distributed with parameters αn and 1/B:

$$W_i \sim B(\alpha n, 1/B), \qquad 1 \le i \le B.$$

(i) Suppose first that h(n) = ω(1). Let α_i denote the density of the sub-instance I_i, 1 ≤ i ≤ B. There exists an i with α_i ≥ α, and therefore α_i > r^{*}_k. It follows that I_i is UNSAT w.h.p., and hence so is I.

The case h(n) = O(1) follows in particular from Theorem 3.2.(a).(iii) (to be proved below).

(ii) In this case $B(n) = \Theta(1)$. Without loss of generality assume B(n) = B is fixed. For $\gamma > 0$, let

$$W_{<\gamma} = \bigcap_{i=1}^{B} \left\{ W_i < \gamma \cdot h(n) \right\}.$$
⁽²⁾

Let α' be an arbitrary fixed number, strictly between α and r_k . Denoting by \overline{A} the complement of an event A, we have:

$$P(U = 1) = P(W_{<\alpha'}) P(U = 1 | W_{<\alpha'}) + P(\overline{W}_{<\alpha'}) P(U = 1 | \overline{W}_{<\alpha'}) \geq P(W_{<\alpha'}) P(U = 1 | W_{<\alpha'}).$$
(3)

By the weak law of large numbers for the binomial random variables W_i ,

$$\frac{W_i}{n/B} \xrightarrow[n \to \infty]{P} \alpha, \qquad 1 \le i \le B,$$

and therefore

$$P\left(\overline{W}_{<\alpha'}\right) = P\left(\bigcup_{i=1}^{B} \{W_i \ge \alpha' \cdot n/B\}\right)$$
$$\leq \sum_{i=1}^{B} P\left(W_i \ge \alpha' \cdot n/B\right)$$
$$= B \cdot P\left(W_1 \ge \alpha' \cdot n/B\right) \xrightarrow[n \to \infty]{} 0.$$

Hence

$$P\left(W_{<\alpha'}\right) = 1 - P\left(\overline{W}_{<\alpha'}\right) \xrightarrow[n \to \infty]{} 1.$$

Now consider the second factor on the right-hand side of (3). Clearly, the more clauses any \mathcal{I}_i contains, the less likely it is to be satisfiable, and therefore

$$P(U = 1 | W_{<\alpha'}) \ge \prod_{i=1}^{B} P(U_i = 1 | W_i = \alpha' \cdot n/B).$$
(4)

As $\alpha' < r_k$, each of the sub-instances \mathcal{I}_i is SAT w.h.p., so that each of the factors in the product on the right-hand side of (4) converges to 1 as $n \to \infty$. Since, by our assumption, *B* is fixed, so does the whole product. Hence \mathcal{I} is SAT w.h.p.

As mentioned in Section 3, the proofs of Theorem 3.2 and Theorem 3.6 make use of some results concerning the balls and bins problem. Let L be the maximum load for M balls and B bins. By [37], for any $\delta > 0$,

$$L \ge \begin{cases} \frac{\ln B}{\ln \frac{B \ln B}{M}}, & \text{if } \frac{B}{\text{polylog}(B)} < M = o(B \ln B), \\ (d_c - \delta) \ln B, & \text{if } M = c \cdot B \ln B, \end{cases}$$
(5)

w.h.p. for an appropriate constant $d_c > c$.

Remark 4.1. The constant d_c , in the second part of (5), is the unique solution of the equation

$$1 + x(\ln c - \ln x + 1) - c = 0$$

in (c, ∞) (see [37, Lemma 3]). A routine calculation shows that $d_c = c + c \cdot u(1/c)$, where the function u is the unique non-negative function defined implicitly by the equation

$$-u(w) + (1 + u(w))\ln(1 + u(w)) = w, \qquad (w \ge 0).$$

The function u(w) has been studied in [26, pp. 101–102], and in particular expressed as a power series in \sqrt{w} near 0.

The fact that $d_c > c$ is the reason that the threshold in Theorem 3.2.(d) is strictly less than r_k^* . One can easily bound $d_c - c$ from below. In fact, write $d_c = c + \varepsilon$. Then

$$1 = -(c+\varepsilon)(\ln c - \ln(c+\varepsilon) + 1) + c$$

= $-(c+\varepsilon)\ln c + (c+\varepsilon)\ln(c+\varepsilon) - \varepsilon$
< $(c+\varepsilon)\cdot\varepsilon/c - \varepsilon = \varepsilon^2/c$,

and hence $d_c > c + \sqrt{c}$.

Proof of Theorem 3.2: We follow the notations used in the proof of Proposition 3.1. Recall that \mathcal{I}_i is the sub-instance formed of those clauses in \mathcal{I} consisting of variables solely from \mathcal{C}_i , and W_i is the number of clauses in \mathcal{I}_i , $1 \le i \le B$. Denote $W_{\max} = \max\{W_1, \ldots, W_B\}$.

Note that, while we have not assumed that T = 1 in parts (a).(iii), (b), and (c) of the theorem, we may make this assumption without loss of generality in these parts as well. In fact, suppose

that any of these three parts has been proven for the case T = 1, and consider the general case. If the probability of (k) in P is p, then w.h.p. there will be at least $p \cdot m(n)/2$ clauses of type (k). To see this, denote by $\mathcal{I}^{(k)}$ the sub-instance of \mathcal{I} obtained by taking the clauses of type (k) and by m'(n) the number of clauses in $\mathcal{I}^{(k)}$. Clearly, m'(n) is distributed binomially with parameters m(n) and p. Employing Chernoff's bound we obtain a lower bound of $p \cdot m(n)/2$ on m'(n) w.h.p. It follows that m'(n) has the same lower bound assumed on m(n) (namely, it is $\omega(n^{1-1/2^k})$ in part (a).(iii), it is $\Omega\left(\frac{n}{\text{polylog}(n)}\right)$ in part (b), and it is $\Omega\left(n \cdot e^{-\beta \cdot \ln n/h(n)}\right)$ in part (c)). As we have assumed the correctness of these parts for T = 1, the instance $\mathcal{I}^{(k)}$ is UNSAT w.h.p., and hence certainly so is the original instance, which contains it. Thus, we may indeed assume in all parts that T = 1.

Since T = 1, each clause has all its literals from the same community. Hence, the selection of a clause corresponds to the selection of a community. Consider clauses as balls, and communities as bins. The process of selecting the clauses, as far as the community to which the variables in each clause belong, is analogous to that of placing m(n) balls in B bins uniformly at random. The idea of the proof in parts (b)-(d) will be to prove that w.h.p. we have $W_{\text{max}}/h(n) > r_k^*$. This will imply that there is at least one sub-instance \mathcal{I}_i with density larger than r_k^* . Thus, already \mathcal{I}_i is UNSAT w.h.p., and consequently so is \mathcal{I} .

- (a) Without loss of generality, assume that h(n) = h > 0 is fixed.
 - (i) By Theorem 3.5.(a), there is no sub-instance with more than $2^k 1$ clauses w.h.p. Since instances with less than 2^k clauses are certainly satisfiable, all \mathcal{I}_i -s are SAT, and hence so is \mathcal{I} .
 - (ii) Here, we may assume that $m(n) = \theta \cdot n^{1-1/2^k}$ for some constant $\theta > 0$. By Theorem 3.5.(c), the probability that there is an \mathcal{I}_i , $1 \le i \le B$, with at least 2^k clauses, is bounded away from 0. Assume, say, that $W_1 \ge 2^k$. Then, with probability at least

$$\left(1/\binom{h}{k}\right)^{2^k} \cdot (2^k)!/2^{k2^k},$$

all 2^k distinct clauses consisting of the variables v_1, \ldots, v_k have been drawn. As the instance is UNSAT if it contains all these 2^k clauses, the probability for our instance to be UNSAT is bounded away from 0. Now, assume that $P = P_{(k)}$, for some k > 0. Now, by Theorem 3.5.(c) there is no sub-instance with more than $2^k - 1$ clauses with probability bounded away from 0. Thus, similarly to part (i), \mathcal{I} is SAT with probability bounded away from 0.

- (iii) Follows from the previous part and Lemma 3.4.
- (b) In view of part (a).(iii), we may assume $h(n) \to \infty$. We may also assume that $m(n) = n/\ln^{\theta} n$ for some $\theta \ge 1$. Clearly, $m(n) \le B$. On the other hand,

$$m(n) = \frac{n}{\ln^{\theta} n} \ge \frac{B}{(2\ln B)^{\theta}} = \frac{B}{\operatorname{polylog}(B)}$$

Thus, by (5), w.h.p., the maximum load is at least

$$\frac{\ln B}{\ln \frac{B \ln B}{m(n)}} \ge \frac{\frac{1}{2} \cdot \ln n}{\ln \left(\frac{n \cdot \ln n}{n/\ln^{\theta} n}\right)} \ge \frac{\ln n}{2(\theta+1)\ln\ln n}.$$
(6)

Now, there are $h(n) = o(\ln n / \ln \ln n)$ variables in each community. By (6), w.h.p., the density of the sub-instance \mathcal{I}_i with the maximal number of clauses is at least

$$\frac{W_{\max}}{h(n)} \ge \frac{\frac{1}{2(\theta+1)} \cdot \ln n / \ln \ln n}{o (\ln n / \ln \ln n)} \xrightarrow[n \to \infty]{} \infty.$$

Hence, this \mathcal{I}_i is UNSAT w.h.p., and therefore so is \mathcal{I} .

(c) By (5), w.h.p., the number of clauses in the sub-instance \mathcal{I}_i with the maximal number of clauses is at least

$$W_{\max} \ge \frac{\ln B}{\ln \frac{B \ln B}{m(n)}} = \frac{\ln n(1 - o(1))}{\ln \frac{B \ln B}{m(n)}}$$

For a large enough n

$$\ln \frac{B \ln B}{m(n)} \le \ln \left(\frac{\frac{n}{h(n)} \cdot \ln n}{n \cdot e^{-\beta \ln n/h(n)}} \right)$$
$$= \ln \frac{\ln n}{h(n)} + \beta \cdot \frac{\ln n}{h(n)}.$$

 \sim

As $\beta < 1/r_k^*$, for large enough x we have $\ln x + \beta x < x/r_k^*$. Hence, for large enough n we have

$$\ln \frac{B \ln B}{m(n)} \le \frac{1}{r_k^*} \cdot \frac{\ln n}{h(n)}.$$

This implies that the density of the sub-instance \mathcal{I}_i with the maximal number of clauses is at least

$$\frac{W_{\max}}{h(n)} \ge \frac{1}{\ln \frac{B \ln B}{m(n)}} \cdot \frac{\ln n(1-o(n))}{h(n)} > r_k^*,$$

and thus UNSAT w.h.p. Consequently, so is \mathcal{I} .

(d) In view of the previous part, we may assume that $h(n) = \theta \ln n$ for some $\theta > 0$. Choose c_0 such that

$$d_{c_0} = \theta r_k^*,$$

where d_c is as in (5). Let $\alpha > c_0/\theta$, and put $c = \alpha \theta$. Thus, $c > c_0$ and $d_c > d_{c_0}$. Let $\delta < d_c - \theta r_k^*$. We have

$$m(n) = n \cdot \alpha = \frac{nc}{\theta} = (1 + o(1)) \cdot \frac{nc\ln B}{\theta \ln n}$$
$$= (c + o(1)) \cdot B \ln B.$$

By (5), the size of the largest sub-instance is $W_{\text{max}} \ge (d_c - \delta) \ln B$ w.h.p. Hence, w.h.p. the density of this sub-instance is

$$\frac{W_{\max}}{h(n)} \ge \frac{(d_c - \delta) \ln B}{h(n)} = \frac{(d_c - \delta) \cdot (1 - o(1)) \ln n}{\theta \ln n}$$
$$= \frac{d_c - \delta}{\theta} - o(1) = r_k^* + \frac{d_c - \delta - \theta r_k^*}{\theta} - o(1).$$

Letting $\varepsilon_0 = r_k^* - c_0/\theta$, we get our claim.

Proof of Lemma 3.4: Denote the random instance in F(n, m'(n), B, P) by \mathcal{I}' . Denote the instance obtained from the first m(n) clauses of \mathcal{I}' by \mathcal{I}'_1 , the instance obtained from the next m(n) clauses of \mathcal{I}' by \mathcal{I}'_2, \ldots , the instance obtained from the last m(n) clauses of \mathcal{I}' by $\mathcal{I}'_{b(n)}$ (with b(n) = m'(n)/m(n)). According to our assumption, there exists an $\varepsilon > 0$ such that

$$P(\mathcal{I}'_i \text{ is SAT}) \leq 1 - \varepsilon, \qquad i = 1, 2, \dots, b(n).$$

Now, the events $\{\mathcal{I}'_i \text{ is SAT}\}, 1 \leq i \leq b(n)$, are independent, and we clearly have

$$\{\mathcal{I}' ext{ is SAT}\} \subseteq igcap_{i=1}^{b(n)} \{\mathcal{I}'_i ext{ is SAT}\}.$$

Since $b(n) \to \infty$:

$$P\left(\mathcal{I}' \text{ is SAT}\right) \leq P\left(\mathcal{I}'_i \text{ is SAT}, 1 \leq i \leq b(n)\right) \leq \left(1 - \varepsilon\right)^{b(n)} \xrightarrow[n \to \infty]{} 0.$$

In the proof of Theorem 3.6 we will use the following result from [9]. There exist some $0 < \varepsilon_0 < 1$ and $\lambda_0 > 0$ such that the satisfiability probability of a random 2-SAT instance \mathcal{I} with $m = n \cdot (1 + \varepsilon)$ clauses is

$$P\left(\mathcal{I} \text{ is SAT}\right) = \begin{cases} 1 - \Theta\left(\frac{1}{n\left|\varepsilon\right|^{3}}\right), & -\varepsilon_{0} \leq \varepsilon \leq -\lambda_{0}n^{-1/3}, \\\\ \Theta\left(1\right), & -\lambda_{0}n^{-1/3} < \varepsilon < \lambda_{0}n^{-1/3}, \\\\ \exp\left(-\Theta\left(n\varepsilon^{3}\right)\right), & \lambda_{0}n^{-1/3} \leq \varepsilon \leq \varepsilon_{0}. \end{cases}$$
(7)

Actually, in the sequel, we will encounter only the first two cases. Note that in the case $m = n \cdot (1 - \varepsilon)$ with $\lambda_0 n^{-1/3} \le \varepsilon \le \varepsilon_0$, we have

$$1 - \frac{\theta_1}{n \cdot \varepsilon^3} \le P\left(\mathcal{I} \text{ is SAT}\right) \le 1 - \frac{\theta_2}{n \cdot \varepsilon^3} \tag{8}$$

for some constants $\theta_1, \theta_2 > 0$.

We will also use an additional result regarding the balls and bins problem. Let L be the maximum load for M balls and B bins. By [37], w.h.p.

$$L \le \frac{M}{B} + \sqrt{\frac{2M\ln B}{B}}, \qquad M = \omega(B\ln^3 B).$$
(9)

Given a sequence $(X_i)_{i=1}^{\infty}$ of random variables and a probability law \mathcal{L} , we write $X_i \xrightarrow[i \to \infty]{\mathcal{L}} \mathcal{L}$ if the sequence converges to \mathcal{L} in distribution.

Proof of Theorem 3.6: We follow the notations used in the proof of Theorem 3.2. Also, for $\gamma > 0$, let

$$I_{<\gamma} = \Big\{ (m_1, \dots, m_B) : m_1 + \dots + m_B = \alpha n, m_i < \gamma \cdot h(n) \forall 1 \le i \le B \Big\},\$$

and let $I_{>\gamma}$ and $I_{\geq\gamma}$ be analogously understood. More generally, for $0 \le p \le 1$, let $I_{<\gamma,p}$ denote the set of *B*-tuples (m_1, \ldots, m_B) with at least *pB* entries $m_i, 1 \le i \le B$, for which $m_i < \gamma \cdot h(n)$. (Thus, $I_{<\gamma} = I_{<\gamma,1}$.)

Note that the set $W_{<\gamma}$, defined in (2), may now be written in the form

$$W_{<\gamma} = \bigcup_{(m_1,\dots,m_B)\in I_{<\gamma}} \bigcap_{i=1}^B \{W_i = m_i\}$$

We use similar notations, for example $W_{>\gamma}$, $W_{\geq\gamma,p}$ and $W_{<\gamma,p}$, analogously.

(a) Let δ , p be sufficiently small positive numbers, to be determined later. Let ε_0 be as in (7). We have

$$P(U = 1) = P(W_{<1+\delta} \cap W_{>1-\varepsilon_{0},p}) \cdot P(U = 1 | W_{<1+\delta} \cap W_{>1-\varepsilon_{0},p}) + P(W_{<1+\delta} \cap \overline{W}_{>1-\varepsilon_{0},p}) \cdot P(U = 1 | W_{<1+\delta} \cap \overline{W}_{>1-\varepsilon_{0},p}) + P(\overline{W}_{<1+\delta}) P(U = 1 | \overline{W}_{<1+\delta}) \leq P(U = 1 | W_{<1+\delta} \cap W_{>1-\varepsilon_{0},p}) + P(W_{<1+\delta} \cap \overline{W}_{>1-\varepsilon_{0},p}) + P(U = 1 | \overline{W}_{<1+\delta}).$$

$$(10)$$

Consider the first term on the right-hand side of (10). The event $W_{<1+\delta} \cap W_{>1-\varepsilon_0,p}$ implies that $W_j = m_j$, $1 \le j \le B$, for some $(m_1, \ldots, m_B) \in I_{<1+\delta} \cap I_{>1-\varepsilon_0,p}$. We note that, conditioned on the event $\bigcap_{i=1}^{B} \{W_i = m_i\}$, the events $\{U_i = 1\}, 1 \le i \le B$, are independent. Also, for each $1 \le i \le B$ with $m_i > (1 - \varepsilon_0) h(n)$ we have

$$P(U_i = 1 | W_i = m_i) \le P(U_i = 1 | W_i = (1 - \varepsilon_0) h(n))$$

Thus

$$P(U = 1 | W_{<1+\delta} \cap W_{>1-\varepsilon_0,p}) \leq \prod_{i:W_i > (1-\varepsilon_0)h(n)} P(U_i = 1 | W_i = (1-\varepsilon_0)h(n))$$

$$\leq P(U_1 = 1 | W_1 = (1-\varepsilon_0)h(n))^{pB}.$$
(11)

In view of Theorem 3.2.(a).(iii), we may assume that $h(n) \to \infty$. By (8), for some $\theta > 0$

$$P\left(U=1 | W_{<1+\delta} \cap W_{>1-\varepsilon_0,p}\right) \le \left(1 - \frac{\theta}{h(n) \cdot \varepsilon_0^3}\right)^{pB} = \left(1 - \frac{\theta/\varepsilon_0^3}{h(n)}\right)^{\frac{h(n) \cdot pn}{h^2(n)}}.$$
 (12)

As

$$\left(1 - \frac{\theta/\varepsilon_0^3}{h(n)}\right)^{h(n)} \xrightarrow[n \to \infty]{} e^{-\theta/\varepsilon_0^3}, \qquad \frac{pn}{h^2(n)} \xrightarrow[n \to \infty]{} \infty,$$

we obtain from (11) and (12)

$$\lim_{n \to \infty} P\left(U = 1 \left| W_{< 1+\delta} \cap W_{> 1-\varepsilon_0, p} \right. \right) = 0.$$
(13)

Now we claim that the event $W_{<1+\delta} \cap \overline{W}_{>1-\varepsilon_0,p}$ in the second term on the right-hand side of (10) is empty. In fact, the event $W_{<1+\delta}$ means that all sub-instances \mathcal{I}_i are of density less than $1 + \delta$, and the event $\overline{W}_{>1-\varepsilon_0,p}$ means that most of them are of density at most $1 - \varepsilon_0$. Since the overall density is $\alpha > 1 - \varepsilon_0$, the two events do not meet for sufficiently small δ, p . Thus

$$P\left(W_{<1+\delta} \cap \overline{W}_{>1-\varepsilon_0,p}\right) = 0.$$
(14)

We turn to the last term on the right-hand side of (10). The condition $\overline{W}_{<1+\delta}$ implies that there is at least one $1 \le j \le B$ such that the density of \mathcal{I}_j is at least $1+\delta$. Since the threshold of 2-SAT is 1, this \mathcal{I}_j is UNSAT w.h.p., and in particular \mathcal{I} is such. Hence:

$$\lim_{n \to \infty} P\left(U = 1 \left| \overline{W}_{<1+\delta} \right. \right) = 0.$$
(15)

By (10), (13), (14) and (15), *I* is UNSAT w.h.p.

- (b) In this part we employ the approach of part (a) with minor changes. We may assume $h(n) = \theta_1 \sqrt{n}$ for some $\theta_1 > 0$.
 - (i) Consider (10). In the first term on the right-hand side, as $pn/h^2(n) \le \theta_2$ for some $\theta_2 > 0$, by (12) we have

$$\overline{\lim}_{n \to \infty} P\left(U = 1 \left| W_{< 1+\delta} \cap W_{> 1-\varepsilon_0, p} \right. \right) \le e^{-\theta \cdot \theta_2/\varepsilon_0^3}.$$
(16)

(14) and (15) still hold in this case. Thus, by (10), (14), (15) and (16),

$$\overline{\lim}_{n \to \infty} P(U=1) \le e^{-\theta \cdot \theta_2/\varepsilon_0^3} < 1.$$

In the other direction, let α' be strictly between α and 1. Similarly to (10),

$$P\left(U=1\right) \ge P\left(W_{<\alpha'}\right) P\left(U=1 \left|W_{<\alpha'}\right.\right)$$
(17)

First, consider the second factor on the right-hand side of (17). Given that $W_{<\alpha'}$ has occurred, for some $(m_1, \ldots, m_k) \in I_{<\alpha'}$ the event $\bigcap_{i=1}^{B} \{W_i = m_i\}$ has occurred. Similarly to (11),

$$P\left(U=1 | W_{<\alpha'}\right) \ge \prod_{i=1}^{B} P\left(U_{i}=1 | W_{i}=\alpha' \cdot h(n)\right) = P\left(U_{1}=1 | W_{1}=\alpha' \cdot h(n)\right)^{B}$$

By (8), for some $\theta_3, \theta_4 > 0$

$$\underline{\lim}_{n \to \infty} P\left(U = 1 \left| W_{<\alpha'} \right.\right) \ge \lim_{n \to \infty} \left(1 - \frac{\theta_3}{h(n) \cdot (1 - \alpha')^3} \right)^{n/h(n)}$$
$$= \lim_{n \to \infty} \left(1 - \frac{\theta_1^{-1} \cdot \theta_3 \cdot (1 - \alpha')^{-3}}{\sqrt{n}} \right)^{\sqrt{n}/\theta_1} = e^{-\theta_4} > 0.$$

Now consider the first factor on the right-hand side of (17). By (9), w.h.p. the number of clauses in the sub-instance \mathcal{I}_i with the maximal number of clauses is bounded above by

$$\frac{m}{B} \cdot (1 + o(1)) \cdot \alpha \cdot h(n).$$

Thus the density of all \mathcal{I}_j -s is bounded by

$$\frac{(1+o(1))\cdot\alpha\cdot h(n)}{h(n)}\xrightarrow[n\to\infty]{}\alpha<\alpha',$$

namely

$$P\left(W_{<\alpha'}\right) \xrightarrow[n \to \infty]{} 1. \tag{18}$$

By (17)-(18)

$$\underline{\lim}_{n \to \infty} P(U=1) \ge 1 \cdot e^{-\theta_4} > 0.$$

Therefore, \mathcal{I} is SAT with probability bounded away from both 0 and 1.

(ii) Similarly to (10), and with p > 0 to be determined later,

$$P(U=1) \le P(U=1|W_{\ge 1,p}) + P(\overline{W}_{\ge 1,p}).$$
 (19)

Consider the first addend on the right-hand side of (19). Similarly to (11),

$$P(U = 1 | W_{\geq 1,p}) \le \prod_{i=1}^{pB} P(U_i = 1 | W_i = h(n)) = P(U_1 = 1 | W_1 = h(n))^{pB}.$$

By (7), for some $0 < \theta_2 < 1$ and $\theta_3 > 0$

$$P\left(U=1 | W_{\geq 1,p}\right) \le \theta_2^{p\sqrt{n}/\theta_3} \xrightarrow[n \to \infty]{} 0.$$
(20)

Consider the second addend on the right-hand side of (19). Define the variables X_j , $1 \le j \le n$, as follows: $X_j = 1$ if the *j*-th clause consists of variables from the first community, and $X_j = 0$ otherwise. Thus, $X_j \sim \text{Ber}(1/B)$. The variables X_1, \ldots, X_n are independent, $|X_j| \le 1$ for $1 \le j \le n$ and

$$\sum_{j=1}^{n} V(X_j) = n \cdot \frac{1}{B} \left(1 - \frac{1}{B} \right) = h(n) \left(1 - \frac{h(n)}{n} \right) \xrightarrow[n \to \infty]{} \infty.$$

Thus, by a version of the Central Limit Theorem [27, Corollary 2.7.1]

$$\frac{\sum_{j=1}^{n} X_j - E\left(\sum_{j=1}^{n} X_j\right)}{\sqrt{\sum_{j=1}^{n} V\left(X_j\right)}} \xrightarrow[n \to \infty]{\mathcal{D}} N(0, 1).$$

Clearly, $E\left(\sum_{j=1}^{n} X_{j}\right) = E(W_{1}) = h(n)$. Thus, for large *n* we have

$$P(W_1 \ge h(n)) = P\left(\frac{W_1 - h(n)}{\sqrt{h(n)(1 - h(n)/n)}} \ge 0\right) \approx \Phi(0) = \frac{1}{2}.$$
 (21)

(We mention in passing that, in fact, we do not need the Central Limit Theorem for our purpose. By [21, Theorem 1], as $W_1 \sim B(n, h(n)/n)$ and h(n)/n > 1/n

$$P(W_1 \ge h(n)) = P(W_1 \ge E(W_1)) > \frac{1}{4}.$$

This inequality is weaker than (21), but suffices for the proof.) Define the variables

$$D_i = \begin{cases} 1, & W_i \ge h(n), \\ 0, & \text{otherwise,} \end{cases} \quad 1 \le i \le B.$$

The D_i -s are Ber (p_0) -distributed, where $p_0 = P(W_1 \ge h(n))$. Let $D = \sum_{i=1}^B D_i$. By (21)

$$E(D) = B \cdot P(W_1 \ge h(n)) > B/3.$$

Consider the proportion of sub-instances with at least h(n) clauses. We want to find a p > 0 such that $P(D > pB) \xrightarrow[n \to \infty]{} 1$. By [15, Lemma 2], the variables D_i are negatively correlated, and hence

$$V(D) = \sum_{i=1}^{B} V(D_i) + 2 \sum_{1 \le i < j \le B} \operatorname{Cov}(D_i, D_j)$$
$$\le B \cdot V(D_1) = B \cdot p_0(1 - p_0) \le B/4.$$

By the one-sided Chebyshev inequality for any $p_1 > 0$

$$P(D - E(D) \ge -p_1B) \ge 1 - \frac{V(D)}{V(D) + p_1^2B^2} \ge 1 - \frac{B/4}{p_1^2B^2} = 1 - \frac{1}{4p_1^2B} \xrightarrow[n \to \infty]{} 1.$$

Thus

$$P(D \ge E(D) - p_1B) = P(D \ge B/3 - p_1B) \xrightarrow[n \to \infty]{} 1.$$

Therefore for p = 1/6 w.h.p. D > B/6. Thus

$$P\left(\overline{W}_{\geq 1,1/6}\right) \xrightarrow[n \to \infty]{} 0.$$
(22)

(c)

(i) Let $\alpha' \in (1 - \varepsilon_0, \alpha)$. Similarly to (10)

$$P(U=0) \le P\left(U=0 \left| W_{<\alpha'} \right.\right) + P\left(\overline{W}_{<\alpha'} \right).$$
(23)

Consider the first term on the right-hand side of (23). Similarly to the proofs of the previous parts, given that the event $W_{<\alpha'}$ has occurred, the density of each sub-instance \mathcal{I}_i is less than α' , and thus,

$$P\left(U_{i}=0\left|W_{<\alpha'}\right.\right) \leq P\left(U_{i}=0\left|W_{i}=\alpha'\cdot h(n)\right.\right), \qquad 1\leq i\leq B.$$

Employing the union bound

$$P\left(U=0\left|W_{<\alpha'}\right) \leq \sum_{i=1}^{B} P\left(U_{i}=0\left|W_{i}=\alpha'\cdot h(n)\right)\right)$$
$$= B \cdot P\left(U_{1}=0\left|W_{1}=\alpha'\cdot h(n)\right).$$
(24)

By (8), as $\alpha' > 1 - \varepsilon_0$, for some $\theta > 0$

$$P\left(U_{1}=0 \mid W_{1}=\alpha' \cdot h(n)\right) < \frac{\theta}{h(n) \cdot (1-\alpha')^{3}}.$$
(25)

By (24) and (25), and as $h(n) = \omega(\sqrt{n})$

$$P\left(U=0\left|W_{<\alpha'}\right.\right) \le \frac{n}{h^2(n)} \cdot \frac{\theta}{\left(1-\alpha'\right)^3} \xrightarrow[n\to\infty]{} 0.$$
(26)

By (26), (18) and (23), \mathcal{I} is SAT w.h.p

(ii) Start from (19). Consider the first term on the right-hand side of (19). As h(n) = o(n), similarly to (20) we have

$$P\left(U=1 \mid W_{\geq 1,p}\right) \le \theta_1^{pn/h(n)} \xrightarrow[n \to \infty]{} 0.$$

By (22), for sufficiently small p, the second term on the right-hand side of (19) will vanish. Thus, \mathcal{I} is UNSAT w.h.p.

- (i) In part (c).(i) we only used the fact that $h(n) = \omega(\sqrt{n})$, so that the proof there applies here as well.
- (ii) In this case we may assume that $B = B_0$ is fixed. By (9), the density of the sub-instance \mathcal{I}_i with the maximal number of clauses is bounded above by

$$\frac{1}{h(n)} \cdot \left(\frac{m}{B} + \sqrt{\frac{2m\ln B}{B}}\right) = 1 + \sqrt{\frac{2\ln B_0}{n/B_0}}$$

Thus, denoting $\alpha'(n) = 1 + \sqrt{2B_0 \ln B_0/n}$, we have

$$P\left(W_{\leq \alpha'(n)}\right) \xrightarrow[n \to \infty]{} 1.$$
(27)

By (17),

$$P\left(U=1\right) \ge P\left(W_{\le \alpha'(n)}\right) P\left(U=1 \left|W_{\le \alpha'(n)}\right.\right)$$

By (7), and similarly to (10), for some $\theta > 0$,

$$P\left(U=1\left|W_{\leq\alpha'(n)}\right)\geq\theta^{B_0}>0.$$
(28)

Thus, by (27)-(28), \mathcal{I} is SAT with probability bounded away from 0. In the other direction, there is at least one sub-instance \mathcal{I}_i with density at least 1. Without loss of generality assume that the density of the first sub-instance \mathcal{I}_1 is at least 1 and thus, for the same θ as above

$$P\left(U_1=1\right) \le \theta < 1.$$

Therefore,

$$P(U = 1) = \prod_{i=1}^{B} P(U_i = 1) \le P(U_1 = 1) < 1.$$

Thus, \mathcal{I} is SAT with probability bounded away from both 0 and 1.

The proof of Theorem 3.8 follows Chvátal and Reed [10]. The case p = 0 follows from Proposition 3.1. We will thus assume that p > 0.

We first recall two definitions and their relevance to the satisfiability/unsatisfiability of an instance.

Definition 4.2. [10] A bicycle is a formula that consists of at least two distinct variables v_1, \ldots, v_s and clauses C_0, C_1, \ldots, C_s with the following structure: there are literals l_1, \ldots, l_s such that each l_i is either v_i or \overline{v}_i , we have $C_i = \overline{l}_i \bigvee l_{i+1}$ for all $1 \le i \le s - 1$, $C_0 = u \bigvee l_1$, and $C_s = \overline{l}_s \bigvee v$ where $u, v \in \{v_1, \ldots, v_s, \overline{v}_1, \ldots, \overline{v}_s\}$.

Chvátal and Reed [10] proved that every unsatisfiable formula contains a bicycle.

Definition 4.3. [10] A snake is a sequence of distinct literals l_1, \ldots, l_s such that no l_i is the complement of another.

Chvátal and Reed [10] proved that, for a snake A consisting of the literals l_1, \ldots, l_s , the formula F_A , consisting of the s + 1 clauses $\overline{l}_i \bigvee l_{i+1}$ for all $0 \le i \le s$ with $l_0 = l_{s+1} = \overline{l}_t$, is unsatisfiable.

Proof of Theorem 3.8: Suppose that

$$\lim_{n \to \infty} \frac{m}{n} = r.$$

We have to show that for r < 1 our formula is satisfiable w.h.p., while for r > 1 it is unsatisfiable w.h.p.

First suppose that r < 1. Let p' be the probability that our formula contains a bicycle. We will derive an upper bound for p'. To derive this upper bound, we will add up the probabilities of our formula containing each specific bicycle. Thus, first take some specific bicycle, consisting of s + 1clauses C_0, C_1, \ldots, C_s as in Definition 4.2 for some $2 \le s \le n$. Also, suppose that exactly j out of the clauses $C_1, C_2, \ldots, C_{s-1}$ consist of two variables from the same community. There are at most m^{s+1} choices as to which of the m clauses will make up the clauses $C_0, C_1, C_2, \ldots, C_s$ in our formula. The probability of a clause in the formula to be some specific clause, with both variables in the same community, in our bicycle is

$$\frac{1-p}{4B\binom{n/B}{2}},$$

whereas if the variables are in different communities then this probability is

$$\frac{p}{\binom{B}{2}\left(\frac{2n}{B}\right)^2}$$

Then the probability that our formula will contain this specific bicycle is bounded above by

$$\frac{(1-p)^{j}p^{s-1-j}m^{s+1}}{\left(\binom{2n}{2}\right)^{2}\left(4B\binom{n/B}{2}\right)^{j}\left(\binom{B}{2}\left(\frac{2n}{B}\right)^{2}\right)^{s-1-j}} = \frac{B^{s-1}(1-p)^{j}p^{s-1-j}m^{s+1}}{n^{2s-j}(2n-1)^{2}(2n-2B)^{j}(2(B-1))^{s-1-j}}.$$

Now we count the number of possible bicycles. Suppose we restrict ourselves to bicycles such that exactly j clauses out of $C_1, C_2, \ldots, C_{s-1}$ as defined above each consists of two variables from the same community. There are $\binom{s-1}{j}$ ways to choose these clauses. Then if we pick the literals for the bicycle one at a time, we have 2n choices for the first literal since we have n Boolean variables in total. If C_1 is supposed to contain both variables from the same community, then there are $\frac{2n}{B} - 2$ choices for the second literal. On the other hand, if C_1 is supposed to contain variables from different communities, then there are $\frac{2(B-1)n}{B}$ choices for the second literal. Continuing in this way, we see that there are less than $2n \left(\frac{2n}{B}\right)^j \left(\frac{2(B-1)n}{B}\right)^{s-1-j}$ choices for the literals in the bicycle. Also, there are at most s^2 choices for u and v. Hence, assuming 2n > B, we have

$$\begin{aligned} p' &< \sum_{s=2}^{n} s^{2} \sum_{j=0}^{s-1} {\binom{s-1}{j}} (2n)^{s} \left(\frac{1}{B}\right)^{j} \left(\frac{B-1}{B}\right)^{s-1-j} \cdot \frac{B^{s-1}(1-p)^{j}p^{s-1-j}m^{s+1}}{n^{2s-j}(2n-1)^{2}(2n-2B)^{j}(2(B-1))^{s-1-j}} \\ &= \sum_{s=2}^{n} s^{2} \sum_{j=0}^{s-1} {\binom{s-1}{j}} \frac{2^{j+1}n^{j-s}(1-p)^{j}p^{s-1-j}m^{s+1}}{(2n-1)^{2}(2n-2B)^{j}} \\ &= \sum_{s=2}^{n} \frac{2s^{2}p^{s-1}m^{s+1}}{n^{s}(2n-1)^{2}} \sum_{j=0}^{s-1} {\binom{s-1}{j}} \left(\frac{2n(1-p)}{p(2n-2B)}\right)^{j} \\ &= \frac{2m^{2}}{n(2n-1)^{2}} \sum_{s=2}^{n} s^{2} \left(\frac{2nm-2pBm}{n(2n-2B)}\right)^{s-1}. \end{aligned}$$

By a geometric series argument, the sum above is finite, and so $p' = O\left(\frac{1}{n}\right)$. Thus, the satisfiability threshold is at least 1.

Now suppose r > 1. We will also assume that p < 1. (If p = 1, the proof is actually simpler.) For each $n \in \mathbb{N}$, choose a $t = t(n) \in \mathbb{N}$ in such a way that

$$\lim_{n \to \infty} t/\log n = \infty, \qquad \lim_{n \to \infty} t/n^{1/9} = 0.$$
 (29)

Let s = 2t - 1. We will show that our formula contains a formula F_A of a snake A consisting of s literals w.h.p. Thus, our formula will be unsatisfiable w.h.p. We use the second moment method. Let $X = \sum X_A$, where $X_A = 1$ if our formula contains each clause of F_A exactly once, and $X_A = 0$ otherwise. We will prove that

$$E(X^2) \le (1+o(1))E(X)^2,$$
(30)

from which the desired result may be deduced using Chebyshev's inequality. Consider an arbitrary snake A. Suppose that F_A contains exactly t_1 clauses each consisting of a pair of variables from different communities and exactly t_2 clauses each consisting of variables from the same community. We have $E(X_A) = f(t_1, t_2)$, where

$$f(x_1, x_2) = \sum_{i=x_1}^{m-x_2} p^i (1-p)^{m-i} {m \choose i} {i \choose x_1} {m-i \choose x_2} x_1! x_2! \left(\frac{1}{4\binom{B}{2} \frac{n^2}{B^2}}\right)^{x_1} \left(\frac{1}{4B\binom{n/B}{2}}\right)^{x_2}$$
$$\cdot \left(1 - \frac{x_1}{4\binom{B}{2} \frac{n^2}{B^2}}\right)^{i-x_1} \left(1 - \frac{x_2}{4B\binom{n/B}{2}}\right)^{m-i-x_2}$$
$$= (1-p)^{x_2} p^{x_1} \left(\frac{1}{4\binom{B}{2} \frac{n^2}{B^2}}\right)^{x_1} \left(\frac{1}{4B\binom{n/B}{2}}\right)^{x_2} \frac{m!}{(m-x_1-x_2)!}$$
$$\cdot \left((1-p) \left(1 - \frac{x_2}{4B\binom{n/B}{2}}\right) + p \left(1 - \frac{x_1}{4\binom{B}{2} \frac{n^2}{B^2}}\right)\right)^{m-x_1-x_2}.$$

Take two snakes A and A', where F_A contains exactly t_1 clauses with variables from different communities and exactly t_2 clauses with variables from the same community, and $F_{A'}$ contains exactly t_3 clauses with variables from different communities and exactly t_4 clauses with variables from the same community. Also, suppose F_A and $F_{A'}$ share precisely i_1 clauses with variables in different communities and precisely i_2 clauses with variables from the same community. Then $E(X_A X_{A'}) = f(t_1 + t_3 - i_1, t_2 + t_4 - i_2)$. Since m = O(n), we have

$$f(x_1, x_2) = (1 + o(1)) \left(\frac{Bpm}{2(B-1)n^2}\right)^{x_1} \left(\frac{B(1-p)m}{2n^2}\right)^{x_2}$$
(31)

uniformly in both cases if we assume that $x_1, x_2 = O(n^{\alpha})$ where $\alpha < 1/2$.

Now let us count the snakes A such that F_A contains exactly t_1 clauses with variables from different communities and exactly t_2 clauses with variables from the same community. We denote the set of all such snakes as S_{t_1,t_2} . First, we may view F_A as a directed graph with vertices y_1, \ldots, y_s (where each y_i is the variable such that l_i is y_i or \overline{y}_i) and edges $y_i y_{i+1}, 0 \le i \le s$, with $y_0 = y_{s+1} = y_t$. This directed graph consists of two direct cycle graphs, each consisting of t vertices and having exactly one vertex in common (the vertex $y_0 = y_{s+1} = y_t$). Each edge corresponds to a clause in F_A . Consider the t_2 edges corresponding to the t_2 clauses with variables in different communities. Let j_1 and j_2 be the number of such edges in each of the two cycle graphs that make up the whole graph. We can see that $j_1, j_2 \neq 1$. For k = 1, 2 for the cycle with the j_k edges, there will be $\binom{t}{j_k}$ ways to choose these j_k edges. These j_k edges will then partition the set of vertices into $\max\{1, j_k\}$ groups, where the variables corresponding to the vertices in a group will belong to the same community. Thus, the number of ways of choosing the community of each of the variables corresponding to the vertices in this cycle graph is the chromatic number of the cycle graph consisting of $\max\{1, j_k\}$ vertices in B colours or $(B-1)^{j_k} + (j_k-1)(-1)^{j_k}$. After choosing all of these communities for each cycle graph, we are left with choosing the variables from these communities, and there will be at least $\frac{n}{B} - s$ choices per vertex after making the choice

for the v_t variable. Putting it altogether, the number of such snakes will be bounded below by

$$\frac{2n}{B} \left(\frac{2n}{B} - 2s\right)^{s-1} \sum_{\substack{j_1 + j_2 = t_1 \\ j_1 \neq 1, j_2 \neq 1}} \binom{t}{j_1} \binom{t}{j_2} \cdot \frac{((B-1)^{j_1} + (B-1)(-1)^{j_1})((B-1)^{j_2} + (B-1)(-1)^{j_2})}{B}$$

and bounded above by

$$\left(\frac{2n}{B}\right)^{s} \sum_{\substack{j_1+j_2=t_1\\j_1\neq 1, j_2\neq 1}} \binom{t}{j_1}\binom{t}{j_2} \cdot \frac{\left((B-1)^{j_1}+(B-1)(-1)^{j_1}\right)\left((B-1)^{j_2}+(B-1)(-1)^{j_2}\right)}{B}$$

By (29), the latter is asymptotic to the actual number of such snakes as $n \to \infty$. By (31), we thus have:

$$\begin{split} E(X) &\sim \sum_{t_1=0}^{2t} \left(\frac{Bpm}{2(B-1)n^2} \right)^{t_1} \left(\frac{B(1-p)m}{2n^2} \right)^{2t-t_1} \left(\frac{2n}{B} \right)^{2t-1} \\ &\quad \cdot \sum_{j_1+j_2=t_1} {t \choose j_1} {t \choose j_2} \cdot \frac{((B-1)^{j_1} + (B-1)(-1)^{j_1}) \left((B-1)^{j_2} + (B-1)(-1)^{j_2} \right)}{B} \\ &= \frac{1}{B} \left(\frac{2n}{B} \right)^{2t-1} \left(\frac{B(1-p)m}{2n^2} \right)^{2t} \left(\sum_{j=0}^{t} {t \choose j} \left(\frac{p}{(B-1)(1-p)} \right)^{j} \left((B-1)^{j} + (B-1)(-1)^{j} \right) \right)^{2} \\ &= \frac{1}{B} \left(\frac{2n}{B} \right)^{2t-1} \left(\frac{B(1-p)m}{2n^2} \right)^{2t} \left(\left(1 + \frac{p}{(1-p)} \right)^{t} + (B-1) \left(1 - \frac{p}{(B-1)(1-p)} \right)^{t} \right)^{2} \\ &= \frac{1}{2n} \left(\left(\frac{m}{n} \right)^{t} + (B-1) \left(\frac{(1-p)m}{n} - \frac{mp}{n(B-1)} \right)^{t} \right)^{2} \end{split}$$
(32)

By (29), we have

$$E(X_A X_{A'}) = (1 + o(1))E(X_A)E(X_{A'}) \left(\frac{2(B-1)n^2}{Bpm}\right)^{i_1} \left(\frac{2n^2}{B(1-p)m}\right)^{i_2}$$

uniformly in the range $0 \le i_1, i_2 \le 2t$. In particular, if F_A and $F_{A'}$ have no clauses in common, then $E(X_A X_{A'}) = (1 + o(1))E(X_A)E(X_{A'})$. Thus, to prove (30), our main concern will be when F_A and $F_{A'}$ have clauses in common. To deal with this case, we will derive an upper bound for

$$\sum_{|F_A \cap F_{A'}|=i} E\left(X_A X_{A'}\right)$$

(where $F_A \cap F_{A'}$ denotes the set of common clauses of F_A and $F_{A'}$) for each $1 \le i \le 2t$. First consider how we can construct two snakes A and A' such that F_A and $F_{A'}$ have i clauses in common and account for its contribution to the above sum. Viewing F_A and $F_{A'}$ as graphs as above, we let

 $F_{AA'}$ be their intersection, with isolated vertices removed. Suppose that $F_{AA'}$ contains *i* edges and *j* vertices. To construct the possible snakes *A* and *A'*, we create a procedure with five steps:

- 1) Choose j terms of A for membership in $F_{AA'}$.
- 2) Assign variables to these *j* terms.
- 3) Choose which positions in the snake A' will be filled with terms in $F_{AA'}$.
- 4) Assign variables to the positions in A' picked out in step 2).
- 5) Assign variables to all other positions in A and A'.

For 1), we can select our j terms of A as follows. We first decide if the edge y_0y_1 is in $F_{AA'}$ or not, and then, for each $1 \le r \le s$, we place a marker at y_r if exactly one of $y_{r-1}y_r$ and y_ry_{r+1} is in $F_{AA'}$. The total number of markers will be between 2(j - i) - 1 and 2(j - i) + 2, and so the total number of choices for the j terms is at most $2\binom{s+3}{2j-2i+2}$. Thus, the total number of choices for 3) will also be at most $2\binom{s+3}{2j-2i+2}$. Also, we have at most $tk!2^k$ choices for step 4), where k is the number of components in $F_{AA'}$.

For step 2), if we impose the restriction that i_1 edges among the j vertices correspond to the clauses with variables in different communities, then the number of ways to assign such variables is $\binom{i}{i_1}\binom{2n}{B}^j B^k(B-1)^{i_1}$. As well, for step 5), if we impose the restrictions that, of the remaining 2t-i clauses in A, there are exactly t_1 with variables in different communities, and that of the remaining 2t - i clauses in A' there are exactly t_2 clauses with variables in different communities, then the number of ways to assign such variables is bounded above by $\binom{2t-i}{t_1}\binom{2n}{t_2}\binom{2n-j}{B}^{2s-2j}(B-1)^{t_1+t_2}$.

First suppose that $1 \le i \le t - 1$. Then none of the components of $F_{AA'}$ may contain loops, so that k = j - i. Putting it all together, weighing all of the possible pairs of snakes A and A', appropriately using (29), we obtain

$$\sum_{|F_A \cap F_{A'}|=i} E\left(X_A X_{A'}\right) < \sum_{j \ge i+1} \frac{9}{2} \binom{s+3}{2j-2i+2}^2 t \cdot (j-i)! (2B)^{j-i} \\ \cdot \left(\left(\frac{2n}{B}\right)^j \sum_{i_1=0}^i \binom{i}{i_1} \left(\frac{Bpm}{2n^2}\right)^{i_1} \left(\frac{B(1-p)m}{2n^2}\right)^{i-i_1}\right) \\ \cdot \left(\left(\frac{2n}{B}\right)^{s-j} \sum_{t_1=0}^{2t-i} \binom{2t-i}{t_1} \left(\frac{Bpm}{2n^2}\right)^{t_1} \left(\frac{B(1-p)m}{2n^2}\right)^{2t-i-t_1}\right)^2 \\ < \frac{9B^2 t (2t+2)^4}{8n^2} \left(\frac{m}{n}\right)^{4t-i} \sum_{j \ge i+1} \left(\frac{B^3 (2t+2)^4}{n}\right)^{j-i}$$

for sufficiently large n. Thus by (32) we have for sufficiently large n

$$\frac{\sum_{|F_A \cap F_{A'}|=i} E\left(X_A X_{A'}\right)}{E(X)^2} < 5t(2t+2)^4 \left(\frac{n}{m}\right)^i \sum_{j \ge i+1} \left(\frac{B^3(2t+2)^4}{n}\right)^{j-i} < \frac{2600B^3 t^9}{n} \left(\frac{n}{m}\right)^i.$$

Now suppose that $t \le i \le 2t$. We have two possibilities for the components of $F_{AA'}$. Either none of them contains loops or exactly one of them contains a loop and the number of loops in this

component is exactly 1 or 2, where the possible loops are y_0, y_1, \ldots, y_t and $y_t, y_{t+1}, \ldots, y_{s+1}$. In either case we have $k \leq j - i + 2$. Thus,

$$\sum_{|F_A \cap F_{A'}|=i} E\left(X_A X_{A'}\right) < \sum_{j \ge i+1} \frac{9}{2} \binom{s+3}{2j-2i+2}^2 t \cdot (j-i)! (2B)^{j-i} B^2 \cdot \left(\left(\frac{2n}{B}\right)^j \sum_{i_1=0}^i \binom{i}{i_1} \left(\frac{Bpm}{2n^2}\right)^{i_1} \left(\frac{B(1-p)m}{2n^2}\right)^{i-i_1}\right) \cdot \left(\left(\frac{2n}{B}\right)^{s-j} \sum_{t_1=0}^{2t-i} \binom{2t-i}{t_1} \left(\frac{Bpm}{2n^2}\right)^{t_1} \left(\frac{B(1-p)m}{2n^2}\right)^{2t-i-t_1}\right)^2 < \frac{9B^6 t (2t+2)^4}{2n^2} \left(\frac{m}{n}\right)^{4t-i} \sum_{j \ge i+1} \left(\frac{B^3 (2t+2)^4}{n}\right)^{j-i}.$$

By (32), for sufficiently large n

$$\frac{\sum_{|F_A \cap F_{A'}|=i} E\left(X_A X_{A'}\right)}{E(X)^2} < 20B^4 t (2t+2)^4 \left(\frac{n}{m}\right)^i \sum_{j \ge i+1} \left(\frac{B^3 (2t+2)^4}{n}\right)^{j-i} < \frac{10400B^7 t^9}{n} \left(\frac{n}{m}\right)^i.$$

Thus

$$\sum_{i=1}^{2t} \frac{\sum_{|F_A \cap F_{A'}|=i} E\left(X_A X_{A'}\right)}{E(X)^2} < \sum_{i=1}^{2t} \frac{10400 B^7 t^9}{n} \left(\frac{n}{m}\right)^i = o(1),$$

from which we can deduce (30).

5 Empirical results

To test the question posed after Theorem 3.8, we have conducted the following experiment. We have taken $n = 10^6$, and $m = n + c \cdot n^{2/3}$, with c = -1, 0, 1, 2. (This non-symmetric range was due to preliminary simulations, that showed that the interesting window is actually centered somewhat above n. For each such m, we generated 10^5 random instances from $F(n, m, 2, P_{(1,1)})$ and $F(n, m, 1, P_{(2)})$ (which is just the random model), tested each instance using the SAT solver SAT4J, described in [8], and calculated the percentage of satisfiable instances in each group. To complete the picture, we did the same for the model $F(n, m, 2, P_{(2)})$.

The results are presented in Table 3. The first two models show remarkably similar results. Unsurprisingly, the third model leads to lower satisfiability probabilities.

6 Conclusions

We have dealt with the satisfiability threshold of a particular model of SAT. This model highlights one of the features in which so-called community-structured SAT instances differ from classical

F m	$0.99 \cdot 10^{6}$	10^{6}	$1.01 \cdot 10^{6}$	$1.02 \cdot 10^{6}$
$F\left(n,m,2,P_{(1,1)}\right)$	0.980	0.909	0.641	0.201
$F\left(n,m,1,P_{(2)}\right)$	0.980	0.908	0.644	0.203
$F\left(n,m,2,P_{(2)}\right)$	0.946	0.827	0.521	0.142

Table 3: Percentage of satisfiable instances (out of 10^5 instances) for $n = 10^6$.

SAT instances. Namely, the set of variables decompose into several disjoint subsets-communities. The significance of these communities stems from the fact that clauses tend to contain variables from the same community. We have shown, roughly speaking, that the satisfiability threshold of such instances tends to be lower than for regular instances. Moreover, if the communities are very small, the threshold may even vanish.

The paper leaves a lot to study for industrial SAT instances. To begin with, there are other features considered in the literature as being characteristic of industrial instances. For example, in the scale-free structure, the variables are selected by some heavy-tailed distribution. Moreover, even regarding the issue of communities, there is more to be done. We assumed here that all communities are of the same size. Obviously, there is no justification for this assumption beyond the fact that it simplifies significantly the analysis of the model. What can be said about the threshold if there are both small and large communities? Even prior to that, what would be reasonable to assume regarding the probability of a variable to be selected from each of the communities?

Acknowledgments

We would like to express our gratitude to David Wilson for helpful information regarding the topic of this paper, and to the two anonymous referees for their comments on the first version of the paper.

References

- [1] Dimitris Achlioptas and Yuval Peres, "The threshold for random k-SAT is $2^k \ln 2 O(k)$ ", Journal of the American Mathematical Society 17.4 (2004), 947–973.
- [2] Malcolm Adams, and Victor Guillemin, "Measure theory and probability", Corrected reprint of the 1986 original, Birkhäuser Boston, Inc., Boston, MA, (1996).
- [3] Noga Alon and Eyal Lubetzky, "Poisson approximation for non-backtracking random walks", Israel Journal of Mathematics 174.1 (2009), 227–252.
- [4] Carlos Ansótegui, Maria L. Bonet, Jesús Giráldez-Cru, and Jordi Levy, "On the Classification of Industrial SAT Families", Artificial Intelligence Research and Development - Proc. of the 18th International Conference of the Catalan Association for Artificial Intelligence, Valencia, Catalonia, Spain, October 21–23 (2015), 163–172.
- [5] Carlos Ansótegui, María L. Bonet, and Jordi Levy, "Towards industrial-like random SAT instances", *Proc. of the 21st International Joint Conference on Artificial Intelligence, IJCAI 2009* (2009).
- [6] Carlos Ansótegui, María L. Bonet, and Jordi Levy, "On the Structure of Industrial SAT Instances", *Principles and Practice of Constraint Programming - CP 2009*, Lecture Notes in Comput. Sci., 5732, I. P. Gent, Ed., Springer Berlin Heidelberg (2009), 127–141.
- [7] Carlos Ansótegui, Jesús Giráldez-Cru, and Jordi Levy, "The community structure of SAT formulas", Proc. of Theory and Applications of Satisfiability Testing–SAT 2012: 15th International Conference, Trento, Italy, June 17–20, 2012, A. Cimatti, R. Sebastiani, Eds., 410–423.
- [8] Daniel Le Berre and Anne Parrain, "The Sat4j library, release 2.2, system description", Journal on Satisfiability, Boolean Modeling and Computation 7 (2010), 59–64.
- [9] Béla Bollobás, Christian Borgs, Jennifer T. Chayes, Jeong H. Bim, and David B. Wilson, "The scaling window of the 2-SAT transition", Random Structures & Algorithms 18 (2001), 3, 201–256.
- [10] Vaclav Chvátal and B. Reed, "Mick gets some (the odds are on his side)", *Proc. 33rd Symp. Foundations of Computer Science* (1992), 620–627. DINA : CHECK REEDS FIRST NAME
- [11] Stephen A. Cook, "The complexity of theorem proving procedures", *Proc. of the Third Annual ACM STOC* (1971), 151–158.
- [12] Amin Coja-Oghlan and Konstantinos Panagiotou, "The asymptotic *k*-SAT threshold", Advances in Mathematics 288 (2016), 985–1068.
- [13] Jian Ding, Allan Sly, and Nike Sun, "Proof of the satisfiability conjecture for large k", (English summary) STOC'15—Proc. of the 2015 ACM Symposium on Theory of Computing (2015), 59–68.

- [14] Josep Díaz, Lefteris Kirousis, Dieter Mitsche, and Xavier Pérez-Giménez, "On the satisfiability threshold of formulas with three literals per clause", Theoretical Computer Science 410 (2009) 30, 2920–2934.
- [15] Devdatt Dubhashi and Desh Ranjan, "Balls and bins: a study in negative dependence", Random Structures & Algorithms 13 (1998) 2, 99–124.
- [16] Wenceslas Fernandez de la Vega, On random 2-SAT, unpublished manuscript (1992).
- [17] John V. Franco and Marvin C. Paull, "Probabilistic analysis of the Davis–Putnam procedure for solving the satisfiability problem", Discrete Applied Mathematics 5(1) (1983), 77–87.
- [18] Jesús Giráldez-Cru and Jordi Levy, "A modularity-based random SAT instances generator", Proc. of the 24th International Joint Conference on Artificial Intelligence, IJCAI'15 (2015), 1952–1958.
- [19] Jesús Giráldez-Cru and Jordi Levy, "Generating SAT instances with community structure", Artificial Intelligence 238 (2016), 119–134.
- [20] Jesús Giráldez-Cru and Jordi Levy, "Locality in Random SAT Instances", *Proc. of the 26th International Joint Conference on Artificial Intelligence, IJCAI'17* (2017) 638–644.
- [21] Spencer Greenberg and Mehryar Mohri, "Tight lower bound on the probability of a binomial exceeding its expectation", Statistics & Probability Letters 86 (2014), 91–98.
- [22] Andreas Goerdt, "A threshold for unsatisfiability", Proc. Mathematical Foundations of Computer Science 1992: 17th International Symp., Prague, Czechoslovakia, August 24–28, 1992, 629 Springer, Berlin (1992), 264–274.
- [23] M. Hajiaghayi ans G. B. Sorkin, "The satisfiability threshold of random 3-SAT is at least 3.52", Research Report RC22942, IBM, October 2003.
- [24] Alexis C. Kaporis, Lefteris M. Kirousis, and Efthimios G. Lalas. "The Probabilistic Analysis of a Greedy Satisfiability Algorithm", Random Structures & Algorithms 28(4) (2006), 444– 480.
- [25] Lefteris M. Kirousis, Evangelos Konstantinou Kranakis, Danny Krizanc, and Y. C. Stamatiou, "Approximating the unsatisfiability threshold of random formulas", Random Structures & Algorithms 12(3) (1998), 253–269.
- [26] Valentin F. Kolchin, Boris A. Sevastýanov, and Vladimir P. Chistyakov, "Random allocations", V. H. Winston & Sons, Washington, D.C., (1978).
- [27] Eric Leo Lehmann, "Elements of large-sample theory", Springer-Verlag, New York, (1999).
- [28] Jia Hui Liang, Chanseok Oh, Minu Mathew, Ciza Thomas, Chunxiao Li, and Vijay Ganesh, "Machine learning-based restart policy for CDCL SAT solvers", *Theory and applications of satisfiability testing – SAT 2018*, Lecture Notes in Comput. Sci., 10929, Springer, Cham (2018), 94–110.

- [29] Stephan Mertens, Marc Mézard, and Riccardo Zecchina, "Threshold values of random *k*-SAT from the cavity method", Random Structures & Algorithms 28 (2006), 340–373.
- [30] Marc Mézard and Riccardo Zecchina, "Random *k*-satisfiability problem: From analytic solution to an efficient algorithm", Physical Review E 66 (2002), 056126.
- [31] Mark E. J. Newman, "Finding community structure in networks using the eigenvectors of matrices", Physical Review E74(3) (2006), 036104.
- [32] Zack Newsham, Vijay Ganesh, Sebastian Fischmeister, Gilles Audemard, and Laurent Simon, "Impact of community structure on SAT solver performance", *Theory and applications of satisfiability testing – SAT 2014*, Lecture Notes in Comput. Sci., 8561, Springer (2014), 252–268.
- [33] Chanseok Oh, "Between SAT and UNSAT: the fundamental difference in CDCL SAT, *Theory* and applications of satisfiability testing SAT 2015, Lecture Notes in Comput. Sci., 9340, Springer, Cham (2015), 307–323.
- [34] Chanseok Oh, "Improving SAT solvers by exploiting empirical characteristics of CDCL", PHD thesis (Doctoral dissertation), New York University, https://cs.nyu.edu/media/publications/oh_chanseok.pdf (2016).
- [35] Juyong Park and Mark E. J. Newman, "Origin of degree correlations in the Internet and other networks", Physical Review E68 (2003), 026112.
- [36] Raymond E. A. C. Paley and Antoni Zygmund, "On some series of functions, (3)", Mathematical Proceedings of the Cambridge Philosophical Society, 28 (1932) 2, 190–205.
- [37] Martin Raab and Angelika Steger, "Balls into bins–a simple and tight analysis", J. D. P. Rolim, M. Serna and M. Luby, Eds., Randomization and Approximation Techniques in Computer Science, 1518, Springer, Berlin (1998), 159–170.
- [38] David B. Wilson, http://dbwilson.com/2sat-data/(1998).

A Proof of Theorem 3.5

In the proof of Theorem 3.5 we shall use the following lemma, which is analogous to Lemma 3.4.

Lemma A.1. Consider the balls and bins problem with B bins and M(B) balls, and also with B bins and M'(B) balls, where $M'(B) = \omega(M(B))$. If the maximum load for M(B) balls is at least $s \ge 1$ with probability bounded away from 0, then the maximum load for M'(B) is at least s w.h.p.

Proof: Assume we part the balls into b(B) = M'(B)/M(B) disjoint batches of M(B) balls each. Suppose we toss the balls in each batch into the bins separately, and check the maximum load for each batch. Let L_i be the maximum load for batch $i, 1 \le i \le b(B)$. According to our assumption, there exists an $\varepsilon > 0$ such that

$$P(L_i \ge s) \ge \varepsilon, \qquad i = 1, 2, \dots, b(B).$$

Let L be the maximum load in the case we place all the M'(B) balls into the B bins. The events $\{L_i \ge s\}, 1 \le i \le b(B)$, are independent, and we clearly have

$$\{L < s\} \subseteq \bigcap_{i=1}^{b(B)} \{L_i < s\}.$$

Hence:

$$P(L \ge s) = 1 - P(L < s) \ge 1 - (1 - \varepsilon)^{b(B)} \xrightarrow[B \to \infty]{} 1.$$

The next proof will make use of the notion of negative association of random variables [15]: Denote $[k] = \{1, ..., k\}$ for k > 0. Random variables $X_1, ..., X_k$ are *negatively associated* if for every two index sets $I, J \subseteq [k]$, with $I \cap J = \emptyset$,

$$E\Big(f_1(X_i; i \in I)f_2(X_j; j \in J)\Big) \le E\Big(f_1(X_i; j \in I)\Big)E\Big(f_2(X_j; j \in J)\Big),$$

for every two functions $f_1 : \mathbf{R}^{|I|} \to \mathbf{R}$ and $f_2 : \mathbf{R}^{|J|} \to \mathbf{R}$, which are both non-decreasing or both non-increasing.

In the proof of Theorem 3.5, we will make use of the following result, concerning the balls and bins problem. Let Y_i denote the number of balls placed in the *i*-th bin, $1 \le i \le B$. Let g_i : $\mathbf{R} \to \mathbf{R}$ be non-decreasing functions, $1 \le i \le B$. By [15, Lemma 2], the variables Y_1, \ldots, Y_B are negatively associated, and in particular the $g_i(Y_i)$ -s are negatively correlated.

Proof of Theorem 3.5: Let Y_1, \ldots, Y_B be as above. We clearly have

$$Y_i \sim B(M(B), 1/B), \qquad 1 \le i \le B.$$

Define the variables

$$S_i = \begin{cases} 1, & Y_i \ge s, \\ 0, & \text{otherwise,} \end{cases} \quad 1 \le i \le B.$$

The S_i -s are Ber(p)-distributed, where $p = P(Y_1 \ge s)$. Let $S = \sum_{i=1}^{B} S_i$.

(a) We use the first moment method. Obviously:

$$P(S > 0) = P(S \ge 1) \le E(S) = Bp.$$

Let us index the balls from 1 to M(B), and let $M_j = 1$ if the *j*-th ball entered the first bin and $M_j = 0$ otherwise, $1 \le j \le M(B)$. Thus, the M_j -s are Ber(1/B)-distributed. Let $\mathcal{J} = {\binom{[M(B)]}{s}}$ denote the set of subsets of size *s* of [M(B)]. By the union bound and symmetry:

$$p = P(Y_1 \ge s) = P\left(\bigcup_{J \in \mathcal{J}} \bigcap_{j \in J} \{M_j = 1\}\right)$$
$$\leq \binom{M(B)}{s} \cdot P\left(M_1 = \ldots = M_s = 1\right) = \binom{M(B)}{s} \left(\frac{1}{B}\right)^s.$$

Since $M(B) = o(B^{1-1/s})$,

$$P(S > 0) \le B \cdot \binom{M(B)}{s} \left(\frac{1}{B}\right)^s \le B \cdot \frac{M(B)^s}{B^s} = \left(\frac{M(B)}{B^{1-1/s}}\right)^s \xrightarrow[B \to \infty]{} 0.$$

Thus, w.h.p. the maximum load does not exceed s - 1.

(b) We employ the second moment method. First, if $M(B) \ge Bs$, then there must be at least one bin with at least s balls in it. Thus we may assume that $\frac{M(B)}{B} < s$. We have

$$\begin{split} E(S) &= B \cdot P\left(Y_1 \ge s\right) = B \cdot \sum_{j=s}^{M(B)} \binom{M(B)}{j} \left(\frac{1}{B}\right)^j \left(1 - \frac{1}{B}\right)^{M(B)-j} \\ &\ge B \cdot \binom{M(B)}{s} \left(\frac{1}{B}\right)^s \left(1 - \frac{1}{B}\right)^{M(B)-s}. \end{split}$$

For sufficiently large B we have $M(B) \ge 2s$, and therefore

$$E(S) \ge B \cdot \frac{(M(B)/(2B))^s}{s!} \cdot \left(\left(1 - \frac{1}{B}\right)^B \right)^{(M(B)-s)/B}$$
$$\ge B \cdot \frac{(M(B)/(2B))^s}{s!} \cdot e^{-2M(B)/B}.$$

Thus we have

$$s!E(S) \ge B \cdot e^{-2M(B)/B} \cdot \left(\frac{M(B)}{2B}\right)^s \ge e^{-2s} \left(\frac{M(B)}{2B^{1-1/s}}\right)^s$$
 (33)

By [15], the variables Y_1, \ldots, Y_B are negatively associated. Since each S_i is a non-decreasing function of Y_i , this yields $\text{Cov}(S_i, S_j) \leq 0$ for $i \neq j$. Hence:

$$V(S) = \sum_{i=1}^{B} V(S_i) + 2 \sum_{1 \le i < j \le B} \operatorname{Cov}(S_i, S_j)$$
$$\le B \cdot V(S_1) = B \cdot p(1-p) < B \cdot p = E(S).$$

As $S \ge 0$, the Paley–Zygmund inequality [36] yields

$$P(S > 0) \ge \frac{E^2(S)}{E(S^2)} = \frac{E^2(S)}{V(S) + E^2(S)} > \frac{E^2(S)}{E(S) + E^2(S)} = \frac{E(S)}{1 + E(S)}.$$
(34)

By (33), we have $E(S) \xrightarrow[B \to \infty]{} \infty$. Also, by (33) and (34), we have

$$P\left(S>0\right) > \frac{E(S)}{1+E(S)}$$

and so $P\left(S>0\right)\xrightarrow[B\to\infty]{}$ 1. Thus, w.h.p. the maximum load is at least s.

(c) The first statement follows from parts (a) and (b), applied with s + 1 and s - 1, respectively, instead of s. For the convergence part, suppose (1) holds. Observe that there are B^M possible ways to distribute the M balls into the B bins. Obviously, $X_B = \sum_{i=1}^{B} \mathbb{1}_{y_i=s}$. Let $1 \le t \le B$. We will prove that

$$\lim_{B \to \infty} E\binom{X}{t} = \frac{C^{st}}{(s!)^t t!}.$$
(35)

Specify t bins, say i_1, i_2, \ldots, i_t out of the B bins. The number of balls in bins i_1, i_2, \ldots, i_t , and all of the other bins combined forms a multinomial distribution. It follows that

$$E\binom{X_B}{t} = \frac{\binom{B}{t}\binom{M}{s}\binom{M-s}{s}\binom{M-2s}{s}\cdots\binom{M-(t-1)s}{s}(B-t)^{M-ts}}{B^M}.$$

As $B \to \infty$, we thus have

$$\begin{split} E\binom{X_B}{t} &= (1+o(1)) \, \frac{M^{st}(B-t)^{M-ts}}{t!s!^t B^{M-t}} \\ &= (1+o(1)) \, \frac{C^{st}B^{st-t}(B-t)^{M-ts}}{t!s!^t B^{M-t}} \\ &= (1+o(1)) \, \frac{C^{st}B^{st}(B-t)^{M-ts}}{t!s!^t B^M} \\ &= (1+o(1)) \, \frac{C^{st}(1-t/B)^{-ts}(1-t/B)^M}{t!(s!)^t}. \end{split}$$

From (1), we have

$$\lim_{B \to \infty} \frac{M}{B} = 0$$

and so (35) holds. The desired result follows from Brun's sieve, which is stated in Theorem 2.1 of [3].