# A Model of Random Industrial SAT 

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#### Abstract

One of the most studied models of SAT is random SAT. In this model, instances are composed from clauses chosen uniformly randomly and independently of each other. This model may be unsatisfactory in that it fails to describe various features of SAT instances, arising in real-world applications. Various modifications have been suggested to define models of industrial SAT. Here, we focus mainly on the aspect of community structure. Namely, here the set of variables consists of a number of disjoint communities, and clauses tend to consist of variables from the same community. Thus, we suggest a model of random industrial SAT, in which the central generalization with respect to random SAT is the additional community structure.

There has been a lot of work on the satisfiability threshold of random $k$-SAT, starting with the calculation of the threshold of 2-SAT, up to the recent result that the threshold exists for sufficiently large $k$.

In this paper, we endeavor to study the satisfiability threshold for the proposed model of random industrial SAT. Our main result is that the threshold in this model tends to be smaller than its counterpart for random SAT. Moreover, under some conditions, this threshold even vanishes.


## 1 Introduction

For both historical and practical reasons, the Boolean satisfiability problem (SAT) is one of the most important problems in theoretical computer science. It was the first problem proven to be NPcomplete [11]. Since its introduction, there has been growing interest in the problem, and many aspects of the problem have been researched.

In this problem, one is required to determine whether a certain Boolean formula is satisfiable. An instance of the problem consists of a Boolean formula in several variables $v_{1}, \ldots, v_{n}$. The formula is usually given in conjunctive normal form (CNF). The basic building block of the formula is a literal, which is either a variable $v_{j}$ or its negation $\bar{v}_{j}$. A clause is a disjunction of the form $l_{1} \vee \ldots \vee l_{k}$ of several distinct literals. Thus, altogether, the formula looks like $C_{1} \wedge C_{2} \wedge \ldots \wedge C_{m}$,

[^0]where each $C_{i}$ is a clause, say $C_{i}=l_{i, 1} \vee \ldots \vee l_{i, k_{i}}$. Given a formula, one may assign a TRUE/FALSE value to each of the variables $v_{1}, \ldots, v_{n}$. The formula is satisfiable, or SAT, if there exists an assignment under which the formula is TRUE, and is unsatisfiable, or UNSAT, otherwise.

The $k$-satisfiability ( $k$-SAT) problem is a special case of SAT, in which each clause is a disjunction of up to $k$ literals. Some authors restrict $k$-SAT to instances with exactly $k$ literals in each clause, which terminology we will follow here. Given $n$ and $k$, let $\Omega(n, k)$ denote the set of all $\binom{n}{k} 2^{k}$ possible clauses of length $k$ over $n$ Boolean variables. A random $k$-SAT instance with $m$ clauses is a uniformly random element of $(\Omega(n, k))^{m}$. Namely, it consists of $m$ clauses, selected uniformly randomly and independently from $\Omega(n, k)$. Thus, clause repetitions are allowed, and two instances, differing in the order of the clauses only, are considered as distinct.

The ratio $m / n$ is the density and denoted by $\alpha$. This parameter turns out to be very important. If $\alpha$ is sufficiently small, then a large random instance with density $\alpha$ is SAT with high probability, whereas if it is sufficiently large then a large random instance is UNSAT with high probability. Despite its loose name, the notion of "with high probability" is well defined. Let $\left(E_{j}\right)_{j=1}^{\infty}$ be a sequence of events. The event $E_{j}$ occurs with high probability (w.h.p.) if $P\left(E_{j}\right) \underset{j \rightarrow \infty}{ } 1$. In our case, we take larger and larger random instances with some fixed density, and inquire whether they are SAT or UNSAT. For $k \geq 2$, denote [1]:

$$
\begin{aligned}
& r_{k} \equiv \sup \{\alpha: \text { A random density- } \alpha \text { instance is SAT w.h.p. }\}, \\
& r_{k}^{*} \equiv \inf \{\alpha: \text { A random density- } \alpha \text { instance is UNSAT w.h.p. }\} .
\end{aligned}
$$

For $k=2$, it was proved long ago [10, 16, 22] that $r_{2}=r_{2}^{*}=1$. The Satisfiability Threshold Conjecture claims that, in fact, $r_{k}=r_{k}^{*}$ for all $k$ [10]. This conjectured common value is the satisfiability threshold. It has been a subject of interest among researchers, theoretically and empirically, to prove the conjecture for $k \geq 3$ and find the threshold. Recently, the conjecture has been proved for large enough $k$ [13].

As part of this research, lower and upper bounds were obtained on $r_{k}$ and $r_{k}^{*}$ for $k \geq 3$. In [17] it was proven that $r_{k}^{*} \leq 2^{k} \ln 2$. This has been improved in [25] to $r_{k}^{*} \leq 2^{k} \ln 2-\frac{1}{2}(1+\ln 2)+\varepsilon_{k}$. From the other side, a sequence of successive improvements led finally to the bound $r_{k} \geq 2^{k} \ln 2-$ $\frac{1}{2}(1+\ln 2)+\varepsilon_{k}[12]$. Thus, with the satisfiability conjecture settled in [13] for large $k$, it follows that $r_{k}=r_{k}^{*}=2^{k} \ln 2-\frac{1}{2}(1+\ln 2)+\varepsilon_{k}$ for such $k$. For small values of $k$, more specific results were obtained. For $k=3$, the best bounds seem to be $r_{3} \geq 3.52$ [23, 24], and $r_{3}^{*} \leq 4.4898$ [14]. Experiments and other results of heuristics, based on statistical physics considerations, indicate that $r_{3} \approx 4.26$ [29, 30], $r_{4} \approx 9.93, r_{5} \approx 21.12, r_{6} \approx 43.37, r_{7} \approx 87.79$ [29].

Much more is known about 2-SAT. First, unlike $k$-SAT for $k \geq 3$, which is an NP-complete problem, 2-SAT instances may be solved by a linear time algorithm [10, 22]. Also, there is quite precise information about 2-SAT for density very close to the threshold $r_{2}=1$ [9, 38].

It has been argued that instances of random $k$-SAT do not in fact represent real-world, or industrial, instances [28, 33, 34]. One of the major differences between industrial and random SAT instances is that the set of variables in industrial instances often consists of a disjoint union of subsets, referred to as communities; clauses tend to comprise variables from the same community, with only a minority of clauses containing variables from distinct communities [7, 32]. There are several additional variations [4, 6]. For example, the variables may be selected non-uniformly (say, according to a power-law distribution $[5,20]$ ), and/or the clauses may be of non-constant length.

In this paper we work with a (generalization of a) model introduced by [18] . Our model is similar to the random model, except for the partition of the variables into communities. These communities are of the same size. There are several clause types (defined precisely in the next section), differing in the number of variables from the same or distinct communities in each clause.

Our focus is on the satisfiability threshold in this model. The question has been studied in [18], mostly experimentally, for the model suggested there. We show that the findings in that paper, whereby the threshold tends to be smaller when there are many single-community clauses (i.e., clauses consisting of variables from the same random community) remain true in the general model. In fact, if the communities are small, the threshold may even be 0 .

We present our model in Section 2, The main results are stated in Section 3, and the proofs follow in Section 4. In Section 5 we present some simulation results.

## 2 A Model of Random Industrial SAT

In industrial SAT, the strength of the community structure of an instance is usually measured by its modularity [7, 19, 35]. Roughly speaking, given a graph, its modularity gives an indication of the tendency of the vertices to be connected to other vertices, which are similar to them in some way. In our case, an instance defines the following undirected graph. The set of nodes is the set of variables $\left\{v_{1}, \ldots, v_{n}\right\}$. There is an edge $\left(v_{i}, v_{j}\right)$ for $i \neq j$ if there exists a clause in the instance, containing both variables $v_{i}$ (or its negation) and $v_{j}$ (or its negation). More precisely, we view this object as a multi-graph; if both $v_{i}$ and $v_{j}$ (or their negations) appear in several clauses, there are several edges connecting them. Given an instance, high modularity indicates that there exists a partition of the set of variables into subsets, such that a large portion of the edges connect vertices of the same subset, compared to a random graph with the same number of vertices and same degrees [31, 35] .

As in the regular model, we have $n$ Boolean variables and $m$ clauses in an instance. Each clause is chosen independently of the others. Each variable in each clause is negated with a probability of $1 / 2$, independently of the other variables. The model differs from the regular model in several aspects: There is a community structure on the set of variables, and we also do not necessarily assume all clauses to be of the same length. Specifically, the set of variables $\left\{v_{1}, \ldots, v_{n}\right\}$ is partitioned into $B$ disjoint (sets of variables referred to as) communities $\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{B}$. For simplicity, we assume all communities to be of the same size $h$, so that $n=B \cdot h$. Without loss of generality, we will assume that $\mathcal{C}_{i}=\left\{v_{j}:(i-1) h+1 \leq j \leq i h\right\}, 1 \leq i \leq B$. As $n$ grows, so do usually both $B$ and $h$ (although at times one of them may remain fixed), and we will write $B(n)$ and $h(n)$ when we want to relate to their dependence on $n$. For an $\ell$-tuple $\mathbf{K}=\left(k_{1}, \ldots, k_{\ell}\right)$ with non-increasing, positive integer entries, denote by $\Omega_{B}(n, \mathbf{K})$ the set of all clauses of length $k_{1}+\cdots+k_{\ell}$, formed of $k_{1}$ variables from some community $\mathcal{C}_{i_{1}}, k_{2}$ from another community $\mathcal{C}_{i_{2}}, \ldots, k_{\ell}$ from some $\ell$-th community $\mathcal{C}_{i_{\ell}}$, where the indices $i_{j}$ are mutually distinct. We will refer to such a clause as a clause of type K. We will always implicitly assume that $k_{i} \leq h$ for each $i$, so that we can indeed choose the required number of variables from the various communities. Similarly, we implicitly assume that $\ell \leq B$. Let $P_{\mathbf{K}}$ be the uniform measure on $\Omega_{B}(n, \mathbf{K})$.

Example 2.1. (a) Let $n=1000, B=10$ and $h=100$. The clauses

$$
\left(\bar{v}_{237} \vee \bar{v}_{250} \vee v_{911} \vee \bar{v}_{917} \vee v_{939}\right),
$$

and

$$
\left(v_{401} \vee \bar{v}_{423} \vee v_{427} \vee \bar{v}_{450} \vee v_{500}\right)
$$

are of types $(3,2)$ and $(5)$ (single-community clause), respectively. (In general, the type of single-community clauses of length $k$ will be written as ( $k$ ).) The clauses above belong to the spaces $\Omega_{10}(1000,(3,2))$ and $\Omega_{10}(1000,(5))$, respectively.
(b) The space $\Omega_{10}(1000,(3,2))$ consists of $a=10 \cdot 9\binom{100}{3}\binom{100}{2} \cdot 2^{5}$ clauses, and $\Omega_{10}(1000,(5))$ consists of $b=10\binom{100}{5} \cdot 2^{5}$ clauses.
(c) Under the measure $P_{(3,2)}$, each clause in $\Omega_{10}(1000,(3,2))$ is chosen with probability $1 / a$; under $P_{(5)}$, each clause in $\Omega_{10}(1000,(5))$ is chosen with probability $1 / b$.

The random instances we will be dealing with are of the following structure. There is some number $T \geq 1$ of clause types $\mathbf{K}_{1}, \ldots, \mathbf{K}_{T}$. Each $\mathbf{K}_{t}, 1 \leq t \leq T$, is a vector $\mathbf{K}_{t}=\left(k_{1 t}, \ldots, k_{\ell t}\right)$. These vectors are mutually distinct. Each clause in the instance is of one of these types. The probability of each clause to be of type $\mathbf{K}_{t}$ is $p_{t}$, where $p_{t}, 1 \leq t \leq T$, are arbitrary fixed real numbers, with $\sum_{t=1}^{T} p_{t}=1$. More formally, we select independently $m$ clauses from the space $\bigcup_{t=1}^{T} \Omega_{B}\left(n, \mathbf{K}_{t}\right)$ according to the measure $P=\sum_{t=1}^{T} p_{t} \cdot P_{\mathbf{K}_{t}}$. Using similar notations to [18], denote by $F(n, m, B, P)$ the probability space of instances. Namely, the sample space of $F(n, m, B, P)$ is the $m$-fold Cartesian product $\left(\bigcup_{t=1}^{T} \Omega_{B}\left(n, \mathbf{K}_{t}\right)\right)^{m}$ of the space corresponding to the selection of a single clause, endowed with the product measure $P^{m}$. (For more on the notions of a product of measure spaces and of the product measure, see, for example, [2, Sec. 2.5].) For concreteness, in Algorithm 1 we present the exact mechanism for selecting an instance of $F(n, m, B, P)$.

Note that, when employing Algorithm 1, we care about the order of choices, so that each clause may be obtained in several ways. This is easier to implement and has no bearing on the probability of obtaining each possible clause.

Thus, the regular model of random $k$-SAT is, with the notations above, $F\left(n, m, 1, P_{(k)}\right)$. Instances in the model presented in [18] include clauses of length $k$ of two types: ( $i$ ) singlecommunity clauses - all $k$ variables belong to the same community ( $B\binom{h}{k} \cdot 2^{k}$ possible choices), and (ii) the $k$ variables belong to $k$ distinct communities $\binom{B}{k} \cdot(2 h)^{k}$ possible choices). For some $0<p<1$, each clause is of type ( $i$ ) with probability $p$ and of type ( $i i$ ) with probability $1-p$. With the above notations, their probability space is

$$
F(n, m, B, p \cdot P_{(k)}+(1-p) \cdot P(\underbrace{1, \ldots, 1}_{k}))
$$

for some $k$.
Example 2.2. With $n, B$ and $h$ as in Example 2.1] and $k=3$, the instance

$$
\left(\bar{v}_{423} \vee v_{459} \vee v_{496}\right) \wedge\left(v_{156} \vee \bar{v}_{437} \wedge v_{626}\right)
$$

is an instance in

$$
F\left(1000,2,10,0.2 \cdot P_{(3)}+0.8 \cdot P_{(1,1,1)}\right) .
$$

```
Algorithm 1: Choosing an instance in \(F(n, m, B, P)\)
    Input: \(n, m, B, \mathbf{K}_{1}, \ldots, \mathbf{K}_{T}, p_{1}, \ldots, p_{T}\).
    Output: An instance \(\mathcal{I}\)
    \(\mathcal{I} \leftarrow \emptyset\);
    for \(i \leftarrow 1\) to \(m\) do
        \(C \leftarrow \emptyset ;\)
        Choose a clause type - each \(\mathbf{K}_{t}\) has probability \(p_{t}\);
        Suppose \(\mathbf{K}_{t}=\left(k_{1}, \ldots, k_{\ell}\right)\);
        Select \(\ell\) distinct integers \(i_{1}, \ldots, i_{\ell}\) in the range \([1, B]\) (with the same probability
            \(\frac{1}{B(B-1) \cdots(B-\ell+1)}\) for each possible choice);
        for \(j \leftarrow 1\) to \(\ell\) do
            Choose \(k_{j}\) distinct integers \(a_{1}, \ldots, a_{k_{j}}\) in the range \([1, h]\) (with the same
                probability \(\frac{1}{h(h-1) \cdots\left(h-k_{j}+1\right)}\) for each possible choice);
                for \(d \leftarrow 1\) to \(k_{j}\) do
                    \(x \leftarrow\left(i_{j}-1\right) \cdot h+a_{d} ;\)
                Negate \(v_{x}\) with probability \(1 / 2\);
                \(C \leftarrow C \vee v_{x} ;\)
        \(\mathcal{I} \leftarrow \mathcal{I} \wedge C ;\)
    return \(\mathcal{I}\)
```

The first clause is of type (3) as all three variables $v_{423}, v_{459}, v_{496}$ belong to the same community $\mathcal{C}_{5}=\left\{v_{i}: 401 \leq i \leq 500\right\}$, while the second clause is of type $(1,1,1)$, as the variables $v_{156}, v_{437}, v_{626}$ belong to three distinct communities: $\mathcal{C}_{2}, \mathcal{C}_{5}$ and $\mathcal{C}_{7}$, respectively.

As our interest in this paper is in instances constructed as above, from this point on we will use the term "community-structured" instead of the more general "industrial".

## 3 The Main Results

As explained above, the clauses in an community-structured instance tend to include variables from the same community. In this paper, moreover, we usually deal with the case where one (or more) of the clause types is a single-community type, namely $\mathbf{K}_{t}=(k)$ for some $1 \leq t \leq T$ and $k \leq h$. In some results, we will further restrict ourselves to the case $T=1$, where the only clause type is a single-community type (equivalently, $P=P_{(k)}$ for some $k$ ).

In [18] it was observed empirically that, when the modularity of the variable incidence graph of the instance increases, the threshold decreases. Now, the modularity in our case is larger when more clauses consist of variables from the same community and when the communities are small. Our first result is quite straightforward, but it already hints that instances in the model suggested in Section 2 tend to be no more satisfiable than random $k$-SAT instances. Note that the first part of the proposition is one of the initial results for random SAT [17] .

Proposition 3.1. Let $\mathcal{I}$ be a random instance in $F(n, \alpha n, B, P)$.
(a) Suppose that for each $\mathbf{K}_{t}, 1 \leq t \leq T$, the clause length is at most $k$. If $\alpha>2^{k} \ln 2$, then $\mathcal{I}$ is UNSAT w.h.p.
(b) Let $T=1$ and $P=P_{(k)}$ for some $k \geq 2$.
(i) If $\alpha>r_{k}^{*}$, then $\mathcal{I}$ is UNSAT w.h.p.
(ii) If $h(n)=\Theta(n)$ and $\alpha<r_{k}$, then $\mathcal{I}$ is SAT w.h.p.

Our next result points out a significant difference between random instances and communitystructured ones. One might expect the threshold to be different for community-structured instances, but it turns out that this difference may be not just quantitative. The following result shows that, surprisingly, under certain conditions the satisfiability threshold is 0 . To this end, we will consider $m$ as some function of $n$, not necessarily $m=\alpha n$, and write $m(n)$ instead of $m$.

For real functions $f$ and $g$, we write $f=\Omega(g)$ if $g=O(f)$, and $f=\omega(g)$ if $g=o(f)$. We also write $f=\operatorname{poly} \log (g)$ if $f=O\left(\ln ^{\theta} g\right)$ for some $\theta$.

Theorem 3.2. Let $\mathcal{I}$ be a random instance in $F(n, m(n), B, P)$, where $\mathbf{K}_{t}=(k)$ for some $1 \leq t \leq T$.
(a) Let $h(n)=O(1)$.
(i) If $T=1$ (so that $\left.P=P_{(k)}\right)$ and $m(n)=o\left(n^{1-1 / 2^{k}}\right)$, then $\mathcal{I}$ is SAT w.h.p.
(ii) If $m(n)=\Theta\left(n^{1-1 / 2^{k}}\right)$, then $\mathcal{I}$ is SAT with probability bounded away from 1. If, moreover, $T=1$, then $\mathcal{I}$ is SAT with probability bounded away from both 0 and 1 .
(iii) If $m(n)=\omega\left(n^{1-1 / 2^{k}}\right)$, then $\mathcal{I}$ is UNSAT w.h.p.
(b) If $h(n)=o\left(\frac{\ln n}{\ln \ln n}\right)$ and $m(n)=\Omega\left(\frac{n}{\operatorname{poly} \log (n)}\right)$, then $\mathcal{I}$ is UNSAT w.h.p.
(c) If $h(n)=o(\ln n)$ and $m(n)=\Omega\left(n \cdot e^{-\beta \cdot \ln n / h(n)}\right)$ for some $\beta<1 / r_{k}^{*}$, then $\mathcal{I}$ is UNSAT w.h.p.
(d) Let $h(n)=O(\ln n)$ and $T=1$. Then there exists some $\varepsilon_{0}>0$ such that, if $m(n)=\alpha n$ with $\alpha>r_{k}^{*}-\varepsilon_{0}$, then $\mathcal{I}$ is UNSAT w.h.p.

Remark 3.3. (a) The $\varepsilon_{0}$ in part (d) is effective. Namely, as will follow from the proof, one can present such an $\varepsilon_{0}$ explicitly (in terms of the implicit constant in the equality $h(n)=$ $O(\ln n)$ ).
(b) Still in case (d), one can deal with the general case of arbitrary $T$ as long as the weight of $P_{(k)}$ in $P$, namely the probability that a clause is of type $P_{(k)}$, is sufficiently large.

In Theorem 3.2 there are four types of results for the asymptotic satisfiability of a random community-structured instance with $n$ variables, $m(n)$ clauses, $B$ communities of size $h(n)=$ $n / B$, and probability measure $P$. Namely, either the probability of satisfiability (i) tends to 0 as $n \rightarrow \infty$, or (ii) it tends to 1 , or (iii) it is bounded away from 1 , or (iv) it is bounded away from both 0 and 1. These results are summarized in Table 1. In general, we assume that $\mathbf{K}_{t}=(k)$ for some $1 \leq t \leq T$ and $k \geq 1$. In the third column we place a ' 1 ' or a ' $*$ ', depending on whether

| Parameters |  |  | Result |
| :---: | :---: | :---: | :---: |
| $\boldsymbol{h ( n )}$ | $\boldsymbol{m}(\boldsymbol{n})$ | $\boldsymbol{T}$ |  |
| $O(1)$ | $o\left(n^{1-1 / 2^{k}}\right)$ | 1 | SAT w.h.p. |
| $O(1)$ | $\Theta\left(n^{1-1 / 2^{k}}\right)$ | 1 | $\in(\delta, 1-\delta)$ |
| $O(1)$ | $\Theta\left(n^{1-1 / 2^{k}}\right)$ | $*$ | $\in(0,1-\delta)$ |
| $O(1)$ | $\omega\left(n^{1-1 / 2^{k}}\right)$ | $*$ | UNSAT w.h.p. |
| $o\left(\frac{\ln n}{\ln \ln n}\right)$ | $\Omega(n /$ polylog$(n))$ | $*$ | UNSAT w.h.p. |
| $o(\ln n)$ | $\Omega\left(n^{1-1 /\left(r_{k}^{*}+\varepsilon\right) h(n)}\right)$ | $*$ | UNSAT w.h.p. |
| $O(\ln n)$ | $>\left(r_{k}^{*}-\varepsilon_{0}\right) n$ | 1 | UNSAT w.h.p. |

Table 1: Asymptotic satisfiability of a random instance with small communities in $F(n, m(n), B, P)$.
$T$ is required to be 1 or is arbitrary, respectively. The notation $\in(0,1-\delta)$ indicates a probability bounded away from 1 , and the notation $\in(\delta, 1-\delta)$ indicates a probability bounded away both from 0 and 1.

The proof of Theorem 3.2 will use the following lemma.
Lemma 3.4. Consider the spaces $F(n, m(n), B, P)$ and $F\left(n, m^{\prime}(n), B, P\right)$, where $m^{\prime}(n)=$ $\omega(m(n))$. If a random instance in $F(n, m(n), B, P)$ is UNSAT with probability bounded away from 0 , then a random instance in $F\left(n, m^{\prime}(n), B, P\right)$ is UNSAT w.h.p.

In the proof of Theorem 3.2 (and that of Theorem 3.6), we use some results regarding the classical "balls and bins" problem. In this problem, there are $M$ balls and $B$ bins. Each ball is placed uniformly randomly in one of the bins, independently of the other balls. One quantity of interest is the maximum load, which is the maximum number of balls in any bin. There are several papers studying the size of the maximum load, as well as generalizations of this problem. It seems that [37] contains all previous results. Our next result seem not to be covered by previous results regarding the balls and bins problem. It will be employed in the proof of Theorem 3.2, and is of independent interest.

Given a sequence $\left(X_{n}\right)_{n=1}^{\infty}$ of random variables and a probability law $\mathcal{L}$, we let $X_{n} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{L}$ denote the fact that $X_{n}$ converges to $\mathcal{L}$ in distribution as $n \rightarrow \infty$. Denote by $\operatorname{Po}(\lambda)$ the Poisson distribution with parameter $\lambda$.
Theorem 3.5. Consider the balls and bins problem with $B$ bins and $M=M(B)$ balls, where $B \rightarrow \infty$. Let $s \geq 2$ be an arbitrarily fixed integer.
(a) If $M(B)=o\left(B^{1-1 / s}\right)$, then the maximum load is at most $s-1$ w.h.p.
(b) If $M(B)=\omega\left(B^{1-1 / s}\right)$, then the maximum load is at least s w.h.p.
(c) If $M(B)=\Theta\left(B^{1-1 / s}\right)$, then the maximum load is either $s-1$ or sw.h.p. Moreover, suppose

$$
\begin{equation*}
M(B)=C \cdot B^{1-1 / s}(1+o(1)), \tag{1}
\end{equation*}
$$

and let $X_{B}$ be the number of bins that contain exactly s balls. Then $X_{B} \xrightarrow[B \rightarrow \infty]{\mathcal{D}} \operatorname{Po}\left(C^{s} / s!\right)$.
Theorem 3.5 will be proven in Appendix A.
As noted earlier, random 2-SAT is much better understood than random $k$-SAT for general $k$. This enables us to obtain a stronger result than Theorem3.2 in the case $P=P_{(2)}$.
Theorem 3.6. Let $\mathcal{I}$ be a random instance in $F\left(n, \alpha n, B, P_{(2)}\right)$.
(a) There exists an $0<\varepsilon_{0}<1$ such that, if $h(n)=o(\sqrt{n})$ and $\alpha>1-\varepsilon_{0}$, then $\mathcal{I}$ is UNSAT w.h.p.
(b) For $h(n)=\Theta(\sqrt{n})$ :
(i) If $1-\varepsilon_{0}<\alpha<1$, where $\varepsilon_{0}$ is as in (a), then $\mathcal{I}$ is SAT with probability bounded away from both 0 and 1 .
(ii) If $\alpha=1$ then $\mathcal{I}$ is UNSAT w.h.p.
(c) For $h(n)=\omega(\sqrt{n})$ with $h(n)=o(n)$ :
(i) If $\alpha<1$ then $\mathcal{I}$ is SAT w.h.p.
(ii) If $\alpha=1$ then $\mathcal{I}$ is UNSAT w.h.p.
(d) For $h(n)=\Theta(n)$ :
(i) If $\alpha<1$ then $\mathcal{I}$ is SAT w.h.p.
(ii) If $\alpha=1$ then $\mathcal{I}$ is SAT with probability bounded away from both 0 and 1 .

Remark 3.7. As in Remark [3.3(b), one can deal with the more general case of arbitrary T, as long as one of the clause types $\mathbf{K}_{t}$ is of the form (2) and is of sufficiently large weight.

Similarly to Table 1, we summarize the results of Theorem 3.6 in Table 2, Here, we always assume $k=2, m(n)=\alpha n$ and $T=1$. The notations are as in Table 1.

As we have seen, when clauses tend to be formed of variables in the same community, the instance tends to become unsatisfiable. One may wonder what happens in the opposite case, namely when variables tend to belong to distinct communities. Intuitively, this constraint should usually make little difference, as anyway few clauses may be expected to contain variables from the same community. However, if there are very few communities, this constraint is more significant. Specifically, consider the extreme case of $B=2, P=P_{(1,1)}$. In this case, we disallow about half of the possible clauses of the classical model $B=1, P=P_{(2)}$. Does it affect the satisfiability threshold? Namely, if when moving from the classical case to a case with most clauses from the same community, we tend to make the instance unsatisfiable, will the constraint of having in each clause variables from distinct communities tend to make it "more satisfiable"? The following theorem shows that it makes a very small difference if at all.

| $\boldsymbol{h ( n )}$ | $\in\left(1-\varepsilon_{0}, 1\right)$ | $=1$ |
| :---: | :---: | :---: |
| $=o(\sqrt{n})$ | UNSAT w.h.p | UNSAT w.h.p. |
| $=\Theta(\sqrt{n})$ | $\in(\delta, 1-\delta)$ | UNSAT w.h.p. |
| $\in \omega(\sqrt{n}) \cap o(n)$ | SAT w.h.p. | UNSAT w.h.p |
| $=\Theta(n)$ | SAT w.h.p | $\in(\delta, 1-\delta)$ |

Table 2: Asymptotic satisfiability of a random instance in $F\left(n, \alpha n, B, P_{(2)}\right)$.

Theorem 3.8. Let $0 \leq p \leq 1$ and $B \geq 2$ arbitrary and fixed. The satisfiability threshold in the model

$$
F\left(n, m, B, p P_{(1,1)}+(1-p) P_{(2)}\right)
$$

is 1.
One may still ask whether the regular random model $F\left(n, m, 1, P_{(2)}\right)$ and the model $F\left(n, m, 2, P_{(1,1)}\right)$ display some difference in behaviour near the threshold, namely for $m=n \cdot(1+o(1))$. More precisely, recall that, by [9], for $m$ in some range of size $\Theta\left(n^{2 / 3}\right)$ around $n$, the satisfiability probability for the random model is bounded away from both 0 and 1 . (See (7) below for a more accurate formulation.) Do the two models behave in the same way for $m=n+\theta n^{2 / 3}$ for fixed $\theta$ ?

We studied this question by a large simulation. We detail the experiment in Section 5. The results seem to indicate strongly that the two models behave in the same way also in the window $m=n \pm \Theta\left(n^{2 / 3}\right)$.

## 4 Proofs

## Proof of Proposition 3.1;

(a) We follow the proof in the random model [17]. Fix a truth assignment and consider $\mathcal{I}$. Each clause has at most $k$ literals. The variables are negated with probability $1 / 2$ independently of each other, and hence each clause is satisfied with probability of at most $1-2^{-k}$, independently of the other clauses. The expected number of satisfying truth assignments is therefore at most $2^{n} \cdot\left(1-2^{-k}\right)^{\alpha n}$. As $\alpha>2^{k} \ln 2$, we have

$$
2^{n} \cdot\left(1-2^{-k}\right)^{\alpha n} \underset{n \rightarrow \infty}{\longrightarrow} 0
$$

(Note that we have not used in this part the specific mechanism by which clauses are selected. The variables in each clause may be selected arbitrarily. As long as all clauses are of length at most $k$, and the sign of each variable in each clause is selected uniformly randomly, and independently of all other variables in this clause and all the others, the conclusion holds.) Thus, $\mathcal{I}$ is UNSAT w.h.p.
(b) A random instance $\mathcal{I}$ in $F\left(n, \alpha n, B, P_{(k)}\right)$ decomposes into $B$ sub-instances $\mathcal{I}_{i}, 1 \leq i \leq B$, where each $\mathcal{I}_{i}$ is formed of those clauses consisting of variables solely from $\mathcal{C}_{i}$. Obviously, $\mathcal{I}$ is SAT if and only if all $\mathcal{I}_{i}$-s are such. For $1 \leq i \leq B$, let $U_{i}=1$ if $\mathcal{I}_{i}$ is satisfiable, and $U_{i}=0$ otherwise. The variable $U=\prod_{i=1}^{B} U_{i}$ indicates whether $\mathcal{I}$ is satisfiable. Let $W_{i}$ denote the number of clauses in $\mathcal{I}_{i}$. Since each of the $\alpha n$ clauses consists of variables from $\mathcal{C}_{i}$ with probability $1 / B$, independently of all other clauses, $W_{i}$ is binomially distributed with parameters $\alpha n$ and $1 / B$ :

$$
W_{i} \sim B(\alpha n, 1 / B), \quad 1 \leq i \leq B
$$

(i) Suppose first that $h(n)=\omega(1)$. Let $\alpha_{i}$ denote the density of the sub-instance $\mathcal{I}_{i}, 1 \leq i \leq$ $B$. There exists an $i$ with $\alpha_{i} \geq \alpha$, and therefore $\alpha_{i}>r_{k}^{*}$. It follows that $\mathcal{I}_{i}$ is UNSAT w.h.p., and hence so is $\mathcal{I}$.

The case $h(n)=O(1)$ follows in particular from Theorem 3.2.(a).(iii) (to be proved below).
(ii) In this case $B(n)=\Theta(1)$. Without loss of generality assume $B(n)=B$ is fixed. For $\gamma>0$, let

$$
\begin{equation*}
W_{<\gamma}=\bigcap_{i=1}^{B}\left\{W_{i}<\gamma \cdot h(n)\right\} . \tag{2}
\end{equation*}
$$

Let $\alpha^{\prime}$ be an arbitrary fixed number, strictly between $\alpha$ and $r_{k}$. Denoting by $\bar{A}$ the complement of an event $A$, we have:

$$
\begin{align*}
P(U=1)= & P\left(W_{<\alpha^{\prime}}\right) P\left(U=1 \mid W_{<\alpha^{\prime}}\right) \\
& +P\left(\bar{W}_{<\alpha^{\prime}}\right) P\left(U=1 \mid \bar{W}_{<\alpha^{\prime}}\right)  \tag{3}\\
\geq & P\left(W_{<\alpha^{\prime}}\right) P\left(U=1 \mid W_{<\alpha^{\prime}}\right) .
\end{align*}
$$

By the weak law of large numbers for the binomial random variables $W_{i}$,

$$
\frac{W_{i}}{n / B} \xrightarrow[n \rightarrow \infty]{P} \alpha, \quad 1 \leq i \leq B
$$

and therefore

$$
\begin{aligned}
P\left(\bar{W}_{<\alpha^{\prime}}\right) & =P\left(\bigcup_{i=1}^{B}\left\{W_{i} \geq \alpha^{\prime} \cdot n / B\right\}\right) \\
& \leq \sum_{i=1}^{B} P\left(W_{i} \geq \alpha^{\prime} \cdot n / B\right) \\
& =B \cdot P\left(W_{1} \geq \alpha^{\prime} \cdot n / B\right) \xrightarrow[n \rightarrow \infty]{ } 0 .
\end{aligned}
$$

Hence

$$
P\left(W_{<\alpha^{\prime}}\right)=1-P\left(\bar{W}_{<\alpha^{\prime}}\right) \xrightarrow[n \rightarrow \infty]{ } 1 .
$$

Now consider the second factor on the right-hand side of (3). Clearly, the more clauses any $\mathcal{I}_{i}$ contains, the less likely it is to be satisfiable, and therefore

$$
\begin{equation*}
P\left(U=1 \mid W_{<\alpha^{\prime}}\right) \geq \prod_{i=1}^{B} P\left(U_{i}=1 \mid W_{i}=\alpha^{\prime} \cdot n / B\right) . \tag{4}
\end{equation*}
$$

As $\alpha^{\prime}<r_{k}$, each of the sub-instances $\mathcal{I}_{i}$ is SAT w.h.p., so that each of the factors in the product on the right-hand side of (4) converges to 1 as $n \rightarrow \infty$. Since, by our assumption, $B$ is fixed, so does the whole product. Hence $\mathcal{I}$ is SAT w.h.p.

As mentioned in Section 3, the proofs of Theorem 3.2 and Theorem 3.6 make use of some results concerning the balls and bins problem. Let $L$ be the maximum load for $M$ balls and $B$ bins. By [37], for any $\delta>0$,

$$
L \geq \begin{cases}\frac{\ln B}{\ln \frac{B \ln B}{M},} & \text { if } \frac{B}{\operatorname{polylog}(B)}<M=o(B \ln B),  \tag{5}\\ \left(d_{c}-\delta\right) \ln B, & \text { if } M=c \cdot B \ln B,\end{cases}
$$

w.h.p. for an appropriate constant $d_{c}>c$.

Remark 4.1. The constant $d_{c}$, in the second part of (5), is the unique solution of the equation

$$
1+x(\ln c-\ln x+1)-c=0
$$

in $(c, \infty)$ (see [37, Lemma 3]). A routine calculation shows that $d_{c}=c+c \cdot u(1 / c)$, where the function $u$ is the unique non-negative function defined implicitly by the equation

$$
-u(w)+(1+u(w)) \ln (1+u(w))=w, \quad(w \geq 0)
$$

The function $u(w)$ has been studied in [26, pp. 101-102], and in particular expressed as a power series in $\sqrt{w}$ near 0 .

The fact that $d_{c}>c$ is the reason that the threshold in Theorem $3.2(d)$ is strictly less than $r_{k}^{*}$. One can easily bound $d_{c}-c$ from below. In fact, write $d_{c}=c+\varepsilon$. Then

$$
\begin{aligned}
1 & =-(c+\varepsilon)(\ln c-\ln (c+\varepsilon)+1)+c \\
& =-(c+\varepsilon) \ln c+(c+\varepsilon) \ln (c+\varepsilon)-\varepsilon \\
& <(c+\varepsilon) \cdot \varepsilon / c-\varepsilon=\varepsilon^{2} / c,
\end{aligned}
$$

and hence $d_{c}>c+\sqrt{c}$.
Proof of Theorem 3.2: We follow the notations used in the proof of Proposition 3.1. Recall that $\mathcal{I}_{i}$ is the sub-instance formed of those clauses in $\mathcal{I}$ consisting of variables solely from $\mathcal{C}_{i}$, and $W_{i}$ is the number of clauses in $\mathcal{I}_{i}, 1 \leq i \leq B$. Denote $W_{\max }=\max \left\{W_{1}, \ldots, W_{B}\right\}$.

Note that, while we have not assumed that $T=1$ in parts (a).(iii), (b), and (c) of the theorem, we may make this assumption without loss of generality in these parts as well. In fact, suppose
that any of these three parts has been proven for the case $T=1$, and consider the general case. If the probability of $(k)$ in $P$ is $p$, then w.h.p. there will be at least $p \cdot m(n) / 2$ clauses of type $(k)$. To see this, denote by $\mathcal{I}^{(k)}$ the sub-instance of $\mathcal{I}$ obtained by taking the clauses of type $(k)$ and by $m^{\prime}(n)$ the number of clauses in $\mathcal{I}^{(k)}$. Clearly, $m^{\prime}(n)$ is distributed binomially with parameters $m(n)$ and $p$. Employing Chernoff's bound we obtain a lower bound of $p \cdot m(n) / 2$ on $m^{\prime}(n)$ w.h.p. It follows that $m^{\prime}(n)$ has the same lower bound assumed on $m(n)$ (namely, it is $\omega\left(n^{1-1 / 2^{k}}\right)$ in part (a).(iii), it is $\Omega\left(\frac{n}{\operatorname{poylog}(n)}\right)$ in part (b), and it is $\Omega\left(n \cdot e^{-\beta \cdot \ln n / h(n)}\right)$ in part (c)). As we have assumed the correctness of these parts for $T=1$, the instance $\mathcal{I}^{(k)}$ is UNSAT w.h.p., and hence certainly so is the original instance, which contains it. Thus, we may indeed assume in all parts that $T=1$.

Since $T=1$, each clause has all its literals from the same community. Hence, the selection of a clause corresponds to the selection of a community. Consider clauses as balls, and communities as bins. The process of selecting the clauses, as far as the community to which the variables in each clause belong, is analogous to that of placing $m(n)$ balls in $B$ bins uniformly at random. The idea of the proof in parts (b)-(d) will be to prove that w.h.p. we have $W_{\max } / h(n)>r_{k}^{*}$. This will imply that there is at least one sub-instance $\mathcal{I}_{i}$ with density larger than $r_{k}^{*}$. Thus, already $\mathcal{I}_{i}$ is UNSAT w.h.p., and consequently so is $\mathcal{I}$.
(a) Without loss of generality, assume that $h(n)=h>0$ is fixed.
(i) By Theorem 3.5, (a), there is no sub-instance with more than $2^{k}-1$ clauses w.h.p. Since instances with less than $2^{k}$ clauses are certainly satisfiable, all $\mathcal{I}_{i}$-s are SAT, and hence so is $\mathcal{I}$.
(ii) Here, we may assume that $m(n)=\theta \cdot n^{1-1 / 2^{k}}$ for some constant $\theta>0$. By Theorem 3.5. (c), the probability that there is an $\mathcal{I}_{i}, 1 \leq i \leq B$, with at least $2^{k}$ clauses, is bounded away from 0 . Assume, say, that $W_{1} \geq 2^{k}$. Then, with probability at least

$$
\left(1 /\binom{h}{k}\right)^{2^{k}} \cdot\left(2^{k}\right)!/ 2^{k 2^{k}}
$$

all $2^{k}$ distinct clauses consisting of the variables $v_{1}, \ldots, v_{k}$ have been drawn. As the instance is UNSAT if it contains all these $2^{k}$ clauses, the probability for our instance to be UNSAT is bounded away from 0 . Now, assume that $P=P_{(k)}$, for some $k>0$. Now, by Theorem 3.5.(c) there is no sub-instance with more than $2^{k}-1$ clauses with probability bounded away from 0 . Thus, similarly to part (i), $\mathcal{I}$ is SAT with probability bounded away from 0 .
(iii) Follows from the previous part and Lemma 3.4,
(b) In view of part (a).(iii), we may assume $h(n) \rightarrow \infty$. We may also assume that $m(n)=$ $n / \ln ^{\theta} n$ for some $\theta \geq 1$. Clearly, $m(n) \leq B$. On the other hand,

$$
m(n)=\frac{n}{\ln ^{\theta} n} \geq \frac{B}{(2 \ln B)^{\theta}}=\frac{B}{\operatorname{poly} \log (B)}
$$

Thus, by (5), w.h.p., the maximum load is at least

$$
\begin{equation*}
\frac{\ln B}{\ln \frac{B \ln B}{m(n)}} \geq \frac{\frac{1}{2} \cdot \ln n}{\ln \left(\frac{n \cdot \ln n}{n / \ln ^{\theta} n}\right)} \geq \frac{\ln n}{2(\theta+1) \ln \ln n} \tag{6}
\end{equation*}
$$

Now, there are $h(n)=o(\ln n / \ln \ln n)$ variables in each community. By (6), w.h.p., the density of the sub-instance $\mathcal{I}_{i}$ with the maximal number of clauses is at least

$$
\frac{W_{\max }}{h(n)} \geq \frac{\frac{1}{2(\theta+1)} \cdot \ln n / \ln \ln n}{o(\ln n / \ln \ln n)} \underset{n \rightarrow \infty}{ } \infty
$$

Hence, this $\mathcal{I}_{i}$ is UNSAT w.h.p., and therefore so is $\mathcal{I}$.
(c) By (5), w.h.p., the number of clauses in the sub-instance $\mathcal{I}_{i}$ with the maximal number of clauses is at least

$$
W_{\max } \geq \frac{\ln B}{\ln \frac{B \ln B}{m(n)}}=\frac{\ln n(1-o(1))}{\ln \frac{B \ln B}{m(n)}}
$$

For a large enough $n$

$$
\begin{aligned}
\ln \frac{B \ln B}{m(n)} & \leq \ln \left(\frac{\frac{n}{h(n)} \cdot \ln n}{n \cdot e^{-\beta \ln n / h(n)}}\right) \\
& =\ln \frac{\ln n}{h(n)}+\beta \cdot \frac{\ln n}{h(n)}
\end{aligned}
$$

As $\beta<1 / r_{k}^{*}$, for large enough $x$ we have $\ln x+\beta x<x / r_{k}^{*}$. Hence, for large enough $n$ we have

$$
\ln \frac{B \ln B}{m(n)} \leq \frac{1}{r_{k}^{*}} \cdot \frac{\ln n}{h(n)}
$$

This implies that the density of the sub-instance $\mathcal{I}_{i}$ with the maximal number of clauses is at least

$$
\frac{W_{\max }}{h(n)} \geq \frac{1}{\ln \frac{B \ln B}{m(n)}} \cdot \frac{\ln n(1-o(n))}{h(n)}>r_{k}^{*}
$$

and thus UNSAT w.h.p. Consequently, so is $\mathcal{I}$.
(d) In view of the previous part, we may assume that $h(n)=\theta \ln n$ for some $\theta>0$. Choose $c_{0}$ such that

$$
d_{c_{0}}=\theta r_{k}^{*},
$$

where $d_{c}$ is as in (5). Let $\alpha>c_{0} / \theta$, and put $c=\alpha \theta$. Thus, $c>c_{0}$ and $d_{c}>d_{c_{0}}$. Let $\delta<d_{c}-\theta r_{k}^{*}$. We have

$$
\begin{aligned}
m(n)=n \cdot \alpha=\frac{n c}{\theta} & =(1+o(1)) \cdot \frac{n c \ln B}{\theta \ln n} \\
& =(c+o(1)) \cdot B \ln B
\end{aligned}
$$

By (5), the size of the largest sub-instance is $W_{\max } \geq\left(d_{c}-\delta\right) \ln B$ w.h.p. Hence, w.h.p. the density of this sub-instance is

$$
\begin{aligned}
\frac{W_{\max }}{h(n)} & \geq \frac{\left(d_{c}-\delta\right) \ln B}{h(n)}=\frac{\left(d_{c}-\delta\right) \cdot(1-o(1)) \ln n}{\theta \ln n} \\
& =\frac{d_{c}-\delta}{\theta}-o(1)=r_{k}^{*}+\frac{d_{c}-\delta-\theta r_{k}^{*}}{\theta}-o(1) .
\end{aligned}
$$

Letting $\varepsilon_{0}=r_{k}^{*}-c_{0} / \theta$, we get our claim.

Proof of Lemma 3.4: Denote the random instance in $F\left(n, m^{\prime}(n), B, P\right)$ by $\mathcal{I}^{\prime}$. Denote the instance obtained from the first $m(n)$ clauses of $\mathcal{I}^{\prime}$ by $\mathcal{I}_{1}^{\prime}$, the instance obtained from the next $m(n)$ clauses of $\mathcal{I}^{\prime}$ by $\mathcal{I}_{2}^{\prime}, \ldots$, the instance obtained from the last $m(n)$ clauses of $\mathcal{I}^{\prime}$ by $\mathcal{I}_{b(n)}^{\prime}$ (with $\left.b(n)=m^{\prime}(n) / m(n)\right)$. According to our assumption, there exists an $\varepsilon>0$ such that

$$
P\left(\mathcal{I}_{i}^{\prime} \text { is SAT }\right) \leq 1-\varepsilon, \quad i=1,2, \ldots, b(n)
$$

Now, the events $\left\{\mathcal{I}_{i}^{\prime}\right.$ is SAT $\}, 1 \leq i \leq b(n)$, are independent, and we clearly have

$$
\left\{\mathcal{I}^{\prime} \text { is } \mathrm{SAT}\right\} \subseteq \bigcap_{i=1}^{b(n)}\left\{\mathcal{I}_{i}^{\prime} \text { is } \mathrm{SAT}\right\}
$$

Since $b(n) \rightarrow \infty$ :

$$
P\left(\mathcal{I}^{\prime} \text { is SAT }\right) \leq P\left(\mathcal{I}_{i}^{\prime} \text { is SAT, } 1 \leq i \leq b(n)\right) \leq(1-\varepsilon)^{b(n)} \underset{n \rightarrow \infty}{\longrightarrow} 0 .
$$

In the proof of Theorem 3.6 we will use the following result from [9]. There exist some $0<\varepsilon_{0}<1$ and $\lambda_{0}>0$ such that the satisfiability probability of a random 2-SAT instance $\mathcal{I}$ with $m=n \cdot(1+\varepsilon)$ clauses is

$$
P(\mathcal{I} \text { is SAT })= \begin{cases}1-\Theta\left(\frac{1}{n|\varepsilon|^{3}}\right), & -\varepsilon_{0} \leq \varepsilon \leq-\lambda_{0} n^{-1 / 3},  \tag{7}\\ \Theta(1), & -\lambda_{0} n^{-1 / 3}<\varepsilon<\lambda_{0} n^{-1 / 3}, \\ \exp \left(-\Theta\left(n \varepsilon^{3}\right)\right), & \lambda_{0} n^{-1 / 3} \leq \varepsilon \leq \varepsilon_{0}\end{cases}
$$

Actually, in the sequel, we will encounter only the first two cases. Note that in the case $m=$ $n \cdot(1-\varepsilon)$ with $\lambda_{0} n^{-1 / 3} \leq \varepsilon \leq \varepsilon_{0}$, we have

$$
\begin{equation*}
1-\frac{\theta_{1}}{n \cdot \varepsilon^{3}} \leq P(\mathcal{I} \text { is SAT }) \leq 1-\frac{\theta_{2}}{n \cdot \varepsilon^{3}} \tag{8}
\end{equation*}
$$

for some constants $\theta_{1}, \theta_{2}>0$.
We will also use an additional result regarding the balls and bins problem. Let $L$ be the maximum load for $M$ balls and $B$ bins. By [37], w.h.p.

$$
\begin{equation*}
L \leq \frac{M}{B}+\sqrt{\frac{2 M \ln B}{B}}, \quad M=\omega\left(B \ln ^{3} B\right) \tag{9}
\end{equation*}
$$

Given a sequence $\left(X_{i}\right)_{i=1}^{\infty}$ of random variables and a probability law $\mathcal{L}$, we write $X_{i} \xrightarrow[i \rightarrow \infty]{\mathcal{D}} \mathcal{L}$ if the sequence converges to $\mathcal{L}$ in distribution.

Proof of Theorem 3.6: We follow the notations used in the proof of Theorem 3.2, Also, for $\gamma>$ 0 , let

$$
I_{<\gamma}=\left\{\left(m_{1}, \ldots, m_{B}\right): m_{1}+\ldots+m_{B}=\alpha n, m_{i}<\gamma \cdot h(n) \forall 1 \leq i \leq B\right\}
$$

and let $I_{>\gamma}$ and $I_{\geq \gamma}$ be analogously understood. More generally, for $0 \leq p \leq 1$, let $I_{<\gamma, p}$ denote the set of $B$-tuples $\left(m_{1}, \ldots, m_{B}\right)$ with at least $p B$ entries $m_{i}, 1 \leq i \leq B$, for which $m_{i}<\gamma \cdot h(n)$. (Thus, $I_{<\gamma}=I_{<\gamma, 1}$.)

Note that the set $W_{<\gamma}$, defined in (2), may now be written in the form

$$
W_{<\gamma}=\bigcup_{\left(m_{1}, \ldots, m_{B}\right) \in I_{<\gamma}} \bigcap_{i=1}^{B}\left\{W_{i}=m_{i}\right\} .
$$

We use similar notations, for example $W_{>\gamma}, W_{\geq \gamma, p}$ and $W_{<\gamma, p}$, analogously.
(a) Let $\delta, p$ be sufficiently small positive numbers, to be determined later. Let $\varepsilon_{0}$ be as in (7). We have

$$
\begin{align*}
P(U=1)= & P\left(W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}\right) \cdot P\left(U=1 \mid W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}\right) \\
& +P\left(W_{<1+\delta} \cap \bar{W}_{>1-\varepsilon_{0}, p}\right) \cdot P\left(U=1 \mid W_{<1+\delta} \cap \bar{W}_{>1-\varepsilon_{0}, p}\right) \\
& +P\left(\bar{W}_{<1+\delta}\right) P\left(U=1 \mid \bar{W}_{<1+\delta}\right)  \tag{10}\\
\leq & P\left(U=1 \mid W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}\right)+P\left(W_{<1+\delta} \cap \bar{W}_{>1-\varepsilon_{0}, p}\right) \\
& +P\left(U=1 \mid \bar{W}_{<1+\delta}\right) .
\end{align*}
$$

Consider the first term on the right-hand side of (10). The event $W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}$ implies that $W_{j}=m_{j}, 1 \leq j \leq B$, for some $\left(m_{1}, \ldots, m_{B}\right) \in I_{<1+\delta} \cap I_{>1-\varepsilon_{0}, p}$. We note that, conditioned on the event $\bigcap_{i=1}^{B}\left\{W_{i}=m_{i}\right\}$, the events $\left\{U_{i}=1\right\}, 1 \leq i \leq B$, are independent. Also, for each $1 \leq i \leq B$ with $m_{i}>\left(1-\varepsilon_{0}\right) h(n)$ we have

$$
P\left(U_{i}=1 \mid W_{i}=m_{i}\right) \leq P\left(U_{i}=1 \mid W_{i}=\left(1-\varepsilon_{0}\right) h(n)\right) .
$$

Thus

$$
\begin{align*}
P\left(U=1 \mid W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}\right) & \leq \prod_{i: W_{i}>\left(1-\varepsilon_{0}\right) h(n)} P\left(U_{i}=1 \mid W_{i}=\left(1-\varepsilon_{0}\right) h(n)\right)  \tag{11}\\
& \leq P\left(U_{1}=1 \mid W_{1}=\left(1-\varepsilon_{0}\right) h(n)\right)^{p B} .
\end{align*}
$$

In view of Theorem 3.2.(a).(iii), we may assume that $h(n) \rightarrow \infty$. By (8), for some $\theta>0$

$$
\begin{equation*}
P\left(U=1 \mid W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}\right) \leq\left(1-\frac{\theta}{h(n) \cdot \varepsilon_{0}^{3}}\right)^{p B}=\left(1-\frac{\theta / \varepsilon_{0}{ }^{3}}{h(n)}\right)^{\frac{h(n) \cdot p n}{h^{2}(n)}} . \tag{12}
\end{equation*}
$$

As

$$
\left(1-\frac{\theta / \varepsilon_{0}^{3}}{h(n)}\right)^{h(n)} \underset{n \rightarrow \infty}{ } e^{-\theta / \varepsilon_{0}^{3}}, \quad \frac{p n}{h^{2}(n)} \underset{n \rightarrow \infty}{\longrightarrow} \infty
$$

we obtain from (11) and (12)

$$
\begin{equation*}
\lim _{n \rightarrow \infty} P\left(U=1 \mid W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}\right)=0 . \tag{13}
\end{equation*}
$$

Now we claim that the event $W_{<1+\delta} \cap \bar{W}_{>1-\varepsilon_{0}, p}$ in the second term on the right-hand side of (10) is empty. In fact, the event $W_{<1+\delta}$ means that all sub-instances $\mathcal{I}_{i}$ are of density less than $1+\delta$, and the event $\bar{W}_{>1-\varepsilon_{0}, p}$ means that most of them are of density at most $1-\varepsilon_{0}$. Since the overall density is $\alpha>1-\varepsilon_{0}$, the two events do not meet for sufficiently small $\delta, p$. Thus

$$
\begin{equation*}
P\left(W_{<1+\delta} \cap \bar{W}_{>1-\varepsilon_{0}, p}\right)=0 . \tag{14}
\end{equation*}
$$

We turn to the last term on the right-hand side of (10). The condition $\bar{W}_{<1+\delta}$ implies that there is at least one $1 \leq j \leq B$ such that the density of $\mathcal{I}_{j}$ is at least $1+\delta$. Since the threshold of 2-SAT is 1 , this $\mathcal{I}_{j}$ is UNSAT w.h.p., and in particular $\mathcal{I}$ is such. Hence:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} P\left(U=1 \mid \bar{W}_{<1+\delta}\right)=0 \tag{15}
\end{equation*}
$$

By (10), (13), (14) and (15), $\mathcal{I}$ is UNSAT w.h.p.
(b) In this part we employ the approach of part (a) with minor changes. We may assume $h(n)=$ $\theta_{1} \sqrt{n}$ for some $\theta_{1}>0$.
(i) Consider (10). In the first term on the right-hand side, as $p n / h^{2}(n) \leq \theta_{2}$ for some $\theta_{2}>0$, by (12) we have

$$
\begin{equation*}
\varlimsup_{n \rightarrow \infty} P\left(U=1 \mid W_{<1+\delta} \cap W_{>1-\varepsilon_{0}, p}\right) \leq e^{-\theta \cdot \theta_{2} / \varepsilon_{0}^{3}} \tag{16}
\end{equation*}
$$

(14) and (15) still hold in this case. Thus, by (10), (14), (15) and (16),

$$
\varlimsup_{n \rightarrow \infty} P(U=1) \leq e^{-\theta \cdot \theta_{2} / \varepsilon_{0}^{3}}<1
$$

In the other direction, let $\alpha^{\prime}$ be strictly between $\alpha$ and 1 . Similarly to (10),

$$
\begin{equation*}
P(U=1) \geq P\left(W_{<\alpha^{\prime}}\right) P\left(U=1 \mid W_{<\alpha^{\prime}}\right) . \tag{17}
\end{equation*}
$$

First, consider the second factor on the right-hand side of (17). Given that $W_{<\alpha^{\prime}}$ has occurred, for some $\left(m_{1}, \ldots, m_{k}\right) \in I_{<\alpha^{\prime}}$ the event $\bigcap_{i=1}^{B}\left\{W_{i}=m_{i}\right\}$ has occurred. Similarly to (11),

$$
P\left(U=1 \mid W_{<\alpha^{\prime}}\right) \geq \prod_{i=1}^{B} P\left(U_{i}=1 \mid W_{i}=\alpha^{\prime} \cdot h(n)\right)=P\left(U_{1}=1 \mid W_{1}=\alpha^{\prime} \cdot h(n)\right)^{B} .
$$

By (8), for some $\theta_{3}, \theta_{4}>0$

$$
\begin{aligned}
\underline{\lim }_{n \rightarrow \infty} P\left(U=1 \mid W_{<\alpha^{\prime}}\right) & \geq \lim _{n \rightarrow \infty}\left(1-\frac{\theta_{3}}{h(n) \cdot\left(1-\alpha^{\prime}\right)^{3}}\right)^{n / h(n)} \\
& =\lim _{n \rightarrow \infty}\left(1-\frac{\theta_{1}^{-1} \cdot \theta_{3} \cdot\left(1-\alpha^{\prime}\right)^{-3}}{\sqrt{n}}\right)^{\sqrt{n} / \theta_{1}}=e^{-\theta_{4}}>0
\end{aligned}
$$

Now consider the first factor on the right-hand side of (17). By (9), w.h.p. the number of clauses in the sub-instance $\mathcal{I}_{i}$ with the maximal number of clauses is bounded above by

$$
\frac{m}{B} \cdot(1+o(1)) \cdot \alpha \cdot h(n) .
$$

Thus the density of all $\mathcal{I}_{j}$-s is bounded by

$$
\frac{(1+o(1)) \cdot \alpha \cdot h(n)}{h(n)} \underset{n \rightarrow \infty}{ } \alpha<\alpha^{\prime}
$$

namely

$$
\begin{equation*}
P\left(W_{<\alpha^{\prime}}\right) \xrightarrow[n \rightarrow \infty]{\longrightarrow} 1 . \tag{18}
\end{equation*}
$$

By (17)-(18)

$$
\underline{\lim }_{n \rightarrow \infty} P(U=1) \geq 1 \cdot e^{-\theta_{4}}>0
$$

Therefore, $\mathcal{I}$ is SAT with probability bounded away from both 0 and 1 .
(ii) Similarly to (10), and with $p>0$ to be determined later,

$$
\begin{equation*}
P(U=1) \leq P\left(U=1 \mid W_{\geq 1, p}\right)+P\left(\bar{W}_{\geq 1, p}\right) \tag{19}
\end{equation*}
$$

Consider the first addend on the right-hand side of (19). Similarly to (11),

$$
P\left(U=1 \mid W_{\geq 1, p}\right) \leq \prod_{i=1}^{p B} P\left(U_{i}=1 \mid W_{i}=h(n)\right)=P\left(U_{1}=1 \mid W_{1}=h(n)\right)^{p B} .
$$

By (7), for some $0<\theta_{2}<1$ and $\theta_{3}>0$

$$
\begin{equation*}
P\left(U=1 \mid W_{\geq 1, p}\right) \leq \theta_{2}^{p \sqrt{n} / \theta_{3}} \underset{n \rightarrow \infty}{\longrightarrow} 0 . \tag{20}
\end{equation*}
$$

Consider the second addend on the right-hand side of (19). Define the variables $X_{j}, 1 \leq$ $j \leq n$, as follows: $X_{j}=1$ if the $j$-th clause consists of variables from the first community, and $X_{j}=0$ otherwise. Thus, $X_{j} \sim \operatorname{Ber}(1 / B)$. The variables $X_{1}, \ldots, X_{n}$ are independent, $\left|X_{j}\right| \leq 1$ for $1 \leq j \leq n$ and

$$
\sum_{j=1}^{n} V\left(X_{j}\right)=n \cdot \frac{1}{B}\left(1-\frac{1}{B}\right)=h(n)\left(1-\frac{h(n)}{n}\right) \underset{n \rightarrow \infty}{\longrightarrow} \infty
$$

Thus, by a version of the Central Limit Theorem [27, Corollary 2.7.1]

$$
\frac{\sum_{j=1}^{n} X_{j}-E\left(\sum_{j=1}^{n} X_{j}\right)}{\sqrt{\sum_{j=1}^{n} V\left(X_{j}\right)}} \underset{n \rightarrow \infty}{\mathcal{D}} N(0,1) .
$$

Clearly, $E\left(\sum_{j=1}^{n} X_{j}\right)=E\left(W_{1}\right)=h(n)$. Thus, for large $n$ we have

$$
\begin{equation*}
P\left(W_{1} \geq h(n)\right)=P\left(\frac{W_{1}-h(n)}{\sqrt{h(n)(1-h(n) / n)}} \geq 0\right) \approx \Phi(0)=\frac{1}{2} . \tag{21}
\end{equation*}
$$

(We mention in passing that, in fact, we do not need the Central Limit Theorem for our purpose. By [21, Theorem 1], as $W_{1} \sim B(n, h(n) / n)$ and $h(n) / n>1 / n$

$$
P\left(W_{1} \geq h(n)\right)=P\left(W_{1} \geq E\left(W_{1}\right)\right)>\frac{1}{4} .
$$

This inequality is weaker than (21), but suffices for the proof.)
Define the variables

$$
D_{i}=\left\{\begin{array}{ll}
1, & W_{i} \geq h(n), \\
0, & \text { otherwise },
\end{array} \quad 1 \leq i \leq B\right.
$$

The $D_{i}$-s are $\operatorname{Ber}\left(p_{0}\right)$-distributed, where $p_{0}=P\left(W_{1} \geq h(n)\right)$. Let $D=\sum_{i=1}^{B} D_{i}$. By (21)

$$
E(D)=B \cdot P\left(W_{1} \geq h(n)\right)>B / 3
$$

Consider the proportion of sub-instances with at least $h(n)$ clauses. We want to find a $p>0$ such that $P(D>p B) \xrightarrow[n \rightarrow \infty]{\longrightarrow}$. By [15, Lemma 2], the variables $D_{i}$ are negatively correlated, and hence

$$
\begin{aligned}
V(D) & =\sum_{i=1}^{B} V\left(D_{i}\right)+2 \sum_{1 \leq i<j \leq B} \operatorname{Cov}\left(D_{i}, D_{j}\right) \\
& \leq B \cdot V\left(D_{1}\right)=B \cdot p_{0}\left(1-p_{0}\right) \leq B / 4
\end{aligned}
$$

By the one-sided Chebyshev inequality for any $p_{1}>0$

$$
P\left(D-E(D) \geq-p_{1} B\right) \geq 1-\frac{V(D)}{V(D)+p_{1}^{2} B^{2}} \geq 1-\frac{B / 4}{p_{1}^{2} B^{2}}=1-\frac{1}{4 p_{1}^{2} B} \underset{n \rightarrow \infty}{\longrightarrow} 1
$$

Thus

$$
P\left(D \geq E(D)-p_{1} B\right)=P\left(D \geq B / 3-p_{1} B\right) \xrightarrow[n \rightarrow \infty]{\longrightarrow}
$$

Therefore for $p=1 / 6$ w.h.p. $D>B / 6$. Thus

$$
\begin{equation*}
P\left(\bar{W}_{\geq 1,1 / 6}\right) \underset{n \rightarrow \infty}{ } 0 . \tag{22}
\end{equation*}
$$

(c)
(i) Let $\alpha^{\prime} \in\left(1-\varepsilon_{0}, \alpha\right)$. Similarly to (10)

$$
\begin{equation*}
P(U=0) \leq P\left(U=0 \mid W_{<\alpha^{\prime}}\right)+P\left(\bar{W}_{<\alpha^{\prime}}\right) . \tag{23}
\end{equation*}
$$

Consider the first term on the right-hand side of (23). Similarly to the proofs of the previous parts, given that the event $W_{<\alpha^{\prime}}$ has occurred, the density of each sub-instance $\mathcal{I}_{i}$ is less than $\alpha^{\prime}$, and thus,

$$
P\left(U_{i}=0 \mid W_{<\alpha^{\prime}}\right) \leq P\left(U_{i}=0 \mid W_{i}=\alpha^{\prime} \cdot h(n)\right), \quad 1 \leq i \leq B
$$

Employing the union bound

$$
\begin{align*}
P\left(U=0 \mid W_{<\alpha^{\prime}}\right) & \leq \sum_{i=1}^{B} P\left(U_{i}=0 \mid W_{i}=\alpha^{\prime} \cdot h(n)\right)  \tag{24}\\
& =B \cdot P\left(U_{1}=0 \mid W_{1}=\alpha^{\prime} \cdot h(n)\right) .
\end{align*}
$$

By (8), as $\alpha^{\prime}>1-\varepsilon_{0}$, for some $\theta>0$

$$
\begin{equation*}
P\left(U_{1}=0 \mid W_{1}=\alpha^{\prime} \cdot h(n)\right)<\frac{\theta}{h(n) \cdot\left(1-\alpha^{\prime}\right)^{3}} . \tag{25}
\end{equation*}
$$

By (24) and (25), and as $h(n)=\omega(\sqrt{n})$

$$
\begin{equation*}
P\left(U=0 \mid W_{<\alpha^{\prime}}\right) \leq \frac{n}{h^{2}(n)} \cdot \frac{\theta}{\left(1-\alpha^{\prime}\right)^{3}} \xrightarrow[n \rightarrow \infty]{\longrightarrow} 0 \tag{26}
\end{equation*}
$$

By (26), (18) and (23), $\mathcal{I}$ is SAT w.h.p
(ii) Start from (19). Consider the first term on the right-hand side of (19). As $h(n)=$ $o(n)$, similarly to (20) we have

$$
P\left(U=1 \mid W_{\geq 1, p}\right) \leq \theta_{1}^{p n / h(n)} \xrightarrow[n \rightarrow \infty]{ } 0 .
$$

By (22), for sufficiently small $p$, the second term on the right-hand side of (19) will vanish.Thus, $\mathcal{I}$ is UNSAT w.h.p.
(d)
(i) In part (c).(i) we only used the fact that $h(n)=\omega(\sqrt{n})$, so that the proof there applies here as well.
(ii) In this case we may assume that $B=B_{0}$ is fixed. By (9), the density of the sub-instance $\mathcal{I}_{i}$ with the maximal number of clauses is bounded above by

$$
\frac{1}{h(n)} \cdot\left(\frac{m}{B}+\sqrt{\frac{2 m \ln B}{B}}\right)=1+\sqrt{\frac{2 \ln B_{0}}{n / B_{0}}} .
$$

Thus, denoting $\alpha^{\prime}(n)=1+\sqrt{2 B_{0} \ln B_{0} / n}$, we have

$$
\begin{equation*}
P\left(W_{\leq \alpha^{\prime}(n)}\right) \xrightarrow[n \rightarrow \infty]{ } 1 \tag{27}
\end{equation*}
$$

By (17),

$$
P(U=1) \geq P\left(W_{\leq \alpha^{\prime}(n)}\right) P\left(U=1 \mid W_{\leq \alpha^{\prime}(n)}\right)
$$

By (7), and similarly to (10), for some $\theta>0$,

$$
\begin{equation*}
P\left(U=1 \mid W_{\leq \alpha^{\prime}(n)}\right) \geq \theta^{B_{0}}>0 \tag{28}
\end{equation*}
$$

Thus, by (27)-(28), $\mathcal{I}$ is SAT with probability bounded away from 0. In the other direction, there is at least one sub-instance $\mathcal{I}_{i}$ with density at least 1. Without loss of generality assume that the density of the first sub-instance $\mathcal{I}_{1}$ is at least 1 and thus, for the same $\theta$ as above

$$
P\left(U_{1}=1\right) \leq \theta<1 .
$$

Therefore,

$$
P(U=1)=\prod_{i=1}^{B} P\left(U_{i}=1\right) \leq P\left(U_{1}=1\right)<1 .
$$

Thus, $\mathcal{I}$ is SAT with probability bounded away from both 0 and 1.

The proof of Theorem 3.8 follows Chvátal and Reed [10]. The case $p=0$ follows from Proposition 3.1. We will thus assume that $p>0$.

We first recall two definitions and their relevance to the satisfiability/unsatisfiability of an instance.
Definition 4.2. [10] $A$ bicycle is a formula that consists of at least two distinct variables $v_{1}, \ldots, v_{s}$ and clauses $C_{0}, C_{1}, \ldots, C_{s}$ with the following structure: there are literals $l_{1}, \ldots, l_{s}$ such that each $l_{i}$ is either $v_{i}$ or $\bar{v}_{i}$, we have $C_{i}=\bar{l}_{i} \bigvee l_{i+1}$ for all $1 \leq i \leq s-1, C_{0}=u \bigvee l_{1}$, and $C_{s}=\bar{l}_{s} \bigvee v$ where $u, v \in\left\{v_{1}, \ldots, v_{s}, \bar{v}_{1}, \ldots, \bar{v}_{s}\right\}$.

Chvátal and Reed [10] proved that every unsatisfiable formula contains a bicycle.
Definition 4.3. [10] $A$ snake is a sequence of distinct literals $l_{1}, \ldots, l_{s}$ such that no $l_{i}$ is the complement of another.

Chvátal and Reed [10] proved that, for a snake $A$ consisting of the literals $l_{1}, \ldots, l_{s}$, the formula $F_{A}$, consisting of the $s+1$ clauses $\bar{l}_{i} \bigvee l_{i+1}$ for all $0 \leq i \leq s$ with $l_{0}=l_{s+1}=\bar{l}_{t}$, is unsatisfiable.

Proof of Theorem 3.8: Suppose that

$$
\lim _{n \rightarrow \infty} \frac{m}{n}=r .
$$

We have to show that for $r<1$ our formula is satisfiable w.h.p., while for $r>1$ it is unsatisfiable w.h.p.

First suppose that $r<1$. Let $p^{\prime}$ be the probability that our formula contains a bicycle. We will derive an upper bound for $p^{\prime}$. To derive this upper bound, we will add up the probabilities of our formula containing each specific bicycle. Thus, first take some specific bicycle, consisting of $s+1$ clauses $C_{0}, C_{1}, \ldots, C_{s}$ as in Definition 4.2 for some $2 \leq s \leq n$. Also, suppose that exactly $j$ out of the clauses $C_{1}, C_{2}, \ldots, C_{s-1}$ consist of two variables from the same community. There are at most $m^{s+1}$ choices as to which of the $m$ clauses will make up the clauses $C_{0}, C_{1}, C_{2}, \ldots, C_{s}$ in our formula. The probability of a clause in the formula to be some specific clause, with both variables in the same community, in our bicycle is

$$
\frac{1-p}{4 B\binom{n / B}{2}},
$$

whereas if the variables are in different communities then this probability is

$$
\frac{p}{\binom{B}{2}\left(\frac{2 n}{B}\right)^{2}} .
$$

Then the probability that our formula will contain this specific bicycle is bounded above by

$$
\frac{(1-p)^{j} p^{s-1-j} m^{s+1}}{\left(\binom{2 n}{2}\right)^{2}\left(4 B\binom{n / B}{2}\right)^{j}\left(\binom{B}{2}\left(\frac{2 n}{B}\right)^{2}\right)^{s-1-j}}=\frac{B^{s-1}(1-p)^{j} p^{s-1-j} m^{s+1}}{n^{2 s-j}(2 n-1)^{2}(2 n-2 B)^{j}(2(B-1))^{s-1-j}} .
$$

Now we count the number of possible bicycles. Suppose we restrict ourselves to bicycles such that exactly $j$ clauses out of $C_{1}, C_{2}, \ldots, C_{s-1}$ as defined above each consists of two variables from the same community. There are $\binom{s-1}{j}$ ways to choose these clauses. Then if we pick the literals for the bicycle one at a time, we have $2 n$ choices for the first literal since we have $n$ Boolean variables in total. If $C_{1}$ is supposed to contain both variables from the same community, then there are $\frac{2 n}{B}-2$ choices for the second literal. On the other hand, if $C_{1}$ is supposed to contain variables from different communities, then there are $\frac{2(B-1) n}{B}$ choices for the second literal. Continuing in this way, we see that there are less than $2 n\left(\frac{2 n}{B}\right)^{j}\left(\frac{2(B-1) n}{B}\right)^{s-1-j}$ choices for the literals in the bicycle. Also, there are at most $s^{2}$ choices for $u$ and $v$. Hence, assuming $2 n>B$, we have

$$
\begin{aligned}
p^{\prime} & <\sum_{s=2}^{n} s^{2} \sum_{j=0}^{s-1}\binom{s-1}{j}(2 n)^{s}\left(\frac{1}{B}\right)^{j}\left(\frac{B-1}{B}\right)^{s-1-j} \cdot \frac{B^{s-1}(1-p)^{j} p^{s-1-j} m^{s+1}}{n^{2 s-j}(2 n-1)^{2}(2 n-2 B)^{j}(2(B-1))^{s-1-j}} \\
& =\sum_{s=2}^{n} s^{2} \sum_{j=0}^{s-1}\binom{s-1}{j} \frac{2^{j+1} n^{j-s}(1-p)^{j} p^{s-1-j} m^{s+1}}{(2 n-1)^{2}(2 n-2 B)^{j}} \\
& =\sum_{s=2}^{n} \frac{2 s^{2} p^{s-1} m^{s+1}}{n^{s}(2 n-1)^{2}} \sum_{j=0}^{s-1}\binom{s-1}{j}\left(\frac{2 n(1-p)}{p(2 n-2 B)}\right)^{j} \\
& =\frac{2 m^{2}}{n(2 n-1)^{2}} \sum_{s=2}^{n} s^{2}\left(\frac{2 n m-2 p B m}{n(2 n-2 B)}\right)^{s-1} .
\end{aligned}
$$

By a geometric series argument, the sum above is finite, and so $p^{\prime}=O\left(\frac{1}{n}\right)$. Thus, the satisfiability threshold is at least 1.

Now suppose $r>1$. We will also assume that $p<1$. (If $p=1$, the proof is actually simpler.) For each $n \in \mathbb{N}$, choose a $t=t(n) \in \mathbb{N}$ in such a way that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} t / \log n=\infty, \quad \lim _{n \rightarrow \infty} t / n^{1 / 9}=0 \tag{29}
\end{equation*}
$$

Let $s=2 t-1$. We will show that our formula contains a formula $F_{A}$ of a snake $A$ consisting of $s$ literals w.h.p. Thus, our formula will be unsatisfiable w.h.p. We use the second moment method. Let $X=\sum X_{A}$, where $X_{A}=1$ if our formula contains each clause of $F_{A}$ exactly once, and $X_{A}=0$ otherwise. We will prove that

$$
\begin{equation*}
E\left(X^{2}\right) \leq(1+o(1)) E(X)^{2}, \tag{30}
\end{equation*}
$$

from which the desired result may be deduced using Chebyshev's inequality. Consider an arbitrary snake $A$. Suppose that $F_{A}$ contains exactly $t_{1}$ clauses each consisting of a pair of variables from different communities and exactly $t_{2}$ clauses each consisting of variables from the same community. We have $E\left(X_{A}\right)=f\left(t_{1}, t_{2}\right)$, where

$$
\begin{aligned}
f\left(x_{1}, x_{2}\right)= & \sum_{i=x_{1}}^{m-x_{2}} p^{i}(1-p)^{m-i}\binom{m}{i}\binom{i}{x_{1}}\binom{m-i}{x_{2}} x_{1}!x_{2}!\left(\frac{1}{4\binom{B}{2} \frac{n^{2}}{B^{2}}}\right)^{x_{1}}\left(\frac{1}{4 B\binom{n / B}{2}}\right)^{x_{2}} \\
& \cdot\left(1-\frac{x_{1}}{4\binom{B}{2} \frac{n^{2}}{B^{2}}}\right)^{i-x_{1}}\left(1-\frac{x_{2}}{4 B\binom{n / B}{2}}\right)^{m-i-x_{2}} \\
= & (1-p)^{x_{2}} p^{x_{1}}\left(\frac{1}{4\binom{B}{2} \frac{n^{2}}{B^{2}}}\right)^{x_{1}}\left(\frac{1}{4 B\binom{n / B}{2}}\right)^{x_{2}} \frac{m!}{\left(m-x_{1}-x_{2}\right)!} \\
& \cdot\left((1-p)\left(1-\frac{x_{2}}{4 B\binom{n / B}{2}}\right)+p\left(1-\frac{x_{1}}{4\binom{B}{2} \frac{n^{2}}{B^{2}}}\right)\right)^{m-x_{1}-x_{2}}
\end{aligned}
$$

Take two snakes $A$ and $A^{\prime}$, where $F_{A}$ contains exactly $t_{1}$ clauses with variables from different communities and exactly $t_{2}$ clauses with variables from the same community, and $F_{A^{\prime}}$ contains exactly $t_{3}$ clauses with variables from different communities and exactly $t_{4}$ clauses with variables from the same community. Also, suppose $F_{A}$ and $F_{A^{\prime}}$ share precisely $i_{1}$ clauses with variables in different communities and precisely $i_{2}$ clauses with variables from the same community. Then $E\left(X_{A} X_{A^{\prime}}\right)=f\left(t_{1}+t_{3}-i_{1}, t_{2}+t_{4}-i_{2}\right)$. Since $m=O(n)$, we have

$$
\begin{equation*}
f\left(x_{1}, x_{2}\right)=(1+o(1))\left(\frac{B p m}{2(B-1) n^{2}}\right)^{x_{1}}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{x_{2}} \tag{31}
\end{equation*}
$$

uniformly in both cases if we assume that $x_{1}, x_{2}=O\left(n^{\alpha}\right)$ where $\alpha<1 / 2$.
Now let us count the snakes $A$ such that $F_{A}$ contains exactly $t_{1}$ clauses with variables from different communities and exactly $t_{2}$ clauses with variables from the same community. We denote the set of all such snakes as $S_{t_{1}, t_{2}}$. First, we may view $F_{A}$ as a directed graph with vertices $y_{1}, \ldots, y_{s}$ (where each $y_{i}$ is the variable such that $l_{i}$ is $y_{i}$ or $\bar{y}_{i}$ ) and edges $y_{i} y_{i+1}, 0 \leq i \leq s$, with $y_{0}=y_{s+1}=y_{t}$. This directed graph consists of two direct cycle graphs, each consisting of $t$ vertices and having exactly one vertex in common (the vertex $y_{0}=y_{s+1}=y_{t}$ ). Each edge corresponds to a clause in $F_{A}$. Consider the $t_{2}$ edges corresponding to the $t_{2}$ clauses with variables in different communities. Let $j_{1}$ and $j_{2}$ be the number of such edges in each of the two cycle graphs that make up the whole graph. We can see that $j_{1}, j_{2} \neq 1$. For $k=1,2$ for the cycle with the $j_{k}$ edges, there will be $\binom{t}{j_{k}}$ ways to choose these $j_{k}$ edges. These $j_{k}$ edges will then partition the set of vertices into $\max \left\{1, j_{k}\right\}$ groups, where the variables corresponding to the vertices in a group will belong to the same community. Thus, the number of ways of choosing the community of each of the variables corresponding to the vertices in this cycle graph is the chromatic number of the cycle graph consisting of $\max \left\{1, j_{k}\right\}$ vertices in $B$ colours or $(B-1)^{j_{k}}+\left(j_{k}-1\right)(-1)^{j_{k}}$. After choosing all of these communities for each cycle graph, we are left with choosing the variables from these communities, and there will be at least $\frac{n}{B}-s$ choices per vertex after making the choice
for the $v_{t}$ variable. Putting it altogether, the number of such snakes will be bounded below by

$$
\frac{2 n}{B}\left(\frac{2 n}{B}-2 s\right)^{s-1} \sum_{\substack{j_{1}+j_{2}=t_{1} \\ j_{1} \neq 1, j_{2} \neq 1}}\binom{t}{j_{1}}\binom{t}{j_{2}} \cdot \frac{\left((B-1)^{j_{1}}+(B-1)(-1)^{j_{1}}\right)\left((B-1)^{j_{2}}+(B-1)(-1)^{j_{2}}\right)}{B}
$$

and bounded above by

$$
\left(\frac{2 n}{B}\right)^{s} \sum_{\substack{j_{1}+j_{2}=t_{1} \\ j_{1} \neq 1, j_{2} \neq 1}}\binom{t}{j_{1}}\binom{t}{j_{2}} \cdot \frac{\left((B-1)^{j_{1}}+(B-1)(-1)^{j_{1}}\right)\left((B-1)^{j_{2}}+(B-1)(-1)^{j_{2}}\right)}{B} .
$$

By (29), the latter is asymptotic to the actual number of such snakes as $n \rightarrow \infty$. By (31), we thus have:

$$
\begin{align*}
E(X) \sim & \sum_{t_{1}=0}^{2 t}\left(\frac{B p m}{2(B-1) n^{2}}\right)^{t_{1}}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{2 t-t_{1}}\left(\frac{2 n}{B}\right)^{2 t-1} \\
& \cdot \sum_{j_{1}+j_{2}=t_{1}}\binom{t}{j_{1}}\binom{t}{j_{2}} \cdot \frac{\left((B-1)^{j_{1}}+(B-1)(-1)^{j_{1}}\right)\left((B-1)^{j_{2}}+(B-1)(-1)^{j_{2}}\right)}{B} \\
= & \frac{1}{B}\left(\frac{2 n}{B}\right)^{2 t-1}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{2 t}\left(\sum_{j=0}^{t}\binom{t}{j}\left(\frac{p}{(B-1)(1-p)}\right)^{j}\left((B-1)^{j}+(B-1)(-1)^{j}\right)\right)^{2} \\
= & \frac{1}{B}\left(\frac{2 n}{B}\right)^{2 t-1}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{2 t}\left(\left(1+\frac{p}{(1-p)}\right)^{t}+(B-1)\left(1-\frac{p}{(B-1)(1-p)}\right)^{t}\right)^{2} \\
= & \frac{1}{2 n}\left(\left(\frac{m}{n}\right)^{t}+(B-1)\left(\frac{(1-p) m}{n}-\frac{m p}{n(B-1)}\right)^{t}\right)^{2} \\
& \sim \frac{1}{2 n}\left(\frac{m}{n}\right)^{2 t} . \tag{32}
\end{align*}
$$

By (29), we have

$$
E\left(X_{A} X_{A^{\prime}}\right)=(1+o(1)) E\left(X_{A}\right) E\left(X_{A^{\prime}}\right)\left(\frac{2(B-1) n^{2}}{B p m}\right)^{i_{1}}\left(\frac{2 n^{2}}{B(1-p) m}\right)^{i_{2}}
$$

uniformly in the range $0 \leq i_{1}, i_{2} \leq 2 t$. In particular, if $F_{A}$ and $F_{A^{\prime}}$ have no clauses in common, then $E\left(X_{A} X_{A^{\prime}}\right)=(1+o(1)) E\left(X_{A}\right) E\left(X_{A^{\prime}}\right)$. Thus, to prove (30), our main concern will be when $F_{A}$ and $F_{A^{\prime}}$ have clauses in common. To deal with this case, we will derive an upper bound for

$$
\sum_{\left|F_{A} \cap F_{A^{\prime}}\right|=i} E\left(X_{A} X_{A^{\prime}}\right)
$$

(where $F_{A} \cap F_{A^{\prime}}$ denotes the set of common clauses of $F_{A}$ and $F_{A^{\prime}}$ ) for each $1 \leq i \leq 2 t$. First consider how we can construct two snakes $A$ and $A^{\prime}$ such that $F_{A}$ and $F_{A^{\prime}}$ have $i$ clauses in common and account for its contribution to the above sum. Viewing $F_{A}$ and $F_{A^{\prime}}$ as graphs as above, we let
$F_{A A^{\prime}}$ be their intersection, with isolated vertices removed. Suppose that $F_{A A^{\prime}}$ contains $i$ edges and $j$ vertices. To construct the possible snakes $A$ and $A^{\prime}$, we create a procedure with five steps:

1) Choose $j$ terms of $A$ for membership in $F_{A A^{\prime}}$.
2) Assign variables to these $j$ terms.
3) Choose which positions in the snake $A^{\prime}$ will be filled with terms in $F_{A A^{\prime}}$.
4) Assign variables to the positions in $A^{\prime}$ picked out in step 2).
5) Assign variables to all other positions in $A$ and $A^{\prime}$.

For 1), we can select our $j$ terms of $A$ as follows. We first decide if the edge $y_{0} y_{1}$ is in $F_{A A^{\prime}}$ or not, and then, for each $1 \leq r \leq s$, we place a marker at $y_{r}$ if exactly one of $y_{r-1} y_{r}$ and $y_{r} y_{r+1}$ is in $F_{A A^{\prime}}$. The total number of markers will be between $2(j-i)-1$ and $2(j-i)+2$, and so the total number of choices for the $j$ terms is at most $2\left(\begin{array}{c}\left(\begin{array}{c}s j-2 i+2\end{array}\right) \text {. Thus, the total number of choices for }\end{array}\right.$ 3) will also be at most $2\binom{s+3}{2 j-2 i+2}$. Also, we have at most $t k!2^{k}$ choices for step 4 ), where $k$ is the number of components in $F_{A A^{\prime}}$.

For step 2), if we impose the restriction that $i_{1}$ edges among the $j$ vertices correspond to the clauses with variables in different communities, then the number of ways to assign such variables is $\binom{i}{i_{1}}\left(\frac{2 n}{B}\right)^{j} B^{k}(B-1)^{i_{1}}$. As well, for step 5), if we impose the restrictions that, of the remaining $2 t-i$ clauses in $A$, there are exactly $t_{1}$ with variables in different communities, and that of the remaining $2 t-i$ clauses in $A^{\prime}$ there are exactly $t_{2}$ clauses with variables in different communities, then the number of ways to assign such variables is bounded above by $\binom{2 t-i}{t_{1}}\binom{2 t-i}{t_{2}}\left(\frac{2 n}{B}\right)^{2 s-2 j}(B-1)^{t_{1}+t_{2}}$.

First suppose that $1 \leq i \leq t-1$. Then none of the components of $F_{A A^{\prime}}$ may contain loops, so that $k=j-i$. Putting it all together, weighing all of the possible pairs of snakes $A$ and $A^{\prime}$, appropriately using (29), we obtain

$$
\begin{aligned}
& \sum_{\left|F_{A} \cap F_{A^{\prime}}\right|=i} E\left(X_{A} X_{A^{\prime}}\right)<\sum_{j \geq i+1} \frac{9}{2}\binom{s+3}{2 j-2 i+2}^{2} t \cdot(j-i)!(2 B)^{j-i} \\
& \cdot\left(\left(\frac{2 n}{B}\right)^{j} \sum_{i_{1}=0}^{i}\binom{i}{i_{1}}\left(\frac{B p m}{2 n^{2}}\right)^{i_{1}}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{i-i_{1}}\right) \\
& \cdot\left(\left(\frac{2 n}{B}\right)^{s-j} \sum_{t_{1}=0}^{2 t-i}\binom{2 t-i}{t_{1}}\left(\frac{B p m}{2 n^{2}}\right)^{t_{1}}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{2 t-i-t_{1}}\right)^{2} \\
&<\frac{9 B^{2} t(2 t+2)^{4}}{8 n^{2}}\left(\frac{m}{n}\right)^{4 t-i} \sum_{j \geq i+1}\left(\frac{B^{3}(2 t+2)^{4}}{n}\right)^{j-i}
\end{aligned}
$$

for sufficiently large $n$. Thus by (32) we have for sufficiently large n

$$
\begin{aligned}
\frac{\sum_{\left|F_{A} \cap F_{A^{\prime}}\right|=i} E\left(X_{A} X_{A^{\prime}}\right)}{E(X)^{2}} & <5 t(2 t+2)^{4}\left(\frac{n}{m}\right)^{i} \sum_{j \geq i+1}\left(\frac{B^{3}(2 t+2)^{4}}{n}\right)^{j-i} \\
& <\frac{2600 B^{3} t^{9}}{n}\left(\frac{n}{m}\right)^{i}
\end{aligned}
$$

Now suppose that $t \leq i \leq 2 t$. We have two possibilities for the components of $F_{A A^{\prime}}$. Either none of them contains loops or exactly one of them contains a loop and the number of loops in this
component is exactly 1 or 2 , where the possible loops are $y_{0}, y_{1}, \ldots, y_{t}$ and $y_{t}, y_{t+1} \ldots, y_{s+1}$. In either case we have $k \leq j-i+2$. Thus,

$$
\begin{aligned}
& \sum_{\left|F_{A} \cap F_{A^{\prime}}\right|=i} E\left(X_{A} X_{A^{\prime}}\right)<\sum_{j \geq i+1} \frac{9}{2}\binom{s+3}{2 j-2 i+2}^{2} t \cdot(j-i)!(2 B)^{j-i} B^{2} \\
& \cdot\left(\left(\frac{2 n}{B}\right)^{j} \sum_{i_{1}=0}^{i}\binom{i}{i_{1}}\left(\frac{B p m}{2 n^{2}}\right)^{i_{1}}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{i-i_{1}}\right) \\
& \cdot\left(\left(\frac{2 n}{B}\right)^{s-j} \sum_{t_{1}=0}^{2 t-i}\binom{2 t-i}{t_{1}}\left(\frac{B p m}{2 n^{2}}\right)^{t_{1}}\left(\frac{B(1-p) m}{2 n^{2}}\right)^{2 t-i-t_{1}}\right)^{2} \\
&<\frac{9 B^{6} t(2 t+2)^{4}}{2 n^{2}}\left(\frac{m}{n}\right)^{4 t-i} \sum_{j \geq i+1}\left(\frac{B^{3}(2 t+2)^{4}}{n}\right)^{j-i}
\end{aligned}
$$

By (32), for sufficiently large $n$

$$
\begin{aligned}
\frac{\sum_{\left|F_{A} \cap F_{A^{\prime}}\right|=i} E\left(X_{A} X_{A^{\prime}}\right)}{E(X)^{2}} & <20 B^{4} t(2 t+2)^{4}\left(\frac{n}{m}\right)^{i} \sum_{j \geq i+1}\left(\frac{B^{3}(2 t+2)^{4}}{n}\right)^{j-i} \\
& <\frac{10400 B^{7} t^{9}}{n}\left(\frac{n}{m}\right)^{i}
\end{aligned}
$$

Thus

$$
\sum_{i=1}^{2 t} \frac{\sum_{\left|F_{A} \cap F_{A^{\prime}}\right|=i} E\left(X_{A} X_{A^{\prime}}\right)}{E(X)^{2}}<\sum_{i=1}^{2 t} \frac{10400 B^{7} t^{9}}{n}\left(\frac{n}{m}\right)^{i}=o(1)
$$

from which we can deduce (30).

## 5 Empirical results

To test the question posed after Theorem 3.8, we have conducted the following experiment. We have taken $n=10^{6}$, and $m=n+c \cdot n^{2 / 3}$, with $c=-1,0,1,2$. (This non-symmetric range was due to preliminary simulations, that showed that the interesting window is actually centered somewhat above $n$. For each such $m$, we generated $10^{5}$ random instances from $F\left(n, m, 2, P_{(1,1)}\right)$ and $F\left(n, m, 1, P_{(2)}\right)$ (which is just the random model), tested each instance using the SAT solver SAT4J, described in [8], and calculated the percentage of satisfiable instances in each group. To complete the picture, we did the same for the model $F\left(n, m, 2, P_{(2)}\right)$.

The results are presented in Table 3. The first two models show remarkably similar results. Unsurprisingly, the third model leads to lower satisfiability probabilities.

## 6 Conclusions

We have dealt with the satisfiability threshold of a particular model of SAT. This model highlights one of the features in which so-called community-structured SAT instances differ from classical

| $\boldsymbol{F}$ | $\boldsymbol{m}$ | $0.99 \cdot 10^{6}$ | $10^{6}$ | $1.01 \cdot 10^{6}$ |
| :--- | :---: | :---: | :---: | :---: |
| $F\left(n, m, 2, P_{(1,1)}\right)$ | 0.980 | 0.909 | 0.641 | 0.201 |
| $F\left(n, m, 1, P_{(2)}\right)$ | 0.980 | 0.908 | 0.644 | 0.203 |
| $F\left(n, m, 2, P_{(2)}\right)$ | 0.946 | 0.827 | 0.521 | 0.142 |

Table 3: Percentage of satisfiable instances (out of $10^{5}$ instances) for $n=10^{6}$.

SAT instances. Namely, the set of variables decompose into several disjoint subsets-communities. The significance of these communities stems from the fact that clauses tend to contain variables from the same community. We have shown, roughly speaking, that the satisfiability threshold of such instances tends to be lower than for regular instances. Moreover, if the communities are very small, the threshold may even vanish.

The paper leaves a lot to study for industrial SAT instances. To begin with, there are other features considered in the literature as being characteristic of industrial instances. For example, in the scale-free structure, the variables are selected by some heavy-tailed distribution. Moreover, even regarding the issue of communities, there is more to be done. We assumed here that all communities are of the same size. Obviously, there is no justification for this assumption beyond the fact that it simplifies significantly the analysis of the model. What can be said about the threshold if there are both small and large communities? Even prior to that, what would be reasonable to assume regarding the probability of a variable to be selected from each of the communities?

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## A Proof of Theorem 3.5

In the proof of Theorem [3.5 we shall use the following lemma, which is analogous to Lemma 3.4,
Lemma A.1. Consider the balls and bins problem with $B$ bins and $M(B)$ balls, and also with $B$ bins and $M^{\prime}(B)$ balls, where $M^{\prime}(B)=\omega(M(B))$. If the maximum load for $M(B)$ balls is at least $s \geq 1$ with probability bounded away from 0 , then the maximum load for $M^{\prime}(B)$ is at least $s$ w.h.p.

Proof: Assume we part the balls into $b(B)=M^{\prime}(B) / M(B)$ disjoint batches of $M(B)$ balls each. Suppose we toss the balls in each batch into the bins separately, and check the maximum load for each batch. Let $L_{i}$ be the maximum load for batch $i, 1 \leq i \leq b(B)$. According to our assumption, there exists an $\varepsilon>0$ such that

$$
P\left(L_{i} \geq s\right) \geq \varepsilon, \quad i=1,2, \ldots, b(B)
$$

Let $L$ be the maximum load in the case we place all the $M^{\prime}(B)$ balls into the $B$ bins. The events $\left\{L_{i} \geq s\right\}, 1 \leq i \leq b(B)$, are independent, and we clearly have

$$
\{L<s\} \subseteq \bigcap_{i=1}^{b(B)}\left\{L_{i}<s\right\}
$$

Hence:

$$
P(L \geq s)=1-P(L<s) \geq 1-(1-\varepsilon)^{b(B)} \underset{B \rightarrow \infty}{ } 1
$$

The next proof will make use of the notion of negative association of random variables [15]: Denote $[k]=\{1, \ldots, k\}$ for $k>0$. Random variables $X_{1}, \ldots, X_{k}$ are negatively associated if for every two index sets $I, J \subseteq[k]$, with $I \cap J=\emptyset$,

$$
E\left(f_{1}\left(X_{i} ; i \in I\right) f_{2}\left(X_{j} ; j \in J\right)\right) \leq E\left(f_{1}\left(X_{i} ; j \in I\right)\right) E\left(f_{2}\left(X_{j} ; j \in J\right)\right)
$$

for every two functions $f_{1}: \mathbf{R}^{|I|} \rightarrow \mathbf{R}$ and $f_{2}: \mathbf{R}^{|J|} \rightarrow \mathbf{R}$, which are both non-decreasing or both non-increasing.

In the proof of Theorem [3.5, we will make use of the following result, concerning the balls and bins problem. Let $Y_{i}$ denote the number of balls placed in the $i$-th bin, $1 \leq i \leq B$. Let $g_{i}$ : $\mathbf{R} \rightarrow \mathbf{R}$ be non-decreasing functions, $1 \leq i \leq B$. By [15, Lemma 2], the variables $Y_{1}, \ldots, Y_{B}$ are negatively associated, and in particular the $g_{i}\left(Y_{i}\right)$-s are negatively correlated.

Proof of Theorem 3.5: Let $Y_{1}, \ldots, Y_{B}$ be as above. We clearly have

$$
Y_{i} \sim B(M(B), 1 / B), \quad 1 \leq i \leq B
$$

Define the variables

$$
S_{i}=\left\{\begin{array}{ll}
1, & Y_{i} \geq s, \\
0, & \text { otherwise },
\end{array} \quad 1 \leq i \leq B\right.
$$

The $S_{i}$-s are $\operatorname{Ber}(p)$-distributed, where $p=P\left(Y_{1} \geq s\right)$. Let $S=\sum_{i=1}^{B} S_{i}$.
(a) We use the first moment method. Obviously:

$$
P(S>0)=P(S \geq 1) \leq E(S)=B p
$$

Let us index the balls from 1 to $M(B)$, and let $M_{j}=1$ if the $j$-th ball entered the first bin and $M_{j}=0$ otherwise, $1 \leq j \leq M(B)$. Thus, the $M_{j}$-s are $\operatorname{Ber}(1 / B)$-distributed. Let $\mathcal{J}=\binom{[M(B)]}{s}$ denote the set of subsets of size $s$ of $[M(B)]$. By the union bound and symmetry:

$$
\begin{aligned}
p=P\left(Y_{1} \geq s\right) & =P\left(\bigcup_{J \in \mathcal{J}} \bigcap_{j \in J}\left\{M_{j}=1\right\}\right) \\
& \leq\binom{ M(B)}{s} \cdot P\left(M_{1}=\ldots=M_{s}=1\right)=\binom{M(B)}{s}\left(\frac{1}{B}\right)^{s} .
\end{aligned}
$$

Since $M(B)=o\left(B^{1-1 / s}\right)$,

$$
P(S>0) \leq B \cdot\binom{M(B)}{s}\left(\frac{1}{B}\right)^{s} \leq B \cdot \frac{M(B)^{s}}{B^{s}}=\left(\frac{M(B)}{B^{1-1 / s}}\right)^{s} \xrightarrow[B \rightarrow \infty]{ } 0
$$

Thus, w.h.p. the maximum load does not exceed $s-1$.
(b) We employ the second moment method. First, if $M(B) \geq B s$, then there must be at least one bin with at least $s$ balls in it. Thus we may assume that $\frac{M(B)}{B}<s$. We have

$$
\begin{aligned}
E(S) & =B \cdot P\left(Y_{1} \geq s\right)=B \cdot \sum_{j=s}^{M(B)}\binom{M(B)}{j}\left(\frac{1}{B}\right)^{j}\left(1-\frac{1}{B}\right)^{M(B)-j} \\
& \geq B \cdot\binom{M(B)}{s}\left(\frac{1}{B}\right)^{s}\left(1-\frac{1}{B}\right)^{M(B)-s}
\end{aligned}
$$

For sufficiently large $B$ we have $M(B) \geq 2 s$, and therefore

$$
\begin{aligned}
E(S) & \geq B \cdot \frac{(M(B) /(2 B))^{s}}{s!} \cdot\left(\left(1-\frac{1}{B}\right)^{B}\right)^{(M(B)-s) / B} \\
& \geq B \cdot \frac{(M(B) /(2 B))^{s}}{s!} \cdot e^{-2 M(B) / B}
\end{aligned}
$$

Thus we have

$$
\begin{equation*}
s!E(S) \geq B \cdot e^{-2 M(B) / B} \cdot\left(\frac{M(B)}{2 B}\right)^{s} \geq e^{-2 s}\left(\frac{M(B)}{2 B^{1-1 / s}}\right)^{s} \tag{33}
\end{equation*}
$$

By [15], the variables $Y_{1}, \ldots, Y_{B}$ are negatively associated. Since each $S_{i}$ is a non-decreasing function of $Y_{i}$, this yields $\operatorname{Cov}\left(S_{i}, S_{j}\right) \leq 0$ for $i \neq j$. Hence:

$$
\begin{aligned}
V(S) & =\sum_{i=1}^{B} V\left(S_{i}\right)+2 \sum_{1 \leq i<j \leq B} \operatorname{Cov}\left(S_{i}, S_{j}\right) \\
& \leq B \cdot V\left(S_{1}\right)=B \cdot p(1-p)<B \cdot p=E(S)
\end{aligned}
$$

As $S \geq 0$, the Paley-Zygmund inequality [36] yields

$$
\begin{equation*}
P(S>0) \geq \frac{E^{2}(S)}{E\left(S^{2}\right)}=\frac{E^{2}(S)}{V(S)+E^{2}(S)}>\frac{E^{2}(S)}{E(S)+E^{2}(S)}=\frac{E(S)}{1+E(S)} \tag{34}
\end{equation*}
$$

By (33), we have $E(S) \xrightarrow[B \rightarrow \infty]{ } \infty$. Also, by (33) and (34), we have

$$
P(S>0)>\frac{E(S)}{1+E(S)}
$$

and so $P(S>0) \underset{B \rightarrow \infty}{\longrightarrow}$. Thus, w.h.p. the maximum load is at least $s$.
(c) The first statement follows from parts (a) and (b), applied with $s+1$ and $s-1$, respectively, instead of $s$. For the convergence part, suppose (1) holds. Observe that there are $B^{M}$ possible ways to distribute the $M$ balls into the $B$ bins. Obviously, $X_{B}=\sum_{i=1}^{B} \mathbb{1}_{y_{i}=s}$. Let $1 \leq t \leq B$. We will prove that

$$
\begin{equation*}
\lim _{B \rightarrow \infty} E\binom{X}{t}=\frac{C^{s t}}{(s!)^{t} t!} \tag{35}
\end{equation*}
$$

Specify $t$ bins, say $i_{1}, i_{2}, \ldots, i_{t}$ out of the $B$ bins. The number of balls in bins $i_{1}, i_{2}, \ldots, i_{t}$, and all of the other bins combined forms a multinomial distribution. It follows that

$$
E\binom{X_{B}}{t}=\frac{\binom{B}{t}\binom{M}{s}\binom{M-s}{s}\binom{M-2 s}{s} \cdots\binom{M-(t-1) s}{s}(B-t)^{M-t s}}{B^{M}} .
$$

As $B \rightarrow \infty$, we thus have

$$
\begin{aligned}
E\binom{X_{B}}{t} & =(1+o(1)) \frac{M^{s t}(B-t)^{M-t s}}{t!s!^{t} B^{M-t}} \\
& =(1+o(1)) \frac{C^{s t} B^{s t-t}(B-t)^{M-t s}}{t!s!^{t} B^{M-t}} \\
& =(1+o(1)) \frac{C^{s t} B^{s t}(B-t)^{M-t s}}{t!s!^{t} B^{M}} \\
& =(1+o(1)) \frac{C^{s t}(1-t / B)^{-t s}(1-t / B)^{M}}{t!(s!)^{t}}
\end{aligned}
$$

From (1), we have

$$
\lim _{B \rightarrow \infty} \frac{M}{B}=0
$$

and so (35) holds. The desired result follows from Brun's sieve, which is stated in Theorem 2.1 of [3].


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