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A Model of the Geodynamo

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Summary

The problem of the generation of the geomagnetic field by convection in the Earth's core is considered. The complete hydromagnetic problem including the Lorentz force is solved in the case of a cylindrical configuration. Because of the dominant effect of rotation this configuration incorporates the essential features of the dynamo process in the Earth's core. The results show general agreement with the known properties of geomagnetism and the Earth's core. It is found that the toroidal field in the core is of the same order of magnitude as the poloidal field. This result is consistent with the basic assumption of theory that the Lorentz force is small compared to the Coriolis force.

1. Introduction

In the past decades it has become generally accepted that the geomagnetic field is generated by motions within the liquid outer part of the Earth's core. The mechanism by which motions in an electrically conducting fluid generate magnetic fields is described by the dynamo equation

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\,.\,\nabla\right)\mathbf{B}-\lambda\nabla^2\mathbf{B}=\mathbf{B}\,.\,\nabla\mathbf{v}, \tag{1.1}$$

where $\mathbf{v}$ is the velocity vector and λ is the magnetic diffusivity.

Although the first mathematically rigorous solutions of this equation for a growing magnetic field $\mathbf{B}$ were obtained only relatively recently (Backus 1958; Herzenberg 1958) a large number of solutions has become known since then. For details on the kinematic dynamo problem we refer to review articles by Roberts (1971) and Gubbins (1974). A more difficult problem is posed by the combination of (1.1) with the equations of motion in which the Lorentz forces are taken into account. This problem is called the non-linear or hydrodynamic dynamo problem. The action of the Lorentz forces is essential for the determination of the amplitude of the magnetic field which cannot be derived from the linear homogeneous equation (1.1). In the context of this paper the solutions for the non-linear dynamo problem by Busse (1973) and Soward (1974) are of particular interest.

While the basic features of the dynamo problem have been well understood, the problem of the geodynamo has remained largely unsolved, mainly because the origin and form of the motions in the Earth's core are not well known. The two most likely candidates for the energy source of the geodynamo are the precession of the Earth and convection in the Earth's core caused either by heat or non-thermal sources (Bullard 1949; Malkus 1963, 1968). Recently the subject of motions in the Earth's core has become more complicated by the hypothesis of Higgins & Kennedy (1971) that the outer core is stably stratified. This hypothesis has stimulated a number of alternative suggestions for the energy source, none of which appears to be convincing (Busse 1975). In their second paper Kennedy & Higgins (1973) allow, however, for a region of about 800 km thickness outside the inner core in which unstable stratification could occur. While this proposal would favour convection motions which are known to occur predominantly in the region close to the inner core (Busse 1970) the possibility of precession-induced motions cannot be disregarded, in view of the uncertainties entering the Higgins–Kennedy hypothesis. Although we use convection in our model of the geodynamo in this paper, this does not imply that a precession-driven geodynamo would not be possible. On the contrary, the similarity of the motions in both cases may eventually allow for the extension of the present analysis to the case of precession.

The model described in this paper is based on the assumption that the Lorentz force does not upset the basic geostropic balance which would hold in the case with vanishing magnetic field. Although it is usually assumed that the Lorentz force equals the Coriolis force in the Earth's core, we do not regard such a balance as likely (Busse 1973). It appears that only that part of the Coriolis force which cannot be balanced by the pressure gradient may be available for a balance with the Lorentz force. A large magnitude of the latter force is also unlikely because of the large amount of energy dissipation associated with a toroidal field of a few hundred Gauss in the Earth's core. The rather stringent energy budget for the magnetic field has been emphasized by Malkus (1963, 1973). The results of our model tend to support our expectation that the toroidal field is of the same order of magnitude as the poloidal field.

In the mathematical treatment of the dynamo problem we shall not attempt to accurately represent the geometrical configuration of the Earth's core. In order to estimate the typical features of the dynamo process most clearly we shall use a cylindrical annulus, which allows an analytical treatment of the problem in the limit of a small gap approximation. Because of the dominant effect of rotation the geometrical change does not affect the qualitative nature of the problem.

The mathematical analysis of the model is described in Sections 3, 4 and 6, after the basic equations have been introduced in Section 2. A heuristic description of the dynamo process is given in Section 5. In Section 7 the model is extended to include a parameter range appropriate to the Earth's core. A discussion of the convection-driven geodynamo follows in Section 8.

2. Mathematical formulation

We consider a homogeneous fluid in a system rotating with constant angular velocity Ω about a direction fixed in space and described by the unit vector $\mathbf{k}$. We assume that the basic equations of motion and the heat equation permit a static solution in which the temperature increases in the direction of gravity. The state of convection will be described as a perturbation of the basic static state. The equations of motion for the velocity vector $\mathbf{v}$ and the heat equation for the deviation θ of the temperature field from the static solution T are given by

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}+2\Omega\mathbf{k}\times\mathbf{v} = -\nabla\pi-\beta\mathbf{g}\theta+\nu\nabla^2\mathbf{v}+\frac{1}{\mu\rho_0}(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{2.1a}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2.1b}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\theta = -\mathbf{v}\cdot\nabla T+\kappa\nabla^2\theta, \tag{2.1c}$$

where $\mathbf{g}$ is the vector of gravity, ν is the kinematic viscosity, and κ is the thermal diffusivity. We have used the Boussinesq approximation in which an incompressible fluid is assumed and the dependence of density on temperature

$$\rho = \rho_0[1-\beta(T-T_0+\theta)]$$

is taken into account in the gravity term only. All other material properties are assumed to be constant. π is the deviation of the pressure from the static pressure divided by the density. In systems such as the Earth's core compressibility is important. The Boussinesq approximation, however, provides a good approximation if ∇T is interpreted to be the excess of the temperature gradient over the adiabatic temperature gradient.

Equations (2.1) together with the dynamo equation (1.1) and appropriate boundary conditions form the basis for a complete description of the problem of generation of magnetic fields by convection. The non-linear terms in the equations prohibit an analytical solution of the problem even when a geometrically simple system is assumed. We shall solve the problem for this reason in four steps, with the assumption that the amplitude of convection as well as the amplitude of the magnetic field are sufficiently small that the non-linear terms can be regarded as perturbations.

In the first step the hydrodynamic problem will be solved for vanishing $\mathbf{B}$. Using the solution for the velocity field $\mathbf{v}$ the linear dynamo equation (1.1) will be solved in the second step. Among all possible solutions of this equation, the non-decaying solution for $\mathbf{B}$ corresponding to the lowest amplitude of convection will be assumed as physically realized. The respective Lorentz force causes a modification of the order $|\mathbf{B}|^2$ of the velocity field according to the equation of motion (2.1a). After the modified velocity field has been calculated in the third step, it is used in the fourth step in the dynamo equation in order to determine the equilibrium amplitude of the magnetic field. Actually, we shall find the last step is not needed, since the main effect of the Lorentz force is to reduce the amplitude of the velocity field. Thus, the non-linear aspect of the dynamo problem which will be considered in Section 6 is surprisingly simple.

3. The hydrodynamic problem

Stationary motions in a rapidly rotating system are governed by the Taylor–Proudman Theorem which states that these motions are independent of the co-ordinate in the direction of the axis of rotation in the limit of vanishing frictional forces. (For more details on this and other properties of rotating fluids mentioned in the following, we refer to Greenspan's 1968 book.) The importance of friction is measured by the Ekman number, which is defined by

$$E = \frac{\nu}{\Omega L^2}$$

where L is a typical length scale of the system in the direction of the axis of rotation. In general the boundary conditions do not permit the velocity field to be entirely independent of the direction of the axis of rotation, which we shall identify with the direction of the z-co-ordinate. Because of their z-dependence, motions will become slightly time-dependent and the scales of the velocity field perpendicular to the axis of rotation may become so small that frictional effects cannot be neglected even in the limit of vanishing Ekman number. Still, the columnar nature of motions remains a characteristic feature in rapidly rotating systems.

A consequence of this property is the fact that the problem of convection between two concentric spherical boundaries is essentially the same as in a cylindrical annulus with spherical top and bottom boundaries and an axis in the z-direction. The

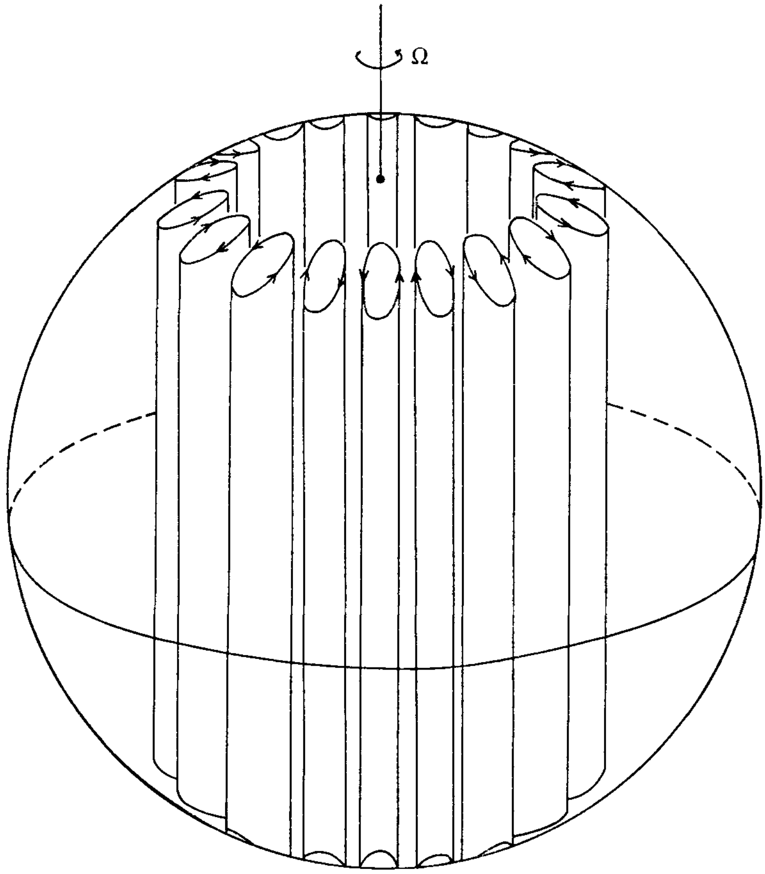

FIG. 1. Qualitative sketch of onset of convection in a fluid sphere according to the linear analysis in Paper I.

dynamical similarity between spherical and cylindrical geometry is emphasized by the property that the motions are predominantly directed perpendicular to the axis of rotation. Accordingly the convection motions are little affected by the z-components of gravity and the static temperature gradient. This permits us to neglect these components in our mathematical analysis and to assume an annulus configuration. For a detailed treatment of the problem of convection in rotating systems we refer to an earlier paper (Busse 1970), which we shall refer to as Paper I.

The fact that the component of gravity in the direction perpendicular to the axis of rotation provides the buoyancy force for convection in the Earth's core suggests the use of the centrifugal force in a laboratory simulation experiment. Since this force has the opposite direction, the temperature gradient must also be reversed. The laboratory observations, which were obtained in collaboration with C. R. Carrigan and which will be described in more detail elsewhere, show the onset of convection as predicted by the theory in Paper I (Fig. 1). For larger temperature gradients the region of convection columns expands as shown in Fig. 2 until nearly the entire volume is filled outside the cylindrical surface touching the inner sphere at the equator. The spatial inhomogeneity of convection in a spherical annulus is inconvenient for the mathematical analysis. For this reason we shall use a homogeneous inclination

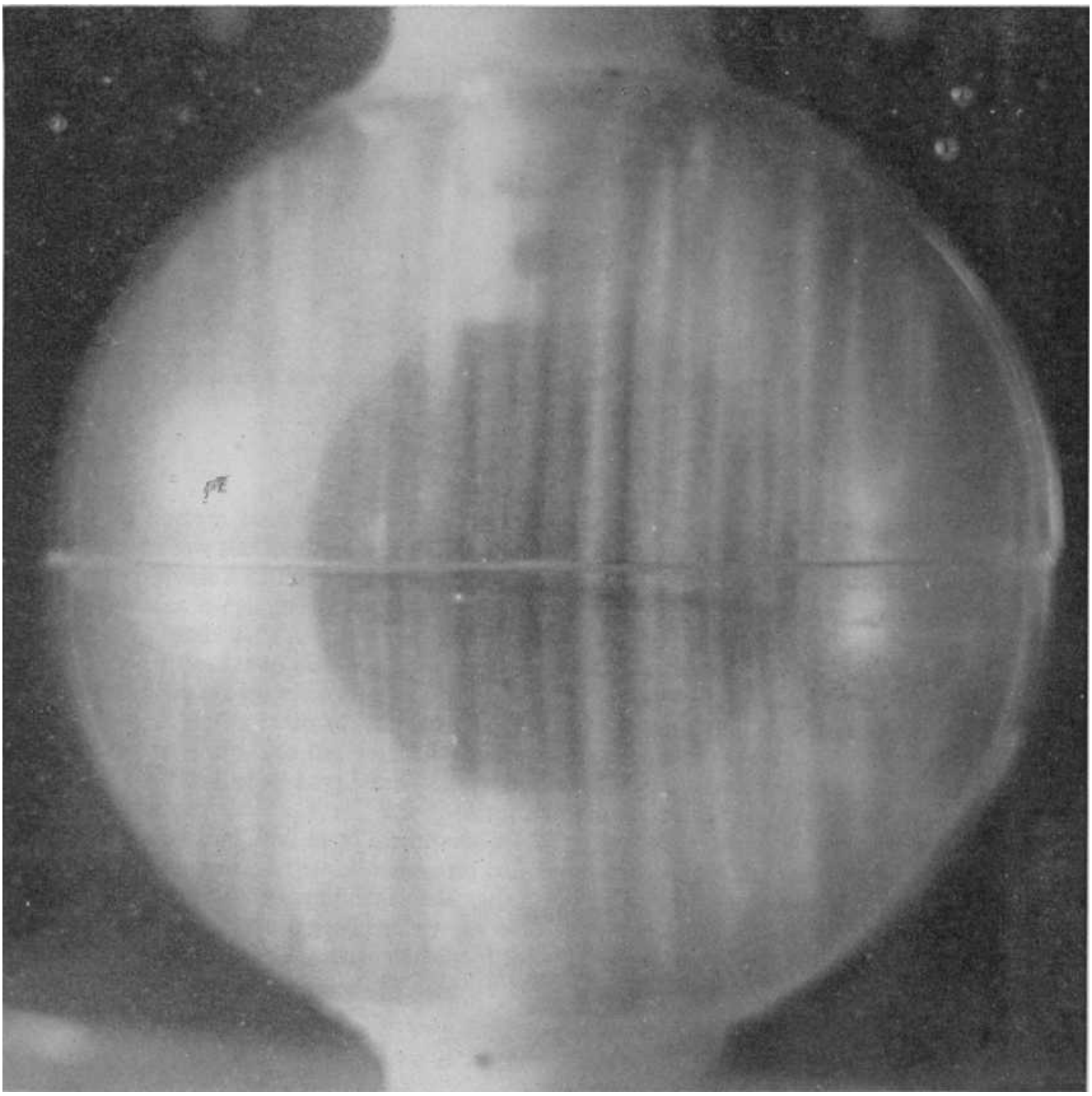

FIG. 2. Convection columns in a rotating spherical fluid shell heated from the outside and cooled from within. The flow is visualized by small floating particles which become oriented with the shear of the convection columns. Inner radius of the shell is $r_i = 2 \cdot 54$ cm, the outer radius is $r_0 = 4 \cdot 765$ cm.

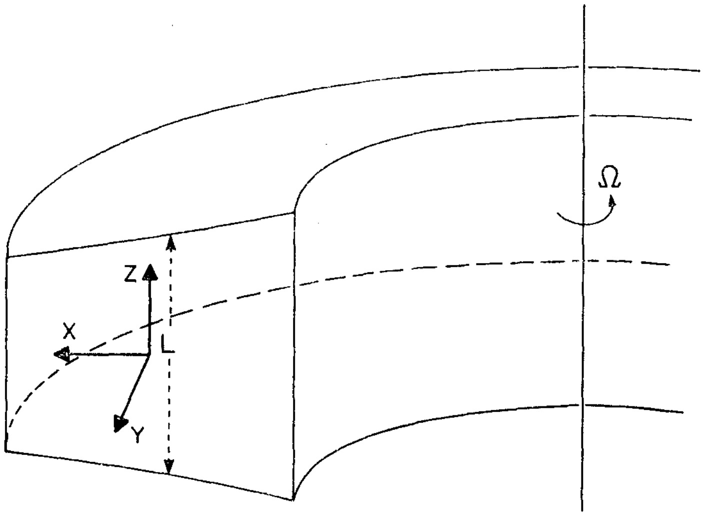

Fig. 3. Qualitative sketch of the geometric configuration used in the theoretical analysis.

of top and bottom boundaries in our annulus model. The mathematical analysis will closely follow the treatment in Paper I. However, some extensions will be made in order to provide a suitable basis for the dynamo model.

We consider a cylindrical annulus as shown in Fig. 3 and assume that the temperatures T_1 and $T_2 (T_2 > T_1)$ are prescribed on the outer and inner cylinders, respectively. In order to non-dimensionalize the equations we use the average height L of the annulus as length scale, Ω^{-1} as time scale, and $(T_2 - T_2)/D$ as temperature scale where D is the ratio of width to height of the annulus. For mathematical convenience we introduce the small gap approximation in which the width of the annulus is assumed small compared with its radius. This allows us to use a Cartesian system of co-ordinates (x, y, z). The corresponding unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are pointed in the radial, azimuthal, and axial directions, respectively. Since only non-dimensional variables will be used in the following, no confusion will arise when we use the same symbols for them as for the dimensional variables. Assuming in addition that gravity and the static temperature gradient are constant vectors directed in the inward radial direction we obtain as the non-dimensional form of equations (2.1) without Lorentz force,

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\mathbf{v} + 2\mathbf{k} \times \mathbf{v} = -\nabla\pi + B\theta\mathbf{i} + E\nabla^2\mathbf{v}, \tag{3.1a}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{3.1b}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\theta = \mathbf{i} \cdot \mathbf{v} + P^{-1} E\nabla^2\theta. \tag{3.1c}$$

In addition to the Ekman number we have introduced the Prandtl number P and the buoyancy number B.

$$P \equiv \frac{\nu}{\kappa}, \quad B \equiv \frac{\beta(T_2 - T_1)g}{\Omega^2 DL}.$$

The boundary condition for the normal component of the velocity at the inner and the outer cylindrical surface of the annulus is given by

$$\mathbf{i} \cdot \mathbf{v} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}D. \tag{3.2}$$

The boundary condition at the top and bottom surface is given by

$$\pm\mathbf{k} \cdot \mathbf{v} + \eta e^{-2\eta x}\, \mathbf{i} \cdot \mathbf{v} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}e^{-2\eta x}. \tag{3.3}$$

We shall regard η as a perturbation parameter which satisfies the conditon $\eta \ll \min(1, 1/D)$. For the purpose of this section we could replace the factor $\exp\{-2\eta x\}$ by unity as we have done in Paper I and as we shall do in the following sections. It is of interest, however, for the later discussion of higher order effects to see that fewer approximations are needed in the derivation of equation (3.9) when this factor is taken into account.

In the following we shall restrict our attention first to the linear part of equations (3.1) by assuming that the amplitude of convection is sufficiently small that the non-linear terms can be neglected. We shall also regard the buoyancy force as small and look for nearly stationary solutions. Since the dominant Coriolis force can be balanced only by the pressure in this case, the velocity field must satisfy the geostrophic balance and obey the Taylor–Proudman Theorem in zeroth order

$$2\mathbf{k} \times \mathbf{v}_0 = -\nabla\pi.$$

The solution of this equation satisfying $\nabla \cdot \mathbf{v}_0 = 0$ is

$$\mathbf{v}_0 = -\tfrac{1}{2}\nabla \times \mathbf{k}\pi_0 \tag{3.4}$$

where π_0 is an as yet undetermined function of x and y independent of z. Only the boundary condition

$$\pi_0 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}D$$

restricts the dependence on x. Since we are investigating a linear problem which is homogeneous in the y-direction we can assume without losing generality

$$\pi_0 = -2\psi_0(x) \exp\{i\alpha y\}.$$

Solution (3.4) nearly satisfies condition (3.3). We anticipate that the modification of solution (3.4) owing to a finite value of η will generate a time-dependence which can be assumed in the form $\exp\{i\omega t\}$ where ω is small compared to unity. In order to determine the solution for small values of η we shall introduce an expansion in powers of η starting with solution (3.4). Before we do so, eliminate the need for equation (3.1b) by using the general representation

$$\mathbf{v} = \nabla \times \mathbf{k}\psi + \nabla \times (\nabla \times \mathbf{k}\phi) \tag{3.5}$$

for the solenoidal vector field $\mathbf{v}$. The expansion for ψ, ϕ, and θ can be written in the form

$$\left.\begin{aligned} \psi &= \{\psi_0 + \eta\psi_1 + \eta^2\psi_2 + \ldots\}\exp\{i\omega t + i\alpha y\}, \\ \phi &= \{\eta\phi_1 + \eta^2\phi_2 + \ldots\}\exp\{i\omega t + i\alpha y\}, \\ \theta &= \{\theta_0 + \eta\theta_1 + \eta^2\theta_2 + \ldots\}\exp\{i\omega t + i\alpha y\}. \end{aligned}\right\} \tag{3.6}$$

By taking the z-components of the curl and of the curl curl of equation (3.1a) we obtain as equations for ϕ and ψ

$$(E\nabla^2 - i\omega)\,\Delta_2\psi + Bi\alpha\theta + 2\mathbf{k} \cdot \nabla\Delta_2\phi = 0, \tag{3.7a}$$

$$(E\nabla^2 - i\omega)\,\nabla^2\Delta_2\phi + B\frac{\partial^2}{\partial x\partial z}\theta - 2\mathbf{k} \cdot \nabla\Delta_2\psi = 0, \tag{3.7b}$$

where Δ_2 is defined by

$$\Delta_2 \equiv \frac{\partial^2}{\partial x^2} - \alpha^2.$$

In zeroth order we recover from (3.7b) the z-independence of ψ_0,

$$\mathbf{k} \,.\, \nabla \Delta_2 \psi_0 = 0.$$

From (3.1c) it follows that θ_0 is z-independent as well

$$(EP^{-1}\,\Delta_2 - i\omega)\,\theta_0 = -i\alpha\psi, \tag{3.8}$$

since we anticipate that the typical length scales of the solution in the x, y-directions are small compared to unity. The boundary conditions for θ_0 at $z = \pm\frac{1}{2}\exp\{-2\eta x\}$ will require in general thin boundary layers which have negligible effect, and will be disregarded for this reason. By integration of (3.7a) over the interval $|z| \leqslant \frac{1}{2}\exp\{-2\eta x\}$ and using condition (3.3) we find

$$\left[E\left(\frac{\partial^2}{\partial x^2} - \alpha^2\right) - i\omega\right]\left(\frac{\partial^2}{\partial x^2} - \alpha^2\right)\psi_0 + Bi\alpha\theta_0 + 4\eta i\alpha\psi_0 = 0. \tag{3.9}$$

Equations (3.8), (3.9) are solved by

$$\psi_0 = A\sin\gamma(x+\tfrac{1}{2}), \quad \theta_0 = \frac{i\alpha}{EP^{-1}\,a^2 + i\omega}\,\psi_0 \tag{3.10}$$

where a^2 is defined by

$$a^2 \equiv \gamma^2 + \alpha^2.$$

Condition (3.2) and the boundary condition $\theta_0 = 0$ at $x = \pm\frac{1}{2}D$ are fulfilled for

$$\gamma = \frac{n\pi}{D}, \quad n = 1, 2, \ldots. \tag{3.11}$$

The values of the parameters ω and B for which the solution (3.10) can be obtained are given by

$$\omega_0 = \frac{-4\eta\alpha}{a^2\,(1+P)}, \quad B_0 = \left(E^2\frac{a^6}{\alpha^2} + \left(\frac{4\eta P}{1+P}\right)^2 \Big/ \alpha^2\right) P^{-1}. \tag{3.12}$$

The minimum of B_0 as a function of α and γ,

$$B_c = 12\left(\frac{P\eta}{1+P}\right)^{\frac{4}{3}} E^{\frac{2}{3}}/P \tag{3.13}$$

is reached for

$$\alpha = \alpha_c \equiv \sqrt{2}\left(\frac{P}{1+P}\right)^{\frac{1}{3}}\left(\frac{\eta}{E}\right)^{\frac{1}{3}}, \quad \gamma \ll \alpha_c. \tag{3.14}$$

Since we have disregarded the effects of dissipation in the Ekman layers at the top and bottom boundaries we must require

$$E^{\frac{1}{2}} \ll \eta \tag{3.15}$$

as has been shown in Paper I.

As long as the aspect ratio D of the annulus is very large compared to α_c^{-1} and the condition

$$\eta \ll D\alpha_c/\pi \tag{3.16}$$

is satisfied, the critical value of B will not vary for modes with different values (3.11) of the radial wavenumber γ. We shall assume in the following that the values of n satisfying condition (3.16) are sufficiently large so that γ can be regarded as a continuous parameter and that the averaging procedure to be introduced in Section 4 is permissable. This assumption is justified by the fact that in a large gap annulus the radial dependence is described by Bessel functions or similar functions exhibiting a radial scale tending to zero as the azimuthal wavelength tends to zero. Since the exact value of γ is not important in the following, we shall not pursue the discussion further at this point.

From equation (3.7a) it is obvious that ϕ_1 depends linearly on z. Using the boundary condition (3.3) we obtain

$$\phi_1 = -2i\alpha z\phi_0/a^2. \tag{3.17}$$

We shall delay the calculation of higher order terms to Section 7 and discuss instead the influence of the boundary condition on the tangential component of the velocity field, which has been neglected so far. In a rotating system this boundary condition is of lesser importance than in a non-rotating system. At boundaries parallel to the z-axis Stewartson layers are formed with a thickness of the order $E^{\frac{1}{4}}$ which do not influence the solution derived above. At other boundaries Ekman layers with a thickness of the order $E^{\frac{1}{2}}$ accomplish the transition from the interior solution to the vanishing velocity of the boundary. The Ekman layer causes a slight modification of the boundary condition for the normal component of the interior velocity field,

$$\mathbf{n}\,.\,\mathbf{v} = -\tfrac{1}{2}E^{\frac{1}{2}}\mathbf{n}\,.\,\nabla\times\left\{\left(\mathbf{n}\times\mathbf{v}+\frac{\mathbf{n}\,.\,\mathbf{k}}{|\mathbf{n}\,.\,\mathbf{k}|}\mathbf{v}\right)(\mathbf{n}\,.\,\mathbf{k})^{-\frac{1}{2}}\right\}. \tag{3.18}$$

For the derivation of this expression we refer to Greenspan's (1968) book. In order to take the Ekman suction effect into account a double power series in powers of $E^{\frac{1}{2}}$ as well as of η will have to be assumed in place of (3.6) in principle. However, since we shall be interested only in the effect of lowest order we simply add inside the wavy bracket of expression (3.6) the term

$$\hat{\phi}_0 = -E^{\frac{1}{2}}z\psi_0, \tag{3.19}$$

which is generated by the modification (3.18) of the boundary condition (3.3). It is obvious that $\hat{\phi}_0$ is small compared to $\eta\phi_1$ because of condition (3.15) even if it is taken into account that α is of the order $(\eta/E)^{\frac{1}{4}}$. However, in contrast to expression (3.17), $\hat{\phi}_0$ is in phase with ψ_0, with the consequence that it makes a non-vanishing contribution to the helicity

$$W(z) \equiv \tfrac{1}{4}\overline{(\Delta_2\psi^+\,\Delta_2\phi+\text{c.c.})} \tag{3.20}$$

of the velocity field. In expression (3.20) ψ^+ denotes the complex conjugate (c.c.) of ψ and the bar indicates the average over the x, y co-ordinates.

So far we have restricted our attention to the onset of convection columns with infinitesimal amplitudes. As the buoyancy parameter B exceeds its critical value B_c the amplitude A of the convective motions is expected to increase. Since the form of the convection is identical in zeroth order with the form of convection rolls in a plane layer except for the time dependence, a perturbation analysis can be carried out analogous to that given by Malkus & Veronis (1958) or Schlüter, Lortz & Busse

(1965). As in the case of rolls with stress-free boundary condition the term $\mathbf{v} \cdot \nabla\mathbf{v}$ vanishes in lowest order and we obtain

$$B = B_0\left(1+\tfrac{1}{8}\frac{\alpha^2 a^2 A^2}{(a^2 E/P)^2+\omega_0{}^2}+\dots\right). \tag{3.21}$$

This result reduces to the corresponding result for rolls in a plane layer for $\omega_0 = 0$.

4. The kinematic dynamo problem

In this section we shall solve equation (1.1) for the velocity field $\mathbf{v}$ derived in the preceding section. In order to non-dimensionalize the problem we introduce

$$\mathbf{H} = \mathbf{B}/\sqrt{(\rho_0\,\mu)}L\Omega$$

as the non-dimensional magnetic field. After defining

$$\tau \equiv \lambda/\Omega L^2$$

we obtain as the non-dimensional form of equation (1.1)

$$\left(\frac{\partial}{\partial t}-\tau\nabla^2\right)\mathbf{H} = \nabla\times(\mathbf{v}\times\mathbf{H}). \tag{4.1}$$

We anticipate that two different scales characterize the solution for the magnetic field, as is the case in many solutions of the dynamo problem. The interaction of the velocity field with a large-scale magnetic field produces a fluctuating magnetic field with the same scale as the velocity field. The interaction of the fluctuating component of the magnetic field with the velocity field gives rise to a large-scale generation term which may balance the Ohmic dissipation of the large-scale field.

The large-scale magnetic field reflects in general the geometrical configuration of the system. In the present case of an annular geometry we expect an axisymmetric large-scale field consisting of two independent components; an azimuthal component which we shall call the toroidal field, and a meridional field. We shall assume that the electrical conductivity vanishes outside the annular region. This property requires that the toroidal field and the curl of the meridional field vanish outside the annulus. Even in the axisymmetric case the description of the large-scale magnetic field is cumbersome in general because of the dependence on two co-ordinates. For this reason we shall introduce the limiting case of a ‘ flat ’ annulus, for which we require

$$D \gg 1 \tag{4.2}$$

in addition to the small gap approximation introduced earlier. The boundary conditions for the large-scale magnetic field at the side walls become unimportant in this case and the spatial dependence is reduced to the z-dependence, at least inside the annulus and in its immediate neighbourhood.

We begin the mathematical analysis of equation (4.1) by introducing the following general representation of the solenoidal vector field $\mathbf{H}$,

$$\mathbf{H} = H_A\{-\mathbf{j}G(z,t)+\mathbf{i}F(z,t)+\nabla\times(\nabla\times\mathbf{k}h)+\nabla\times\mathbf{k}g\} \tag{4.3}$$

where h and g are functions of space and time describing the fluctuating component of the magnetic field, i.e.

$$\bar{h} = \bar{g} = 0.$$

As before, the bar indicates the average over the x, y cross-section of the annulus.

The representation (4.3) does not include a z-component of the mean field, since the condition

$$\frac{\partial}{\partial z}\bar{H}_z = 0$$

arising from $\nabla \cdot \bar{\mathbf{H}} = 0$ requires $\bar{H}_z = 0$ in the annulus unless sources at infinity provide for a homogeneous field in the z-direction. The assumption $\bar{H}_z = 0$ is also consistent with the dynamo equation (4.1). Since the azimuthal field $G(z, t)$ must vanish outside the annulus we obtain as boundary condition

$$G(z, t) = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{4.4}$$

The meridional field $F(z, t)$ does not have to vanish at $z = \pm\frac{1}{2}$. To ensure, however that the meridional field is generated within the annulus

$$F(\tfrac{1}{2}) = -F(-\tfrac{1}{2}) \tag{4.5}$$

must be required. This condition allows for the continuation of the meridional field towards infinity in such a way that it will decay at least as fast as a dipolar field.

By taking the x, y-average of the $x-$ and y-components of (4.1) we obtain two equations for F and G,

$$\left(\frac{\partial}{\partial t} - \tau\frac{\partial^2}{\partial z^2}\right)F = -\frac{\partial}{\partial z}\left[\overline{-\Delta_2\phi\left(\frac{\partial^2}{\partial x\partial z}h + \frac{\partial}{\partial y}g\right)} + \overline{\left(\frac{\partial}{\partial y}\psi + \frac{\partial^2}{\partial x\partial z}\phi\right)\Delta_2 h}\right], \tag{4.6a}$$

$$\left(\frac{\partial}{\partial t} - \tau\frac{\partial^2}{\partial z^2}\right)G = -\frac{\partial}{\partial z}\left[\overline{\Delta_2\phi\left(\frac{\partial^2}{\partial y\partial z}h - \frac{\partial}{\partial x}g\right)} + \overline{\left(\frac{\partial}{\partial x}\psi - \frac{\partial^2}{\partial y\partial z}\phi\right)\Delta_2 h}\right]. \tag{4.6b}$$

The z-component of equation (4.1) and the z-component of the curl of equation (4.1) yield two equations for the unknown functions g and h.

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\Delta_2 g = -\Delta_2\left(G\frac{\partial}{\partial y}\psi - F\frac{\partial}{\partial x}\psi\right) - \Delta_2\left(\frac{\partial}{\partial x}\phi\frac{\partial}{\partial z}G + \frac{\partial}{\partial y}\phi\frac{\partial}{\partial z}F\right) \tag{4.7a}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\Delta_2 h = \left(F\frac{\partial}{\partial x} - G\frac{\partial}{\partial y}\right)\Delta_2\phi. \tag{4.7b}$$

In these equations we have neglected all terms with h and g on the right-hand side since we anticipate that the amplitude of the fluctuating magnetic field is small compared to the mean field. In the following we shall also neglect the second term on the right-hand side of (4.7a) since the velocity field associated with ϕ is of the order η smaller than the basic geostrophic velocity field described by ψ.

We shall solve equations (4.6) and (4.7) in the stationary case when no amplification or decay of the magnetic field takes place. Because of the wavelike time-dependence of ϕ, ψ an analogous time-dependence is introduced for g, h, while the mean field $\mathbf{i}F - \mathbf{j}G$ is time-independent. Since the z-dependence of g, h is small compared to the x-, y-dependence, equations (4.7) can be easily integrated

$$g = \frac{1}{i\omega_0 + \tau a^2}\left(F\frac{\partial}{\partial x}\psi - G\frac{\partial}{\partial y}\psi\right), \tag{4.8a}$$

$$h = \frac{1}{i\omega_0 + \tau a^2}\left(F\frac{\partial}{\partial x}\phi - G\frac{\partial}{\partial y}\phi\right). \tag{4.8b}$$

As in the case of the velocity field thin boundary layers are formed at $z = \pm\frac{1}{2}$ which will be omitted in our analysis since they are not important for the following considerations. In order to use these solutions in equations (4.6) for F and G it must be remembered that only the real part has physical meaning in our problem. Thus we obtain from (4.6)

$$\left.\begin{aligned} \tau\frac{\partial^2}{\partial z^2}F &= \frac{\partial}{\partial z}\left[\tfrac{1}{4}G\left(-\Delta_2\phi^+\overline{\frac{\alpha^2\psi}{i\omega_0+\tau a^2}}+\frac{\partial}{\partial y}\psi^+\overline{\frac{a^2}{i\omega_0+\tau a^2}\frac{\partial}{\partial y}\phi}+\text{c.c.}\right)\right] \\ \tau\frac{\partial^2}{\partial z^2}G &= \frac{\partial}{\partial z}\left[\tfrac{1}{4}F\left(\Delta_2\phi^+\overline{\frac{\gamma^2\psi}{i\omega_0+\tau a^2}}-\frac{\partial}{\partial x}\psi^+\overline{\frac{a^2}{i\omega_0+\tau a^2}\frac{\partial}{\partial x}\phi}+\text{c.c}\right)\right]. \end{aligned}\right\} \quad (4.9)$$

where ϕ^+ indicates the complex conjugate (c.c.) of ϕ. We have neglected all terms which yield a vanishing average because of an antisymmetric x-dependence. In addition we have neglected terms which are quadratic in ϕ, since they are of lesser order than those retained. By integrating equations (4.9) once with respect to z we obtain

$$\left.\begin{aligned} \frac{d}{dz}F &= -\frac{\alpha}{\gamma}f(z)G \\ \frac{d}{dz}G &= \frac{\gamma}{\alpha}f(z)F+c \end{aligned}\right\} \quad (4.10)$$

where $f(z)$ is defined by

$$f(z) = \alpha\gamma(A^2/4\tau)\{[E^{\frac{1}{2}}\alpha^2\tau a^2]z+\ldots\}[\omega_0^2+\tau^2a^4]^{-1}. \quad (4.11)$$

The constant c of integration represents the average electric field in the x-direction. No average electric field in the y-direction is permitted in the stationary case since the corresponding electrostatic potential cannot satisfy the periodic boundary condition in the azimuthal direction of the annulus.

By introducing a new complex variable

$$H \equiv G\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}+iF\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}}$$

problem (4.10) can be written in the form of a single differential equation

$$\frac{d}{dz}H = -if(z)H+\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}c$$

with the general solution

$$H = e^{-i\Phi(z)}\left\{b+\int_0^z e^{i\Phi(\zeta)}\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}c\,d\zeta\right\} \quad (4.12)$$

where $\Phi(z)$ is defined by

$$\Phi(z) \equiv \int_0^z f(\zeta)\,d\zeta.$$

In order to satisfy the boundary condition (4.5) for $F(z)$ the antisymmetric solution corresponding to $b = 0$ must be chosen. $\Phi(z)$ is a symmetric function since $f(z)$ is antisymmetric in z owing to the symmetry of the problem. In calculating expression (4.11) for $f(z)$ we have included only the lowest orders of ϕ, ψ explicitly. Higher-order contributions will be discussed in Section 7. In lowest order $f(z)$ is a linear function of z,

$$f(z) = \mu z$$

which allows for a simple evaluation of the boundary condition (4.4) for $G(z)$. Since c is a real number $G(z)$ can be written in the form,

$$G(z) = c\sqrt{\left(\frac{\pi}{\mu}\right)}\left\{\cos\left(\frac{\mu}{2}z^2\right)C\left(\sqrt{\left(\frac{\mu}{\pi}\right)}z\right)+\sin\left(\frac{\mu}{2}z^2\right)S\left(\sqrt{\left(\frac{\mu}{\pi}\right)}z\right)\right\} \tag{4.13}$$

where C, S denote the Fresnel integrals,

$$C(x) \equiv \int_0^x \cos\frac{\pi}{2}t^2\,dt, \quad S(x) \equiv \int_0^x \sin\frac{\pi}{2}t^2\,dt.$$

Using the tables by Abramowitz & Stegun (1965) the zeros of $G(\frac{1}{2})$ as a function of μ can be readily computed. From the physical point of view only the lowest value μ_1 of μ for which $G(\frac{1}{2})$ vanishes is of interest,

$$\mu_1 = 18.42.$$

Although we have not solved the full time-dependent problem we can apply the analogy to other dynamo problems (Busse 1973) and conclude that for

$$\mu \equiv \frac{\alpha\gamma A^2(E^{\frac{1}{2}}\alpha^2\tau a^2)}{2\tau(\omega_0{}^2+\tau^2 a^4)} \geqslant \mu_1 \tag{4.14}$$

generation of the magnetic field given by solution (4.12) occurs, while for values of μ less than μ_1 magnetic fields must decay within the first-order approximation for which the calculations have been carried out.

For the evaluation of F and G a representation of the form (4.13) is rather inconvenient. We shall use instead the series expansion for Fresnal integrals as given, for example, in Abramowitz & Stegun (1965). This leads to the following representation

$$\left.\begin{aligned} F(\zeta) &= \frac{\alpha}{\gamma}\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}\zeta^{4n+3}}{1\cdot3\cdot\ldots\cdot(4n+3)}, \\ G(\zeta) &= \sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n}\zeta^{4n+1}}{1\cdot3\cdot\ldots\cdot(4n+1)} \end{aligned}\right\} \tag{4.15}$$

where ζ is defined by

$$\zeta \equiv \sqrt{\left(\frac{\mu_1}{\pi}\right)}z = 2{\cdot}42z.$$

Since the amplitude of the magnetic field is not defined in the kinematic dynamo problem we have normalized the solutions by setting $c = (\mu/\pi)^{\frac{1}{2}}$. The series expansion (4.15) is rapidly converging for $|z| \leqslant \frac{1}{2}$. We find that for our purposes the first two terms give a reasonable approximation.

The amplitude $|H_A|$ of the magnetic field will be determined in Section 6, where

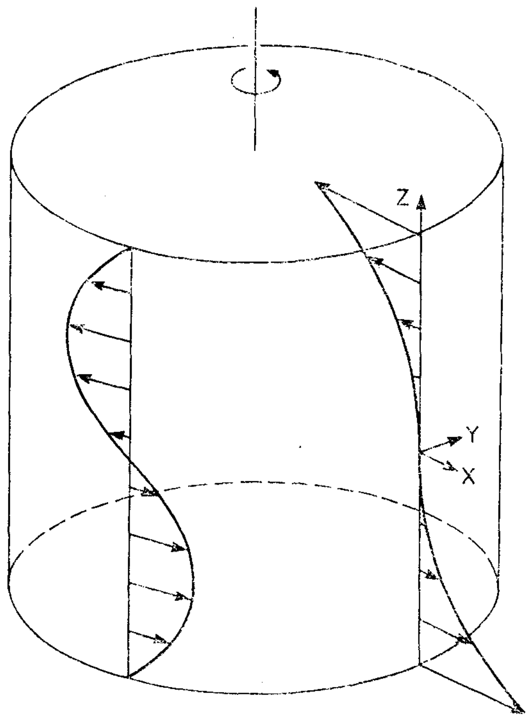

FIG. 4. Qualitative sketch of the axisymmetric part of the magnetic field generated by convection.

the action of the Lorentz force will be considered. The sign of H_A, however, remains undetermined. Both signs are equally possible in accordance with the corresponding invariance of the basic equations. In drawing a qualitative Fig. 4 we have assumed a negative sign, which corresponds to the presently realized polarity of the geomagnetic field.

5. Interpretation of the dynamo mechanism

In this section we shall attempt to interpret the mathematical theory derived in the preceding section. It is appropriate to start with the limitations of the mathematical analysis. The geometrical configuration of the problem and the hydrodynamical solution require

$$E^{\frac{1}{4}} \ll \eta \ll (1/D) \ll 1 \tag{5.1}$$

and

$$AP/E \ll 1. \tag{5.2}$$

Some of these relationships have been introduced for mathematical convenience, others are required by the particular dynamics of the system. Not all of the relationships (5.1), (5.2) must be satisfied in a strong sense, and we expect that the asymptotic theory still provides a fair approximation if the ' $\ll$ ' sign in (5.1), (5.2) is replaced by a ' $\lesssim$ ' sign. This expectation is based on experimental tests of the asymptotic theory (Busse & Carrigan 1974) as well as on numerical calculations. The latter have demonstrated, for instance, in a variety of convection problems (Fromm 1965), that the small amplitude approximation (3.21) for the function $B(A)$ holds within a few percent when AP/E is of the order unity. The assumption that the fluctuating magnetic field is small compared to the mean field leads to a condition similar to (5.2),

$$\frac{A}{4E/(1+P)+\tau} \leqslant 1 \tag{5.3}$$

where we expect again that the theory remains approximately correct, even if an equality sign is used in this condition.

The condition (4.14) for dynamo action does not contain a term proportional to η since the contribution of $\langle \phi_1{}^+ \psi_0 + \phi_1 \psi_0{}^+ \rangle$ to the helicity vanishes. For this reason it becomes necessary to evaluate contributions of higher order in η as will be done in Section 7. The helicity (3.20) is provided by the Ekman suction effect and is antisymmetric with respect to the equatorial plane $z = 0$. We note that the mean field exhibits the same symmetry. The fluctuating fields described by g and h have the opposite symmetry of the corresponding velocity fields described by ψ and ϕ. This symmetry is preserved when effects of higher order in η will be considered in Section 7. In contrast to the first order term in η they also contribute to the helicity.

Condition (5.2) is in accord with the dynamo condition (4.14) if

$$(\kappa/\lambda)\,\alpha\gamma E^{\frac{1}{4}} \geqslant 1,$$

which leaves some freedom for the choice of the parameters κ/λ and γ/α in the limit $E \to 0$. Among the various kinematic dynamo models described in the literature the first case of the dynamos derived by Roberts (1972) resembles the present case most

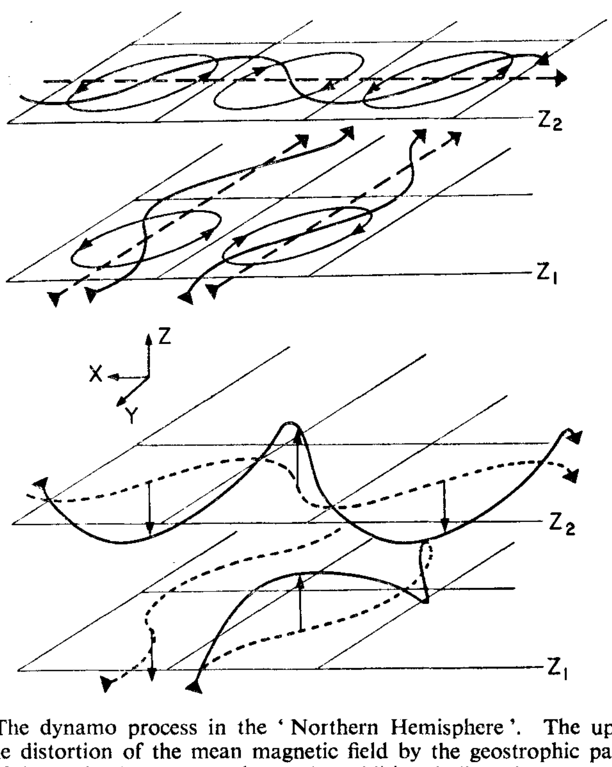

FIG. 5. The dynamo process in the 'Northern Hemisphere'. The upper part shows the distortion of the mean magnetic field by the geostrophic part of the velocity field. The lower part shows the additional distortion caused by the z-component of the velocity field. Thick lines show magnetic field, thinner lines indicate velocity field.

closely. Although the velocity field does not depend on the z-co-ordinate in Roberts' case and although only periodic boundary conditions are satisfied by the magnetic field, the basic mechanism appears to be similar. It is instructive to use the 'frozen field' approximation with $\lambda = 0$ in order to visualize the dynamo process. Even though magnetic diffusion is required for generation of magnetic flux, the geometric aspect of the dynamo process can be exhibited in the limit $\lambda = 0$. Fig. 5 shows a sketch of the dynamo process in the 'Northern hemisphere'. In the upper part of the figure the distortion of the mean field by the geostrophic flow is shown at two different planes $z = z_1, z_2$. At the low value z_1 the mean field is nearly azimuthal, while the meridional component dominates at $z = z_2$. The lower part of the figure shows the additional distortion of the field by the Ekman suction flow. It is seen that a mean azimuthal component is generated by the downward flow, while a mean meridional component is carried upwards from the plane of the original azimuthal field. Thus the two components reinforce each other in the dynamo process.

6. The action of the Lorentz force

In order to take into account the effect of a finite magnetic field in the equations of motion, the term

$$(\nabla \times \mathbf{H}) \times \mathbf{H} \tag{6.1}$$

must be added on the right-hand side of (3.1a). We consider the case of a small amplitude H_A of the magnetic field (4.3) and regard the Lorentz force (6.1) as a perturbation. Using $M = H_A{}^2$ as the perturbation parameter we solve the equations for the velocity field

$$\hat{\mathbf{v}} = \nabla \times \mathbf{k}\hat{\psi} + \nabla \times (\nabla \times \mathbf{k}\hat{\phi})$$

by expanding the variables in a series of powers of M,

$$\hat{\psi} = \psi + M\psi^{(1)} + \ldots$$

$$\hat{B} = B + MB^{(1)} + \ldots$$

$$\hat{\omega} = \omega + M\omega^{(1)} + \ldots$$

where the first term on the right-hand side refers to the solution for vanishing magnetic field obtained in Section 3. The expansions for $\hat{\phi}$, $\hat{\theta}$ are analogous to that for $\hat{\psi}$.

Since the fluctuating field described by h and g is small compared to the mean field we shall neglect all bilinear terms in h and g in the Lorentz force. First we consider the x, y-average of the Lorentz force and find that it is balanced entirely by the pressure perturbation.

$$\overline{(\nabla \times \mathbf{H}) \times \mathbf{H}} = \tfrac{1}{2}\mathbf{k}(d/dz)(G^2 + F^2) = \overline{\nabla \pi^{(1)}}.$$

We note in particular that no mean flow in the azimuthal direction is introduced by the Lorentz force. By taking the z-components of the curl and the curl curl of the equation of motion we obtain in analogy to (3.7)

$$(E\nabla^2 - i\omega_0)\,\Delta_2\psi^{(1)} + B_0\,(\partial/\partial y)\,\theta^{(1)} + 2\mathbf{k}\,.\,\nabla\Delta_2\phi^{(1)}$$
$$= B^{(1)}\,(\partial/\partial y)\theta + i\omega^{(1)}\,\Delta_2\psi + L_1, \tag{6.2a}$$

$$2\mathbf{k}\,.\,\nabla\Delta_2\psi^{(1)} = L_2, \tag{6.2b}$$

when L_1 and L_2 are given by

$$L_1 = \left(G\frac{\partial}{\partial y} - F\frac{\partial}{\partial x}\right)\Delta_2 g + \left(G'\frac{\partial}{\partial x} + F'\frac{\partial}{\partial y}\right)\Delta_2 h, \tag{6.3a}$$

$$L_2 = \left(F\frac{\partial}{\partial x} - G\frac{\partial}{\partial y}\right)\nabla^2 \Delta_2 h - \left(F''\frac{\partial}{\partial x} - G''\frac{\partial}{\partial y}\right)\Delta_2 h. \tag{6.3b}$$

In addition we need the equation for $\theta^{(1)}$,

$$(P^{-1} E\nabla^2 - i\omega_0)\theta^{(1)} + (\partial/\partial y)\psi^{(1)} = i\omega^{(1)}\theta. \tag{6.4}$$

Equations (6.2), (6.4) constitute a system of linear inhomogeneous equations for which a solution of the homogeneous system exists. A solution of the inhomogeneous system can be obtained in this case only if the inhomogeneity satisfies a solvability condition. We obtain the solvability condition by multiplying (6.2a), (6.2b), and (6.4) by ψ_0, $-\eta\phi_1$, and $B_0\theta_0$, respectively. After multiplying the equations with $\exp\{-\omega_0 t - i\alpha y\}$, adding them, and averaging them over the entire annulus, we find by performing some partial integrations that the left-hand side vanishes because of relations (3.8), (3.9), and (3.17). The right-hand side yields, if we neglect terms of higher order in η,

$$0 = i\alpha B^{(1)}\overline{\theta_0\psi_0} - i\omega^{(1)}(a^2\overline{\psi_0\psi_0} - B_0 PE^{-1}\overline{\theta_0\theta_0}) + \int_{-\frac{1}{2}}^{+\frac{1}{2}} \overline{L_1\psi_0}\,dz - \eta\int_{-\frac{1}{2}}^{+\frac{1}{2}} \overline{L_2\phi_1}\,dz. \tag{6.5}$$

This condition can be simplified by neglecting the term involving L_2 and the second term in L_1, since these terms are of higher order in η. Real and imaginary parts of relation (6.5) yield two linear inhomogeneous equations for the determination of $\omega^{(1)}$ and $B^{(1)}$. Of particular interest is the solution for $B^{(1)}$,

$$B^{(1)} = B_0 N \int_{-\frac{1}{2}}^{+\frac{1}{2}} (\gamma^2 F^2(z) + \alpha^2 G^2(z))\,dz \tag{6.6}$$

where N is defined by

$$N \equiv \frac{2\omega_0^2 + (\lambda/\kappa)\{(EP^{-1}a^2)^2 - \omega_0^2 + [(EP^{-1}a^2)^2 + \omega_0^2]^2/B_0\}}{[(EP^{-1}a^2)^2 + \omega_0^2 + B_0](\omega_0^2/a^2 + \tau^2 a^2)}.$$

The main conclusion to be drawn from the result (6.6) is that $B^{(1)}$ is always positive if the expression (3.12) for ω_0 is used. This has the important consequence that an equilibrium solution for the magnetic field exists which is stable with respect to perturbations of the amplitude H_A. This is evident from the following consideration. The dependence of the buoyancy parameter B on the convection amplitude A and on the magnetic field strength H_A is given by

$$B = B_0\left(1 + \tfrac{1}{8}\frac{\alpha^2 a^2 A^2}{(a^2 E/P)^2 + \omega_0^2}\right) + H_A{}^2 B^{(1)}. \tag{6.7}$$

Since B must be regarded as a given fixed parameter of the system the magnetic field can grow only by a corresponding decrease of the amplitude of convection. When the equality sign in relation (4.14) is reached the growth of the magnetic field is stopped and the equilibrium value given by (6.7) with

$$A^2 = \frac{\mu_1(\omega_0^2 + \tau^2 a^4)2\tau}{\alpha\gamma(E^{\frac{1}{2}} a^2 \tau\alpha^2)} \tag{6.8}$$

is attained. If $B^{(1)}$ would be negative an increase (decrease) of the equilibrium value H_A would lead to an increase (decrease) of A and therefore to further growth (decay) of H_A corresponding to an unstable equilibrium.

Because of the non-trivial result of the solvability condition(6.5) there is no need to calculate $\psi^{(1)}$ and $\phi^{(1)}$. Although the corresponding modifications of the velocity field are of the order M they will affect the equilibrium balance (6.7) only in the order M^2. Thus they are of minor importance in comparison to the change of the amplitude of the velocity field caused by the Lorentz force.

7. Higher-order effects

As we have mentioned before terms of higher order in η become important in the expression (4.11) for $f(z)$. For simplicity we assume that τa^2 is large compared to ω_0 which is the case in the Earth's core where the magnetic Prandtl number

$$P_m \equiv (\nu/\lambda) = E/\tau \tag{7.1}$$

is assumed to be small compared to unity. The first order of equation (3.7b) yields

$$2\mathbf{k} \cdot \nabla \Delta_2 \psi_1 = (Ea^2 + i\omega)\, a^2\, 2i\alpha\psi_0 z \tag{7.2}$$

with the solution

$$\psi_1 = -(i\alpha/2)(z^2 - \tfrac{1}{12})(Ea^2 + i\omega)\psi_0. \tag{7.3}$$

We have determined the constant of integration in the solution for ψ_1 by using the normalization condition

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{\psi_n \psi_0}\, dz = 0 \quad \text{for} \quad n \geqslant 1. \tag{7.4}$$

For the purpose of this section it is sufficient to assume top and bottom boundaries at $z \approx \pm\frac{1}{2}$. The factor $\exp\{-2\eta x\}$ was introduced in (3.3) only to avoid higher-order terms from this approximation. The next higher order of equations (3.7a) and (3.1c) yield equations for ϕ_2 and θ_1,

$$(Ea^2 + i\omega_0)\, a^2 \psi_1 + B_0\, i\alpha\theta_1 = -i\omega_1 a^2 \psi_0 + B_1\, i\alpha\theta_0 + \eta 2\mathbf{k} \cdot \nabla a^2 \phi_2, \tag{7.5a}$$

$$(EP^{-1}\, a^2 + i\omega_0)\,\theta_1 - i\alpha\psi_1 = -i\omega_1\,\theta_0. \tag{7.5b}$$

Here we have assumed an expansion for B and ω in powers of η in analogy to (3.6). B_1 and ω_1 are determined by the solvability condition for the linear inhomogeneous system of equation (7.5). This condition is obtained by multiplying (7.5a) and (7.5b) by ψ_0 and $-B_0\theta_0$, respectively. After the equations are added we find by using equation (3.9)

$$-4\eta i\alpha\psi_0\psi_1 = i\omega_1[B_0(\theta_0)^2 - a^2(\psi_0)^2] + B_1\, i\alpha\,\theta_0\,\psi_0 + \eta^2 a^2 \psi_0\,(\partial/\partial z)\,\phi_2. \tag{7.6}$$

The average of this equation over the interval $-\frac{1}{2} \leqslant z \leqslant \frac{1}{2}$ yields an equation of B_1 and ω_1 after the boundary conditions

$$a^2\phi_2 = \mp i\alpha\psi_1 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{7.7}$$

have been used. In another way of looking at equation (7.6) we may regard B_1 and ω_1 as constants to be adjusted in such a way that the solution ϕ_2 of (7.6) satisfies the boundary conditions (7.7). Thus we obtain

$$\phi_2 = -\frac{Ea^2 + i\omega_0}{3a^2}(z^2 + \tfrac{1}{4})z\psi_0\,\alpha^2. \tag{7.8}$$

We note that ψ_1, ϕ_2, B_1, and ω_1 are of the order η smaller than ψ_0, ϕ_1, B_0, and ω_0, respectively, if the critical value (3.14) for α is used. This suggests that a different expansion in η may be more appropriate, since the actual order of magnitude of the terms is not evident in (3.6). However, the particular choice of the series expansion is a matter of convenience as long as the basic assumption of the perturbation theory is fullfilled, that all terms neglected are of smaller order than those retained.

In general the term involving ψ_1, ϕ_2 yields an additional contribution in the expression (4.11) for $f(z)$. To treat the effect of this term separately we shall assume in the following

$$P_m = E/\tau \approx (\omega_0/\tau a^2) \ll 1, \tag{7.9}$$

as well as

$$E^{\frac{1}{2}} \ll E\alpha^2 \eta^2, \tag{7.10}$$

in which case the contribution from the term

$$\tfrac{1}{4}\eta^2(\overline{\phi_2{}^+ \psi_0} + \overline{\phi_1{}^+ \psi_1} + \text{c.c.})$$

exceeds the previously considered lower-order contribution on the right-hand side of (4.7). In place of $f(z)$ we obtain

$$\hat{f}(z) = \alpha\gamma\,(A^2 z/2\tau^2)\,[\tfrac{2}{3}(\tfrac{1}{4}-z^2)\,E\alpha^2\,\eta^2 + \ldots]. \tag{7.11}$$

An explicit solution of the form (4.13) in terms of known functions cannot be obtained in this case. For an order of magnitude estimate we approximate $\frac{1}{4}-z^2$ by its average value $\frac{1}{6}$ and obtain the result that generation of magnetic field occurs for

$$\frac{\alpha\gamma A^2\, P_m{}^2\, \alpha^2\, \eta^2}{18E} \gtrsim \mu_1 \tag{7.12}$$

8. Discussion

The goal of the dynamo model described in this paper was to incorporate all qualitative features of a convection driven geodynamo. Since the model is valid only for a limited range of parameters the question of whether the model resembles the geodynamo realized in the Earth's core remains open even if it can be assumed that convection is the driving mechanism. The difference in the geometrical configuration between the model and the Earth's core is probably of minor importance. We expect that qualitative aspects will not be affected by this difference and that even order of magnitude estimates for the geodynamo can be made on the basis of the model. The fact that the Lorentz force is regarded as perturbation is a more serious limitation. Unless the Lorentz force becomes comparable to the Coriolis force, however, the geostrophic balance and columnar nature of the convective motion will be preserved. On the other hand, quantitative aspects, as for instance the typical wavenumber of the motion are likely to be influenced by the magnetic field. Since the physical parameters of the core are not well known it does not seem worthwhile at this point to do complicated calculations for a quantitatively improved model. In the following we intend to demonstrate that even at the present stage the model is in reasonable agreement with the empirical knowledge of the geodynamo.

Although a comparison with observational evidence is not sufficient to answer the question of whether the model correctly represents the basic properties of the geodynamo, it provides at least a necessary condition. We find that the model exhibits the following major features of the geomagnetic field:

(i) The field outside the core is mainly dipolar.

(ii) The field is nearly aligned with the axis of rotation.

(iii) The mean field is stationary. Small-scale components are time-dependent.

The existence of reversals of the geomagnetic field does not contradict property (iii), since they occur on a much longer time scale than the dynamo time scale, which is identical with the free geomagnetic decay time of a few 10^4 years. The sign of H_A remains undetermined in all dynamo models because of the homogeneity of the problem. Hence it is conceivable that a drastic change in the velocity field of the core may give rise to the appearance of a magnetic field with opposite polarity. According to our model the scale of the fluctuating magnetic field is much too small to be observable at the Earth's surface. However statistical fluctuations of the amplitude of this field may well be related to the phenomenon of secular variation. It is noteworthy that the group velocity is westward according to the dispersion relation (3.12), and its order of magnitude could correspond well to the westward drift of the non-dipole part of the geomagnetic field. We intend to pursue this point in a later publication.

More stringent constraints on a convection-driven geodynamo arise from energetic considerations. Because of heat generation by radioactivity in the crust and mantle only a small fraction of the heat flow observed at the Earth's surface can originate in the core. The current estimates for a heat flux from the core fluctuate between 10^{12} W and 10^{13} W (Malkus 1973). The lower number represents the flux provided by the freezing of the inner core at a uniform rate (Kennedy & Higgins 1973) while the higher value is regarded as an upper bound, from considerations of crustal heat sources (Roy, Blackwell & Birch 1968).

For a quantitative discussion of the convection geodynamo we shall use the values of the physical properties of the core given in Table 1. For references on these values we refer to Gubbins (1974) except for the value of the thermal conductivity, which we take from Frazer (1973), since Gubbins' value appears to be too small in relation the electrical conductivity. We assume that convection columns occur typically at a distance of half the radius of the core from the axis of rotation. This yields a length scale of $L = 6{\cdot}10^8$ cm and the values

$$\eta = 1/\sqrt{3}, \quad E = 4{\cdot}10^{-16}, \quad B_0 = 1{\cdot}7 \,.\, 10^{-11} \tag{8.1}$$

for the basic parameter of our theory. In calculating B_0 we have assumed a value of $5{\cdot}10^2$ cm s^{-2} for the component of gravity perpendicular to the axis of rotation.

Since the magnetic Prandtl number $P_m = 6{\cdot}10^{-7}$ has a very small value, the dynamo action provided by the term (7.11) dominates. We interpret $\gamma\alpha A^2/4$ as the product of the rms values of the radial and the azimuthal velocity fields. The dynamo condition (7.12) yields

$$(\overline{\langle v_x^2\rangle \langle v_y^2\rangle})^{\frac{1}{2}} = \tfrac{1}{4}\gamma\alpha A^2 = 0{\cdot}16\,\mathrm{cm}^2\,\mathrm{s}^{-2} \tag{8.2}$$

which corresponds to a typical velocity of $0{\cdot}4$ cm s^{-1}. This velocity amplitude appears to be somewhat large if compared with other estimates of velocities in the core. It must be remembered, however, that those estimates refer in general to large-scale motions such as differential rotation corresponding to the westward drift of non-dipole part of the geometric field. Whether fluctuating velocities as high as (8.2) are physically realistic can be best tested by calculating the corresponding heat transport.

Table 1

Physical properties of the Earth's core

Kinematic viscosity	$\nu = 0{\cdot}01\,\mathrm{cm}^2\,\mathrm{s}^{-1}$
Thermal conductivity	$k = 0{\cdot}6\,\mathrm{W\,K}^{-1}\,\mathrm{cm}^{-1}$
Thermal diffusivity	$\kappa = 0{\cdot}08\,\mathrm{cm}^2\,\mathrm{s}^{-1}$
Magnetic diffusivity	$\lambda = 1{\cdot}6\times10^4\,\mathrm{cm}^2\,\mathrm{s}^{-1}$
Coeff. of thermal expansion	$\beta = 4{\cdot}5\times10^{-6}\,\mathrm{K}^{-1}$

Since the theory of convection described in Section 3 is limited to small amplitudes, it is not directly applicable to the earth core. Expression (3.1) predicts a linear relation between $B-B_0$ and the convective heat transport or the kinetic energy of the convective motions. In the latter case expression (3.21) may actually provide a fair approximation even at very high values of A. Laboratory experiments (Malkus 1954a; Deardorff & Willis 1967) as well as theoretical considerations (Malkus 1954b) shows a linear relationship between Rayleigh number and kinetic energy in a plane convection layer. The proportionality constant is close to that given by the small amplitude expansion result. With this justification we use (3.21) to estimate the critical value B_g for the dynamo process according to (7.12),

$$\frac{B_g-B_0}{B_0} \approx 0{\cdot}75\ .\ 10^9. \tag{8.3}$$

In obtaining this estimate we have assumed the value 1/3 for the ratio γ/α. Although we expect that the actual value of this ratio is closer to unity in the core we have chosen the ratio 1/3 in order to satisfy approximately the theoretical assumption $\gamma^2 \ll \alpha^2$.

The convective heat transport increases more rapidly than the kinetic energy of convection, and expression (3.21) can provide at best a lower bound for the heat transport. The Nusselt number Nu which represents the ratio between the heat transport in the case of convection to the heat transport in the absence of convection would become independent of B asymptotically according to (3.21). This is unrealistic, and we must resort to laboratory experience, which provides an empirical law of the form (Globe & Dropkin 1959)

$$Nu \approx 0.07R^{\frac{1}{3}} \tag{8.4a}$$

if we neglect the Prandtl number dependence, which appears to be negligible in most cases. Recent experiments (Chu & Goldstein 1973; Fitzjarrald 1975) suggest a lower exponent of 0·3 for the Rayleigh number R. The 1/3-law offers the advantage, however, that it can be related to theoretical considerations (Busse 1967) which yield

$$Nu \approx (R/2R_c)^{\frac{1}{3}} \tag{8.4b}$$

where R_c refers to the critical value of the Rayleigh number. Relation (8.4b) has the advantage that it takes into account the effect of different boundary conditions. For rigid boundaries (8.4b) agrees with the laboratory value. Since Rayleigh number R and buoyancy parameter B differ only by a constant factor, we find by using (8.3) in (8.4b)

$$Nu = 7.10^2 \tag{8.5}$$

for convection in the core. The total heat flux through the cylindrical annulus is given by

$$h = -k(\overline{\partial T}/\partial x)\, Nu\, Lr_0\pi = 8{\cdot}10^{12}\ W \tag{8.6}$$

where $r_0/2 = 1{\cdot}74\ .\ 10^8$ cm is the radius of the convection annulus and where the relation

$$-\frac{\overline{\partial T}}{\partial x} = \frac{T_2-T_1}{DL} = B\frac{\Omega^2}{\beta g} = 3{\cdot}10^{-8}\ \mathrm{K\,cm^{-1}} \tag{8.7}$$

has been used. Even if we disregard all other uncertainties in our considerations the estimate (8.6) may well be too large by a factor 2 or 4 due to the high Nusselt number (8.5) alone. Since the convection has an oscillatory time-dependence it is likely to be less efficient in transporting heat than we have assumed. The secondary mode of convection which sets in and increases the heat transport in laboratory systems when Nu drops below the value (8.4) will be inhibited by the effect of rotation. Thus we conclude that the heat transport required by a convection-driven geodynamo is not in obvious disagreement with the observational evidence.

The value (8.7) represents the negative temperature gradient in excess of the adiabatic gradient because of the Boussinesq approximation in our theory. Since the adiabatic gradient is much larger than the value (8.7) the temperature distribution will not be changed significantly by convection. A serious objection to the convection model we have considered so far comes from the hypothesis by Higgins & Kennedy (1971) that the core is stably stratified because the melting temperature depends less on the pressure than the adiabat. This property was also suggested by Clark (1968). It is unlikely that a process for the generation of the geomagnetic field can be found in the case of a stratified outer core (Busse 1975). In their second paper Kennedy & Higgins (1973) allow, however, for a region of 800 km thickness adjacent to the inner core where the adiabatic gradient may be exceeded. Even though the radial dimension is significantly reduced in this case, a convection-driven dynamo may still be possible. Assuming for simplicity that the stratified region is replaced by solid material we find $\eta = 0{\cdot}76$ and $L = 3.2 \cdot 10^8$ cm. Although the Ekman number is increased, the increase in η has a compensatory effect and actual numbers are not much changed except for the heat transport h which is reduced by one half because of the smaller dimensions. On the other hand, the mathematical model becomes less applicable in this case and the calculated values are even less reliable than in the case without stratification.

From expression (6.4) one may expect that the term involving A and the magnetic term are roughly of the same order of magnitude. This expectation is fulfilled by the model if a typical value of 5 G is assumed for the strength of the meridional field in the core. A similar estimate may be used to establish that the Lorentz force is small compared to the Coriolis force, in agreement with the basic assumption of the model.

While our model of the geodynamo is internally consistent and does not show any obvious disagreement with present knowledge of the properties of the Earth's core, it does not exclude the possibilities of quite different geodynamos. The property that poloidal and toroidal fields in the core are of the same order of magnitude is probably the most distinguishing result of this model. Unfortunately there is no unambiguous method to infer the strength of the toroidal field in the core. Traditionally it has been assumed that a toroidal field of a few hundred Gauss exists in the core which is believed to be generated by a differential rotation acting on the poloidal field. The present model does not exhibit a differential rotation except possibly in higher orders in η which have not been investigated. It is likely that a small differential rotation is introduced by the inhomogeneous geometrical configuration of the actual Earth's core. We do not expect, however, that this differential rotation will change the order of magnitude of the toroidal field.

9. Concluding remarks

The model of the geodynamo considered in this paper is characterized by two length scales. A large length scale describes the dependence in the direction of the axis of rotation, while the length scale in the perpendicular directions is of the other $(E/\eta)^{\frac{1}{3}}$ smaller. For this reason the condition (4.14) for dynamo action cannot be interpreted in terms of a single magnetic Reynolds number. Rather, the parameter μ may be interpreted as the product of two magnetic Reynolds numbers corresponding to the different length scales.

A dynamo for which the smaller length scale is increased would require a lesser value of the velocity amplitude. The convective heat transport would increase in this case, however, as can be concluded from the fact that expression (8.6) increases with $E^{10/9}$. Hence we may conclude that a small length scale or, alternatively, a sufficiently small value of the viscosity is required by the upper bound on the heat transport from the Earth's core.

According to our model the convective heat transport should exhibit considerable deviation from spherical symmetry. A latitudinal dependence of the observed surface heat flux on a latitudinal dependence of the temperature at the core-mantle boundary inferred from seismic data may ultimately help to decide whether convection occurs in the core. At the present time the scatter of the data does not permit any definite statement on this problem. The fact that convective motions are less likely to occur inside the cylindrical surface bounding the inner core at the equator could also manifest itself in the form of the geomagnetic field and its secular variation. This possibility will be the subject of future extensions of the present work.

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