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A MODEL THEORY FOR Γ-CONTRACTIONS

J. AGLER and N.J. YOUNG

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ABSTRACT. A Γ-*contraction* is a pair of commuting operators on Hilbert space for which the symmetrised bidisc

$$\Gamma \stackrel{\text{def}}{=} \{(z_1 + z_2, z_1 z_2) : |z_1| \leqslant 1, |z_2| \leqslant 1\} \subset \mathbb{C}^2$$

is a spectral set. We develop a model theory for such pairs which parallels a part of the well-known Nagy-Foiaş model for contractions. In particular we show that any Γ-contraction is unitarily equivalent to the restriction to a joint invariant subspace of the orthogonal direct sum of a Γ-unitary and a "model Γ-contraction" of the form $(T_\psi, T_{\overline{z}})$ where $T_\psi, T_{\overline{z}}$ are suitable block-Toeplitz operators on a vectorial Hardy space, and Γ-unitaries are defined to be pairs of operators of the form $(U_1 + U_2, U_1 U_2)$ for some pair U_1, U_2 of commuting unitaries.

KEYWORDS: *Model operator, spectral set, symmetrised bidisc.*

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0. INTRODUCTION

In this paper we present some operator theory which is an offshoot of a problem originally posed by engineers. The function theory of the set

$$\Gamma \stackrel{\text{def}}{=} \{(z_1 + z_2, z_1 z_2) : |z_1| \leqslant 1, |z_2| \leqslant 1\} \subset \mathbb{C}^2$$

plays a part in some interpolation problems that arise in H^∞ control theory ([12], [16], [13]). One such is the *spectral Nevanlinna-Pick problem* ([10]); it is a hard variant of a classical problem, and leads (in a special case) to the problem of analytic interpolation from the unit disc to Γ ([5]). Given the effectiveness of Sarason's generalized interpolation technique ([17]) for some classical interpolation problems it is natural to look for an operator-theoretic approach to the function theory of Γ. A measure of success has come from the study of the family of commuting pairs of operators for which the symmetrised bidisc Γ is a spectral set. An understanding of this family has led to the solution of a special case of the spectral Nevanlinna-Pick problem ([5], [7]) and also to the discovery of some surprising facts about the complex geometry of Γ ([6]).

Any commuting pair of operators having Γ as a spectral set will be called a Γ-*contraction*. In this paper we concentrate on the operator theory of the family of Γ-contractions rather than function-theoretic or geometric aspects. Many of the fundamental results in the theory of contractions have close parallels for Γ-contractions. There are Γ-analogues of unitaries, isometries, the Wold decomposition and completely non-unitary contractions, and there is an analogue of at least a part of the Sz.-Nagy–Foiaş functional model ([19]). There have been numerous earlier developments of model theories for families of commuting tuples of operators associated with other sets in $\mathbb{C}^n$ ([2], [8], [9]); what is novel here, we believe, is that the set Γ is both non-convex and inhomogeneous, yet we are nevertheless able to obtain detailed results.

A Γ-contraction can be obtained by symmetrising any pair of commuting contractions, just as points of Γ are obtained by applying the "symmetrisation map"

$$\pi : \mathbb{C}^2 \to \mathbb{C}^2, \quad (z_1, z_2) \mapsto (z_1 + z_2, z_1 z_2)$$

to the bidisc. However, an important subtlety is that not all Γ-contractions are obtained in this way (see Examples 1.7 and 2.3). A related fact is that continuous functions into Γ do not all factor continuously through the bidisc, and indeed functions that do not so factor are of interest in the applications to interpolation. Here our main result (Theorem 3.2) provides a model for Γ-contractions. In brief, every Γ-contraction is the restriction to a common invariant subspace of a Γ-co-isometry, and every Γ-co-isometry is expressible as the orthogonal direct sum of a Γ-unitary and a pure Γ-co-isometry, which has a model on a vectorial Hardy space parametrised by operators of numerical radius less than or equal to one. We leave open, however, the problem of constructively describing the common invariant subspaces of Γ-co-isometries in terms of a characteristic operator function or its analogue.

We denote by $\mathbb{D}$ and $\overline{\mathbb{D}}$ the open and closed unit discs in the complex plane $\mathbb{C}$. Note that $\Gamma = \pi(\overline{\mathbb{D}}^2)$. We usually denote a typical point of Γ by (s, p), the variables chosen to suggest "sum" and "product". We shall also use the notation (S, P) for a pair of commuting operators associated in some way with Γ. In this paper an *operator* will always be a bounded linear operator on a Hilbert space. Consider a commuting pair (S, P) of operators. We shall say that Γ is a *spectral set* for (S, P), or that (S, P) is a Γ-*contraction*, if, for every polynomial f in two variables,

$$\|f(S, P)\| \leqslant \sup_{\Gamma} |f|. \tag{0.1}$$

Furthermore, Γ is said to be a *complete spectral set* for (S,P), or (S,P) to be a *complete* Γ*-contraction*, if, for every matricial polynomial f in two variables,

$$\|f(S,P)\| \leqslant \sup_{z\in\Gamma} \|f(z)\|.$$

Here, if S and P act on H and the matricial polynomial f is given by $f=[f_{ij}]$ of type $m \times n$, where each f_{ij} is a scalar polynomial, then $f(S,P)$ denotes the operator from H^n to H^m with block matrix $[f_{ij}(S,P)]$.

We denote the unit circle by $\mathbb{T}$. Note that the *distinguished boundary* of Γ, defined to be the Šilov boundary of the algebra of functions which are continuous on Γ and analytic on the interior of Γ, is $\pi(\mathbb{T}^2)$. We shall use some spaces of vector- and operator-valued functions. Let E be a separable Hilbert space. We denote by $\mathcal{L}(E)$ the space of operators on E, with the operator norm. $H^2(E)$ will be the usual Hardy space of analytic E-valued functions on $\mathbb{D}$ and $L^2(E)$ the Hilbert space of square-integrable E-valued functions on $\mathbb{T}$, with their natural inner products. $H^\infty\mathcal{L}(E)$ denotes the space of bounded analytic $\mathcal{L}(E)$-valued functions on $\mathbb{D}$, $L^\infty\mathcal{L}(E)$ the space of bounded measurable $\mathcal{L}(E)$-valued functions on $\mathbb{T}$, each with the appropriate version of the supremum norm. For $\varphi \in L^\infty\mathcal{L}(E)$ we denote by T_φ the Toeplitz operator with symbol φ, given by

$$T_\varphi f = P_+(\varphi f), \quad f \in H^2(E),$$

where $P_+ : L^2(E) \to H^2(E)$ is the orthogonal projector. In particular T_z is the unilateral shift operator on $H^2(E)$ (we denote the identity function on $\mathbb{T}$ by z) and $T_{\overline{z}}$ is the backward shift on $H^2(E)$.

We have defined Γ-contractions by the requirement that the inequality (0.1) hold for all *polynomial* functions f in two variables; it might be thought more natural to require (0.1) to hold for all functions f analytic in a neighbourhood of Γ. In fact this would give an equivalent condition, by virtue of the polynomial convexity of Γ ([3], Lemma 2.1). Suppose that (S,P) is a Γ-contraction on a Hilbert space H. It is elementary to show that, because of polynomial convexity, the polynomial joint spectrum $\sigma_{\rm pol}(S,P)$ is contained in Γ. Here $\sigma_{\rm pol}(S,P)$ is defined to be the joint spectrum of (S,P) relative to the algebra $\mathcal{A}$ ([15], 3.5.4), where $\mathcal{A}$ is the closed subalgebra of $\mathcal{L}(H)$ generated by S,P and the identity operator on H. Hence, if f is analytic on a neighbourhood of Γ then f is also analytic on a neighbourhood of $\sigma_{\rm pol}(S,P)$, and so $f(S,P)$ is defined by any version of the functional calculus for tuples of commuting operators, e.g. 3.5.9 in [15]. Moreover, it is easy to see that f can be approximated uniformly on a neighbourhood of Γ by polynomials (equivalently, any symmetric analytic function on a neighbourhood of the closed bidisc is approximable uniformly on a symmetric neighbourhood of the closed bidisc by symmetric polynomials, as follows easily from Cauchy's integral formula). It follows that inequality (0.1) holds for f. The slightly delicate issues surrounding the various notions of joint spectrum and functional calculus are not relevant to this paper, simply because of the polynomial convexity of Γ.

1. Γ AND Γ-CONTRACTIONS

We begin by recapitulating from earlier papers some facts about the set Γ and Γ-contractions. We shall need the operator-valued function ρ of commuting pairs of operators given by

$$\begin{aligned}\rho(S,P) &= 2(1-P^*P) - S + S^*P - S^* + P^*S \\ &= \frac{1}{2}\left\{(2-S)^*(2-S) - (2P-S)^*(2P-S)\right\}.\end{aligned}$$

Note that $(s,p) \in \Gamma$ if and only if the zeros of the polynomial $z^2 - sz + p$ both lie in $\overline{\mathbb{D}}$. We are thus in the territory of classical zero location therorems (e.g. [18]). In fact there are several dissimilar characterizations of Γ.

Theorem 1.1. *Let $(s,p) \in \mathbb{C}^2$. The following are equivalent:*
(i) *$(s,p) \in \Gamma$;*
(ii) *$|s - \bar{s}p| + |p|^2 \leqslant 1$ and $|s| \leqslant 2$;*
(iii) *$2|s - \bar{s}p| + |s^2 - 4p| + |s|^2 \leqslant 4$;*
(iv) *$\rho(\alpha s, \alpha^2 p) \geqslant 0$ for all $\alpha \in \mathbb{D}$;*
(v) *$|p| \leqslant 1$ and there exists $\beta \in \mathbb{C}$ such that $|\beta| \leqslant 1$ and $s = \beta p + \bar{\beta}$;*
(vi) *$|s| \leqslant 2$ and, for all $\alpha \in \mathbb{D}$,*

$$\left|\frac{2\alpha p - s}{2 - \alpha s}\right| \leqslant 1;$$

(vii) *for all $\alpha \in \mathbb{D}$, $1 - \bar{\alpha}s + \bar{\alpha}^2 p \neq 0$ and*

$$\left|\frac{p - \alpha s + \alpha^2}{1 - \bar{\alpha}s + \bar{\alpha}^2 p}\right| \leqslant 1.$$

Proof. (i) ⇔ (iv) is Theorem 2.2 in [3], (i) ⇔ (iii) is Theorem 1.6 in [6] and (i) ⇔ (vii) is contained in Theorem 1.5 of [5].

(i) ⇒ (ii) Let $(s,p) \in \Gamma$. Clearly $|s| \leqslant 2$. Let $0 < r < 1$; then $(rs, r^2p) \in \operatorname{int}\Gamma$, the interior of Γ, which is $\pi(\mathbb{D}^2)$. By Schur's theorem,

$$\begin{bmatrix} 1 - r^4|p|^2 & -r\bar{s} + r^2\bar{p}s \\ -r\bar{s} + r^2\bar{p}s & 1 - r^4|p|^2 \end{bmatrix} > 0$$

and hence

$$1 - r^4|p|^2 > |-r\bar{s} + r^2\bar{p}s|.$$

Let $r \to 1$ to deduce that (ii) holds.

(ii) ⇒ (i) Suppose (ii). There are two cases.

Case 1. $|s| < 2$. We have, for all $\omega \in \mathbb{T}$, $1 - |p|^2 - \operatorname{Re}\{\omega(s - \bar{s}p)\} \geqslant 0$, whence (see (ii))

$$\frac{1}{4}\left\{(2 - \overline{\omega s})(2 - \omega s) - (2\overline{p\omega} - \bar{s})(2p\omega - s)\right\} \geqslant 0.$$

Since $|s| < 2$, $(2 - \omega s)^{-1}$ exists for $\omega \in \mathbb{T}$ and so $\left|\frac{2p\omega - s}{2 - \omega s}\right| \leqslant 1$ for all $\omega \in \mathbb{T}$. It follows by the Maximum Modulus Theorem that

$$\left|\frac{2p\alpha - s}{2 - \alpha s}\right| \leqslant 1$$

for all $\alpha \in \mathbb{D}$. Hence $|2-\alpha s|^2 - |2p\alpha - s|^2 \geqslant 0$ for all $\alpha \in \mathbb{D}$, which is to say that (iv) holds. Hence $(s,p) \in \Gamma$.

Case 2. $|s| = 2$. Write $s = 2\omega$, $|\omega| = 1$, and $h = 1 - p\overline{\omega}^2$. Then we have $|2\omega - 2\overline{\omega}p| + |p\overline{\omega}^2|^2 \leqslant 1$, that is $2|h| + |1-h|^2 \leqslant 1$, which simplifies to $|h|^2 \leqslant 2(\operatorname{Re} h - |h|)$, and this clearly implies $|h| = 0$ since $\operatorname{Re} h - |h| \leqslant 0$. Thus $s = 2\omega$ and $p = \omega^2$, so $(s,p) = \pi(\omega,\omega) \in \Gamma$. We have shown that (ii) $\Leftrightarrow$ (i).

(ii) $\Leftrightarrow$ (v) Suppose (v). Clearly $|s| \leqslant 2$ and

$$s - \overline{s}p = \beta p + \overline{\beta} - \overline{\beta}|p|^2 - \beta p = \overline{\beta}(1-|p|^2),$$

whence $|s - \overline{s}p| \leqslant 1 - |p|^2$. Thus (v) $\Rightarrow$ (ii).

Conversely, suppose (ii). If $|p| < 1$ we may define

$$\beta = \frac{s - \overline{s}p}{1-|p|^2}.$$

Then $|\beta| \leqslant 1$ and $\beta p + \overline{\beta} = s$, so that (v) holds. On the other hand, if $|p| = 1$ we may put $p = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in \mathbb{R}$. Note that

$$|s - \overline{s}p| \leqslant 1 - |p|^2 = 0,$$

so that $s = \overline{s}p$ and hence $s\mathrm{e}^{-\mathrm{i}\theta/2}$ is real. Since $|s| \leqslant 2$ we may write $s\mathrm{e}^{-\mathrm{i}\theta/2} = 2\cos\gamma$ for some $\gamma \in \mathbb{R}$. Let $\beta = \mathrm{e}^{\mathrm{i}(\gamma - \theta/2)}$. Then $|\beta| = 1$ and $s = \beta p + \overline{\beta}$. Hence (ii) $\Rightarrow$ (v).

(vi) $\Rightarrow$ (iv) is immediate, and (ii) $\Rightarrow$ (vi) is essentially the same as the proof that (ii) $\Rightarrow$ (i) above. ∎

Note that in proving Case 1 above we established the following refinement of (i) $\Leftrightarrow$ (iv).

THEOREM 1.2. *Let $s,p \in \mathbb{C}$ and suppose $|s| < 2$. Then $(s,p) \in \Gamma$ if and only if, for all $\omega \in \mathbb{T}$, $\rho(\omega s, \omega^2 p) \geqslant 0$.*

We shall also need characterizations of the distinguished boundary of Γ.

THEOREM 1.3. *Let $s,p \in \mathbb{C}$. The following are equivalent:*
(i) *(s,p) is in the distinguished boundary of Γ;*
(ii) *$|p| = 1$ and $\overline{s} = \overline{p}s$ and $|s| \leqslant 2$;*
(iii) *$(s,p) = (2x\mathrm{e}^{\mathrm{i}\theta/2}, \mathrm{e}^{\mathrm{i}\theta})$ for some $\theta \in \mathbb{R}$ and some $x \in [-1,1]$.*

Proof. (i) $\Leftrightarrow$ (iii) Suppose $s = \lambda_1 + \lambda_2$, $p = \lambda_1\lambda_2$ where $|\lambda_1| = |\lambda_2| = 1$. Then $|p| = 1$ and so $p = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in \mathbb{R}$, and $\lambda_2 = p\overline{\lambda}_1 = \mathrm{e}^{\mathrm{i}\theta}\overline{\lambda}_1$. Hence

$$s = \lambda_1 + \lambda_2 = \lambda_1 + \mathrm{e}^{\mathrm{i}\theta}\overline{\lambda}_1 = \mathrm{e}^{\mathrm{i}\theta/2}2\operatorname{Re}\left\{\mathrm{e}^{-\mathrm{i}\theta/2}\lambda_1\right\} = 2x\mathrm{e}^{\mathrm{i}\theta/2}$$

for some $x \in [-1,1]$. Thus (i) $\Rightarrow$ (iii). (iii) $\Rightarrow$ (ii) is obvious. Suppose (ii) holds. If $s = 0$ then $(s,p) = (0,p)$ and (i) holds. Otherwise write $s = |s|\mathrm{e}^{\mathrm{i}\theta}$ and note that $p = s/\overline{s} = \mathrm{e}^{\mathrm{i}2\theta}$. The equations $\lambda_1 + \lambda_2 = s$, $\lambda_1\lambda_2 = p$ imply

$$(\lambda_1 - \lambda_2)^2 = s^2 - 4p = -\mathrm{e}^{\mathrm{i}2\theta}(4 - |s|^2),$$

and one may solve to obtain

$$\lambda_1, \lambda_2 = \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\left\{|s| \pm \mathrm{i}\sqrt{4-|s|^2}\right\},$$

and clearly $|\lambda_1| = |\lambda_2| = 1$. Thus (ii) $\Rightarrow$ (i). ∎

Corollary 1.4. *The distinguished boundary of* Γ *is homeomorphic to a Möbius band.*

Proof. The characterization (iii) in the theorem gives the representation

$$(2x\mathrm{e}^{\mathrm{i}\theta/2}, \mathrm{e}^{\mathrm{i}\theta}) \in \Gamma \leftrightarrow (x, \theta)$$

of the distinguished boundary of Γ, where $-1 \leqslant x \leqslant 1, 0 \leqslant \theta \leqslant 2\pi$ and the points $(x, 0)$ and $(-x, 2\pi)$ are identified. This correspondence clearly gives a continuous bijective mapping of the Möbius band (as a quotient space of a rectangle in $\mathbb{R}^2$) onto the distinguished boundary of Γ, with the topology induced by $\mathbb{C}^2$, and since the Möbius band is compact it follows that the correspondence is a homeomorphism. ∎

We remark that Γ is not convex. The points $(2,1) = \pi(1,1)$ and $(2\mathrm{i}, -1) = \pi(\mathrm{i}, \mathrm{i})$ both lie in Γ, but their mid-point $(1+\mathrm{i}, 0) = \pi(1+\mathrm{i}, 0)$ is not in Γ. It would be interesting to know whether $\operatorname{int}\Gamma$ is holomorphically equivalent to a convex set.

The next theorem summarises the main results on Γ-contractions established in [3] and [4].

Theorem 1.5. *Let* (S, P) *be a pair of commuting operators on a Hilbert space* H*. The following statements are equivalent:*

(i) (S, P) *is a* Γ*-contraction;*

(ii) (S, P) *is a complete* Γ*-contraction;*

(iii) $\rho(\alpha S, \alpha^2 P) \geqslant 0$ *for all* $\alpha \in \mathbb{D}$*;*

(iv) *there exist Hilbert spaces* H_-, H_+ *and a commuting pair of normal operators* $(\widetilde{S}, \widetilde{P})$ *on* $K \stackrel{\text{def}}{=} H_- \oplus H \oplus H_+$ *such that the algebraic joint spectrum* $\sigma(\widetilde{S}, \widetilde{P})$ *is contained in the distinguished boundary of* Γ *and* $\widetilde{S}, \widetilde{P}$ *are expressible by operator matrices of the form*

$$\widetilde{S} \sim \left[\begin{pmatrix} * & * & * \\ 0 & S & * \\ 0 & 0 & * \end{pmatrix}\right] \quad \text{and} \quad \widetilde{P} \sim \left[\begin{pmatrix} * & * & * \\ 0 & P & * \\ 0 & 0 & * \end{pmatrix}\right]$$

with respect to the orthogonal decomposition $K = H_- \oplus H \oplus H_+$*;*

(v) *for all* $\alpha \in \mathbb{D}$,

$$\|(2\alpha P - S)(2 - \alpha S)^{-1}\| \leqslant 1;$$

(vi) *for all* $\alpha \in \mathbb{D}$, $1 - \overline{\alpha} S + \overline{\alpha}^2 P$ *is invertible and*

$$\|(P - \alpha S + \alpha^2)(1 - \overline{\alpha} S + \overline{\alpha}^2 P)^{-1}\| \leqslant 1.$$

Moreover, if the spectral radius of S *is less than* 2 *then the following statement is also equivalent to* (i) –(vi):

$$\text{(iii}'\text{)} \qquad \rho(\omega S, \omega^2 P) \geqslant 0 \quad \text{for all } \omega \in \mathbb{T}.$$

Proof. The equivalence of (i) to (v) is contained in Theorem 1.5 of [4] while the equivalence of (i) and (vi) is given in Theorem 1.5 of [5]. The final statement is proved just as in Case 1 of (ii) ⇒ (i) in Theorem 1.1. Indeed, suppose S has spectral radius less than 2 and (iii′) holds. We have

$$(2 - \omega S)^*(2 - \omega S) - (2\omega^2 P - \omega S)^*(2\omega^2 P - \omega S) \geqslant 0.$$

Since $2-\omega S$ is invertible it follows that $\|(2\omega^2 P-\omega S)(2-\omega S)^{-1}\| \leqslant 1$ for all $\omega \in \mathbb{T}$. Again by the Maximum Modulus Principle, $\|(2\alpha^2 P-\alpha S)(2-\alpha S)^{-1}\| \leqslant 1$ for all $\alpha \in \mathbb{D}$, and this may be re-expanded to give $\rho(\alpha S, \alpha^2 P) \geqslant 0$ for all $\alpha \in \mathbb{D}$. Thus (iii′) ⇒ (iii).

Clearly (iii) ⇒ (iii′), and so the statements are equivalent.

Note that the equivalence of (iii) and (v) is immediate from the factorization (ii). ∎

Remark 1.6. (i) Statement (iv) in Theorem 1.5 is sometimes expressed: (S,P) has a normal dilation to the distinguished boundary of Γ.

(ii) Without the spectral radius assumption on S, (iii) and (iii′) would not be equivalent, even for scalar S and P. If $S = 2+1/2$, $P = 2\times 1/2$ then (iii) is false but (iii′) is true.

Example 1.7. (Symmetrisation of pairs of contractions) An easy way to construct a Γ-contraction is to take $S=A+B$, $P=AB$ where A, B are commuting contractions. One might wonder if all Γ-contractions arise in this way. In fact they do not. Such Γ-contractions have the property that S^2-4P has a square root which commutes with S and P (indeed, this characterizes them). If P is a contraction it follows from condition (iii′) that $(0,P)$ is a Γ-contraction, but if P has no square root then $(0,P)$ cannot be of the stated form.

A more interesting example of this phenomenon is given below in Example 2.3.

Recall that the *numerical radius* of an operator T on a Hilbert space H is defined to be

$$w(T) = \sup\{|\langle Tx, x\rangle| : \|x\|_H \leqslant 1\}.$$

Corollary 1.8. *Let S be an operator. $(S,0)$ is a Γ-contraction if and only if $w(S)\leqslant 1$.*

Proof. By (i) ⇔ (iii) of Theorem 1.5, $(S,0)$ is a Γ-contraction if and only if $2-2\mathrm{Re}\,(\alpha S)\geqslant 0$ for all $\alpha\in\mathbb{D}$, which is to say $\mathrm{Re}\,\langle \alpha Sx, x\rangle \leqslant 1$ for all $\alpha \in \mathbb{D}$ and unit vectors x, and this is equivalent to $w(S)\leqslant 1$. ∎

More generally, condition (iii′) can be expressed in terms of the numerical radius whenever $\|P\|<1$, for then we may conjugate $\rho(\omega S, \omega^2 P)$ by $(1-P^*P)^{-1/2}$ to get the equivalent condition

$$2-2\mathrm{Re}\left\{\omega(1-P^*P)^{-1/2}(S-S^*P)(1-P^*P)^{-1/2}\right\}\geqslant 0$$

for all $\omega\in\mathbb{T}$. We obtain the following:

Corollary 1.9. *Let (S,P) be a commuting pair of operators such that $\|P\|<1$ and the spectral radius of S is less than 2. Then (S,P) is a Γ-contraction if and only if*

$$w\big((1-P^*P)^{-1/2}(S-S^*P)(1-P^*P)^{-1/2}\big)\leqslant 1.$$

2. Γ-UNITARIES AND Γ-ISOMETRIES

Unitaries, isometries and co-isometries are important special types of contractions. There are natural analogues of these classes for Γ-contractions. To define them we introduce, for any pair S, P of operators on a Hilbert space H, the notation $C^*(S,P)$ for the C^*-subalgebra of $\mathcal{L}(H)$ generated by S, P and the identity operator. If S, P are commuting normal operators, then by Fuglede's theorem $C^*(S,P)$ is a commutative C^*-algebra, and for such S, P we denote by $\sigma(S,P)$ the joint spectrum of (S,P) relative to the algebra $C^*(S,P)$.

Definition 2.1. Let S, P be commuting operators on a Hilbert space H. We say that the pair (S,P) is

(i) a Γ-*unitary* if S and P are normal operators and the joint spectrum $\sigma(S,P)$ of (S,P) is contained in the distinguished boundary of Γ;

(ii) a Γ-*isometry* if there exists a Hilbert space K containing H and a Γ-unitary $(\widetilde{S},\widetilde{P})$ on K such that H is invariant for both $\widetilde{S}$ and $\widetilde{P}$, and $S = \widetilde{S}|H$, $P = \widetilde{P}|H$;

(iii) a Γ-*co-isometry* if (S^*, P^*) is a Γ-isometry.

It is indeed true that the unitary operators (in the usual sense) are precisely the normal operators whose spectra in the C^*-algebras they generate lie in the unit circle, and so definition (i) above appears a natural generalization. On the other hand, one might expect an analogue of the standard polynomial-type definition of a unitary operator: $U^*U = UU^* = 1$. The following result shows there is no conflict here.

Theorem 2.2. *Let S, P be commuting operators on a Hilbert space H. The following are equivalent:*

(i) *(S,P) is a Γ-unitary;*

(ii) *$P^*P = 1 = PP^*$ and $P^*S = S^*$ and $\|S\| \leqslant 2$;*

(iii) *there exist commuting unitary operators U_1, U_2 on H such that*

$$S = U_1 + U_2, \quad P = U_1U_2.$$

Proof. (i) ⇒ (iii) Let (S,P) be a Γ-unitary. By the Spectral Theorem for commuting normal operators there is a spectral measure $E(\cdot)$ on $\sigma(S,P)$ such that

$$S = \int_{\sigma(S,P)} z_1 E(\mathrm{d}z), \quad P = \int_{\sigma(S,P)} z_2 E(\mathrm{d}z),$$

where z_1, z_2 are the co-ordinate functions on $\mathbb{C}^2$. Pick a measurable right inverse τ of the restriction of π to $\mathbb{T}^2$, so that τ maps the distinguished boundary of Γ to $\mathbb{T}^2$. Write $\tau = (\tau_1, \tau_2)$, and let

$$U_j = \int_{\sigma(S,P)} \tau_j(z) E(\mathrm{d}z), \quad j = 1, 2.$$

Then U_1, U_2 are commuting unitary operators on H and

$$U_1 + U_2 = \int_{\sigma(S,P)} (\tau_1 + \tau_2)(z) E(\mathrm{d}z) = \int_{\sigma(S,P)} z_1 E(\mathrm{d}z) = S.$$

Similarly $U_1U_2 = P$. Thus (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii) is obvious. Suppose (ii) holds. Then P is normal, and since $S^* = P^{-1}S$, so is S. Thus S and P are commuting normal operators and they generate a commutative C^*-algebra $C^*(S,P)$. The Gelfand representation identifies $C^*(S,P)$ with $C(\sigma(S,P))$, and $\widehat{S}, \widehat{P}$ are the restrictions to $\sigma(S,P)$ of the co-ordinate functions on $\mathbb{C}^2$. Consider any point $z = (s,p)$ of $\sigma(S,P)$. Then $\widehat{S}(z) = s, \widehat{P}(z) = p$. By (ii) and properties of the Gelfand transform,

$$(\widehat{P})^- \widehat{P} = 1 = \widehat{P}(\widehat{P})^-, \quad (\widehat{P})^- \widehat{S} = (\widehat{S})^-, \quad \|\widehat{S}\| \leqslant 2.$$

Applying these relations at the point z we obtain

$$|p| = 1, \quad \overline{p}s = \overline{s}, \quad |s| \leqslant 2.$$

By Theorem 1.3 it follows that z lies in the distinguished boundary of Γ. Thus (ii) $\Rightarrow$ (i). ∎

The equivalence of (i) and (iii) in Theorem 2.2 amounts to saying that the Γ-unitaries are simply the symmetrisations of commuting unitary pairs. Does an analogous statement hold for Γ-isometries? We know from Example 1.7 that it does not for Γ-contractions. Certainly, if V_1, V_2 are commuting isometries then $\pi(V_1, V_2)$ is a Γ-isometry, but the following shows that not all Γ-isometries arise in this way.

Example 2.3. (Symmetric H^2) Let H be the subspace of the Hardy space H^2 of the bidisc comprising the symmetric functions. Let S, P be the operations on H of multiplication by $z_1 + z_2$, z_1z_2 respectively. It is clear that that (S,P) is a Γ-isometry on H, being the restriction of an obvious Γ-unitary on $L^2(\mathbb{T}^2)$ to a common invariant subspace. However, (S,P) cannot be written in the form $\pi(T_1,T_2)$ for any pair of commuting operators. For suppose $S = T_1+T_2$, $P = T_1T_2$. Then

$$S^2 - 4P = (T_1 - T_2)^2.$$

Let $X = T_1 - T_2$: then X commutes with S and P, and the last equation implies that X^2 is multiplication by $(z_1 - z_2)^2$. Commutation with S and P implies that X is multiplication by the bounded symmetric analytic function $\psi = X1$, and hence we have $\psi^2 = (z_1 - z_2)^2$. However, there is no continuous symmetric function ψ on the bidisc such that $\psi^2 = (z_1 - z_2)^2$ (consider the sets $E_\pm = \{(z_1,z_2) \in \mathbb{D}^2 : \psi(z_1,z_2) = \pm(z_1 - z_2)\}$). Thus there can be no such pair (T_1,T_2).

Recall that an isometry on a Hilbert space H is said to be a *pure isometry* if there is no non-trivial subspace of H on which it acts as a unitary operator. Pure isometries are unitarily equivalent to shift operators (of arbitrary multiplicity), and the Wold decomposition theorem asserts that every isometry is the orthogonal direct sum of a unitary and a pure isometry ([19], Theorem I.1.1). We shall say that a commuting pair (S,P) is a *pure Γ-isometry* if (S,P) is a Γ-isometry and P is a pure isometry. Pure Γ-isometries can be modelled by Toeplitz operators, as follows.

Theorem 2.4. *Let* (S, P) *be commuting operators on a separable Hilbert space* H. (S, P) *is a pure* Γ*-isometry if and only if there exist a separable Hilbert space* E*, a unitary operator* $U : H \to H^2(E)$ *and an operator* A *on* E *such that* $w(A) \leqslant 1$ *and*

$$S = U^*T_\varphi U, \quad P = U^*T_z U \tag{2.1}$$

where

$$\varphi(z) = A + A^*z, \quad z \in \mathbb{D}. \tag{2.2}$$

Proof. Suppose that S, P are the restrictions to a common invariant subspace H of a Γ-unitary $(\widetilde{S}, \widetilde{P})$ on a superspace K of H. By Theorem 2.2, $\widetilde{P}^*\widetilde{S} = \widetilde{S}^*$, and by compression to H it follows that $P^*S = S^*$; likewise, the fact that $\|\widetilde{S}\| \leqslant 2$ tells us that $\|S\| \leqslant 2$. Since P is a pure isometry and H is separable, we may identify H with $H^2(E)$, for some separable Hilbert space E, and P (up to unitary equivalence) with the shift operator T_z on $H^2(E)$. Since S commutes with the shift operator it has the form $S = T_\varphi$ for some $\varphi \in H^\infty\mathcal{L}(E)$. The relations $P^*S = S^*$ and $\|S\| \leqslant 2$ yield

$$T_{\overline{z}}T_\varphi = T_\varphi^*, \quad \|\varphi\|_\infty \leqslant 2.$$

The former relation implies that, for all $z \in \mathbb{T}$, $\overline{z}\varphi(z) = \varphi(z)^*$, and consideration of Fourier series shows that $\varphi(z) = A + A^*z$ for some operator A on E. For any $\theta \in \mathbb{R}$,

$$\|2\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\theta}A)\| = \|\mathrm{e}^{\mathrm{i}\theta}A + \mathrm{e}^{-\mathrm{i}\theta}A^*\| = \|A + A^*\mathrm{e}^{-\mathrm{i}2\theta}\| \leqslant 2,$$

whence $w(A) \leqslant 1$.

Conversely, suppose S, P are given by equations (2.1) and (2.2), where $w(A) \leqslant 1$. We may assume that U is the identity. Let M_φ, M_z be the multiplication operators on $L^2(E)$ with symbols φ, z respectively; then it is easy to see from Theorem 2.2 that (M_φ, M_z) is a Γ-unitary. S, P are the restrictions to the common invariant subspace $H^2(E)$ of M_φ, M_z, and hence (S, P) is a Γ-isometry. Since P is a shift, (S, P) is a pure Γ-isometry. ∎

Our next theorem contains analogues of both the Wold decomposition and the above characterization of Γ-unitaries. First we need a simple observation.

Lemma 2.5. *Let* U, V *be a unitary and a pure isometry on Hilbert spaces* H_1, H_2 *respectively, and let* $T : H_1 \to H_2$ *be an operator such that* $TU = VT$. *Then* $T = 0$.

Proof. By iteration we have, for any positive integer $n, TU^n = V^nT$ and hence $U^{*n}T^* = T^*V^{*n}$. Thus T^* vanishes on $\ker V^{*n}$, and since $\bigcup_n \ker V^{*n}$ is dense in H_2 we have $T^* = 0$. ∎

THEOREM 2.6. *Let S, P be commuting operators on a Hilbert space H. The following statements are equivalent:*

(i) *(S,P) is a Γ-isometry;*

(ii) *there is an orthogonal decomposition $H = H_1 \oplus H_2$ into common reducing subspaces of S and P such that $(S|H_1, P|H_1)$ is Γ-unitary and $(S|H_2, P|H_2)$ is a pure Γ-isometry;*

(iii) *$P^*P = 1$ and $P^*S = S^*$ and $\|S\| \leqslant 2$;*

(iv) *$\|S\| \leqslant 2$ and, for all $\omega \in \mathbb{T}$, $\rho(\omega S, \omega^2 P) = 0$.*

Proof. (i) ⇒ (iii) Suppose that $(\widetilde{S}, \widetilde{P})$ is a Γ-unitary on a space $K \supset H$, H is a common invariant subspace of $\widetilde{S}$ and $\widetilde{P}$ and S, P are the restrictions of $\widetilde{S}, \widetilde{P}$ to H. By Theorem 2.2,

$$\widetilde{P}^*\widetilde{P} = 1, \quad \widetilde{P}^*\widetilde{S} = \widetilde{S}^*, \quad \widetilde{S}^*\widetilde{S} \leqslant 4.$$

On compressing to H we obtain

$$P^*P = 1, \quad P^*S = S^*, \quad S^*S \leqslant 4.$$

Thus (i) ⇒ (iii).

(iii) ⇒ (iv) is obvious. If $\rho(\omega S, \omega^2 P) = 0$ for all $\omega \in \mathbb{T}$ then on integrating with respect to ω we obtain $1 - P^*P = 0$ and thence also $S^* - P^*S = 0$. Thus (iii) ⇔ (iv).

(iii) ⇒ (ii). It is easy to reduce to the case that H is separable. Suppose (iii) holds. By the Wold decomposition we may write $P = U \oplus V$ on $H = H_1 \oplus H_2$ where H_1, H_2 are reducing subspaces for P, U is unitary and V is a pure isometry. With respect to this decomposition let

$$S \sim \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

The relation $SP = PS$ shows that $S_{21}U = VS_{21}$. Hence, by Lemma 2.5, $S_{21} = 0$. Since $P^*S = S^*$,

$$\begin{bmatrix} U^*S_{11} & U^*S_{12} \\ 0 & V^*S_{22} \end{bmatrix} = \begin{bmatrix} S_{11}^* & 0 \\ S_{12}^* & S_{22}^* \end{bmatrix}.$$

It follows that $S_{12} = 0$, and so H_1, H_2 are reducing for S. We have $US_{11} = S_{11}U$, U is unitary, $U^*S_{11} = S_{11}^*$ and $\|S_{11}\| \leqslant 2$. Hence, by Theorem 2.2, (S_{11}, U) is Γ-unitary — that is, $(S|H_1, P|H_1)$ is Γ-unitary.

We claim that (S_{22}, V) is a pure Γ-isometry on H_2. Indeed, since V is a pure isometry, we can identify it with the shift operator T_z on a vectorial H^2 space, $H^2(E)$ say, for some separable Hilbert space E. Since S_{22} commutes with $V \equiv T_z$, S_{22} has the form T_φ for some $\varphi \in H^\infty\mathcal{L}(E)$. The relation $V^*S_{22} = S_{22}^*$ then gives $T_{\overline{z}}T_\varphi = T_{\varphi^*}$, whence, for all $z \in \mathbb{T}$,

$$\overline{z}\varphi(z) = \varphi(z)^*. \tag{2.3}$$

It follows from consideration of Fourier series that $\varphi(z) = A + A^*z$ for some operator A on E, and from the fact that

$$\|\varphi\|_\infty = \|S_{22}\| \leqslant \|S\| \leqslant 2$$

we can infer that $w(A) \leqslant 1$. Hence, by Theorem 2.4 (S_{22}, V) is a pure isometry. That is, $(S|H_2, P|H_2)$ is a pure isometry. Thus (iii) ⇒ (ii).

It is trivial that (ii) ⇒ (i). ∎

Corollary 2.7. *Let S, P be commuting operators. (S,P) is a Γ-co-isometry if and only if*

$$PP^* = 1, \quad PS^* = S \quad \text{and} \quad \|S\| \leqslant 2.$$

Any contraction can be expressed as the orthogonal direct sum of a unitary operator and a completely non-unitary contraction ([19], Theorem I.3.2). We shall now show that, for any Γ-contraction (S,P), if we split P up in this way, then S decomposes into the direct sum of operators on the same subspaces.

Theorem 2.8. *Let (S,P) be a Γ-contraction on a Hilbert space H. Let H_1 be the maximal subspace of H which reduces P and on which P is unitary. Let $H_2 = H \ominus H_1$. Then H_1 and H_2 reduce S, $(S|H_1, P|H_1)$ is a Γ-unitary and $(S|H_2, P|H_2)$ is a Γ-contraction for which $P|H_2$ is completely non-unitary.*

Proof. Let $S = [S_{ij}]_{i,j=1}^2$, $P = \operatorname{diag}\{P_1, P_2\}$ with respect to the decomposition $H = H_1 \oplus H_2$, so that P_1 is unitary and P_2 is completely non-unitary. It follows that if $x \in H_2$ and

$$\|P_2^n x\| = \|x\| = \|P_2^{*n} x\|, \quad n = 1, 2, \ldots$$

then $x = 0$.

The fact that S and P commute tells us that

$$S_{11}P_1 = P_1 S_{11}, \quad S_{12}P_2 = P_1 S_{12}, \tag{2.4}$$

$$S_{21}P_1 = P_2 S_{21}, \quad S_{22}P_2 = P_2 S_{22}. \tag{2.5}$$

By Theorem 1.5, for all $\omega \in \mathbb{T}$,

$$\begin{aligned} 0 \leqslant \rho(\omega S, \omega^2 P) = 2\begin{bmatrix} 0 & 0 \\ 0 & 1 - P_2^* P_2 \end{bmatrix} &- \omega \begin{bmatrix} S_{11} - S_{11}^* P_1 & S_{12} - S_{21}^* P_2 \\ S_{21} - S_{12}^* P_1 & S_{22} - S_{22}^* P_2 \end{bmatrix} \\ &- \overline{\omega} \begin{bmatrix} S_{11}^* - P_1^* S_{11} & S_{21}^* - P_1^* S_{12} \\ S_{12}^* - P_2^* S_{21} & S_{22}^* - P_2^* S_{22} \end{bmatrix}. \end{aligned} \tag{2.6}$$

Consideration of the $(1,1)$ block reveals that $S_{11} = S_{11}^* P_1$. Since (S,P) is a Γ-contraction, $\|S\| \leqslant 2$ and hence also $\|S_{11}\| \leqslant 2$. By Theorem 2.2, (S_{11}, P_1) is a Γ-unitary.

Now examine the $(1,2)$ block in the inequality (2.6). It yields

$$\omega(S_{12} - S_{21}^* P_2) + \overline{\omega}(S_{21}^* - P_1^* S_{12}) = 0$$

for all $\omega \in \mathbb{T}$, and hence

$$S_{12} = S_{21}^* P_2, \quad S_{21}^* = P_1^* S_{12}. \tag{2.7}$$

Thus $S_{21} = S_{12}^* P_1$, and together with the first equation in (2.5), this implies that

$$S_{12}^* P_1^2 = S_{21} P_1 = P_2 S_{21} = P_2 S_{12}^* P_1,$$

and hence

$$S_{12}^* P_1 = P_2 S_{12}^*. \tag{2.8}$$

By iterating the equations in (2.4) and (2.8) we find that, for any $n \geqslant 1$,

$$S_{12} P_2^n = P_1^n S_{12}, \quad S_{12} P_2^{*n} = P_{12}^{*n} S_{12}.$$

Thus

$$S_{12}P_2^nP_2^{*n} = P_1^nS_{12}P_2^{*n} = P_1^nP_2^{*n}S_{12} = S_{12},$$
$$S_{12}P_2^{*n}P_2^n = P_2^{*n}S_{12}P_1^n = P_2^{*n}P_1^nS_{12} = S_{12},$$

and so we have

$$P_2^nP_2^{*n}S_{12}^* = S_{12}^* = P_2^{*n}P_2^nS_{12}^*.$$

It follows that, for any $x \in H_1$ and $n \geqslant 1$,

$$\|P_2^{*n}S_{12}^*x\| = \|S_{12}^*x\| = \|P_2^nS_{12}^*x\|.$$

Since P_2 is completely non-unitary, we must have $S_{12}^*x = 0$, and so $S_{12} = 0$. By (2.7), $S_{21} = 0$ too. Thus H_1 and H_2 reduce S as claimed. All that remains to prove is the statement that (S_{22}, P_2) is a Γ-contraction; it is immediate from the definition that the restriction of a Γ-contraction to any common reducing subspace is again a Γ-contraction. ∎

In view of this theorem there is no need to introduce "completely non-Γ-unitary Γ-contractions": they coincide with Γ-contractions (S, P) for which P is completely non-unitary in the usual sense. Since Γ-unitaries correspond (by Theorem 2.2) to pairs of commuting unitaries, the study of the general Γ-contraction is reduced to the study of those for which P is completely non-unitary.

3. A MODEL FOR Γ-CONTRACTIONS

An important ingredient in Nagy-Foiaş model theory is the fact that every contraction has a co-isometric extension. An analogous statement holds for Γ-contractions.

THEOREM 3.1. *Let (S, P) be a Γ-contraction on a Hilbert space H. There exists a Hilbert space K containing H and a Γ-co-isometry $(S^\flat, P^\flat)$ on K such that H is invariant under $S^\flat$ and $P^\flat$, and $S = S^\flat|H, P = P^\flat|H$.*

Proof. It is immediate from the definition of Γ-contractions that (S^*, P^*) is also a Γ-contraction. By Theorem 1.5 there exist Hilbert spaces H_-, H_+ and a Γ-isometry $(\widetilde{S}, \widetilde{P})$ on $H_- \oplus H \oplus H_+$ such that

$$\widetilde{S} \sim \begin{bmatrix} * & * & * \\ 0 & S^* & * \\ 0 & 0 & * \end{bmatrix}, \quad \widetilde{P} \sim \begin{bmatrix} * & * & * \\ 0 & P^* & * \\ 0 & 0 & * \end{bmatrix}.$$

The space $H_- \oplus H$ is invariant under $\widetilde{S}$ and $\widetilde{P}$, and so $(\widetilde{S}|H_- \oplus H, \widetilde{P}|H_- \oplus H)$ is a Γ-isometry. Let $S^\flat = (\widetilde{S}|H_- \oplus H)^*$, and $P^\flat = (\widetilde{P}|H_- \oplus H)^*$. Then $(S^\flat, P^\flat)$ is a Γ-co-isometry on $H_- \oplus H$, and

$$S^\flat \sim \begin{bmatrix} * & 0 \\ * & S \end{bmatrix}, \quad P^\flat \sim \begin{bmatrix} * & 0 \\ * & P \end{bmatrix}.$$

Thus H is invariant under $S^\flat$ and $P^\flat$, and $S = S^\flat|H$, $P = P^\flat|H$ as required. ∎

We can now give a model for Γ-contractions analogous to the well-established models of contractions (e.g. [19]). Roughly speaking, every Γ-contraction is the restriction to a common invariant subspace of the orthogonal direct sum of a Γ-unitary and the adjoint of a pure Γ-isometry (T_φ, T_z), as described in Theorem 2.4.

Theorem 3.2. *Let (S, P) be a Γ-contraction on a Hilbert space H. There exist a Hilbert space K containing H, a Γ-co-isometry $(S^\flat, P^\flat)$ on K and an orthogonal decomposition $K_1 \oplus K_2$ of K such that:*

(i) *H is a common invariant subspace of $S^\flat$ and $P^\flat$, and $S = S^\flat|H, P = P^\flat|H$;*

(ii) *K_1 and K_2 reduce both $S^\flat$ and $P^\flat$;*

(iii) *$(S^\flat|K_1, P^\flat|K_1)$ is a Γ-unitary;*

(iv) *there exist a Hilbert space E and an operator A on E such that $w(A) \leqslant 1$ and $(S^\flat|K_2, P^\flat|K_2)$ is unitarily equivalent to $(T_\psi, T_{\bar{z}})$ acting on $H^2(E)$, where $\psi \in L^\infty\mathcal{L}(E)$ is given by*

$$\psi(z) = A^* + A\bar{z}, \quad z \in \mathbb{T}. \tag{3.1}$$

Proof. Theorem 3.1 guarantees the existence of K and of a Γ-co-isometry $(S^\flat, P^\flat)$ satisfying (i). Apply Theorem 2.6 to the Γ-isometry $(S^{\flat *}, P^{\flat *})$ on K: by the equivalence of (i) and (ii) there is an orthogonal decomposition $K = K_1 \oplus K_2$ into common reducing subspaces of $S^\flat$ and $P^\flat$ so that $(S^{\flat *}|K_1, P^{\flat *}|K_1)$ is a Γ-unitary, and $(S^{\flat *}|K_2, P^{\flat *}|K_2)$ is a pure Γ-isometry. On applying Theorem 2.4 to $(S^{\flat *}|K_2, P^{\flat *}|K_2)$ we obtain

$$S^\flat|K_2 \sim T_\psi, \quad P^\flat|K_2 \sim T_{\bar{z}},$$

acting on $H^2(E)$, for suitable E and ψ, as in statement (iv). ∎

This theorem may be regarded as the analogue for Γ-contractions of the first of the two stages in the construction of the Nagy-Foiaş model of contractions. To carry out the second stage, and so obtain a genuine functional model for the general Γ-contraction, one would need to provide a description in suitably concrete terms of the common invariant subspaces of Γ-coisometries, perhaps along the lines of that given in the Nagy-Foiaş theory by the characteristic operator function. Consider for example the special case of a Γ-contraction (S, P) which extends to a *pure* Γ-coisometry $(S^\flat, P^\flat)$ (so that $K_1 = \{0\}$ in the decomposition in Theorem 3.2). Here $P^\flat$ is a coisometric extension of P, but there is no reason to think it is minimal, and so one should not expect E and H to be given by the characteristic operator function of P. Identifying K $(= K_2)$ with $H^2(E)$, we observe that H is a subspace of $H^2(E)$ invariant under the backward shift, and so is expressible in the form $H = H^2(E) \ominus \Phi H^2(E_*)$ for some separable Hilbert space E_* and some $\mathcal{L}(E_*, E)$-valued inner function Φ. Since H is invariant under T_ψ, with ψ given by equation (3.1), it must be that $\Phi H^2(E_*)$ is invariant under T_ψ^*, that is,

$$(A + A^* z)\Phi(z) = \Phi(z)F(z) \tag{3.2}$$

for some $F \in H^\infty\mathcal{L}(E_*)$. Conversely, if E and E_* are separable Hilbert spaces, $A \in \mathcal{L}(E)$ satisfies $w(A) \leqslant 1$, Φ is an inner $\mathcal{L}(E_*, E)$-valued function and the equation (3.3) holds for some $F \in H^\infty\mathcal{L}(E_*)$, then we obtain a Γ-contraction by restricting $(T_{A^*+A\bar{z}}, T_{\bar{z}})$ to $H = H^2(E) \ominus \Phi H^2(E_*)$. To obtain a satisfactory description of Γ-contractions in the non-residual case $(K_1 = \{0\})$ one would need a characterization of all possible 4-tuples (E, E_*, A, Φ) satisfying the above conditions. We do not at present have a constructive description of such 4-tuples.

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J. AGLER
Department of Mathematics
University of California San Diego
La Jolla, CA 92093
USA

E-mail: jagler@ucsd.edu

N.J. YOUNG
Department of Mathematics
University of Newcastle upon Tyne
Merz Court
Newcastle upon Tyne NE1 7RU
ENGLAND

E-mail: N.J.Young@ncl.ac.uk

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