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A Model with Persistent Vacuum* **

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Abstract. It is shown that there exists a selfadjoint Hamilton operator corresponding to the interaction $H_0^{(a)} + H_0^{(b)} + \int \Phi_b^+(x)\, \Phi_a(x)\, \Phi_b^-(x)\, d^3x$, where a and b denote two types of scalar particles. We discuss the scattering theory of this operator.

I. Introduction

It is one of the aims of quantum field theory to describe in a mathematically precise fashion the scattering of relativistic quantized particles. This aim has been reached with different degrees of imperfection for several models of varying complexity. In this article we show that the results known for the Lee model remain essentially valid for a more complex model type to be defined in Section II.

The model treated in this paper is not a relativistic model. It shows, however, the problem of an increasing number of particles and the problem of a logarithmically divergent mass renormalization as the momentum cutoff is removed. The kinematics is relativistic and the interaction is translation invariant. There is no vacuum polarization, and in this sense the model is less ambitious than $\lambda : \Phi^4 :$, and it is at this price that we get more detailed results.

We shall see that this model allows essentially the same conclusions about the existence of scattering states as the Lee model which is very well understood [11] but in which the problem of multiple particle creation is not present. The model we shall treat is very similar to the Nelson model for which much is known about the total Hamiltonian and the n-point functions [9, 1]. The Nelson model is somewhat simpler than the present model because the mass renormalization is a logarithmically diverging constant, whereas it is an operator in the case of the persistent model. In Section II we state the definition of the model and give a basic estimate. Section III contains the proof for the selfadjointness of the renormalized total Hamiltonian. The difficult part is the summation of the renormalized Born series for the resolvent. In Section IV we

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construct scattering states. The main point is the construction of the intertwining operator T satisfying $HT = TH_0$ on the one-particle states. T is given by the Friedrichs expansion.

II. Definitions and Basic Estimates

1. Notations for Second-Quantized Operators

We begin with the construction of the Fock space $\mathscr{F}$ for one scalar field of mass $\theta > 0$, in s space dimensions. All arguments k, l, p, q, x, y and all integration variables are to be considered as elements of $\mathbb{R}^s$. As usual, $\theta(k) = (\theta^2 + k \cdot k)^{\frac{1}{2}}$. We define the Fock space $\mathscr{F}$ as $\mathscr{F} = \mathscr{F}(\theta) = \bigoplus_{n=0}^{\infty} \mathscr{F}^{(n)}$. $\varphi \in \mathscr{F}$ means

$$\varphi = \left\{ \varphi^{(n)} \in \mathscr{F}^{(n)};\ \|\varphi\|^2 \equiv \sum_{n=0}^{\infty} \|\varphi^{(n)}\|^2 < \infty \right\}.$$

Here, $\mathscr{F}^{(n)}$ is the space of symmetric functions in $L_2(\mathbb{R}^{sn})$. We write the scalar product in $\mathscr{F}^{(n)}$ as

$$(\varphi^{(n)}, \psi^{(n)}) = \int d^s k_1 \ldots d^s k_n\, \overline{\varphi^{(n)}}(k_1, \ldots k_n)\, \psi^{(n)}(k_1, \ldots k_n).$$

We shall write from now on dk instead of $d^s k$.

On $\mathscr{F}$ we have a representation of the canonical commutation relations

$$[a(k), a(l)] = 0 \quad \text{and} \quad [a(k), a^*(l)] = \delta(k - l).$$

The annihilation operator $a(k)$ is defined by

$$(a(k)\, \varphi)^{(n-1)}(k_1, \ldots k_{n-1}) = n^{\frac{1}{2}}\, \varphi^{(n)}(k, k_1, \ldots k_{n-1}).$$

$a^*(k)$ is the adjoint of $a(k)$. The number of particles operator N is defined to be

$$\{N \varphi^{(n)}\} = \{n\, \varphi^{(n)}\}.$$

We define *Wick monomials* W by

$$W = \int dk_1 \ldots dk_d\, w(k_1, \ldots k_d)\, a^*(k_1) \ldots a^*(k_i)\, a(k_{i+1}) \ldots a(k_d), \tag{1}$$

$0 \leq i \leq d$. d is called the degree of W, w is called the (numerical) *kernel* of W. By convention, a small letter will always denote the numerical kernel of the Wick monomial with the same capital letter.

We suppose that the reader is familiar with Wick's theorem (see e.g. [6]). We shall use sometimes a graphical representation described in [4] and in [6]. A Wick-ordered expression of the form (1) is visualized by a vertex of a graph, where the a^*'s are depicted by legs pointing to the left (and going actually to infinity), and the a's are depicted by legs pointing to the right. So the operator W of Eq. (1) is drawn as

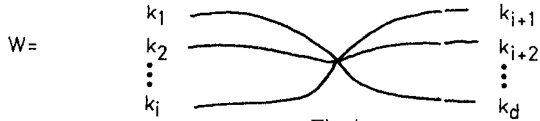
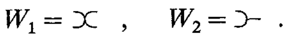


Fig. 1

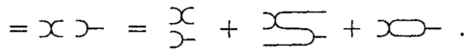
Wick's theorem is then easily depicted by forming all contraction schemes between the legs of the factors.

Example. Let



Then the Wick ordered expansion of $W_1 W_2$ is

$$W_1 W_2 = :\underbrace{W_1\ W_2}_{0}: + :\underbrace{W_1\ W_2}_{1}: + :\underbrace{W_1\ W_2}_{2}:$$



It is understood that the symbol $\underbrace{\quad}_{i}$ and the graphs imply already the combinatorial factors which might occur.

We define finally the time zero free field operator $\Phi(x)$ by

$$\Phi(x) = \Phi^+(x) + \Phi^-(x),$$

$$\Phi^+(x) = \Phi^-(x)^* = \int dk\, \theta(k)^{-\frac{1}{2}}\, a^*(k)\, e^{ikx}, \qquad x \in \mathbb{R}^s.$$

An important technical tool for constructive quantum field theory, the *cutoffs*, will appear only in the form of a momentum cutoff σ. We define

$$W_\sigma = \int_{|k_j| \leqq \sigma,\, j=1,\ldots d} dk_1 \ldots dk_d\, w(k_1, \ldots k_d)\, a^*(k_1) \ldots a(k_d). \tag{2}$$

Evidently $W_\infty = W$. For reasons explained later, we need no space cutoff.

2. *Definition of the Model and a Basic Estimate*

Persistent models are theories with two types of particles which are called *nucleons* and *mesons* (irrespective of the actual statistics). The main characteristic of the Hamiltonians of such models is that these Hamiltonians commute with the nucleon particle-number operator, i.e. the *nucleon number is conserved.* The Fock space underlying the theory (which will be defined below) can therefore be decomposed into an infinite direct sum of superselecting *sectors* with fixed nucleon number. As a further characteristic the Hamiltonian annihilates the Fock vacuum; therefore the *free and the physical vacuum are identical.* This in turn implies that no space cutoff will be necessary to make the Hamiltonian a well-defined operator (Haag's theorem does not apply), and that one can have therefore a translation invariant interaction. However, the difficulties due to high momenta or to large meson number, known from relativistic quantum field theory, can be present.

We leave now this general context and start the definitions leading to the model which we shall treat below. Both the *mesons and the nucleons will be bosons* in all the following discussions; many results can be carried over to the case where the nucleon is a fermion.

We define a Fock space $\mathscr{H}$ for the two types of particles:

$$\mathscr{H} = \mathscr{F}(\omega) \otimes \mathscr{F}(\mu) = \mathscr{F} \otimes \mathscr{F} .$$

Here, $\mathscr{F}$ is the Fock space for one particle type defined in Section II.1. It will be convenient to distinguish several subspaces of $\mathscr{H}$:

$$\mathscr{H}^{(n)} = \mathscr{F}^{(n)} \otimes \mathscr{F} ,$$

the space with fixed nucleon number (often called a sector), and

$$\mathscr{H}^{(n,m)} = \mathscr{F}^{(n)} \otimes \mathscr{F}^{(m)} ,$$

the space with nucleon and meson number fixed.

We will apply consistently the following conventions:

— $a^\#$ denotes a meson creation or annihilation operator: $a^* = \rightsquigarrow$;
— $b^\#$ denotes a nucleon creation or annihilation operator: $b^* = \longrightarrow$;
— a or b used as index or as superscript between brackets denotes that the corresponding quantity is used for mesons or nucleons respectively;
— n denotes the nucleon number, N_b denotes the nucleon number operator;
— m denotes the meson number, N_a denotes the meson number operator;
— $\omega(k)$ is used for nucleons, $\omega(k) = (\omega^2 + k^2)^{\frac{1}{2}}$, $\omega > 0$;
— $\mu(k)$ is used for mesons, $\mu(k) = (\mu^2 + k^2)^{\frac{1}{2}}$, $\mu > 0$.

With these notations, we can now define the Hamiltonian H:

$$H = H_0 + V.$$

H_0, the *free Hamiltonian*, is defined by

$$H_0 = H_0^{(a)} + H_0^{(b)} ,$$

$$H_0^{(a)} = \int \mu(k)\, a^*(k)\, a(k)\, dk ,$$

$$H_0^{(b)} = \int \omega(k)\, b^*(k)\, b(k)\, dk .$$

The *interaction Hamiltonian* V is given by the formal expression

$$V = \int \Phi_b^+(x)\, \Phi_b^-(x)\, \Phi_a(x)\, dx .$$

By construction, H maps $\mathscr{H}^{(n)} \cap \mathscr{D}(H)$ into $\mathscr{H}^{(n)}$ for each $n = 0, 1, 2 \ldots$, so it makes sense to speak of $H|_n$, the restriction of H to $\mathscr{H}^{(n)}$. We shall write

$$H = \bigoplus_{n=0}^{\infty} H|_n \quad \text{on} \quad \mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}^{(n)} .$$

Our aims are to give a meaning to this formal Hamiltonian in $s = 3$ space dimensions and to derive some of its properties. Before starting this work in the subsequent chapters, we close this section by the following remarks.

The conclusions of Haag's theorem become irrelevant for persistent models because the physical and the free vacuum coincide. Therefore one can work with models which have no space cutoff. This motivates the preceding definitions. The following result is typical for interactions which conserve the particle number of one particle type:

Lemma 1. *Let W be a Wick monomial of the form*

$$W = \int b^*(p_1)\, b(p_2)\, a^*(k_1) \ldots a^*(k_i)\, a(k_{i+1}) \ldots a(k_d)$$
$$\cdot w(p_1, p_2, k_1, \ldots k_d)\, \delta(p_1 - p_2 + k_1 + \cdots + k_i - k_{i+1} - \cdots - k_d)\, dp_1\, dp_2\, dk_1 \ldots dk_d .$$

If

$$\sup_p \int |w(p, p + k_1 + \cdots + k_i - k_{i+1} - \cdots - k_d, k_1, \ldots k_d)|^2\, dk_1 \ldots dk_d = Y$$

is finite, then W is defined on $\mathscr{D}(N_b (N_a + \mathbb{1})^{d/2})$ and

$$\| W (N_a + \mathbb{1})^{-d/2}|_n \| \leqq n\, Y^{\frac{1}{2}} \cdot \text{const.},$$

where the constant depends only on d.

Proof. Let $\varphi \in \mathscr{H}^{(n, m+d-i)}$, $\psi \in \mathscr{H}^{(n, m+i)}$. Let $p + \Sigma \equiv p + k_1 + \cdots + k_i - k_{i+1} - \cdots - k_d$. Then

$$(\psi, W\varphi) = \left(\frac{(m+i)!}{m!} \; \frac{(m+d-i)!}{m!} \right)^{\frac{1}{2}} n$$
$$\cdot \int \overline{\psi}(q_1, \ldots q_{n-1}, p; l_1, \ldots l_m, k_1 \ldots k_i)\, \varphi(q_1, \ldots q_{n-1}, p + \Sigma; l_1, \ldots l_m, k_{i+1}, \ldots k_d)$$
$$\cdot w(p, p + \Sigma, k_1, \ldots k_d)\, dp\, dk_1 \ldots dk_d\, dq_1 \ldots dq_{n-1}\, dl_1 \ldots dl_m .$$

The numerical factor can be estimated by

$$n(\ldots)^{\frac{1}{2}} \leqq n c^d m^{d/2} ,$$

where c is a constant and the integral can be estimated by the Schwartz inequality

$$\int \overline{\psi} w \varphi \leqq \int dp\, \|\psi\|_{2'}\, \|w\|_{2'}\, \|\varphi\|_{2'} ,$$

where $\| \; \|_{2'}$ means the L_2-norm over all variables of the functions except p.

Using now

$$(\int dp \, \|\psi\|_{2'}^2)^{\frac{1}{2}} = \|\psi\|_2$$

and again the Schwartz inequality, we get

$$\int dp \, \|\psi\|_{2'} \, \|w\|_{2'} \, \|\varphi\|_{2'} \leqq \int dp \, \|\psi\|_{2'} \, \|\varphi\|_{2'} \, Y \leqq \|\psi\|_2 \, \|\varphi\|_2 \, Y$$

and this proves the Lemma.

III. Existence of the Renormalized Hamiltonian

1. The Hamiltonian with Momentum Cutoff

In this section we shall see that, with cutoff, V is a relatively bounded perturbation of H_0.

Let V_σ denote the momentum cutoff operator V (see Eq. (2)), that is

$$\begin{aligned} V_\sigma &= V_\sigma^c + V_\sigma^a, \qquad V_\sigma^a = (V_\sigma^c)^* \\ V_\sigma^c &= \int_{|k|,|l|,|k-l| \leqq \sigma} b^*(k-l)\, a^*(l)\, b(k)\, \omega(k-l)^{-\frac{1}{2}} \mu(l)^{-\frac{1}{2}} \omega(k)^{-\frac{1}{2}} d^3k d^3l\,. \end{aligned} \tag{3}$$

c and a denote creation and annihilation of the meson respectively. Let $\omega_\sigma^{-1}(k) = \omega^{-1}(k)$ if $|k| \leqq \sigma$, zero otherwise.

Lemma 2. *Let $\sigma < \infty$, $n \geqq 0$, $\lambda \in \mathbb{R}$. Then $V_\sigma|_n$ is small in the sense of Kato with respect to $H_0|_n$.*

Proof. By Lemma 1,

$$\begin{aligned} &\|V_\sigma(N_a + \mathbb{1})^{-\frac{1}{2}}|_n\|^2 \\ &\leqq 4n^2 \sup_p \int |\omega_\sigma^{-\frac{1}{2}}(p+q)\, \omega_\sigma^{-\frac{1}{2}}(p)\, \mu_\sigma^{-\frac{1}{2}}(q)|^2 \, dq = n^2 C_\sigma^2\,; \qquad C_\sigma < \infty \end{aligned}$$

and therefore, for $\varphi^{(n)} \in \mathscr{H}^{(n)} \cap \mathscr{D}(H_0)$,

$$\|\lambda V_\sigma \varphi\| \leqq n \cdot |\lambda| \cdot C_\sigma \, \|(N_a + \mathbb{1})^{\frac{1}{2}} \varphi\| \leqq \varepsilon \, \|H_0 \varphi\| + C(n \cdot |\lambda| \cdot C_\sigma, \varepsilon) \, \|\varphi\|$$

for any $\varepsilon > 0$. The last inequality above is a well-known relation between N and H_0 (see e.g. [9]), and thus Lemma 2 is proved.

We are now prepared to state the

Theorem 3. *For all $\sigma < \infty$ and all $\lambda \in \mathbb{R}$ and any number of space dimensions, the operator $H_0 + \lambda V_\sigma$ is selfadjoint on $\mathscr{D}(H_0) \subset \mathscr{H}$.*

Proof. The selfadjointness of

$$H|_n \equiv H_0|_n + \lambda V_\sigma|_n$$

follows by Lemma 2 from the theorem of Kato (Theorem V.4.3 in [8]). In fact, λV_σ is a symmetric operator by definition and bounded with respect to H_0 by a bound $\varepsilon < 1$, due to Lemma 2.

It is then easy (see e.g. [3]) to conclude that

$$H = \bigoplus_{n=0}^{\infty} H|_n$$

is a selfadjoint operator.

2. *Convergence of the Renormalized Born Series*

A convenient way to study the selfadjointness of an operator which is in its cutoff version a small perturbation of H_0 is to look at the Born series for the resolvent

$$\begin{aligned} R_\sigma(z) &= (z - H_{\text{ren},\sigma})^{-1} \\ &= R_0(z) \sum_{k=0}^{\infty} \{(\lambda V_\sigma + \lambda^2 M_\sigma) R_0(z)\}^k \end{aligned} \tag{4}$$

where $R_0(z) = (z - H_0)^{-1}$; V_σ is defined in Eq. (3) and M_σ is given by

$$M_\sigma = \int b^*(k)\, b(k) \int_{\substack{|q| \leq \sigma \\ |k-q| \leq \sigma}} \frac{\mu(q)^{-1}\, \omega(k-q)^{-1}}{\mu(q) + \omega(k-q)}\, d^3q\, d^3k$$

and

$$H_{\text{ren},\sigma} = H_0 + \lambda V_\sigma + \lambda^2 M_\sigma .$$

For our persistent model the choice of M_σ is given by the construction of a dense domain for $H_0 + \lambda V_\sigma$ by means of a dressing transformation (see [3] or [6]).

Schrader [11] has shown the convergence of the Born series for the Lee model which has the same momentum divergence as our model in three dimensions. Hepp has given in [6] a somewhat more elegant proof for a similar relativistic model. If his estimates are refined by the Lemma 5 given below, they can be applied to the model we are considering now.

The two principal ideas for the convergence proof of the Born series are a resummation (according to the powers of the coupling constant λ) and the fact that the norm of the k-th order term can be estimated by a product of norms. The absolute convergence of the series will justify a posteriori the resummation.

We shall consider from now on a fixed sector $n \geq 1$. We state now the main result of this section.

Theorem 4. *The renormalized Born series* (4) *for* $R_\sigma(z)|_n$ *converges (if properly reordered) in norm for all* z *satisfying* $\operatorname{Re} z < -\delta(\lambda, n)$, *uniformly in* $\sigma \leq \infty$. $\operatorname*{norm-lim}_{\sigma \to \infty} R_\sigma|_n$ *exists.*

Proof. The resummation mentioned above is done as follows. We write

$$R_\sigma(z) = \sum_{k=0}^{\infty} \lambda^k \sum_{i_1, \ldots i_k = 1}^{2} (R_0(z)\, V_{i_1} R_0(z) \ldots V_{i_k} R_0(z))_{\text{ren}} \tag{5}$$

where $V_1 = V_\sigma^a$, $V_2 = V_\sigma^c$ and $(\ )_{\text{ren}}$ means that every factor $V_1 R_0(z) V_2$ is replaced by

$$:V_1 \underbrace{R_0(z)}_{0} V_2: + :V_1 \underbrace{R_0(z)}_{1} V_2: + (V_1 \underbrace{R_0(z)}_{2} V_2)_{\text{ren}}$$

with

$$(V_1 \underbrace{R_0(z)}_{2} V_2)_{\text{ren}} = V_1 \underbrace{R_0(z)}_{2} V_2 - M_\sigma .$$

Here, R_0 is contracted with V_1 and V_2.

This resummation allows a factoring of the corresponding norms which is expressed in the estimates of Lemma 6.

An important estimate relating H_0 and V is the following

Lemma 5. *Let $n > 0$ be fixed, $\tau \geqq \frac{1}{2}$. Let r^a be a continuous function. Let*

$$R^a = \int r^a(k_1, k_2, k_3)\, \delta(k_1 - k_2 - k_3)\, b^*(k_1)\, b(k_2)\, a(k_3)\, dk_1\, dk_2\, dk_3 .$$

Then

$$\|R^a(H_0^{(a)} + \mathbb{1})^{-\tau}|_n\|^2 \leqq n^2 \int dp\, \mu(p)^{-2\tau} \sup_q |r^a(p, q, p-q)|^2 .$$

Proof. This lemma has first been given by Glimm [5], Lemma 2.4.1, with a slight technical error in the proof. A correct and extended proof can be found in [7], Lemma 50 (p. 137).

Remark. The main point of Lemma 5 is that $H_0^{-\frac{1}{2}-\varepsilon}$, $\varepsilon > 0$, compensates at the same time the unboundedness of the annihilation operator *and* gives an improvement in momentum behaviour. Lemma 5 holds as well for the adjoint operators. Lemma 5 in turn implies the following estimates on certain fragments of the Born series:

Lemma 6. *Let $0 < \varepsilon < \frac{1}{4}$. For $\operatorname{Re} z \leqq -1$ there exist constants C_n and C_ε (independent of $0 < \sigma \leqq \infty$) such that*

$$\|R_0(z)\| \leqq \text{const.}\ \text{dist}(z, [0, \infty))^{-1} , \tag{6}$$

$$\|V_\sigma^a R_0(z)^{(1+\varepsilon)/2}|_n\| \leqq C_\varepsilon C_n , \tag{7}$$

$$\|R_0(z)^{(1+\varepsilon)/2} V_\sigma^c|_n\| \leqq C_\varepsilon C_n , \tag{8}$$

$$\|R_0(z)^{\varepsilon/2} V_\sigma^a \underbrace{R_0(z)}_{0} V_\sigma^c R_0(z)^{\varepsilon/2}|_n\| \leqq C_\varepsilon^2 C_n^2 , \tag{9}$$

$$\|R_0(z)^{\varepsilon/2} V_\sigma^a \underbrace{R_0(z)}_{1} V_\sigma^c R_0(z)^{\varepsilon/2}|_n\| \leqq C_\varepsilon^2 C_n^2 , \tag{10}$$

$$\|R_0(z)\, (V_\sigma^a \underbrace{R_0(z)}_{2} V_\sigma^c)_{\text{ren}}|_n\| \leqq C_\varepsilon^2 C_n^2 |z|^{-1+\varepsilon} , \tag{11}$$

$$\|V_\sigma^a R_0(z)\, (V_\sigma^a \underbrace{R_0(z)}_{2} V_\sigma^c)_{\text{ren}}\, R_0(z)\, V_\sigma^c|_n\| \leqq C_\varepsilon^4 C_n^4 |z|^{-1+2\varepsilon} . \tag{12}$$

Proof of Lemma 6. In principle, Lemma 6 is the same as formulas 3.66 to 3.71 of Theorem 3.5 in [6]. However, in some situations, the infinite meson number problem which is typical for the *persistent model* but absent in the *Lee model* has to be taken care of. This is done by improving the estimates of Hepp in the sense of our Lemma 5: H_0^{-1} does not only furnish an additional negative power but is also compensates one meson creation or annihilation operator.

We give the necessary additional remarks for each of the formulae (6) to (12).

(6): This is well-known (see e.g. [8]).

(7): We apply Lemma 5:

$$\|V_\sigma^a(z-H_0)^{-(1+\varepsilon)/2}\|^2 \leqq \|V_\sigma^a(H_0^{(a)}+\mathbb{1})^{-(1+\varepsilon)/2}\|^2$$
$$\leqq n^2 \int dp\, \mu(p)^{-1-\varepsilon} \sup_q \omega(p)^{-1}\omega(q)^{-1}\mu(p-q)^{-1}.$$

But

$$\sup_q \omega(q)^{-1}\mu(p-q)^{-1} \leqq \text{const.}\,\omega(p)^{-1},$$

thus the integral converges and (7) follows.

(8): This is the adjoint of (7).

(9): To obtain (9), we write

$$R_0^{\varepsilon/2} : \underbrace{V_\sigma^a R_0 V_\sigma^c}_{0} : R_0^{\varepsilon/2}$$
$$= : \underbrace{V_\sigma^a R_0^{1+\varepsilon} V_\sigma^c}_{0} : + : [R_0^{\varepsilon/2}, \underbrace{V_\sigma^a]\, R_0^{1+\varepsilon/2} V_\sigma^c}_{0} :$$
$$+ : \underbrace{V_\sigma^a R_0^{1+\varepsilon/2} [V_\sigma^c}_{0}, R_0^{\varepsilon/2}] : + : [R_0^{\varepsilon/2}, \underbrace{V_\sigma^a]\, R_0 [V_\sigma^c}_{0}, R_0^{\varepsilon/2}] :$$

and (9) can easily be obtained by estimates of the type of (7) and (8).

(10): This is obtained in the same manner as (9).

(11) and (12): These two relations are not trivial. They follow, however, directly from the proof given by Hepp for the analogical case of the Lee model, because the two meson legs of V^a and V^c are contracted. This means that the meson number problem does not arise and pure energy estimates – as given by Hepp – are sufficient.

This completes the proof of Lemma 6.

The rest of the proof of Theorem 4 is now identical to the proof of Theorem 3.5 in [6], which consists in splitting appropriately the powers of R_0 in order to write each term of the Born series as a product of terms occurring inside one of the norms of Lemma 6. It is evident by the choice of the relations (6) to (12) that this is always possible. One finds then that

$$\|(R_0(z)\, V_{i_1} R_0(z) \ldots R_0(z)\, V_{i_k} R_0(z))_{\text{ren}}|_n\| \leqq C_\varepsilon^k C_n^k |z|^{-\tau k} 3^{k/2}, \qquad (13)$$

for some $\tau > 0$. The factor $3^{k/2}$ comes from the definition of $(\ )_{\text{ren}}$. One gets then

$$\|R_\sigma(z)\| \leqq \sum_{k=0}^{\infty} K^k |z|^{-\tau k} |\lambda|^k \quad \text{with} \quad K = 2 \cdot 3^{\frac{1}{2}} C_\varepsilon C_n, \quad \operatorname{Re} z < -1$$

and this proves Theorem 4.

Remark. The proof can also be done in another way by estimating

a) the norm of each Wick-ordered contribution of order k (after resummation) by the methods used for any superrenormalizable theory [2] and

b) by estimating the number of such terms.

We do not present this proof here because much more space would be needed than for the version given above.

3. Existence of a Selfadjoint Operator in the Limit without Cutoff

In this section we state without proof the consequences of Theorem 4. For details the reader is referred to the work of Schrader [11] or Hepp [6]. Theorem 4 is indeed strong enough (together with some additional facts which are very easy to verify) to ensure

Theorem 7. *Let $R_\infty(z)$ denote the sum*

$$\lim_{\sigma \to \infty} \left(R_0(z) \sum_{k=0}^{\infty} [(\lambda V_\sigma + \lambda^2 M_\sigma) R_0(z)]^k \right)_{\text{ren}}$$

of Theorem 4. Let $H|_n = z - R_\infty(z)^{-1}|_n$. $H|_n$ is a selfadjoint, linear operator, bounded below.

Proof. Schrader [11], Corollary 1, Lemma 5, Lemma 6 and Theorem 4.

Remark. The selfadjoint operator H is the closure of the essentially selfadjoint operator defined by a dressing transformation (for details, see [3]).

IV. Results on Scattering

It is known [6] that scattering states can be constructed as soon as one has been able to give an operator T which intertwines the one-particle states and which is densely defined. The main problem of this section will therefore be to construct the operator T. The construction of the scattering states follows the standard lines described in [6].

It will only be possible to define one-particle states for bounded momentum and small coupling constants, because otherwise one loses control over the regularity properties of R_0 or equivalently of $\Gamma_\pm$, the Γ-operation of Friedrichs.

Theorem 8. *Let* $0 \leqq R < \infty$, *let* $H = H_0 + \lambda V^c + \lambda V^a$. *There exists a mass renormalization* S *and a densely defined operator* T *such that on the vacuum* φ_0,

$$(H+S)\, T b^*(p)\, \varphi_0 = T H_0 b^*(p)\, \varphi_0$$

holds for all $|p| \leqq R$ *and for* $|\lambda| \leqq \lambda_0(R)$. $\lambda_0(R) > 0$. S *and* T *depend on* λ *and* R.

Remarks. $-S$ is equal to $\lambda^2 \underbrace{V^a \Gamma V^c}_{2} + S'$ with S' an infinite sum of finite mass renormalizations.

The treatment of the infinite renormalization has to be understood in the same sense as for the summation of the Born series. We will not mention this problem again.

Proof of the Theorem. A formal solution to the problem $H T' b^* \varphi_0 = T' H_0 b^* \varphi_0$ is given by the perturbation theory of Friedrichs [4] in a form described by Hepp [6].

T' is formally given by $T' = :\exp \Gamma Q':$;

$$Q' = \left[\sum_{n=1}^{\infty} (-1)^n Q'_{(n)}\right]_1 ; \qquad Q'_{(1)} = \lambda(V^c + V^a)\,; \qquad Q'_{(k)} = Q'_{(1)} \Gamma(Q'_{(k-1)})\,.$$

Here $[\ ]_1$ denotes the sum over all connected graphs with exactly one nucleon annihilator and some creation operators. Γ denotes the $\Gamma_\pm$ operations of Friedrichs which coincide here because of regularity.

The operators of Q' are of the form $\left(\Sigma \equiv \sum_{i=1}^{q} k_i\right)$

$$W^{(q)} = \int_{|k_0| \leqq R} b^*(k_0 - \Sigma) \prod_{i=1}^{q} a^*(k_i)\, b(k_0)\, w(k_0, \ldots k_q)\, dk_0 \ldots dk_q\,, \qquad q = 0, 1, 2, \ldots\,. \tag{15}$$

Then $\Gamma = \Gamma^1$, where we define Γ^α $(0 < \alpha \leqq 1)$ by

$$\Gamma^\alpha(W^{(q)}) = \int_{|k_0| \leqq R} b^*(k_0 - \Sigma) \prod_{i=1}^{q} a^*(k_i)\, b(k_0)\, w(k_0, \ldots k_q) \cdot \left(\omega(k_0 - \Sigma) + \sum_{i=1}^{q} \mu(k_i) - \omega(k_0)\right)^{-\alpha} \cdot dk_0 \ldots dk_q\,.$$

The expansion of Q' contains terms of the form $W^{(0)}$ on which the Γ operation is not defined. We call them *mass graphs.* To make sense out of T we will have to modify the definition of $Q'_{(1)}$ so that Q' contains no mass graphs. We omit primes in the new definition.

Let $S_{1,1} = -\lambda V_R^c$; $S_{1,i} = 0, i = 0, 2, 3, 4, \ldots$; let

$$S_{n,i} = -\lambda \underbrace{V^c \Gamma S_{n-1,i-1}}_{1} - \lambda \underbrace{V^a \Gamma S_{n-1,i+1}}_{2} - \sum_{k=1}^{(n-i)/2} \underbrace{M_{2k} \Gamma S_{n-2k,i}}_{1}, \qquad 0 < i \leqq n; \quad n \geqq 2$$

$$S_{n,i} = 0 \quad \text{if} \quad i > n \quad \text{or} \quad i = 0;$$

$M_n = -\lambda \underbrace{V^a \Gamma S_{n-1,1}}_{2}$. n denotes the order, i the number of meson creators, and

$$V_R^c = \int_{|k_0| \leqq R} b^*(k_0 - k_1)\, a^*(k_1)\, b(k_0)\, \omega(k_0 - k_1)^{-\frac{1}{2}}\, \mu(k_1)^{-\frac{1}{2}}\, \omega(k_0)^{-\frac{1}{2}}\, dk_0\, dk_1 .$$

We define now $Q_{(1)}$, Q and T:

$$\begin{aligned} Q_{(1)} &= \lambda V^c + \lambda V^a + \sum_{l=1}^{\infty} M_{2l} \equiv \lambda V^c + \lambda V^a + S, \\ Q_{(k)} &= Q_{(1)} \Gamma(Q_{(k-1)}), \quad k = 2, 3, \ldots, \\ Q &= \left[\sum_{k=1}^{\infty} (-1)^k Q_{(k)} \right]_{1,R}, \end{aligned} \tag{16}$$

where the R indicates restriction of all Wick ordered terms $W^{(q)}$ to $|k_0| \leqq R$.

It is easy to verify that Q contains no mass terms: Write

$$Q = \sum_{n=0}^{\infty} \sum_{i=0}^{n} R_{n,i}$$

where n and i have the same meaning as above. Evidently $R_{n,i} = S_{n,i}$ if $n \geqq i \geqq 1$ and $R_{n,0} = -\lambda \underbrace{V^a \Gamma R_{n-1,1}}_{2} - M_n$ which is zero by the definition of M_n if n is even and by connectivity otherwise.

Note that in the series for Q mass graphs cancel only if they form the rightmost factor of the expression. This will be a complication in the subsequent estimates. We define *skeletons:* a skeleton of an expression and its corresponding graph is the same expression with all terms occurring in some M_n replaced by $\mathbb{1}$, for $n \geqq 4$.

Example. The skeleton of:

$$\Gamma(V^a \Gamma(V^a \Gamma(V^c \Gamma(V^c \Gamma(V^c \Gamma(V^a \Gamma(V^a \Gamma(V^c \Gamma(V^c))) \Gamma(V^c)))))))$$

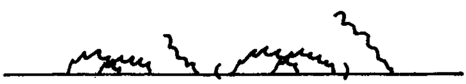
$=$ 

Fig. 2

is $\Gamma(V^a \Gamma(V^a \Gamma(V^c \Gamma(V^c \Gamma(V^c \Gamma(V^c))))))$

$=$ 

Fig. 3

(we have omitted the usual contraction notation $\smile_2\smile$). The removed terms are called *insertions*. If an insertion is itself not a skeleton, one can speak of its own insertion, which will then be said to be of *depth* two, and so on.

The problem connected with the insertions is that their Γ operations do not act on the variables of the surrounding skeleton. We will therefore first estimate the skeletons and we will treat later on the total graphs as a collection of inserted skeletons of varying depths.

Definition. $E_R^{(k)} = \{(p_1, \dots p_k) \in \mathbb{R}^{3k}, |p_i| \leqq R, i = 1, \dots k\}; k = 1, 2, \dots$.

Lemma 9. *Let* $W^{(q)}$ *be as in Eq.* (15). *Let* $\varphi \in \mathscr{H}^{(1)}$, $\operatorname{supp}\varphi \in E_R^{(1)}$, $q \geqq 1$, $\alpha > 0$. *Then*

$$\|\Gamma^\alpha(W^{(q)})\,\varphi\| \leqq D(R)^\alpha \,\|(H_0^{(q)} + \mathbb{1})^{-\alpha}\, W^{(q)}\,\varphi\| \,. \qquad D(R) < \infty \,.$$

Proof. $(\Gamma^\alpha(W^{(q)})\,\varphi)(p, k_1, \dots k_q) = ((H_0 - \omega(p))^{-\alpha}\, W^{(q)}\,\varphi)(p, k_1, \dots k_c)$; but

$$\|((H_0 - \omega(p))^{-\alpha}\, W^{(q)}\,\varphi)\,(p, k_1, \dots k_q)\|$$
$$\leqq \left(\frac{\Omega(R) + 1}{\Omega(R) - \omega(R)}\right)^\alpha \|(H_0^{(q)} + \mathbb{1})^{-\alpha}\, W^{(q)}\,\varphi\|$$

where $\Omega(R) = (R^2 + (\mu(0) + \omega(0))^2)^{\frac{1}{2}}$. This proves the assertion.

It will be more convenient to count graphs, in contrast to the case of the Born series. We therefore introduce a "kernel norm" $\|\ \|_s$, defined by

$$\|W^{(q)}\,\varphi^{(1,0)}\|^2 \leqq q!\, \|W^{(q)}\|_s^2\, \|\varphi^{(1,0)}\|^2$$

where

$$\|W^{(q)}\|_s^2 = \sup_{|k_0| \leqq R} \int |w(k_0, \dots k_q)|^2\, dk_1 \dots dk_q$$

(see also Lemma 1). Then one gets by methods similar to those used in Lemma 6 the following

Lemma 10. *Let* $W^{(p)}$ *be as in Eq.* (15), $0 < \varepsilon < \frac{1}{4}$, $\varphi = \varphi^{(1,0)}$, $\operatorname{supp}\varphi \in E_R^{(1)}$, $\|\varphi\| = 1$. *Let*

$$\tilde{V}^c = (\tilde{V}^a)^* = \int b^*(k_0 - k_1)\, a^*(k_1)\, b(k_0)\, \omega^{-\frac{1}{2}}(k_0 - k_1)\, \mu^{-\frac{1}{2}}(k_1)\, \omega^{-\frac{1}{2}}(k_0)$$
$$\cdot\, \mu^{-\frac{1}{2} - \varepsilon}(k_1)\, dk_0\, dk_1 \,.$$

Then

$$\|\Gamma^{\frac{1}{2}+\varepsilon}(\underbrace{V^c\ W^{(p)}}_{1})\,\varphi\| \leqq (p+1)!^{\frac{1}{2}}\,(p+1)^{-\frac{1}{2}}\,\|\tilde{V}^c\|_s\,\|W^{(p)}\|_s\,C(R),$$

$$\|\underbrace{V^a\ \Gamma^{\frac{1}{2}+\varepsilon}}_{1}(W^{(p)})\,\varphi\| \leqq (p-1)!^{\frac{1}{2}}\,p^{-\frac{1}{2}}\,\|\tilde{V}^a\|_s\,\|W^{(p)}\|_s\,C(R),$$

$$\|\Gamma^{\varepsilon}(\underbrace{V^a\Gamma(V^c}_{0,1}\underbrace{\quad(W^{(p)}}_{2})))\,\varphi\| \leqq p!^{\frac{1}{2}}\,\|\tilde{V}^a\|_s\,p^{-1}\,\|\tilde{V}^c\|_s\,\|W^{(p)}\|_s\,C(R)^2,$$

$$\|\Gamma(\underbrace{V^a\Gamma(V^c}_{2}\underbrace{\ W^{(p)}}_{1}))\varphi - \Gamma(\underbrace{V^a\Gamma(V^c)}_{2}\underbrace{\ W^{(p)}}_{1})\,\varphi\| \leqq p!^{\frac{1}{2}}\,\|W^{(p)}\|_s\,C(R)^2\,K_1,$$

$$\|\underbrace{V^a\Gamma(\underbrace{V^a\Gamma(V^c}_{2}\Gamma(}_{2}\underbrace{V^c\ W^{(p)}}_{1}))) - \underbrace{V^a\Gamma(\underbrace{V^a\Gamma(V^c)}_{2}\Gamma(}_{2}\underbrace{V^c\ W^{(p)}}_{1}))\,\varphi\|$$

$$\leqq p!^{\frac{1}{2}}\,\|\tilde{V}^a\|_s\,\|\tilde{V}^c\|_s\,K_1\cdot p^{-1+\varepsilon}\,\|W^{(p)}\|_s\,C(R)^4, \qquad (K_1<\infty,\ C(R)<\infty).$$

It is now easy to prove

Lemma 11. *Let $\tilde{Q}$ be the sum of all skeletons of ΓQ, $\tilde{Q}=\sum_{i=1}^{\infty}\tilde{Q}^{(i)}$, where i denotes the number of meson creators. For every $0<\varepsilon<\frac{1}{4}$, there exists a $\lambda_1(R)>0$, and a $K_0<\infty$ such that*

$$\|\tilde{Q}^{(i)}\|_s \leqq K_0(i!)^{-1+\varepsilon}$$

for all $|\lambda|<\lambda_1(R)$.

Proof. We write $\tilde{Q}^{(i)}=\sum_{n=1}^{\infty}\tilde{Q}^{(i)}_n$ where n denotes the order. Let $C_1=\max(1,\|V^c\|_s)\cdot C(R)\cdot\max(K_1,1)\cdot|\lambda|$. We will prove by induction

$$\|\tilde{Q}^{(i)}_n\|_s \leqq 6^n C_1^n\,2^{\frac{1}{2}n}(i!)^{-1+\varepsilon}.$$

Using Lemma 10, we find immediately $\|\tilde{Q}^{(1)}_1\|_s \leqq 6C_1\,2^{\frac{1}{2}}$, $\|\tilde{Q}^{(0)}_n\|_s=0$. Then, taking into account the unused powers of Γ and the number of contractions, one gets

$$\|\tilde{Q}^{(i)}_n\|_s \leqq i^{-1+\varepsilon}C_1\,\|\tilde{Q}^{(i-1)}_{n-1}\|_s + (i+1)^{1+\varepsilon}i^{-1}C_1\,\|\tilde{Q}^{(i+1)}_{n-1}\|_s$$
$$+\,2i^{-1+\varepsilon}C_1^2\,\|\tilde{Q}^{(i)}_{n-2}\|_s + C_1^2\,\|\tilde{Q}^{(i)}_{n-2}\|_s + (i+1)\,i^{-2+2\varepsilon}C_1^4\,\|\tilde{Q}^{(i)}_{n-4}\|_s\,.$$

As an example, we give the derivation of the second term:

$$\|\Gamma V^a Q^{(i+1)}_{n-1}\|_s$$
$$\leqq (i^{-1}(i+1)^{\frac{1}{2}+\varepsilon}(i+1)^{-\frac{1}{2}}(i+1))\,(i+1)^{-1+\varepsilon}(i!)^{-1+\varepsilon}\,C_1\,\|Q^{(i+1)}_{n-1}\|_s$$

where the factors have the following origins: i^{-1} comes from the Γ, $(i+1)^{\frac{1}{2}+\varepsilon}$ comes from the unused Γ to the right (unused, because by construction we are not in one of the special situations), $(i+1)^{-\frac{1}{2}}$ comes from $\|V^a\Gamma\|_s$ and $(i+1)$ comes from the number of contractions. The remaining factors come from the induction assumption.

This completes the induction proof and by summing the geometric series one gets

$$\|\tilde{Q}^{(i)}\|_s \leqq (1-C_0)^{-1}(i!)^{-1+\varepsilon}, \quad \text{if} \quad C_0 = C_1 2^{\frac{1}{2}} 6 < 1,$$

and this proves Lemma 11.

Corollary. *Let G be a graph containing exactly one* M_{2k} *and let* $\tilde{G}$ *be the skeleton of G* (M_{2k} *is a sum of insertions!). Then* $\|G\|_s \leqq (C_1 2^{\frac{1}{2}})^{2k} \|\tilde{G}\|_s$.

Proof. The estimates of Lemma 11 hold for all the Γ's inside the M_{2k} because of the restriction $|k_0| \leqq R$ in the definition of M_{2k}. One can therefore apply Lemma 11 to all Γ's inside M_{2k}. The Γ to the extreme left of M_{2k} is well defined because M_{2k} is an insertion.

This allows now to treat the insertions in full generality:

Lemma 12. *Let Q be defined by Eq.* (16), $\Gamma Q = \sum_{i=1}^{\infty} Q^{(i)}$, *where i denotes the number of meson creators. For every* $0 < \varepsilon < \frac{1}{4}$ *there exists a* $\lambda_0(R) > 0$ *and a* $K < \infty$ *such that*

$$\|Q^{(i)}\|_s \leqq K(i!)^{-1+\varepsilon}, \quad \textit{for all} \quad |\lambda| \leqq \lambda_0(R).$$

Proof. Let

$$\|Q^{(i)}\|_s \leqq \sum_{n=1}^{\infty} \sum_{k=0}^{[n/2]} D_{i,n,k},$$

where $D_{i,n,k}$ is an upper bound for the s-norms coming from the sum of all the graphs of $Q^{(i)}$ with order n and with at least one insertion of depth k and no insertions of depth $k+1$. By Lemma 11, $D_{i,n,0} \leqq C_0^n (i!)^{-1+\varepsilon}$ for $i \geqq 1$, with $C_0 = 2^{\frac{1}{2}} C_1 6$. We define $D_{0,n,0} = C_0^n$ in view of the application of the corollary.

Now, for $k = 1, 2, 3, \ldots$

$$D_{i,n,k} = \sum_{\substack{j_0 + 2j_1 + \cdots + 2j_k = n \\ j_l \geqq 1,\, l = 0, \ldots k}} D_{i,j_0,0} D_{0,2j_1,0} \binom{j_0 \quad + j_1 - 1}{j_1}$$

$$\cdot \prod_{l=2}^{k} \binom{(2j_{l-1} - 1) + j_l - 1}{j_l} D_{0,2j_l,0}.$$

Here j_l, $(l \geqq 1)$ denotes $\frac{1}{2}$ the sum of the orders of the skeletons of the insertions of depth l. $\binom{(2j_{l-1}-1)+j_l-1}{j_l}$ is an upper bound for the number of ways to place these j_l pairs of vertices into a maximum of $2j_{l-1}-1$ "holes" which can be formed by the skeletons of order $2j_{l-1}$ and of depth $l-1$. $D_{0,2j_l,0}$ is an upper bound for the $\| \; \|_s$-norm coming from the sum of all the inserted skeletons of depth l and order $2j_l \cdot j_0$ is treated separately, but accordingly.

Writing $x_0 = j_0$, $x_l = 2j_l$, $l = 1, \dots k$, we get the estimate

$$D_{i,n,k} \leqq D_{i,n,0} \sum_{\substack{x_0+\cdots+x_k=n \\ x_l \geqq 1, l=0,\dots k}} \prod_{l=1}^{k} \binom{x_{l-1}+x_l-1}{x_l} = D_{i,n,0} B_{n,k}\,.$$

Here we have used

$$\binom{x_{l-1}+x_l-1}{x_l} = \binom{2j_{l-1}+2j_l-1}{2j_l} \geqq \binom{(2j_{l-1}-1)+j_l-1}{j_l}$$

for $l \geqq 2$ and a similar estimate in the case $l = 1$.

It is known that $B_n = \sum_{k=0}^{n} B_{n,k}$ denotes the sum over all plane one-rooted trees with $n+1$ vertices and endpoints (including the root). This can also be seen from the structure of the insertion mechanism. Using generating functions, one can show that

$$B_n = \binom{2n}{n} \Big/ (n+1) \qquad [10, \text{p. } 196\text{—}197]\,.$$

We find therefore that

$$D_{i,n} = \sum_{k=0}^{n} D_{i,n,k} \leqq D_{i,n,0} \cdot \binom{2n}{n} \Big/ (n+1) \leqq 4^n \cdot D_{i,n,0}\,.$$

Therefore $\|Q^{(i)}\|_s \leqq K \cdot (i!)^{-1+\varepsilon}$ if $|\lambda| < \lambda_1(R)/4$ and this proves Lemma 12.

We come now to the proof of Theorem 8.

Let $\varphi \in \mathscr{H}^{(n,m)}$.

Then for $T = \,:\!\exp \Gamma(Q)\!:$,

$$T\varphi = \sum_{j=0}^{n} \frac{1}{j!} :\Gamma(Q)^j: \underbrace{}_{j} \varphi\,.$$

Let $\Sigma'_{i,j}$ denote the sum over all partitions of i into a sum of j terms $k_1, \dots k_j \geqq 1$.

Then

$$(T\varphi)^{(n,i+m)} = \sum_{u=0}^{n} \frac{1}{u!} \Sigma'_{i,u} : \prod_{j=1}^{u} Q^{(k_j)} : \underbrace{}_{u} \varphi^{(n,m)}\,.$$

Using

$$\prod_{j=1}^{u} (k_j)! \geqq \text{const}(i!)/n^i; \quad \Sigma'_{i,u} 1 \leqq \binom{i+u-1}{i}; \quad \binom{a}{b} \leqq 2^a$$

we get, using Lemma 12,

$$\|(T\varphi)^{(n,i+m)}\|^2$$

$$\leqq \left[\sum_{u=0}^{n} \frac{1}{u!}\binom{n}{u} \Sigma'_{i,u} \prod_{j=1}^{u} (k_j!)^{-1+\varepsilon} K^u\right]^2 (m+i)!\, n!\, (m!\, n!)^{-1} \|\varphi\|^2$$

$$\leqq \left[\sum_{u=0}^{n} \binom{n}{u}\binom{i+u-1}{i} K^u\right]^2 \left(\frac{i!}{n^i}\right)^{-2+2\varepsilon} \text{const.} \frac{(m+i)!}{m!} \|\varphi\|^2$$

$$\leqq [(K+1)^n 2^{n+i-1} n^i]^2 (i!)^{-1+2\varepsilon} 2^{m+i} \text{const.} \|\varphi\|^2$$

$$\leqq C(n,m,\varepsilon)^i (i!)^{-1+2\varepsilon} \|\varphi\|^2$$

and therefore T is defined on $\varphi^{(n,m)}$ and hence on the dense domain formed by the vectors of $\mathscr{H}$ with a finite number of particles. This completes the proof of Theorem 8.

Theorem 8 is indeed the main ingredient for the construction of scattering states. For details the reader is referred to [6]. We state only the definitions and the result, adding some remarks concerning the proof where it differs from the proof for the Lee model given in [6].

We define the "physical" one-particle creation operators as

$$\hat{a}^*(p) = a(p) \quad \text{and} \quad \hat{b}^*(p) = h(p)^{-\frac{1}{2}} (\mathbb{1} + \underbrace{\Gamma(Q)}_{1}) b^*(p).$$

Here, $h(p)$ is a normalisation of T defined as follows: Let

$$A = (:\exp[\Gamma(Q)^* \Gamma(Q)]_M:)^{-\frac{1}{2}},$$

where $[\ ]_M$ denotes all the mass graphs of $\Gamma(Q)^* \Gamma(Q)$ or in other terms all the completely contracted graphs of $\Gamma(Q)^* \Gamma(Q)$. Now h is defined by

$$A\varphi^{(n,m)}(k_1, \dots k_n, l_1, \dots l_m) = \prod_{i=1}^{n} h(k_i)^{-\frac{1}{2}} \varphi^{(n,m)}(k_1, \dots k_n, l_1, \dots l_m).$$

A and h depend on λ and are both bounded and invertible. Also

$$(Tb^*(p)\varphi_0, Tb^*(q)\varphi_0) = h(p)\,\delta(p-q).$$

With these definitions we can formulate the

Theorem 13. *Let* $H = H_0 + \lambda V + S$. *On*

$$\mathscr{H}^{(n)}(R) = \{\varphi^{(n)} \in \mathscr{H}^{(n)}, \operatorname{supp}\varphi \in E_R^{(n)}$$

in the nucleon variables}

$$\Omega^{\pm} = \underset{t\to\pm\infty}{s\text{-lim}} \exp(iHt)\, TA \exp(-iH_0 t)$$

exists and satisfies

$$\exp(iHt)\,\Omega^{\pm} = \Omega^{\pm} \exp(iH_0 t)\,, \qquad (\Omega^{\pm})^* \,\Omega^{\pm} = \mathbb{1}\,.$$

Proof. The proof of Theorem 13 is essentially identical with the proof of Theorem 3.3 in [6]. The uniformity of the estimates in all orders of T (or Q) can be established by interchanging differentiation and summation which is in turn possible by our Lemma 10. For each term of order n, differentiation yields n terms and this additional factor is absorbed by the fact that $\sum_{n=0}^{\infty} n^k x^n = k!\left(\sum_{n=0}^{\infty} x^n\right)^{k+1}$, that is, it does not spoil the convergence of the geometric series in Lemma 11.

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References

1. Cannon, J. T.: Field theoretic properties of Nelson's model. Princeton University, Thesis 1968.
2. Eckmann, J.-P.: A theorem on kernels of superrenormalizable theories. Preprint.
3. — Hamiltonians of persistent interactions. Université de Genève, Thesis 1970.
4. Friedrichs, K. O.: Perturbation of spectra in Hilbert space. Providence: AMS, 1965.
5. Glimm, J.: Yukawa coupling of quantum fields in two dimensions I. Commun. Math. Phys. **5**, 343 (1967).
6. Hepp, K.: Théorie de la renormalisation; Lecture Notes in Physics Vol. 2. Berlin, Heidelberg, New York: Springer 1969.
7. Jaffe, A.: Constructive quantum field theory. Lecture Notes ETH Zürich (1968).
8. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1966.
9. Nelson, E.: Interaction of nonrelativistic particles with a quantized scalar field. J. Math. Phys. **5**, 1190 (1964).
10. Polya, G.: Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. Acta Math. **68**, 145 (1937).
11. Schrader, R.: On the existence of a local Hamiltonian in the Galilean invariant Lee model. Commun. Math. Phys. **10**, 155 (1968).

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