A modified approach to the Doplicher/Roberts theorem on the construction of the field algebra and the symmetry group in superselection theory

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Sfb 288 Preprint No. 134

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereiches 288 entstanden und als Manuskript vervielfältigt worden.

Berlin, August 1994

A modified approach to the Doplicher/Roberts theorem on the construction of the field algebra and the symmetry group in superselection theory

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July 22, 1994

Abstract

Let $\mathcal{A}$ be a unital C^*-algebra with trivial center $\mathcal{Z}(\mathcal{A}) = \mathbf{C}\,1$. Let $\mathcal{T}$ denote a tensorial category of unital endomorphisms of $\mathcal{A}$ equipped with several properties to be explained in the text.

Doplicher and Roberts have shown, among other things, that there is a C^*-algebra $\mathcal{F} \supset \mathcal{A}$ and a compact group $\mathcal{G}$ of automorphisms of $\mathcal{F}$ such that $\mathcal{F}$ is a Hilbert C^*-system over $\mathcal{A}$ w.r.t. $\mathcal{G}$ where $\mathcal{A}$ is the fixed point algebra w.r.t. $\mathcal{G}$, $\mathcal{G} = \operatorname{stab}\mathcal{A}$ and the objects $\rho \in \operatorname{Ob}\mathcal{T}$ are characterized as the canonical endomorphisms of certain algebraic $\mathcal{G}$ -invariant Hilbert spaces $\mathcal{H}_\rho \subset \mathcal{F}$, see Doplicher/Roberts [1, 2, 3].

The starting point of the approach presented in this paper to point out the mentioned result is an $\mathcal{A}$-leftmodule $\mathcal{F}_0 := \{\sum_{\rho,j} A_{\rho,j}\Phi_{\rho,j}\}$. ρ runs through a full system of irreducible and mutually disjoint objects of $\mathcal{T}$, $j = 1, 2, ..., d(\rho)$, where $d(\rho)$ denotes the statistical dimension of ρ, $\{\Phi_{\rho j}\}_{j=1}^{d(\rho)}$ is an orthonormal basis of a $d(\rho)$-dimensional Hilbert space. The system $\{\Phi_{\rho j}\}_{\rho j}$ forms a leftmodule basis of $\mathcal{F}_0$, the coefficients $A_{\rho j}$ are members of $\mathcal{A}$. The strategy is to equip successively $\mathcal{F}_0$ with a bimodule structure, a product and a $*$-structure and finally with a C^*-norm $\|.\|_*$. The symmetry group $\mathcal{G}$ appears as the group of all automorphisms of the $*$-algebra $\mathcal{F}_0$ leaving the $\mathcal{A}$-scalar product $\langle F, G\rangle := \sum_{\rho,j} d(\rho)^{-1} A_{\rho j} B_{\rho j}^*$ invariant, where $F = \sum_{\rho,j} A_{\rho j}\Phi_{\rho j}$, $G = \sum_{\rho,j} B_{\rho j}\Phi_{\rho j}$. The field algebra is then given by $\mathcal{F} := \operatorname{clo}_{\|.\|_*}\mathcal{F}_0$.

1 Introduction

A few years ago Doplicher and Roberts established a new duality theory for compact groups which is a deepening of the well-known duality theory of Tannaka/Krein. They developed this theory in connection with the superselection problem of the algebraic quantum field theory (see Doplicher/Roberts [4, 5, 1, 2, 3]). In the algebraic quantum field theory the main part of the superselection problem is to construct an extension $\mathcal{F} \supset \mathcal{A}$ of the quasilocal observable algebra $\mathcal{A}$, called field algebra, which should describe the sector structure of $\mathcal{A}$ and the transition rules by a symmetry principle (e.g. a compact group $\mathcal{G}$) on $\mathcal{F}$ such that, for example, the fusion rules are explained in group-theoretical terms of $\mathcal{G}$.

In a recent paper [6] (see also Baumgärtel/Wollenberg [7]) the author described a rather simple construction of the field algebra $\mathcal{F}$ as a Hilbert C^*-system over $\mathcal{A}$ (or a crossed product of $\mathcal{A}$ with a group dual of $\mathcal{G}$), if the group $\mathcal{G}$ is known and if an action of a group dual of $\mathcal{G}$ on $\mathcal{A}$ is given a priori.

The present paper is an attempt to use the method of [6] for a construction of $\mathcal{F}$ and of the group $\mathcal{G}$ (which appears as the invariance group of an $\mathcal{A}$-scalar product). An essential point of this ansatz is to construct Hilbert spaces $\mathcal{H}_\alpha$ in the $\mathcal{A}$-bimodule $\mathcal{F}_0$ to start with, where the Hilbert spaces $\mathcal{H}_\alpha$ correspond bijectively to the objects α of the category $\mathcal{T}$ of endomorphisms of $\mathcal{A}$.

This approach shows directly the intrinsic structure of $\mathcal{F}$ and $\mathcal{G}$. On the other hand, it is not (completely) independent of Doplicher/Roberts deep duality theory. At a crucial point in the treatment (introduction of the product structure in $\mathcal{F}_0$) the author used an important result of Doplicher/Roberts ([2,4.17 Theorem, p.191]) to proceed further.

2 Preliminaries

Let $\mathcal{A}$ be a unital C^*-algebra with center $\mathcal{Z}(\mathcal{A}) = \mathbf{C}1$. By $\rho, \sigma, \ldots$ we denote unital endomorphisms of $\mathcal{A}$. We use the denotation

$$(\rho, \sigma) := \{T \in \mathcal{A} : T\rho(A) = \sigma(A)T,\ A \in \mathcal{A}\}$$

for the linear space of all intertwiners between ρ and σ. The identical endomorphism is denoted by ι. If $R \in (\rho, \rho'), S \in (\sigma, \sigma')$ then $R\rho(S) = \rho'(S)R \in (\rho\sigma, \rho'\sigma')$. We use the notation $R \times S := R\rho(S)$. Note that $(\iota, \iota) = \mathbf{C}\,1$ and $1 \in (\rho, \rho)$ for all ρ, this is indicated by $1 = 1_\rho$.

We consider categories $\mathcal{T}$ of unital endomorphisms equipped with the following properties:

(T1) $\iota \in \mathrm{Ob}\,\mathcal{T}$.

(T2) If $\rho \in \mathrm{Ob}\,\mathcal{T}$ and $U \in (\rho, \sigma)$, U unitary, then $\sigma \in \mathrm{Ob}\mathcal{T}$.

(T3) $\rho, \sigma \in \mathrm{Ob}\mathcal{T}$ implies $\rho\sigma \in \mathrm{Ob}\mathcal{T}$.

(T4) The set of arrows between ρ and σ is given by (ρ, σ).

(T5) If $\rho \in \mathrm{Ob}\mathcal{T}$ and if $0 \neq E \in (\rho, \rho)$ is a selfadjoint projection then there is an isometry $V \in \mathcal{A}$, $VV^* = E$ such that $\sigma \in \mathrm{Ob}\mathcal{T}$ where $\sigma(A) := V^*\rho(A)V$, i.e. $\mathrm{Ob}\,\mathcal{T}$ is closed w.r.t. subobjects.

(T6) If $\rho, \sigma \in \mathrm{Ob}\,\mathcal{T}$ then there are isometries $V, W \in \mathcal{A}$ with $VV^* + WW^* = 1$ such that $\tau \in \mathrm{Ob}\,\mathcal{T}$ where $\tau(A) := V\rho(A)V^* + W\sigma(A)W^*$, i.e. $\mathrm{Ob}\,\mathcal{T}$ is closed w.r.t. direct sums.

(T7) There is a *permutation structure* on $\mathcal{T}$, i.e. there is an assignment $\{\rho, \sigma\} \to \epsilon(\rho, \sigma) \in (\rho\sigma, \sigma\rho)$, $\epsilon(\rho, \sigma)$ unitary, with the following properties:

$$\epsilon(\sigma, \rho)\epsilon(\rho, \sigma) = 1_{\rho\sigma},$$

$$\epsilon(\iota,\rho) = \epsilon(\rho,\iota) = 1_\rho,$$
$$\epsilon(\rho\sigma,\tau) = \epsilon(\rho,\tau) \times 1_\sigma \cdot 1_\rho \times \epsilon(\sigma,\tau),$$
$$\epsilon(\rho',\sigma') \cdot R \times S = S \times R \cdot \epsilon(\rho,\sigma),\ R \in (\rho,\rho'),\ S \in (\sigma,\sigma').$$

(T8) There is a *conjugation structure* on $\mathcal{T}$, i.e. there is an assignment $\mathrm{Ob}\,\mathcal{T} \ni \rho \to \overline{\rho} \in \mathrm{Ob}\,\mathcal{T}$ such that there are corresponding arrows $R_\rho \in (\iota, \overline{\rho}\rho)$, $S_\rho \in (\iota, \rho\overline{\rho})$ with the following properties:

$$S_\rho := \epsilon(\overline{\rho},\rho) R_\rho,$$
$$S_\rho^* \times 1_\rho \cdot 1_\rho \times R_\rho = 1_\rho,$$
$$R_\rho^* \times 1_{\overline{\rho}} \cdot 1_{\overline{\rho}} \times S_\rho = 1_{\overline{\rho}}.$$

The endomorphism ρ is called *irreducible* if $(\rho,\rho) = \mathbf{C}\,1$; ρ and σ are called *disjoint* if $(\rho,\sigma) = \{0\}$, they are called *unitarily equivalent* if there is a unitary $U \in (\rho,\sigma)$.

The theory of $\mathcal{T}$ is well-known, it is developed in Doplicher/Roberts [2,p.164 f.]. We recall several properties: $\overline{\rho}$ is unique up to unitary equivalence, the statistical dimension $d(\rho) \in \mathbf{N}$ given by $d(\rho)1 = R_\rho^* R_\rho$ satisfies $d(\rho_1\rho_2) = d(\rho_1)d(\rho_2)$, $d(W_1\rho_1(\cdot)W_1^* + W_2\rho_2(\cdot)W_2^*) = d(\rho_1) + d(\rho_2)$. If ρ_1, ρ_2 are unitarily equivalent then $d(\rho_1) = d(\rho_2)$, furthermore $d(\rho) = d(\overline{\rho})$. If $0 \neq E \in (\rho,\rho)$ is a selfadjoint projection then $d(V^*\rho(\cdot)V) = R_\rho^* \overline{\rho}(E) R_\rho$, where $VV^* = E$.

Each ρ can be decomposed into a finite direct sum of irreducibles, $\rho(\cdot) = \sum_{j=1}^r W_j\rho_j(\cdot)W_j^*$, $W_j^* W_k = \delta_{jk}$, $\sum_j W_j W_j^* = 1$, ρ_j irreducible; (ρ,ρ) is finite-dimensional. If τ is irreducible, ρ arbitrary then (τ,ρ) is a finite-dimensional algebraic Hilbert space in $\mathcal{A}$, for $A, B \in (\tau,\rho)$ the scalar product (A,B) is given by $(A,B)1 = A^*B$; $\dim(\tau,\rho) =: m(\tau,\rho)$ coincides with the multiplicity of the equivalence class of τ occurring in the decomposition of ρ, i.e.

$$d(\rho) = \sum_\tau m(\tau,\rho) d(\tau).$$

For later use recall that $\rho \in \mathcal{T}$ is called *special* if $\det\rho = \iota$ (see [2,p.174] or [7,p.271]).

3 The $\mathcal{A}$-bimodule $\mathcal{F}_0$ and the Hilbert spaces $\mathcal{H}_\rho$

We choose a full system $\hat{\mathcal{T}}$ of irreducible and mutually disjoint endomorphisms $\tau \in \mathrm{Ob}\,\mathcal{T}$. To each τ we assign a $d(\tau)$-dimensional Hilbert space $\mathcal{H}_\tau$, $\mathcal{H}_\iota := \mathbf{C}\,1$. We choose an orthonormal basis $\Phi_{\tau j}$, $j = 1,2,\ldots,d(\tau)$, of $\mathcal{H}_\tau$, $\Phi_\iota := 1$, and we introduce the $\mathcal{A}$-leftmodule

$$\mathcal{F}_0 := \left\{ \sum_{\tau,j} A_{\tau j}\Phi_{\tau j},\ A_{\tau j} \in \mathcal{A},\ \text{finite sum} \right\}$$

where $\{\Phi_{\tau j}\}_{\tau j}$ is assumed to be a leftmodule basis (then each other system $\{\Psi_{\tau j}\}_{\tau j}$ of orthonormal bases of $\mathcal{H}_\tau$ turns out to be a leftmodule basis). Note that $\mathcal{A} \subset \mathcal{F}_0$. $\mathcal{F}_0$ is a bimodule w.r.t. the definition

$$\Phi A = \tau(A)\Phi, \quad \Phi \in \mathcal{H}_\tau,\ A \in \mathcal{A},$$

and extension by linearity to the whole $\mathcal{F}_0$. Then the relative commutant $\mathcal{A}' \cap \mathcal{F}_0 := \{F \in \mathcal{F}_0 : AF = FA,\ A \in \mathcal{A}\}$ equals $\mathbf{C}\,1$.

Let $\rho \in \mathcal{T}$ and $\tau \in \hat{\mathcal{T}}$. We consider the linear subspace spa $\{(\tau,\rho)\mathcal{H}_\tau\} \subset \mathcal{F}_0$. Obviously, it is a Hilbert space which can be considered as the tensor product $(\tau,\rho)\otimes \mathcal{H}_\tau$ with the scalar product

$$(A\Phi, B\Psi) := (A,B)_{(\tau,\rho)}(\Phi,\Psi)_{\mathcal{H}_\tau}, \quad A,B \in (\tau,\rho);\ \Phi,\Psi \in \mathcal{H}_\tau.$$

The sum

$$\sum_\tau \mathrm{spa}\{(\tau,\rho)\mathcal{H}_\tau\} =: \mathcal{H}_\rho$$

is always a direct sum. Therefore $\mathcal{H}_\rho$ is a Hilbert space with the scalar product

$$\sum_\tau (X_\tau, Y_\tau), \quad X_\tau, Y_\tau \in \mathrm{spa}\{(\tau,\rho)\mathcal{H}_\tau\}.$$

The Hilbert spaces $\mathcal{H}_\tau$ satisfy the following properties:

3.1.PROPOSITION. *The assignment*

$$\mathrm{Ob}\,\mathcal{T} \ni \rho \to \mathcal{H}_\rho \subset \mathcal{F}_0$$

is one-to-one, $X \in \mathcal{H}_\rho$ *iff* $XA = \rho(A)X$ *for all* $A \in \mathcal{A}$ *and the dimension of* $\mathcal{H}_\rho$ *coincides with* $d(\rho)$,

$$\dim \mathcal{H}_\rho = d(\rho), \quad \rho \in \mathcal{T}. \tag{1}$$

Proof. Let $X \in \mathcal{H}_\rho$. Then $X = \sum_\tau A_\tau \Phi_\tau$ with $A_\tau \in (\tau,\rho)$, $\Phi_\tau \in \mathcal{H}_\tau$. Therefore $XA = \sum_\tau A_\tau\Phi_\tau A = \sum_\tau A_\tau \tau(A)\Phi_\tau = \rho(A)\sum_\tau A_\tau\Phi_\tau = \rho(A)X$. Conversely, let $X \in \mathcal{F}_0$, i.e. $X = \sum_{\tau,j} A_{\tau j}\Phi_{\tau j}$, $A_{\tau j} \in \mathcal{A}$ and $XA = \rho(A)X$. This means

$$\sum_{\tau,j} A_{\tau j}\tau(A)\Phi_{\tau j} = \sum_{\tau,j} \rho(A)A_{\tau j}\Phi_{\tau j}, \quad A \in \mathcal{A},$$

or

$$A_{\tau j}\tau(A) = \rho(A)A_{\tau j}, \quad A \in \mathcal{A},$$

i.e $A_{\tau j} \in (\tau,\rho)$.

If $\mathcal{H}_\rho = \mathcal{H}_\sigma =: \mathcal{H}$ then $\rho(A)X = \sigma(A)X$ follows for all $A \in \mathcal{A}$, $X \in \mathcal{H}$, in particular for $X \in \mathcal{H}_\tau$ where $(\tau,\rho) \neq \{0\}$, i.e. $\rho(A)\Phi_{\tau j} = \sigma(A)\Phi_{\tau j}$ or $\rho(A) = \sigma(A)$ for all $A \in \mathcal{A}$, i.e. $\rho = \sigma$. Finally, we have

$$\dim \mathcal{H}_\rho = \sum_\tau \dim(\mathrm{spa}\{(\tau,\rho)\mathcal{H}_\tau\}) = \sum_\tau \dim(\tau,\rho)\dim \mathcal{H}_\tau =$$

$$= \sum_\tau m(\tau,\rho)d(\tau) = d(\rho). \;\square$$

The arrows from $(\rho.\sigma)$ can be interpreted as linear operators from $\mathcal{L}(\mathcal{H}_\rho \to \mathcal{H}_\sigma)$ by the assignment

$$(\rho,\sigma) \ni A \to \phi(A) \in \mathcal{L}(\mathcal{H}_\rho \to \mathcal{H}_\sigma) : \phi(A)X := AX, \quad X \in \mathcal{H}_\rho. \tag{2}$$

Note that the assignment (2) is injective and that $\phi(A^*)$ is the Hilbert space adjoint of $\phi(A)$:

3.2.PROPOSITION. *Let $A_1, A_2 \in (\rho, \sigma)$. If $\phi(A_1) = \phi(A_2)$ then $A_1 = A_2$. Furthermore,*

$$\phi(A^*) = \phi(A)^*, \quad A \in (\rho, \sigma).$$

Proof. According to the assumption we have $A_1 X = A_2 X$ for all $X \in \mathcal{H}_\rho$. This implies $A_1 Y = A_2 Y$ for all $Y \in (\tau, \rho)$, $\tau \in \hat{\mathcal{T}}$, i.e. $(A_1 - A_2)Y = 0$. Choose an orthonormal basis $W_{\tau j} \in (\tau, \rho)$, $j = 1, 2, ..., m(\tau, \rho)$. Then $\sum_j W_{\tau j} W_{\tau j}^* = E_\tau \in (\rho, \rho)$, where E_τ is the central projection assigned to τ. Recall that $\sum_\tau E_\tau = 1$. So first we obtain $(A_1 - A_2)E_\tau = 0$ for all τ occurring in the decomposition of ρ and second $A_1 - A_2 = 0$. Furthermore, with $A \in (\rho, \sigma)$, $A_1 \in (\tau, \sigma)$, $A_2 \in (\tau, \rho)$ we have $(A_1, AA_2)_{(\tau,\sigma)} = A_1^* A A_2 = (A^* A_1)^* A_2 = (A^* A_1, A_2)_{(\tau,\rho)}$. □

4 Construction of special isometries (cocycles) $J(\rho, \sigma) : \mathcal{H}_\rho \otimes \mathcal{H}_\sigma \rightarrow \mathcal{H}_{\rho\sigma}$ and the product structure in $\mathcal{F}_0$

From (1) and $d(\rho\sigma) = d(\rho)d(\sigma)$ we get immediately $\dim \mathcal{H}_\rho \otimes \mathcal{H}_\sigma = \dim \mathcal{H}_{\rho\sigma}$, i.e. the Hilbert spaces $\mathcal{H}_\rho \otimes \mathcal{H}_\sigma$ and $\mathcal{H}_{\rho\sigma}$ are isomorphic. Within the set of all isometries from $\mathcal{H}_\rho \otimes \mathcal{H}_\sigma$ onto $\mathcal{H}_{\rho\sigma}$ we construct special ones (cocycles) $J(\rho, \sigma)$, equipped with the following properties:

(C1) $J(\rho, \iota) = J(\iota, \rho) = \mathrm{id}_{\mathcal{H}_\rho}$,

(C2) $J(\rho\sigma, \tau)(J(\rho, \sigma) \otimes 1_{\mathcal{H}_\tau}) = J(\rho, \sigma\tau)(1_{\mathcal{H}_\rho} \otimes J(\sigma, \tau))$,

(C3) $J(\rho', \sigma')\phi(A) \otimes \phi(B) = \phi(A \times B)J(\rho, \sigma)$, $\quad A \in (\rho, \rho')$, $B \in (\sigma, \sigma')$,

(C4) $\phi(\epsilon(\rho, \sigma))J(\rho, \sigma) = J(\sigma, \rho)\Theta(\mathcal{H}_\rho, \mathcal{H}_\sigma)$,

where $\Theta(\mathcal{H}_\rho, \mathcal{H}_\sigma)$ denotes the flip isomorphism $\mathcal{H}_\rho \otimes \mathcal{H}_\sigma \rightarrow \mathcal{H}_\sigma \otimes \mathcal{H}_\rho$, and where $\epsilon(\cdot, \cdot)$ are the unitaries of the permutation structure in $\mathcal{T}$.

Note that one can introduce an equivalence relation within the set of all isometries $J(\rho, \sigma)$ satisfying (C1)-(C3) by

$$J' \sim J \quad \text{if} \quad J'(\rho, \sigma) = V(\rho\sigma)^{-1} J(\rho, \sigma) V(\rho) \otimes V(\sigma)$$

where $\rho \rightarrow V(\rho) \in \mathcal{L}(\mathcal{H}_\rho)$ is a unitary with the property

$$\phi(A)V(\rho) = V(\sigma)\phi(A), \quad A \in (\rho, \sigma).$$

It turns out that $J(\sigma, \rho)\Theta(\mathcal{H}_\rho, \mathcal{H}_\sigma)J(\rho, \sigma)^{-1}$ is an invariant of the equivalence classes and this invariant satisfies the property (T7) of the permutation structure.

We focus on the construction of $J(\rho, \sigma)$ if $\epsilon(\cdot, \cdot)$ is given. We separate three parts of the construction problem:

First we choose a special $\rho \in \mathcal{T}$, we consider the subcategory consisting of all powers ρ^n, $n = 0, 1, 2, ...$ and we construct $J(\rho^n, \rho^m)$, $n, m = 0, 1, 2, ...$ At this first stage we use an important result of Doplicher/Roberts [2,4.17 Theorem, p.191].

Second we construct corresponding isometries $J(\alpha,\beta)$ for all $\alpha,\beta \in \mathcal{T}$ which are *dominated* by a special (fixed) ρ, i.e. α,β are assumed to be subobjects of direct sums of powers of ρ.

Third, using the second step, we construct a *unique* function $\mathcal{T}\times\mathcal{T} \ni \{\alpha,\beta\} \to J(\alpha,\beta) \in \mathcal{L}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta \to \mathcal{H}_{\alpha\beta})$ satisfying (C1)-(C4).

The first step: Let $\rho \in \mathcal{T}$ be special. We look for auxiliary isometries Φ_n from $\mathcal{H}_{\rho^n}$ onto $\mathcal{H}_\rho^n$, equipped with several properties, where $\mathcal{H}_\rho^n$ denotes the n-fold tensor product of $\mathcal{H}_\rho$.

4.1.PROPOSITION. *Let $\rho \in \mathcal{T}$ be special. Then there is a system $\{\Phi_r\}_{r=1}^\infty$ of isometries $\Phi_r : \mathcal{H}_{\rho^r} \to \mathcal{H}_\rho^r$ such that the isomorphisms $\beta_r^s : \mathcal{L}(\mathcal{H}_{\rho^r} \to \mathcal{H}_{\rho^s}) \to \mathcal{L}(\mathcal{H}_\rho^r \to \mathcal{H}_\rho^s)$ given by*

$$\beta_r^s(X) := \Phi_s \circ X \circ \Phi_r^{-1}, \quad X \in \mathcal{L}(\mathcal{H}_{\rho^r} \to \mathcal{H}_{\rho^s}), \tag{3}$$

satisfies

$$(\beta_{r+s}^{r'+s'} \circ \phi)(A \times B) = (\beta_r^{r'} \circ \phi)(A) \otimes (\beta_s^{s'} \circ \phi)(B) \tag{4}$$

for all $A \in (\rho^r, \rho^{r'})$, $B \in (\rho^s, \rho^{s'})$. Moreover,

$$(\beta_n^n \circ \phi)(\epsilon_\rho^{(n)}(\pi)) = \Theta_\rho^{(n)}(\pi), \quad \pi \in \mathbf{P}_n,\ n = 1,2,... \tag{5}$$

holds, i.e.

$$\Phi_n \circ \phi(\epsilon_\rho^{(n)}(\pi)) = \Theta_\rho^{(n)}(\pi) \circ \Phi_n, \quad \pi \in \mathbf{P}_n.$$

Proof. According to Doplicher/Roberts [2,4.17 Theorem, p.191] there is a closed subgroup $\mathcal{G}$ of SU($\mathcal{H}_\rho$) and a unital monomorphism α of a graded C^*-algebra $C^*(\rho)$, constructed by the intertwiner spaces (ρ^r, ρ^s), which maps $C^*(\rho)$ into the Cuntz algebra $\mathcal{C}_{\mathcal{H}_\rho}$ generated by $\mathcal{H}_\rho$. One has $\alpha(C^*(\rho)) = \mathcal{C}_\mathcal{G}$ where $\mathcal{C}_\mathcal{G}$ denotes the fixed point algebra of $\mathcal{C}_{\mathcal{H}_\rho}$ under the canonical action of $\mathcal{G}$ and

$$\alpha|(\rho^n, \rho^m) = \mathcal{L}_\mathcal{G}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m).$$

Note that $\mathcal{L}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m)$ is interpreted, as usual, as a subspace of $C_{\mathcal{H}_\rho}$ and $\mathcal{L}_\mathcal{G}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m)$ denotes the intersection of $\mathcal{C}_\mathcal{G}$ and $\mathcal{L}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m)$, i.e. the set of all intertwiners w.r.t. the (tensorial) action of $\mathcal{G}$ on $\mathcal{H}_\rho^n$ resp. $\mathcal{H}_\rho^m$. Note further that α satisfies the relations

$$\alpha(A \times B) = \alpha(A) \otimes \alpha(B), \quad A \in (\rho^r, \rho^{r'}),\ B \in (\rho^s, \rho^{s'}), \tag{6}$$

and

$$\alpha(\epsilon_\rho(\pi)) = \Theta_{\mathcal{H}_\rho}(\pi), \quad \pi \in \mathbf{P}_\infty. \tag{7}$$

Now we form the Fock spaces

$$\mathcal{F} := \oplus_{n=0}^\infty \mathcal{H}_{\rho^n},\ \mathcal{H}_\rho^0 := \mathbf{C};\ \mathcal{F}_1 := \oplus_{n=0}^\infty \mathcal{H}_\rho^n,\ \mathcal{H}_\rho^0 := \mathbf{C},\ \text{(Hilbert sums)}.$$

By E_n we denote the projection of $\mathcal{F}$ onto its component $\mathcal{H}_{\rho^n}$; correspondingly F_n is the projection of $\mathcal{F}_1$ onto its component $\mathcal{H}_\rho^n$.

We consider the v.Neumann algebras $\mathcal{M}, \mathcal{M}_1$ on $\mathcal{F}$ resp. $\mathcal{F}_1$, generated by $\phi((\rho^n, \rho^m)) \subset \mathcal{L}(\mathcal{H}_{\rho^n} \to \mathcal{H}_{\rho^m})$ resp. by $\mathcal{L}_\mathcal{G}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m)$, i.e.

$$\mathcal{M} := \left(\bigcup_{n,m=0}^\infty E_m \phi((\rho^n, \rho^m)) E_n \right)'',$$

$$\mathcal{M}_1 := \left(\bigcup_{n,m=0}^{\infty} F_m \mathcal{L}_{\mathcal{G}}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m) F_n \right)''.$$

Then the Doplicher/Roberts-isomorphism α induces an isomorphism β between the algebras $\mathcal{M}$ and $\mathcal{M}_1$, $\beta : \mathcal{M} \to \mathcal{M}_1$, where

$$\beta(E_m \phi(A) E_n) = F_m \alpha(A) F_n, \quad A \in (\rho^n, \rho^m),$$

(see for example Kadison/Ringrose [9, p.427]). It has the property

$$\beta(E_m X E_n) = F_m \beta(X) F_n, \quad X \in \mathcal{M}.$$

Obviously, $\mathcal{M}_1 = U(\mathcal{G})'$ where $U(\mathcal{G})$ denotes the canonical representation of $\mathcal{G}$ on $\mathcal{F}_1$ given by the identical representation on $\mathcal{H}_\rho$: Note that $\mathcal{F}_1 \ni X = \sum_{n,m=0}^{\infty} F_m X F_n$ (strong convergence), hence $X \in U(\mathcal{G})'$ implies $F_m X F_n U(g) = U(g) F_m X F_n$ because $F_n U(g) = U(g) F_n$, for all $g \in \mathcal{G}$. That is, we have $F_m X F_n | \mathcal{H}_\rho^n \in \mathcal{L}_{\mathcal{G}}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m)$ or $F_m X F_n = F_m Y F_n$ with $Y \in \mathcal{L}_{\mathcal{G}}(\mathcal{H}_\rho^n \to \mathcal{H}_\rho^m)$. The other inclusion is trivial.

Thus $\mathcal{M}_1$ is of type I, therefore, as is well-known (see e.g. [9, p.660ff.]) the isomorphism β is spatial, i.e. there is an isometry $\Phi : \mathcal{F} \to \mathcal{F}_1$ such that $\beta(X) = \Phi X \Phi^{-1}$, $X \in \mathcal{M}$. Now we have $\beta(E_n) = F_n \beta(E_n) F_n$, i.e. $F_n \beta(E_n) = \beta(E_n) F_n = \beta(E_n)$ or

$$F_n \cdot \Phi E_n \Phi^{-1} = \Phi E_n \Phi^{-1} \cdot F_n = \Phi E_n \Phi^{-1}.$$

Since $\dim \Phi E_n \Phi^{-1} = \dim E_n = \dim F_n$ we obtain $F_n = \Phi E_n \Phi^{-1}$ or $\Phi E_n = F_n \Phi$. This means that Φ is diagonal,

$$\Phi = \operatorname{diag}\{\Phi_n\}_{n=0}^{\infty}$$

where $\Phi_n : \mathcal{H}_{\rho^n} \to \mathcal{H}_\rho^n$, $n = 0, 1, 2, \ldots$, is isometric.

Therefore, the isomorphisms

$$\beta_n^m := \Phi_m \phi(A) \Phi_n^{-1}, \quad A \in (\rho^n, \rho^m)$$

satisfy the equation

$$\beta_n^m(\phi(A)) = \alpha(A), \quad A \in (\rho^n, \rho^m),$$

Now from (7) we obtain (5). Furthermore, from (6) we obtain (4): Namely, $A \in (\rho^r, \rho^{r'})$, $B \in (\rho^s, \rho^{s'})$ implies $A \times B \in (\rho^{r+s}, \rho^{r'+s'})$, i.e. we have $\phi(A) \in \mathcal{L}(\mathcal{H}_{\rho^r} \to \mathcal{H}_{\rho^{r'}})$, $\phi(B) \in \mathcal{L}(\mathcal{H}_{\rho^s} \to \mathcal{H}_{\rho^{s'}}$, $\phi(A \times B) \in \mathcal{L}(\mathcal{H}_{\rho^{r+s}} \to \mathcal{H}_{\rho^{r'+s'}})$, i.e.

$$\alpha(A \times B) = \beta_{r+s}^{r'+s'}(\phi(A \times B))$$

and

$$\alpha(A) \otimes \alpha(B) = \beta_r^{r'}(\phi(A)) \otimes \beta_s^{s'}(\phi(B)). \quad \square$$

It would be nice to have a proof for the existence of the β_n^m which is independent of the mentioned theorem of Doplicher/Roberts.

Using the isometries Φ_n, we define isometries from $\mathcal{H}_{\rho^r} \otimes \mathcal{H}_{\rho^s}$ onto $\mathcal{H}_{\rho^{r+s}}$ by

$$J(\rho^r, \rho^s) := \Phi_{r+s}(\Phi_r \otimes \Phi_s)^{-1}.$$

Then we obtain

4.2.PROPOSITION. *The isometries* $J(\rho^r, \rho^s)$ *satisfy* (C1)-(C4).

Proof. (C1) is obvious. (C2): We have to prove

$$J(\rho^{r+n}, \rho^s)(J(\rho^r, \rho^n) \otimes 1_{\mathcal{H}_{\rho^s}}) = J(\rho^r, \rho^{n+s})(1_{\mathcal{H}_{\rho^r}} \otimes J(\rho^n, \rho^s)).$$

By the definition of $J(\cdot, \cdot)$ we have

$$\Phi_{r+n+s} = J(\rho^{r+n}, \rho^s)\Phi_{r+n} \otimes \Phi_s = J(\rho^r, \rho^{n+s})(\Phi_r \otimes \Phi_{n+s}) \tag{8}$$

and

$$J(\rho^{r+n}, \rho^s)\Phi_{r+n} \otimes \Phi_s = J(\rho^{r+n}, \rho^s)(\{J(\rho^r, \rho^n)(\Phi_r \otimes \Phi_n)\} \otimes \Phi_s) =$$
$$= J(\rho^{r+n}, \rho^s)(J(\rho^r, \rho^n) \otimes 1_{\mathcal{H}_{\rho^s}})(\Phi_r \otimes \Phi_n \otimes \Phi_s).$$

For the right hand side of (8) we get similarly

$$J(\rho^r, \rho^{n+s})(\Phi_r \otimes \{J(\rho^n, \rho^s)(\Phi_n \otimes \Phi_s)\}) = J(\rho^r, \rho^{n+s})(1_{\mathcal{H}_{\rho^r}} \otimes J(\rho^n, \rho^s))(\Phi_r \otimes \Phi_n \otimes \Phi_s).$$

(C3): For brevity we write simply β instead of $\beta_r^{r'}$ etc. For $A \in (\rho^r, \rho^{r'})$, $B \in (\rho^s, \rho^{s'})$ we have

$$J(\rho^{r'}, \rho^{s'})\phi(A) \otimes \phi(B) = \Phi_{r'+s'}\{(\Phi_{r'}^{-1}\phi(A)) \otimes (\Phi_{s'}^{-1}\phi(B))\} =$$

$$= \Phi_{r'+s'}\{(\beta \circ \phi)(A)\Phi_r^{-1} \otimes (\beta \circ \phi)(B)\Phi_s^{-1}\} = \Phi_{r'+s'}((\beta \circ \phi)(A) \otimes (\beta \circ \phi)(B))\Phi_r^{-1} \otimes \Phi_s^{-1} =$$
$$= \Phi_{r'+s'}((\beta \circ \phi)(A \times B)\Phi_r^{-1} \otimes \Phi_s^{-1}) = \phi(A \times B)\Phi_{r+s}(\Phi_r^{-1} \otimes \Phi_s^{-1}) = \phi(A \times B)J(\rho^r, \rho^s).$$

Finally we prove (C4): We calculate

$$J(\rho^s, \rho^r)\Theta(\mathcal{H}_{\rho^r}, \mathcal{H}_{\rho^s}) = \Phi_{r+s}(\Phi_s^{-1} \otimes \Phi_r^{-1})\Theta(\mathcal{H}_{\rho^r}, \mathcal{H}_{\rho^s}) = \Phi_{r+s}\Theta(\mathcal{H}_\rho^r, \mathcal{H}_\rho^s)(\Phi_r^{-1} \otimes \Phi_s^{-1}) =$$

$$= \Phi_{r+s}\Theta_{\mathcal{H}_\rho}^{(r+s)}((r,s))(\Phi_r^{-1} \otimes \Phi_s^{-1}) = \phi(\epsilon_\rho^{(r+s)}((r,s)))\Phi_{r+s}(\Phi_r^{-1} \otimes \Phi_s^{-1}) =$$
$$= \phi(\epsilon(\rho^r, \rho^s))J(\rho^r, \rho^s). \square$$

The second step: We have to construct $J(\alpha, \beta)$ for direct sums α, β of powers of ρ (where ρ is special) and then we have to verify the properties (C1)-(C4). The same procedure is to establish for subobjects of such direct sums. We begin with direct sums: According to their definition we have the following formulas for direct sums α, β, γ, where the isometries W_j, V_k, U_l have the properties mentioned above:

$$\alpha(\cdot) := \sum_j W_j \rho^{m_j}(\cdot) W_j^*, \ \beta(\cdot) := \sum_k V_k \rho^{n_k}(\cdot) V_k^*, \ \gamma(\cdot) := \sum_l U_l \rho^{r_l}(\cdot) U_l^*,$$

$$W_j \in (\rho^{m_j}, \alpha), \quad V_k \in (\rho^{n_k}, \beta), \quad U_l \in (\rho^{r_l}, \gamma),$$
$$\mathcal{H}_\alpha = \sum_j W_j \mathcal{H}_{\rho^{m_j}}, \quad \mathcal{H}_\beta = \sum_k V_k \mathcal{H}_{\rho^{n_k}}, \quad \mathcal{H}_\gamma = \sum_l U_l \mathcal{H}_{\rho^{r_l}}.$$

We put

$$J(\alpha, \beta) := \sum_{j,k} \phi(W_j \times V_k) J(\rho^{m_j}, \rho^{n_k}) \phi(W_j^*) \otimes \phi(V_k^*), \tag{9}$$

similarly for the other objects $J(\beta, \gamma)$, $J(\alpha\beta, \gamma)$ etc. Then we assert

4.3.LEMMA. *The isometries* (9) *satisfy the properties* (C1)-(C4).

Proof. (C1) is obvious. (C3): We have

$$J(\alpha',\beta') = \sum_{j',k'} \phi(\tilde{W}_{j'} \times \tilde{V}_{k'}) J(\rho^{\tilde{m}_{j'}}, \rho^{\tilde{n}_{k'}}) \phi(\tilde{W}^*_{j'}) \otimes \phi(\tilde{V}^*_{k'})$$

where $\tilde{W}_{j'} \in (\rho^{\tilde{m}_{j'}}, \alpha')$, $\tilde{V}_{k'} \in (\rho^{\tilde{n}_{k'}}, \beta')$. Let $A \in (\alpha, \alpha')$, $B \in (\beta, \beta')$. Then

$$\tilde{W}^*_{j'} A W_j \in (\rho^{m_j}, \rho^{\tilde{m}_{j'}}), \quad \tilde{V}^*_{k'} B V_k \in (\rho^{n_k}, \rho^{\tilde{n}_{k'}}),$$

and we have, according to (C3) for the powers of ρ

$$J(\rho^{\tilde{m}_{j'}}, \rho^{\tilde{n}_{k'}}) \phi(\tilde{W}^*_{j'} A W_j) \otimes \phi(\tilde{V}^*_{k'} B V_k) = \phi(\tilde{W}^*_{j'} A W_j \times \tilde{V}^*_{k'} B V_k) J(\rho^{m_j}, \rho^{n_k}).$$

Now multiplying this equation from the left by $\phi(\tilde{W}_{j'} \times \tilde{V}_{k'})$, from the right by $\phi(W^*_j) \otimes \phi(V^*_k)$ and summing up w.r.t. j', k', j, k, we obtain

$$\sum_{j',k'} \phi(\tilde{W}_{j'} \times \tilde{V}_{k'}) J(\rho^{\tilde{m}_{j'}}, \rho^{\tilde{n}_{k'}}) \phi(\tilde{W}^*_{j'}) \otimes \phi(\tilde{V}^*_{k'}) \cdot \phi(A) \otimes \phi(B) =$$

$$= \phi(A \times B) \sum_{j,k} \phi(W_j \times V_k) J(\rho^{m_j}, \rho^{n_k}) \phi(W^*_j) \otimes \phi(V^*_k)$$

or

$$J(\alpha', \beta') \phi(A) \otimes \phi(B) = \phi(A \times B) J(\alpha, \beta).$$

(C2): Note that

$$\alpha\beta(\cdot) = \sum_{j,k} W_j \times V_k \, \rho^{m_j + n_k}(\cdot)(W_j \times V_k)^*.$$

Then

$$J(\alpha\beta, \gamma) = \sum_{j,k,l} \phi((W_j \times V_k) \times U_l) J(\rho^{m_j+n_k}, \rho^{r_l}) \phi((W_j \times V_k)^*) \otimes \phi(U^*_l)$$

and we have to calculate

$$J(\alpha\beta, \gamma)(J(\alpha, \beta) \otimes 1_{\mathcal{H}_\gamma}) =$$

$$\sum_{j,k,l} \phi(W_j \times V_k \times U_l) J(\rho^{m_j+n_k}, \rho^{r_l}) \phi(W_j \times V_k)^* \otimes \phi(U_l)^* \cdot$$

$$\cdot \sum_{j',k'} \phi(W_{j'} \times V_{k'}) J(\rho^{m_{j'}}, \rho^{n_{k'}})(\phi(W_{j'})^* \otimes \phi(V_{k'})^*) \otimes 1_{\mathcal{H}_\gamma}$$

$$= \sum_{j,k,l} \sum_{j',k'} \phi(W_j \times V_k \times U_l) J(\rho^{m_j+n_k}, \rho^{r_l}) \cdot$$

$$\cdot (\phi((W_j \times V_k)^* W_{j'} \times V_{k'}) J(\rho^{m_{j'}}, \rho^{n_{k'}})(\phi(W_{j'})^* \otimes \phi(V_{k'})^*) \otimes \phi(U_l)^*)$$

$$= \sum_{j,k,l} \phi(W_j \times V_k \times U_l) J(\rho^{m_j+n_k}, \rho^{r_l})(J(\rho^{m_j}, \rho^{n_k}) \otimes 1_{\mathcal{H}_{\rho^{r_l}}})(\phi(W_j)^* \otimes \phi(V_k)^* \otimes \phi(U_l)^*).$$

On the other hand,

$$J(\alpha,\beta\gamma)(1_{\mathcal{H}_\alpha}\otimes J(\beta,\gamma)) =$$

$$\sum_{j,k,l}\phi(W_j\times V_k\times U_l)J(\rho^{m_j},\rho^{n_k+r_l})(1_{\mathcal{H}_{\rho^{m_j}}}\otimes J(\rho^{n_k},\rho^{r_l}))(\phi(W_j)^*\otimes\phi(V_k)^*\otimes\phi(U_l)^*).$$

Hence the validity of (C2) for powers of ρ implies its validity for direct sums. (C4): We assert that

$$J(\beta,\alpha)\Theta(\mathcal{H}_\alpha,\mathcal{H}_\beta) = \phi(\epsilon(\alpha,\beta))J(\alpha,\beta). \tag{10}$$

We calculate the left hand side: it equals

$$\sum_{k,j}\phi(V_k\times W_j)J(\rho^{n_k},\rho^{m_j})\phi(V_k)^*\otimes\phi(W_j)^*\Theta(\mathcal{H}_\alpha,\mathcal{H}_\beta) =$$

$$=\sum_{k,j}\phi(V_k\times W_j)J(\rho^{n_k},\rho^{m_j})\Theta(\mathcal{H}_{\rho^{m_j}},\mathcal{H}_{\rho^{n_k}})\phi(W_j)^*\otimes\phi(V_k)^*$$

$$=\sum_{k,j}\phi(V_k\times W_j)\phi(\epsilon(\rho^{m_j},\rho^{n_k}))J(\rho^{m_j},\rho^{n_k})\phi(W_j)^*\otimes\phi(V_k)^*$$

$$=\sum_{k,j}\phi(\epsilon(\alpha,\beta))\phi(W_j\times V_k)J(\rho^{m_j},\rho^{n_k})\phi(W_j)^*\otimes\phi(V_k)^* = \phi(\epsilon(\alpha,\beta))J(\alpha,\beta),$$

i.e. (10) is proved.□

Now we turn to the case of subobjects of direct sums. Let α,β be direct sums and let α',β' denote subobjects of α,β, correspondingly. Then we have

$$\alpha'(\cdot) = W^*\alpha(\cdot)W,\ \beta'(\cdot) = V^*\beta(\cdot)V,\ W\in(\alpha',\alpha),\ V\in(\beta',\beta),\ W^*W = V^*V = 1.$$

Now we put

$$J(\alpha',\beta') := \phi(W\times V)^*J(\alpha,\beta)\phi(W)\otimes\phi(V). \tag{11}$$

Note that

$$\alpha'\beta'(\cdot) = W^*\times V^*(\alpha\beta)(\cdot)W\times V.$$

Then we have

4.4.LEMMA. *The isometries* (11) *satisfy the properties* (C1)-(C4).

Proof. (C1) is obvious. (C3): Let α_1', β_1' be subobjects of α_1, β_1, respectively:

$$\alpha_1'(\cdot) = W_1^*\alpha_1(\cdot)W_1,\quad \beta_1'(\cdot) = V_1^*\beta_1(\cdot)V_1,\quad W_1\in(\alpha_1',\alpha_1),\quad V_1\in(\beta_1',\beta_1),$$

$$W_1^*W_1 = V_1^*V_1 = 1.$$

Choose $A\in(\alpha',\alpha_1')$, $B\in(\beta',\beta_1')$. Then

$$W_1AW^*\in(\alpha,\alpha_1),\ V_1BV^*\in(\beta,\beta_1)$$

and we have, according to Lemma 4.3.,

$$J(\alpha_1,\beta_1)\phi(W_1AW^*)\otimes\phi(V_1BV^*) = \phi(W_1AW^*\times V_1BV^*)J(\alpha,\beta).$$

Multiplying this equation from the left by $\phi(W_1^* \times V_1^*)$, and from the right by $\phi(W \otimes V)$ we obtain

$$\phi(W_1^* \times V_1^*)J(\alpha_1, \beta_1)\phi(W_1) \otimes \phi(V_1) \cdot \phi(A) \otimes \phi(B) =$$
$$= \phi(A \times B)\phi(W^* \times V^*)J(\alpha, \beta)\phi(W) \otimes \phi(V)$$

or

$$J(\alpha_1', \beta_1')\phi(A) \otimes \phi(B) = \phi(A \times B)J(\alpha, \beta).$$

(C2): Let α', β' be as before and let γ' be a subobject of γ, i.e. $\gamma'(\cdot) = U^*\gamma(\cdot)U$, $U \in (\gamma', \gamma)$, $U^*U = 1$. We calculate

$$J(\alpha'\beta', \gamma')(J(\alpha', \beta') \otimes 1_{\mathcal{H}_{\gamma'}}) =$$

$$\phi((W \times V) \times U)^* J(\alpha\beta, \gamma)(\phi(W \times V) \otimes \phi(U))(\phi(W \times V)^* J(\alpha, \beta)(\phi(W) \otimes \phi(V)) \otimes 1_{\mathcal{H}_{\gamma'}})$$
$$= \phi(W \times V \times U)^* J(\alpha\beta, \gamma)(\phi((W \times V)(W \times V)^*)J(\alpha, \beta)(\phi(W) \otimes \phi(V)) \otimes \phi(U)) =$$
$$= \phi(W \times V \times U)^* J(\alpha\beta, \gamma)(\phi(WW^* \times VV^*)J(\alpha, \beta)(\phi(W) \otimes \phi(V)) \otimes \phi(U))$$
$$= \phi(W \times V \times U)^* J(\alpha\beta, \gamma)(J(\alpha, \beta) \otimes 1_{\mathcal{H}_\gamma})\phi(W) \otimes \phi(V) \otimes \phi(U)$$
$$= \phi(W \times V \times U)^* J(\alpha, \beta\gamma)(1_{\mathcal{H}_\alpha} \otimes J(\beta, \gamma))\phi(W) \otimes \phi(V) \otimes \phi(U)$$

because of Lemma 4.3. and finally this equals $J(\alpha', \beta'\gamma')(1_{\mathcal{H}_{\alpha'}} \otimes J(\beta', \gamma'))$.

(C4): We have to prove

$$J(\beta', \alpha')\Theta(\mathcal{H}_{\alpha'}, \mathcal{H}_{\beta'}) = \phi(\epsilon(\alpha', \beta'))J(\alpha', \beta').$$

So we calculate the left hand side, it equals

$$\phi(V \times W)^* J(\beta, \alpha)\phi(V) \otimes \phi(W)\Theta(\mathcal{H}_{\alpha'}, \mathcal{H}_{\beta'}) = \phi(V \times W)^* J(\beta, \alpha)\Theta(\mathcal{H}_\alpha, \mathcal{H}_\beta)\phi(W) \otimes \phi(V)$$
$$= \phi(V \times W)^* \phi(\epsilon(\alpha, \beta))J(\alpha, \beta)\phi(W) \otimes \phi(V) =$$
$$= \phi(\epsilon(\alpha', \beta'))\phi(W^* \times V^*)J(\alpha, \beta)\phi(W) \otimes \phi(V) = \phi(\epsilon(\alpha', \beta'))J(\alpha', \beta'). \square$$

The third step consists in the construction of a unique function

$$\{\alpha, \beta\} \to J(\alpha, \beta), \quad \alpha, \beta \in \mathcal{T},$$

such that $J(\alpha, \beta)$ satisfies all the properties (C1)-(C4). First let α, β be fixed. We consider the system of *all* isometries Φ from $\mathcal{H}_\alpha \otimes \mathcal{H}_\beta$ onto $\mathcal{H}_{\alpha\beta}$. It is denoted by $M(\alpha, \beta)$. Obviously, $M(\alpha, \beta)$ is a compact space (essentially the unitary group of all $d(\alpha)d(\beta) \times d(\alpha)d(\beta)$-matrices). Then the cartesian product

$$M := \prod_{\alpha, \beta \in \mathcal{T}} M(\alpha, \beta)$$

is a compact space w.r.t the Tychonov topology.

Let $\mathcal{T}_{fin} := \{\alpha_1, \alpha_2, ..., \alpha_r\}$ be a finite set of objects. Let $M(\mathcal{T}_{fin}) \subset M$ denote the subset consisting of all functions $J(\rho, \sigma)$ such that (C1)-(C4) is satisfied if $\rho, \sigma \in \mathcal{T}_{fin}$. The foregoing considerations (the first and the second step) show that $M(\mathcal{T}_{fin}) \neq \emptyset$ for each $\mathcal{T}_{fin}$, because the elements of $\mathcal{T}_{fin}$ are dominated by some special object ρ.

Note that the intersection $\bigcap_{\mathcal{T}_{fin}\in\mathcal{C}} M(\mathcal{T}_{fin})$ for a finite collection $\mathcal{C} = \{\mathcal{T}_{fin}\}$ is also nonempty because

$$\bigcap_{\mathcal{T}_{fin}\in\mathcal{C}} M(\mathcal{T}_{fin}) \supseteq M\left(\bigcup_{\mathcal{T}_{fin}\in\mathcal{C}} \mathcal{T}_{fin}\right)$$

and $\bigcup_{\mathcal{T}_{fin}\in\mathcal{C}} \mathcal{T}_{fin}$ is again a finite subset of objects. Therefore we obtain

4.5.LEMMA. *There is a function $J(\alpha,\beta)$ satisfying all the properties* (C1)-(C4).

Proof. It is obvious because, according to Cantor's theorem,

$$\bigcap_{\mathcal{T}_{fin}} M(\mathcal{T}_{fin}) \neq \emptyset$$

where now $\mathcal{T}_{fin}$ runs through *all* finite subsets of objects. A member $J(\cdot,\cdot) \in \cap M(\mathcal{T}_{fin})$ satisfies the properties (C1)-(C4) for all $\alpha, \beta \in \mathcal{T}$. □

We note that the argument using Cantor's theorem is already used by Doplicher/Roberts in the course of their treatment.

Now the goal of the second part of this section is the introduction of a product structure in $\mathcal{F}_0$, using the special isometries $J(\alpha,\beta) : \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \to \mathcal{H}_{\alpha\beta}$ constructed before.

Let $\Phi \in \mathcal{H}_\rho$, $\Psi \in \mathcal{H}_\sigma$, where ρ, σ are irreducible. Then we define

$$\Phi \cdot \Psi := J(\rho,\sigma)(\Phi \otimes \Psi). \tag{12}$$

The extension of this definition to arbitrary elements of $\mathcal{F}_0$ is given by the distributive law. A straightforward calculation shows then that (12) is also true for elements of arbitrary Hilbert spaces $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$ (see e.g. [6,p.365]).

4.6.PROPOSITION. *The product in $\mathcal{F}_0$, defined by* (12) *and by its extension by the distributive law, is associative.*

Proof. Obvious by the definition, because the tensor product is associative and (C2) holds.□

We remark that corresponding formulas already appear in the treatment of Doplicher/Roberts: the formulas (SM1-SM3) and (6.2) in [2,p.303].

5 The $*$-structure in the algebra $\mathcal{F}_0$

The introduction of a $*$-structure in $\mathcal{F}_0$ consists of two steps. In the first one we introduce the concept of a *conjugated basis* in the Hilbert space $\mathcal{H}_{\overline{\rho}}$ where $\overline{\rho}$ denotes the representer of the *conjugated class* w.r.t. ρ.

Let $R_\rho \in (\iota, \overline{\rho}\rho)$ be a conjugate belonging to the pair $\rho, \overline{\rho}$. Then $R_\rho \in \mathcal{H}_{\overline{\rho}\rho}$ holds and therefore

$$R_\rho = J(\overline{\rho},\rho)u, \quad u \in \mathcal{H}_{\overline{\rho}} \otimes \mathcal{H}_\rho.$$

We choose an orthonormal basis $\{\Phi_k\}_{k=1}^d$ in $\mathcal{H}_\rho$ and an orthonormal basis $\{\Psi_j\}_{j=1}^d$ in $\mathcal{H}_{\overline{\rho}}$, where $d = d(\rho) = d(\overline{\rho})$. Then

$$u = \sum_{j,k} u_{jk}\Psi_j \otimes \Phi_k$$

where $\{u_{jk}\}$ is a $d \times d$-matrix with complex entries. Now we have

5.1.LEMMA. *The $d \times d$-matrix $\{u_{jk}\}$ is unitary.*

Proof. We use the relation

$$R_\rho^* \times 1_{\overline{\rho}} \cdot 1_{\overline{\rho}} \times S_\rho = 1_{\overline{\rho}}$$

where $S_\rho = \epsilon(\overline{\rho}, \rho) R_\rho$. We have $S_\rho = J(\rho, \overline{\rho}) v$ with $v \in \mathcal{H}_\rho \otimes \mathcal{H}_{\overline{\rho}}$ and

$$J(\rho, \overline{\rho}) \Theta(\mathcal{H}_{\overline{\rho}}, \mathcal{H}_\rho) u = \phi(\epsilon(\overline{\rho}, \rho)) J(\overline{\rho}, \rho) u = \epsilon(\overline{\rho}, \rho) R_\rho = S_\rho,$$

therefore $v = \Theta(\mathcal{H}_{\overline{\rho}}, \mathcal{H}_\rho) u$. We have $\phi(1_{\overline{\rho}} \times S_\rho) \in \mathcal{L}(\mathcal{H}_{\overline{\rho}} \to \mathcal{H}_{\overline{\rho}\rho\overline{\rho}})$ hence $\phi(1_{\overline{\rho}} \times S_\rho) x_{\overline{\rho}} = 1_{\overline{\rho}} \times S_\rho \cdot x_{\overline{\rho}}$, $x_{\overline{\rho}} \in \mathcal{H}_{\overline{\rho}}$. Furthermore,

$$\phi(1_{\overline{\rho}} \times S_\rho) 1_{\overline{\rho}} = \phi(1_{\overline{\rho}} \times S_\rho) J(\overline{\rho}, \iota) = J(\overline{\rho}, \rho\overline{\rho}) \phi(1_{\overline{\rho}} \otimes \phi(S_\rho)$$

hence

$$\phi(1_{\overline{\rho}} \times S_\rho) x_{\overline{\rho}} = J(\overline{\rho}, \rho\overline{\rho}) x_{\overline{\rho}} \otimes S_\rho = J(\overline{\rho}, \rho\overline{\rho})(x_{\overline{\rho}} \otimes J(\rho, \overline{\rho}) v)$$
$$= J(\overline{\rho}, \rho\overline{\rho})(1_{\overline{\rho}} \otimes J(\rho, \overline{\rho})) x_{\overline{\rho}} \otimes v = J(\overline{\rho}\rho, \overline{\rho})(J(\overline{\rho}, \rho) \otimes 1_{\overline{\rho}}) x_{\overline{\rho}} \otimes v.$$

Now there is a representation $v = \sum_{j,k} v_{jk} \Phi_j \otimes \Psi_k$ and replacing v by this expression we obtain

$$\phi(1_{\overline{\rho}} \times S_\rho) x_{\overline{\rho}} = J(\overline{\rho}\rho, \overline{\rho})(J(\overline{\rho}, \rho) \otimes 1_{\overline{\rho}}) x_{\overline{\rho}} \otimes \sum_{j,k} v_{jk} \Phi_j \otimes \Psi_k$$

$$= J(\overline{\rho}\rho, \overline{\rho})(J(\overline{\rho}, \rho) \otimes 1_{\overline{\rho}}) \sum_{j,k} v_{jk} x_{\overline{\rho}} \otimes \Phi_j \otimes \Psi_k$$

$$= J(\overline{\rho}\rho, \overline{\rho}) \sum_{j,k} v_{jk} (J(\overline{\rho}, \rho) x_{\overline{\rho}} \otimes \Phi_j) \otimes \Psi_k$$

and

$$\phi(R_\rho^* \times 1_{\overline{\rho}}) \phi(1_{\overline{\rho}} \times S_\rho) x_{\overline{\rho}} =$$

$$\phi(R_\rho^* \times 1_{\overline{\rho}}) J(\overline{\rho}\rho, \overline{\rho}) \sum_{j,k} v_{jk} (J(\overline{\rho}, \rho) x_{\overline{\rho}} \otimes \Phi_j) \otimes \Psi_k$$

$$= J(\iota, \overline{\rho}) \phi(R_\rho^*) \otimes \phi(1_{\overline{\rho}}) \sum_{j,k} v_{jk} (J(\overline{\rho}, \rho) x_{\overline{\rho}} \otimes \Phi_j) \otimes \Psi_k$$

$$= \phi(R_\rho^*) \otimes \phi(1_{\overline{\rho}}) \sum_{j,k} v_{jk} (J(\overline{\rho}, \rho) x_{\overline{\rho}} \otimes \Phi_j) \otimes \Psi_k.$$

Now for $x \in \mathcal{H}_{\overline{\rho}\rho}$, $y \in \mathcal{H}_{\overline{\rho}}$ we have

$$\phi(R_\rho^*) \otimes \phi(1_{\overline{\rho}})(x \otimes y) = \phi(R_\rho^*) x \otimes \phi(1_{\overline{\rho}}) y = (R_\rho, x)_{\mathcal{H}_{\overline{\rho}\rho}} 1 \otimes y = (R_\rho, x)_{\mathcal{H}_{\overline{\rho}\rho}} y$$

because $\phi(R_\rho^*) x = R_\rho^* \cdot x = (R_\rho, x)_{\mathcal{H}_{\overline{\rho}\rho}} 1$. In particular, we put $x := J(\overline{\rho}, \rho)(x_{\overline{\rho}} \otimes \Phi_j)$, $y := \Psi_k$. Then we obtain

$$\phi(R_\rho^*) \otimes \phi(1_{\overline{\rho}})(J(\overline{\rho}, \rho)(x_{\overline{\rho}} \otimes \Phi_j)) \otimes \Psi_k = (R_\rho, J(\overline{\rho}, \rho)(x_{\overline{\rho}} \otimes \Phi_j))_{\mathcal{H}_{\overline{\rho}\rho}} \Psi_k =$$

$$= (J(\overline{\rho},\rho)u, J(\overline{\rho},\rho)(x_{\overline{\rho}}\otimes \phi_j))_{\mathcal{H}_{\overline{\rho}\rho}}\Psi_k = (u, x_{\overline{\rho}}\otimes \Phi_j)_{\mathcal{H}_{\overline{\rho}}\otimes\mathcal{H}_\rho}\Psi_k =$$

$$= \left(\sum_{j',k'} v_{j'k'}\Psi_{k'}\otimes\Phi_{j'}, x_{\overline{\rho}}\otimes\Phi_j\right)_{\mathcal{H}_{\overline{\rho}}\otimes\mathcal{H}_\rho}\Psi_k = \sum_{j',k'}\overline{v}_{j'k'}(\Psi_{k'},x_{\overline{\rho}})_{\mathcal{H}_{\overline{\rho}}}(\Phi_{j'},\Phi_j)_{\mathcal{H}_\rho}\Psi_k =$$

$$= \sum_{k'}\overline{v}_{jk'}(\Psi_{k'},x_{\overline{\rho}})_{\mathcal{H}_{\overline{\rho}}}$$

hence

$$\phi(R_\rho^*\times 1_{\overline{\rho}})\phi(1_{\overline{\rho}}\times S_\rho)x_{\overline{\rho}} =$$

$$\sum_{j,k} v_{jk}\phi(R_\rho^*)\otimes\phi(1_{\overline{\rho}})(J(\overline{\rho},\rho)x_{\overline{\rho}}\otimes\Phi_j)\otimes\Psi_k = \sum_{j,k} v_{jk}\left(\sum_{k'}\overline{v}_{jk'}(\Psi_{k'},x_{\overline{\rho}})_{\mathcal{H}_{\overline{\rho}}}\right)\Psi_k =$$

$$\sum_k\left(\sum_{j,k'} v_{jk}\overline{v}_{jk'}(\Psi_{k'},x_{\overline{\rho}})_{\mathcal{H}_{\overline{\rho}}}\right)\Psi_k = x_{\overline{\rho}} = \sum_k(\Psi_k,x_{\overline{\rho}})_{\mathcal{H}_{\overline{\rho}}}\Psi_k.$$

This implies

$$\sum_{j,k'} v_{jk}\overline{v}_{jk'}(\Psi_{k'},x_{\overline{\rho}})_{\mathcal{H}_{\overline{\rho}}} = (\Psi_k,x_{\overline{\rho}})_{\mathcal{H}_{\overline{\rho}}},\quad x_{\overline{\rho}}\in\mathcal{H}_{\overline{\rho}},$$

i.e.

$$\sum_{j,k'} v_{jk}\overline{v}_{jk'}\Psi_{k'} = \Psi_k,\quad k=1,2,...,d,$$

or

$$\sum_j v_{jk}\overline{v}_{jk'} = \delta_{kk'},$$

i.e. the matrix $\{v_{jk}\}$ is unitary, hence $\{u_{jk}\}$ is also unitary.□

Using the second conjugation relation

$$S_\rho^*\times 1_\rho\cdot 1_\rho\times R_\rho = 1_\rho,$$

one obtains the same result, i.e. the unitarity of the matrix $\{u_{jk}\}$ of the vector $u\in\mathcal{H}_{\overline{\rho}}\otimes\mathcal{H}_\rho$ w.r.t. arbitrary chosen orthonormal bases. However, in this case the unitarity relation reads

$$\sum_k v_{jk}\overline{v}_{j'k} = \delta_{jj'},$$

i.e. the difference is only that between the orthogonality of rows and columns.

Using the unitary matrix $\{u_{jk}\}$ one can define a transformed orthonormal basis in $\mathcal{H}_{\overline{\rho}}$ by the formula

$$\Psi'_j := \sum_{j'} u_{j'j}\Psi_{j'}.$$

Obviously, $\{\Psi'_j\}_{j=1}^d$ is an orthonormal basis in $\mathcal{H}_{\overline{\rho}}$. However, w.r.t. this basis the formula for the vector $u\in\mathcal{H}_{\overline{\rho}}\otimes\mathcal{H}_\rho$ reads now

$$u = \sum_j \Psi'_j\otimes\Phi_j.$$

We define the concept *conjugated basis* by this relation.

5.2.DEFINITION. Let $\{\Phi_j\}_{j=1}^d$ be an orthonormal basis of the Hilbert space $\mathcal{H}_\rho$. An orthonormal basis $\{\Psi_k\}_{k=1}^d$ of $\mathcal{H}_{\overline{\rho}}$ is said to be *conjugated* w.r.t. $\{\Phi_j\}_{j=1}^d$ if

$$u = \sum_{j=1}^{d} \Psi_j \otimes \Phi_j, \quad R_\rho = J(\overline{\rho}, \rho)u.$$

Then the foregoing considerations based on Lemma 5.1. can be formulated as follows:

5.3.LEMMA. *Let $\rho \in \mathcal{T}$ and let $\overline{\rho} \in \mathcal{T}$ be conjugated w.r.t. ρ. Let $\{\Phi_{\rho j}\}_{j=1}^{d(\rho)}$ be an orthonormal basis in $\mathcal{H}_\rho$. Then there exists a conjugated basis $\{\Psi_{\overline{\rho} j}\}_{j=1}^{d(\rho)}$ in $\mathcal{H}_{\overline{\rho}}$, i.e.*

$$R_\rho = J(\overline{\rho}, \rho) \left(\sum_{j=1}^{d(\rho)} \Psi_{\overline{\rho} j} \otimes \Phi_{\rho j} \right).$$

In general a conjugated basis is not unique. However, note that if ρ is irreducible then R_ρ hence u, where $R_\rho = J(\overline{\rho}, \rho)u$, are unique up to a phase factor λ, $|\lambda| = 1$. Therefore we obtain

5.4.LEMMA. *If ρ is irreducible then the conjugated basis is unique up to a phase factor λ, $|\lambda| = 1$.*

Proof. Let ρ be irreducible. Then R_ρ is unique up to a factor λ, $|\lambda| = 1$ (see Doplicher/Roberts [2,Lemma 2.2.,p.165]). Then u is also unique up to this factor. Let Φ_j be an orthonormal basis of $\mathcal{H}_\rho$ and let Ψ_k, Ψ'_k be conjugated bases. Then we have $\sum_j \Psi'_j \otimes \Phi_j = \lambda \sum_j \Psi j \otimes \Phi_j$, $|\lambda| = 1$ hence $\sum_j (\Psi'_j - \lambda \Psi_j) \otimes \Phi_j = 0$ and it follows that $\Psi'_j = \lambda \Psi_j$. □

Now we proceed to the second step, the definition of the $*$-structure in $\mathcal{F}_0$.

5.5.DEFINITION. Let $\rho \in \hat{\mathcal{T}}$ be irreducible and let $\Phi_{\rho j}$, $j = 1, ..., d$ be an orthonormal basis in $\mathcal{H}_\rho$. The conjugated basis in $\mathcal{H}_{\overline{\rho}}$ is denoted by $\Phi_{\overline{\rho} j}$, $j = 1, ..., d$. Then $\Phi^*_{\rho j}$ is defined by

$$\Phi^*_{\rho j} := R^*_\rho \Phi_{\overline{\rho} j}, \quad j = 1, ..., d.$$

Note that this definition makes sense. Namely, if $\Psi_{\overline{\rho} j}$ is a second conjugated basis then $\Psi_{\overline{\rho} j} = \lambda \Phi_{\overline{\rho} j}$, $|\lambda| = 1$, and $\sum_j \Psi_{\overline{\rho} j} \otimes \Phi_{\rho j} = \lambda \sum_j \Phi_{\overline{\rho} j} \otimes \Phi_{\rho j}$ hence $J(\overline{\rho}, \rho) \sum_j \Psi_{\overline{\rho} j} \otimes \Phi_{\rho j} = \lambda J(\overline{\rho}, \rho) \sum_j \Phi_{\overline{\rho} j} \otimes \Phi_{\rho j}$ or $\tilde{R}_\rho = \lambda R_\rho$ and $\tilde{R}^*_\rho \Psi_{\overline{\rho} j} = \overline{\lambda} R^*_\rho \lambda \Phi_{\overline{\rho} j} = |\lambda|^2 R^*_\rho \Phi_{\overline{\rho} j} = R^*_\rho \Phi_{\overline{\rho} j}$.

Additionally we define

$$(A \Phi_{\rho j})^* := \Phi^*_{\rho j} A^*$$

and extend the definition by linearity to the whole algebra $\mathcal{F}_0$. We have

5.6.LEMMA. *The relations*

$$(\Phi^*_{\rho j})^* = \Phi_{\rho j}, \tag{13}$$

$$(\Phi_{\rho j} \cdot \Phi_{\rho' j'})^* = \Phi^*_{\rho' j'} \cdot \Phi^*_{\rho j}, \tag{14}$$

hold.

Proof. First we prove (13): We have

$$(\Phi^*_{\rho j})^* = (R^*_\rho \Phi_{\overline{\rho} j})^* = \Phi^*_{\overline{\rho} j} R_\rho = R^*_{\overline{\rho}} \Phi_{\rho j} R_\rho =$$

$$R^*_{\overline{\rho}} \rho(R_\rho) \Phi_{\rho j} = R^*_{\overline{\rho}} \times 1_\rho \cdot 1_\rho \times R_\rho \cdot \Phi_{\rho j} = \Phi_{\rho j}$$

because of the conjugation relation (T8) for R_ρ, where $R_{\overline{\rho}} := S_\rho$.

Second we prove (14): Here we have

$$\Phi_{\rho j} \cdot \Phi_{\rho' j'} = \sum_k W_k \psi_k, \quad \psi_k := \sum_j \lambda_j \Phi_{\tau_k j} \in \mathcal{H}_{\tau_k}, \quad W_k \in (\tau_k, \rho\rho')$$

where the $\tau_k \in \hat{\mathcal{T}}$ are irreducible. Therefore

$$(\Phi_{\rho j} \cdot \Phi_{\rho' j'})^* = \sum_k \psi_k^* W_k^* = \sum_k R^*_{\tau_k} \overline{\psi}_k W_k^* = \sum_k R^*_{\tau_k} \overline{\tau}_k (W_k^*) \overline{\psi}_k, \tag{15}$$

where $\overline{\psi}_k := \sum_j \overline{\lambda}_j \Phi_{\overline{\tau_k} j}$. On the other hand we get

$$\Phi^*_{\rho' j'} \Phi^*_{\rho j} = R^*_{\rho'} \Phi_{\overline{\rho'} j'} R^*_\rho \Phi_{\overline{\rho} j} = R^*_{\rho'} \overline{\rho'}(R^*_\rho) \Phi_{\overline{\rho'} j'} \Phi_{\overline{\rho} j}$$

and we have

$$\Phi_{\overline{\rho'} j'} \Phi_{\overline{\rho} j} = \sum_k \tilde{W}_k \overline{\psi}_k$$

where $\tilde{W}_k$ is the conjugated arrow w.r.t. W_k, $\tilde{W}_k \in (\overline{\tau_k}, \overline{\rho'}\overline{\rho})$, see [2,p.167] or [6,p.257]. So we obtain

$$\Phi^*_{\rho' j'} \Phi^*_{\rho j} = \sum_k R^*_{\rho'} \overline{\rho'}(R^*_\rho) \tilde{W}_k \overline{\psi}_k. \tag{16}$$

Comparing (15) and (16) we see that it is sufficient to prove

$$R^*_\tau \overline{\tau}(W^*) = R^*_{\rho'} \overline{\rho'}(R^*_\rho) \tilde{W},$$

where τ is irreducible, $W \in (\tau, \rho\rho')$, and $\tilde{W} \in (\overline{\tau}, \overline{\rho'}\overline{\rho})$ is the conjugated arrow. Note that

$$R^*_{\rho'} \overline{\rho'}(R^*_\rho) = R^*_{\rho\rho'}.$$

This means that it is sufficient to prove

$$R^*_\tau \overline{\tau}(W^*) = R^*_\sigma \tilde{W}, \tag{17}$$

where τ is irreducible, σ arbitrary, $W \in (\tau, \sigma)$, $\tilde{W} \in (\overline{\tau}, \overline{\sigma})$ the conjugated arrow. But (17) is the characteristic relation for the conjugated arrow (see Doplicher/Roberts [2,Lemma 2.5.,p.167] or [7,(12),p.257]).□

The relations (13) and (14) ensure that the $*$-structure given by Definition 5.5 satisfies the usual properties, i.e. $\mathcal{F}_0$ is a $*$-algebra.

The orthonormal bases of the Hilbert spaces $\mathcal{H}_\alpha$, $\alpha \in \mathcal{T}$, satisfy important relations.

5.7.PROPOSITION. *Let $\alpha \in \mathcal{T}$ be arbitrary and let $\{\Phi_{\alpha j}\}_{j=1}^{d(\alpha)}$ be an orthonormal basis of $\mathcal{H}_\alpha$. Then the relations*

$$\Phi^*_{\alpha j}\Phi_{\alpha k} = \delta_{jk}1, \tag{18}$$

$$\sum_j \Phi_{\alpha j}\Phi^*_{\alpha j} = 1 \tag{19}$$

hold.

Proof. First we prove (18) for $\rho \in \hat{\mathcal{T}}$. We have $\Phi^*_{\rho j}\Phi_{\rho k} = R^*_\rho \Phi_{\overline{\rho} j}\Phi_{\rho k}$. Note that

$$\phi(R^*_\rho)X = R^*_\rho X = (R_\rho, X)_{\mathcal{H}_{\overline{\rho}\rho}}1, \quad X \in \mathcal{H}_{\overline{\rho}\rho}.$$

Thus

$$\Phi^*_{\rho j}\Phi_{\rho k} = (R_\rho, \Phi_{\overline{\rho} j}\Phi_{\rho k})_{\mathcal{H}_{\overline{\rho}\rho}}1 =$$

$$= \left(\sum_{l=1}^{d(\rho)} \Phi_{\overline{\rho} l}\Phi_{\rho l}, \Phi_{\overline{\rho} j}\Phi_{\rho k}\right)_{\mathcal{H}_{\overline{\rho}\rho}} 1 =$$

$$\sum_{l=1}^{d(\rho)}(\Phi_{\overline{\rho} l}, \Phi_{\overline{\rho} j})_{\mathcal{H}_{\overline{\rho}}}(\Phi_{\rho l}, \Phi_{\rho k})_{\mathcal{H}_\rho}1 = \sum_{l=1}^{d(\rho)} \delta_{lj}\delta_{lk}1 = \delta_{jk}1.$$

Now, using the definition of the Hilbert spaces $\mathcal{H}_\alpha$, $\alpha \in \mathcal{T}$, it is easy to extend the formula to arbitrary $\alpha \in \mathcal{T}$.

Second we prove (19), again first for $\rho \in \hat{\mathcal{T}}$. We calculate

$$\sum_{j=1}^{d(\rho)} \Phi_{\rho j}\Phi^*_{\rho j} = \sum_{j=1}^{d(\rho)} \Phi_{\rho j}R^*_\rho\Phi_{\overline{\rho} j} = \sum_{j=1}^{d(\rho)} \rho(R^*_\rho)\Phi_{\rho j}\Phi_{\overline{\rho} j}$$

$$= \rho(R^*_\rho)\sum_{j=1}^{d(\rho)} \Phi_{\rho j}\Phi_{\overline{\rho} j} = \rho(R^*_\rho)R_{\overline{\rho}} = 1_\rho \times R^*_\rho \cdot R_{\overline{\rho}} \times 1_\rho = 1_\rho,$$

according to the conjugation relation. Again, using the definition of the Hilbert spaces $\mathcal{H}_\alpha$, the formula can be easily extended to arbitrary $\alpha \in \mathcal{T}$. □

Proposition 5.7. means that the Hilbert spaces $\mathcal{H}_\alpha \subset \mathcal{F}_0$, $\alpha \in \mathcal{T}$, are *algebraic* Hilbert spaces of the $*$-algebra $\mathcal{F}_0$. In particular, (19) means that the so-called *support* of $\mathcal{H}_\alpha$ equals 1. Note that the Hilbert spaces $\mathcal{H}_\rho$ for irreducibles $\rho \in \hat{\mathcal{T}}$ together with $\mathcal{A}$ are generating for $\mathcal{F}_0$ w.r.t. $\mathcal{F}_0$ as an $\mathcal{A}$-leftmodule.

Recall that according to Propositions 3.1. and 3.2. we have bijections

$$\mathrm{Ob}\,\mathcal{T} \in \alpha \longleftrightarrow \mathcal{H}_\alpha \subset \mathcal{F}_0,$$

$$\mathcal{A} \supset (\alpha, \beta) \longleftrightarrow \phi((\alpha, \beta)) \subset \mathcal{L}(\mathcal{H}_\alpha \to \mathcal{H}_\beta).$$

Taking into account the results in sections 4 and 5 we can conclude that in fact by these bijections there is established a categorial isomorphism: First we have

$$\mathcal{H}_{\alpha\beta} = \mathcal{H}_\alpha \cdot \mathcal{H}_\beta$$

where $\mathcal{H}_\alpha \cdot \mathcal{H}_\beta$ denotes the algebraic Hilbert space spa $\{\mathcal{H}_\alpha \cdot \mathcal{H}_\beta\}$. Second, according to the original definition of the algebraic Hilbert spaces $\mathcal{H}_\alpha$, we get: If

$$\gamma(\cdot) = V\alpha(\cdot)V^* + W\beta(\cdot)W^*,$$

where $V \in (\alpha, \gamma)$, $W \in (\beta, \gamma)$, $VV^* + WW^* = 1$, $V^*V = W^*W = 1$, then

$$\mathcal{H}_\gamma = V\mathcal{H}_\alpha + W\mathcal{H}_\beta.$$

Third we obtain: the category $\{\mathcal{H}_\alpha\}_\alpha$ is equipped with a conjugation, given by the conjugation $\alpha \to \overline{\alpha}$.

Therefore, at this stage we can apply the Tannaka-Krein result. This means there is a compact group $\mathcal{K}$ such that $\hat{\mathcal{K}}$ is in one-to-one correspondence to $\{\mathcal{H}_\rho\}_\rho$ for a full collection of representers $\rho \in \hat{\mathcal{T}}$. So to each $\alpha \in \mathcal{T}$ we get a unitary representation $U_\alpha(\mathcal{K})$ of $\mathcal{K}$ on $\mathcal{H}_\alpha$ which is irreducible for $\alpha \in \hat{\mathcal{T}}$. Note that $\mathcal{K}$ acts on $\mathcal{H}_{\overline{\alpha}}$, $\alpha \in \mathcal{T}$, by the *conjugated* representation w.r.t. $\mathcal{H}_\alpha$, i.e.

$$U_{\overline{\alpha}}(k) = \overline{U_\alpha(k)}, \quad k \in \mathcal{K},$$

where - means complex conjugation of the representing matrices.

The special structure of $\mathcal{F}_0$ allows to define an action of $\mathcal{K}$ on $\mathcal{F}_0$ by the definition

$$U(k)(\sum_{\rho,j} A_{\rho j}\Phi_{\rho j}) := \sum_{\rho,j} A_{\rho j}U_\rho(k)\Phi_{\rho j}.$$

Obviously we have

$$U(k)(X) = U_\alpha(k)X, \quad X \in \mathcal{H}_\alpha,$$

because from

$$X = \sum_{j=1}^{r} B_j\Phi_j,\ B_j \in (\rho_j, \alpha),\ \Phi_j \in \mathcal{H}_{\rho_j},\ \rho_j \text{ irreducible},$$

we obtain

$$U(k)(X) = \sum_{j=1}^{r} B_j U_{\rho_j}\Phi_j = U_\alpha(k)\sum_{j=1}^{r} B_j\Phi_j = U_\alpha(k)X.$$

Therefore it turns out that $U(k)$ is even an automorphism of $\mathcal{F}_0$, because for $\Phi_1 \in \mathcal{H}_\alpha$, $\Phi_2 \in \mathcal{H}_\beta$ we obtain

$$U(k)(\Phi_1 \cdot \Phi_2) = U_{\alpha\beta}(k)(\Phi_1\Phi_2) = (U_\alpha(k)\Phi_1)(U_\beta(k)\Phi_2) = U(k)\Phi_1 \cdot U(k)\Phi_2.$$

Moreover we have

$$(U(k)(F))^* = U(k)(F^*), \quad F \in \mathcal{F}_0.$$

Namely, let $\{\Phi_{\rho j}\}_{j=1}^{d(\rho)}$ be an orthonormal basis of $\mathcal{H}_\rho$ and

$$U_\rho(k)\Phi_{\rho j} = \sum_{j'=1}^{d(\rho)} u_{j'j}^{(\rho)}(k)\Phi_{\rho j'}.$$

Then we get

$$(U(k)(\Phi_{\rho j}))^* = (U_\rho(k)\Phi_{\rho j})^* = \left(\sum_{j'=1}^{d(\rho)} u^{(\rho)}_{j'j}(k)\Phi_{\rho j'}\right)^*$$

$$= \sum_{j'=1}^{d(\rho)} \overline{u^{(\rho)}_{j'j}(k)}\Phi^*_{\rho j'} = \sum_{j'=1}^{d(\rho)} \overline{u^{(\rho)}_{j'j}} R^*_\rho \Phi_{\overline{\rho} j'} =$$

$$R^*_\rho \sum_{j'=1}^{d(\rho)} \overline{u^{(\rho)}_{j'j}} \Phi_{\overline{\rho} j'} = R^*_\rho (U_{\overline{\rho}}(k)\Phi_{\overline{\rho} j}) =$$

$$R^*_\rho U(k)(\Phi_{\overline{\rho} j}) = U(k)(R^*_\rho \Phi_{\overline{\rho} j}) = U(k)(\Phi^*_{\rho j}).$$

This relation can be easily extended to be valid for arbitrary $F = \sum_{\rho,j} A_{\rho j}\Phi_{\rho j}$.

Furthermore, the fixed point algebra of $U(\mathcal{K})$ within $\mathcal{F}_0$ equals $\mathcal{A}$, because if $\mathcal{F}_0 \ni F = \sum_{\rho,j} A_{\rho j}\Phi_{\rho j}$ and $U(k)(F) = F$ for all $k \in \mathcal{K}$ then we obtain

$$\sum_{\rho,j} A_{\rho j} U_\rho(k)\Phi_{\rho j} = \sum_{\rho,j,j'} A_{\rho j} u^{(\rho)}_{j'j}(k)\Phi_{\rho j'} = \sum_{\rho,j} A_{\rho j}\Phi_{\rho j},$$

hence

$$\sum_{j'=1}^{d(\rho)} u^{(\rho)}_{jj'}(k) A_{\rho j'} = A_{\rho j}$$

follows for all $k \in \mathcal{K}$, $\rho \in \hat{\mathcal{T}}$, $j = 1, 2..., d(\rho)$. If $\rho \neq \iota$, by integration over $\mathcal{K}$ we get $A_{\rho j} = 0$.

In the next section we characterize the group $\mathcal{K}$ as the group of all automorphisms of $\mathcal{F}_0$ leaving a characteristic $\mathcal{A}$-scalar product invariant.

6 The $\mathcal{A}$-scalar product and the symmetry group

Let $F, G \in \mathcal{F}_0$. Then $F = \sum_{\rho,j} A_{\rho j}\Phi_{\rho j}$, $G = \sum_{\rho,j} B_{\rho j}\Phi_{\rho j}$ where $\{\Phi_{\rho j}\}_{j=1}^{d(\rho)}$ denotes, as before, an orthonormal basis in $\mathcal{H}_\rho$; $A_{\rho j}, B_{\rho j} \in \mathcal{A}$. Recall $\Phi_\iota = 1$. The coefficient A_ι of F is called the $\mathcal{A}$-component of F, resp. B_ι is the $\mathcal{A}$-component of G. We define an $\mathcal{A}$-scalar product in the $*$-algebra F_0 by

$$\langle F, G\rangle_{\mathcal{A}} := \sum_{\rho,j} \frac{1}{d(\rho)} A_{\rho j} B^*_{\rho j}.$$

Obviously, $\langle F, G\rangle_{\mathcal{A}} \in \mathcal{A}$ and $\langle F, G\rangle_{\mathcal{A}}$ is independent of the special choice of the orthonormal bases in the Hilbert spaces $\mathcal{H}_\rho$. The $\mathcal{A}$-scalar product $\langle F, G\rangle_{\mathcal{A}}$ satisfies the properties

$$\langle AF, BG\rangle_{\mathcal{A}} = A\langle F, G\rangle_{\mathcal{A}} B^*,$$

$$\langle F, F\rangle_{\mathcal{A}} \geq 0,$$

$$\langle F, F\rangle_{\mathcal{A}} = 0 \quad \text{iff} \quad F = 0.$$

We have

6.1.LEMMA. *Let $F, G \in \mathcal{F}_0$. Then $\langle F, G\rangle_{\mathcal{A}}$ coincides with the $\mathcal{A}$-component of FG^*.*

Proof. It is sufficient to prove this for $F := \Phi_{\alpha j}$, $G := \Phi_{\beta k}$, $\alpha, \beta \in \hat{\mathcal{T}}$ irreducible. Recall that there is a decomposition

$$\Phi_{\alpha j} \cdot \Phi_{\beta k} = \sum_{\tau, l} K^{\tau l}_{\alpha j \beta k} \Phi_{\tau l}$$

of the product $\Phi_{\alpha j} \cdot \Phi_{\beta k}$ where $K^{\tau l}_{\alpha j \beta k} \in (\tau, \alpha\beta)$. We have

$$\Phi_{\alpha j} \Phi^*_{\beta k} = \Phi_{\alpha j} R^*_\beta \Phi_{\overline{\beta} k} = \alpha(R^*_\beta) \Phi_{\alpha j} \Phi_{\overline{\beta} k}.$$

Therefore, the $\mathcal{A}$-component of $\Phi_{\alpha j}\Phi^*_{\beta k}$ coincides with $\alpha(R^*_\beta)K^{\iota}_{\alpha j \overline{\beta} k}$. On the other hand, by definition we have

$$\langle \Phi_{\alpha j}, \Phi_{\beta k}\rangle_{\mathcal{A}} = \frac{1}{d(\alpha)} \delta_{\alpha\beta} \delta_{jk} 1.$$

So it is sufficient to prove

$$\alpha(R^*_\beta) K^{\iota}_{\alpha j \overline{\beta} k} = \frac{1}{d(\alpha)} \delta_{\alpha\beta}\delta_{jk} 1. \tag{20}$$

We check easily that

$$\alpha(R^*_\beta) K^{\iota}_{\alpha j \overline{\beta} k} \in (\beta, \alpha)$$

because $K^{\iota}_{\alpha j \overline{\beta} k} \in (\beta, \alpha\overline{\beta}\beta)$ and $\alpha(R^*_\beta) \in (\alpha\overline{\beta}\beta, \alpha)$. Therefore we have: if $\alpha \neq \beta$ then $\alpha(R^*_\beta) K^{\iota}_{\alpha j \overline{\beta} k} = 0$ and (20) is true. Now let $\alpha = \beta$. Then we have to prove

$$\alpha(R^*_\alpha) K_{\alpha j \overline{\alpha} k} = \frac{1}{d(\alpha)} \delta_{jk} 1.$$

Since the left hand side is a member of (α, α), we have

$$\alpha(R^*_\alpha) K^{\iota}_{\alpha j \overline{\alpha} k} = \lambda_{jk} 1, \quad \lambda_{jk} \in \mathbf{C}.$$

Note that

$$K^{\iota}_{\alpha j \overline{\alpha} k} = J(\alpha, \overline{\alpha}) u_{jk}, \quad u_{jk} \in \mathcal{H}_\alpha \otimes \mathcal{H}_{\overline{\alpha}}, \quad u_{jk} = \sum_{p,q} c^{pq}_{jk} \Phi_{\alpha p} \otimes \Phi_{\overline{\alpha} q}.$$

Let $X \in \mathcal{H}_\alpha$. We calculate

$$\alpha(R^*_\alpha) K^{\iota}_{\alpha j \overline{\alpha} k} X = 1_\alpha \times R^*_\alpha \cdot J(\alpha\overline{\alpha}, \alpha)(K^{\iota}_{\alpha j \overline{\alpha} k} \otimes X)$$

$$= 1_\alpha \times R^*_\alpha \cdot J(\alpha\overline{\alpha}, \alpha)(J(\alpha, \overline{\alpha}) u_{jk} \otimes 1_{\mathcal{H}_\alpha} X)$$

$$= 1_\alpha \times R^*_\alpha \cdot J(\alpha\overline{\alpha}, \alpha)(J(\alpha, \overline{\alpha}) \otimes 1_{\mathcal{H}_\alpha})(u_{jk} \otimes X) =$$

$$1_\alpha \times R^*_\alpha \cdot J(\alpha, \overline{\alpha}\alpha)(1_{\mathcal{H}_\alpha} \otimes J(\overline{\alpha}, \alpha)) \left(\sum_{p,q} c^{pq}_{jk} \Phi_{\alpha p} \otimes \Phi_{\overline{\alpha} q} \otimes X \right) =$$

$$J(\alpha,\iota)(1_{\mathcal{H}_\alpha}\otimes R_\alpha^*)(1_{\mathcal{H}_\alpha}\otimes J(\overline{\alpha},\alpha))\left(\sum_{p,q} c_{jk}^{pq}\Phi_{\alpha p}\otimes\Phi_{\overline{\alpha}q}\otimes X\right)=$$

$$(1_{\mathcal{H}_\alpha}\otimes R_\alpha^* J(\overline{\alpha},\alpha))\left(\sum_{p,q} c_{jk}^{pq}\Phi_{\alpha p}\otimes\Phi_{\overline{\alpha}q}\otimes X\right)=\sum_{p,q} c_{jk}^{pq}\Phi_{\alpha p}(R_\alpha, J(\overline{\alpha},\alpha)\Phi_{\overline{\alpha}q}\otimes X)_{\mathcal{H}_{\overline{\alpha}\alpha}}=$$

$$\sum_{p,q} c_{jk}^{pq}\Phi_{\alpha p}\left(J(\overline{\alpha},\alpha)\sum_{l=1}^{d(\alpha)}\Phi_{\overline{\alpha}l}\otimes\Phi_{\alpha l}, J(\overline{\alpha},\alpha)\Phi_{\overline{\alpha}q}\otimes X\right)_{\mathcal{H}_{\overline{\alpha}\alpha}}=$$

$$\sum_{p,q} c_{jk}^{pq}\Phi_{\alpha p}\sum_{l=1}^{d(\alpha)}(\Phi_{\overline{\alpha}l}\otimes\Phi_{\alpha l},\Phi_{\overline{\alpha}q}\otimes X)_{\mathcal{H}_{\overline{\alpha}}\otimes\mathcal{H}_\alpha}=\sum_{p,q} c_{jk}^{pq}\Phi_{\alpha p}\sum_{l=1}^{d(\alpha)}(\Phi_{\overline{\alpha}l},\Phi_{\overline{\alpha}q})_{\mathcal{H}_{\overline{\alpha}}}(\Phi_{\alpha l},X)_{\mathcal{H}_\alpha}$$

$$=\sum_{p,q} c_{jk}^{pq}\Phi_{\alpha p}(\Phi_{\alpha q},X)_{\mathcal{H}_\alpha}=\lambda_{jk}X.$$

For $X:=\Phi_{\alpha s}$ we obtain the equation

$$\sum_p c_{jk}^{ps}\Phi_{\alpha p}=\lambda_{jk}\Phi_{\alpha s}$$

hence

$$c_{jk}^{ps}=\delta^{ps}\lambda_{jk}$$

follows. That is, we obtain

$$u_{jk}=\lambda_{jk}\sum_p\Phi_{\alpha p}\otimes\Phi_{\overline{\alpha}p}$$

and

$$K_{\alpha j\overline{\alpha}k}^{\iota}=\lambda_{jk}J(\alpha,\overline{\alpha})\left(\sum_p\Phi_{\alpha p}\otimes\Phi_{\overline{\alpha}p}\right)=\lambda_{jk}R_{\overline{\alpha}}.$$

Therefore we can write

$$\Phi_{\alpha j}\cdot\Phi_{\overline{\alpha}k}=\lambda_{jk}R_{\overline{\alpha}}+\sum_{\gamma\neq\iota}\sum_l K_{\alpha j\overline{\alpha}k}^{\gamma l}\Phi_{\gamma l}. \tag{21}$$

In other words, we have: The $\mathcal{A}$-component of $\Phi_{\alpha j}\cdot\Phi_{\overline{\alpha}k}$ equals $\lambda_{jk}R_{\overline{\alpha}}$. From (21) we obtain

$$R_{\overline{\alpha}}^*\cdot\Phi_{\alpha j}\cdot\Phi_{\overline{\alpha}k}=\lambda_{jk}R_{\overline{\alpha}}^*R_{\overline{\alpha}}+\sum_{\gamma\neq\iota}\sum_l R_{\overline{\alpha}}^*K_{\alpha j\overline{\alpha}k}^{\gamma l}\Phi_{\gamma l}.$$

Recall $R_{\overline{\alpha}}\in\mathcal{H}_{\alpha\overline{\alpha}}$ and $\Phi_{\alpha j}\Phi_{\overline{\alpha}k}\in\mathcal{H}_{\alpha\overline{\alpha}}$. Therefore we get

$$R_{\overline{\alpha}}^*\Phi_{\alpha j}\Phi_{\overline{\alpha}k}=(R_{\overline{\alpha}},\Phi_{\alpha j}\Phi_{\overline{\alpha}k})_{\mathcal{H}_{\alpha\overline{\alpha}}}1=\lambda_{jk}d(\alpha)1+\sum_{\gamma\neq\iota}\sum_l R_{\overline{\alpha}}^*K_{\alpha j\overline{\alpha}k}^{\gamma l}\Phi_{\gamma l}.$$

Now we calculate

$$(R_{\overline{\alpha}},\Phi_{\alpha j}\Phi_{\overline{\alpha}k})_{\mathcal{H}_{\alpha\overline{\alpha}}}=\left(J(\alpha,\overline{\alpha})\sum_{l=1}^{d(\alpha)}\Phi_{\alpha l}\otimes\Phi_{\overline{\alpha}l}, J(\alpha,\overline{\alpha})(\Phi_{\alpha j}\otimes\Phi_{\overline{\alpha}k})\right)_{\mathcal{H}_{\alpha\overline{\alpha}}}$$

$$= \sum_{l=1}^{d(\alpha)} (\Phi_{\alpha l}, \Phi_{\alpha j})_{\mathcal{H}_\alpha} (\Phi_{\overline{\alpha} l}, \Phi_{\overline{\alpha} k})_{\mathcal{H}_{\overline{\alpha}}} = \sum_{l=1}^{d(\alpha)} \delta_{lj} \delta_{lk} = \delta_{jk}$$

and we obtain

$$\delta_{jk} 1 = \lambda_{jk} d(\alpha) 1 + \sum_{\gamma \neq \iota} \sum_{l} R^*_{\overline{\alpha}} K^{\gamma l}_{\alpha j \overline{\alpha} k} \Phi_{\gamma l}.$$

Since the system $\{\Phi_{\rho j}\}_{\rho j}$ forms an $\mathcal{A}$-leftmodule basis we finally get

$$\lambda_{jk} = \frac{1}{d(\alpha)} \delta_{jk}$$

and (20) is proved for $\alpha = \beta$, too.□

6.2.COROLLARY. *The relation*

$$\langle X, Y F^* \rangle_{\mathcal{A}} = \langle X F, Y \rangle_{\mathcal{A}}; \quad X, Y, F \in \mathcal{F}_0,$$

holds.

Proof. We already know that $\langle F, G \rangle_{\mathcal{A}}$ equals the $\mathcal{A}$-component of FG^*. Now

$$X(YF^*)^* = XFY^*$$

holds. Therefore their $\mathcal{A}$-components also coincide. □

Next we introduce a C^*-norm on $\mathcal{F}_0$. The corresponding arguments are standard and follow the lines of Doplicher/Roberts, e.g. in [1,p.88 f.]; see also [7,p.217 f.] and [7,p.286 f.].

The first step is to introduce a norm in $\mathcal{F}_0$ by the definition

$$|F|_{\mathcal{A}} := \| \langle F, F \rangle_{\mathcal{A}} \|^{1/2}. \tag{22}$$

Obviously $|A|_{\mathcal{A}} = \|A\|$ for $A \in \mathcal{A}$ because $\|AA^*\|^{1/2} = \|A\|$. Note that $| \cdot |_{\mathcal{A}}$ fails to be a C^*-norm. Therefore in a second step we define the following norm on $\mathcal{F}_0$:

$$\|F\|_* := |\pi(F)|, \quad F \in \mathcal{F}_0, \tag{23}$$

where $(\pi(F))X := XF^*$, $X \in \mathcal{F}_0$ and $|\pi(F)|$ means the operator norm w.r.t. the norm $| \cdot |_{\mathcal{A}}$ on $\mathcal{F}_0$, i.e..

$$|\pi(F)| = \sup_{|X|_{\mathcal{A}} \leq 1} |XF^*|_{\mathcal{A}}.$$

$\| \cdot \|_*$ turns out to be a C^*-norm on $\mathcal{F}_0$. Note that the C^*-property of $\| \cdot \|_*$ follows from Corollary 6.2. which says that

$$\langle X, \pi(F) Y \rangle_{\mathcal{A}} = \langle \pi(F^*) X, Y \rangle_{\mathcal{A}}, \quad X, Y, F \in \mathcal{F}_0.$$

Note further that $\| \cdot \|_*$ on $\mathcal{F}_0$ is an extension of the C^*-norm $\| \cdot \|$ on $\mathcal{A} \subset \mathcal{F}_0$, i.e.

$$\|A\|_* = \|A\|, \quad A \in \mathcal{A}.$$

Moreover, we have

$$|F|_{\mathcal{A}} \leq \|F\|_*, \quad F \in \mathcal{F}_0.$$

Finally we form the C^*-algebra

$$\mathcal{F} := \text{clo}_{\|\cdot\|_*}(\mathcal{F}_0).$$

Now we turn to a special automorphism group $\mathcal{G}$ of $\mathcal{F}$. First we consider again the $*$-subalgebra $\mathcal{F}_0 \subseteq \mathcal{F}$ which is dense in $\mathcal{F}$ and consider automorphisms g of $\mathcal{F}_0$ requiring invariance of the $\mathcal{A}$-scalar product $\langle\cdot,\cdot\rangle_{\mathcal{A}}$:

6.3.DEFINITION. The automorphism $g \in \text{aut}\,\mathcal{F}_0$ is said to be a *symmetry* if

$$\langle gF_1, gF_2\rangle_{\mathcal{A}} = \langle F_1, F_2\rangle_{\mathcal{A}}, \quad F_1, F_2 \in \mathcal{F}_0. \tag{24}$$

The collection of all automorphisms $g \in \text{aut}\,\mathcal{F}_0$ of this type forms a group $\mathcal{G} \neq \emptyset$, because the identical automorphism id: $\text{id}\,F := F$ belongs to $\mathcal{G}$. The group $\mathcal{G}$ is called the *symmetry* group.

6.4.LEMMA. *Each automorphism $g \in \mathcal{G}$ can be uniquely extended to an automorphism of $\mathcal{F}$ so that $\mathcal{G}$ can be considered as a subgroup of* $\text{aut}\,\mathcal{F}$, $\quad \mathcal{G} \subseteq \text{aut}\,\mathcal{F}$.

Proof. From (22) and (24) it follows immediately that $|gF|_{\mathcal{A}} = |F|_{\mathcal{A}}$, $F \in \mathcal{F}_0$. Furthermore, we have $\|gF\|_* = \|F\|_*$ for all $F \in \mathcal{F}_0$ because from (23) we obtain

$$\|gF\|_* = |\pi(gF)| = \sup_{|X|_{\mathcal{A}}\leq 1} |X(gF)^*|_{\mathcal{A}} = \sup_{|X|_{\mathcal{A}}\leq 1} |g\{(g^{-1}X)F^*\}|_{\mathcal{A}}$$

$$= \sup_{|X|_{\mathcal{A}}\leq 1} |(g^{-1}X)F^*|_{\mathcal{A}} = \sup_{|Y|_{\mathcal{A}}\leq 1} |YF^*|_{\mathcal{A}} = |\pi(F)| = \|F\|_*.$$

Therefore, by continuous (isometric) extension each g can be extended to the whole C^*-algebra $\mathcal{F}$, yielding an automorphism of $\mathcal{F}$. □

We emphasize that the symmetry group $\mathcal{G}$ coincides with the *stability group* $\text{stab}\,\mathcal{A} \subset \text{aut}\,\mathcal{F}$ of $\mathcal{F}$. Recall that $\text{stab}\,\mathcal{A} := \{g \in \text{aut}\,\mathcal{F} : gA = A \text{ for all } A \in \mathcal{A}\}$.

First we have

6.5.LEMMA. *Each Hilbert space $\mathcal{H}_\alpha$, $\alpha \in \mathcal{T}$, is invariant w.r.t. $g \in \text{stab}\,\mathcal{A}$.*

Proof. Recall that $X \in \mathcal{H}_\alpha$ iff $\alpha(A)X = XA$ for all $A \in \mathcal{A}$. Therefore we have

$$\alpha(A)\cdot gX = gX \cdot A, \quad X \in \mathcal{H}_\alpha,\, A \in \mathcal{A},\, g \in \text{stab}\,\mathcal{A},$$

i.e. $gX \in \mathcal{H}_\alpha$. □

Now we prove

6.6.LEMMA. *The relation*

$$\mathcal{G} = \text{stab}\,\mathcal{A}$$

holds.

Proof. First we prove $\mathcal{G} \subseteq \text{stab}\,\mathcal{A}$. Let $A \in \mathcal{A}$. Then we have to prove that $gA = A$ for all$g \in \mathcal{G}$. Note that a priori $gA \in \mathcal{F}_0$. We put $\mathcal{F}_0 \ni F = \sum_{\alpha,j}(F)_{\alpha j}\Phi_{\alpha j}$ with fixed orthonormal bases $\{\Phi_{\alpha j}\}_{j=1}^{d(\alpha)}$. For example $(F)_\iota$ is the $\mathcal{A}$-component of F. Recall that $\langle F, 1\rangle_{\mathcal{A}} = (F)_\iota$. Then

$$\langle gA, \Phi_{\alpha j}\rangle_{\mathcal{A}} = \langle A, g^{-1}\Phi_{\alpha j}\rangle_{\mathcal{A}} = A(g^{-1}\Phi_{\alpha j})_\iota^* = 0$$

for $\alpha \neq \iota$, $j = 1, 2, ..., d(\alpha)$ because $g\mathcal{H}_\alpha \subseteq \mathcal{H}_\alpha$, i.e. $gA \in \mathcal{A}$. This implies

$$gA = (gA)_\iota = \langle gA, 1\rangle_{\mathcal{A}} = \langle gA, g1\rangle_{\mathcal{A}} = \langle A, 1\rangle_{\mathcal{A}} = A$$

hence $gA = A$ follows.

Second we prove stab $\mathcal{A} \subseteq \mathcal{G}$. Let $g \in$ stab $\mathcal{A}$. From Lemma 6.5. we get

$$g\Phi_{\alpha j} = \sum_{j'} u^{(\alpha)}_{j'j}(g)\Phi_{\alpha j'},$$

where $\{u^{(\alpha)}_{j'j}(g)\}^{d(\alpha)}_{j',j=1}$ is a unitary $d(\alpha) \times d(\alpha)$-matrix because

$$(g\Phi_{\alpha j})^* g\Phi_{\alpha j'} = g(\Phi^*_{\alpha j}\Phi_{\alpha j'}) = g(\delta_{jj'}1) = \delta_{jj'}1.$$

Now we have

$$\langle gF_1, gF_2\rangle_{\mathcal{A}} = \sum_{\rho,j,\sigma,k} A_{\rho j}\langle g\Phi_{\rho j}, g\Phi_{\sigma k}\rangle_{\mathcal{A}} B^*_{\sigma k}.$$

In the case $\rho \neq \sigma$ we have $\langle g\Phi_{\rho j}, g\Phi_{\sigma k}\rangle_{\mathcal{A}} = 0 = \langle \Phi_{\rho j}, \Phi_{\sigma k}\rangle_{\mathcal{A}}$. In the other case $\rho = \sigma$ we get

$$\langle g\Phi_{\rho j}, g\Phi_{\rho k}\rangle_{\mathcal{A}} = \left\langle \sum_{j'} u_{j'j}\Phi_{\rho j'}, \sum_{k'} u_{k'k}\Phi_{\rho k'} \right\rangle_{\mathcal{A}} = \sum_{j'k'} u_{j'j}\overline{u}_{k'k}\langle \Phi_{\rho j'}, \Phi_{\rho k'}\rangle_{\mathcal{A}}$$

$$= \sum_{j'k'} u_{j'j}\overline{u}_{k'k}\frac{1}{d(\rho)}\delta_{j'k'} = \frac{1}{d(\rho)}\sum_{k'} u_{k'j}\overline{u}_{k'k} = \frac{1}{d(\rho)}\delta_{jk} = \langle \Phi_{\rho j}, \Phi_{\rho k}\rangle_{\mathcal{A}},$$

hence $\langle gF_1, gF_2\rangle_{\mathcal{A}} = \langle F_1, F_2\rangle_{\mathcal{A}}$ follows. □

Further it turns out that $\mathcal{G}$ is compact.

6.7.PROPOSITION. *The symmetry group $\mathcal{G}$ is compact w.r.t. the pointwise norm topology.*

Proof. The Hilbert spaces $\mathcal{H}_\alpha$, dim $\mathcal{H}_\alpha = d(\alpha)$, are invariant w.r.t. $\mathcal{G}$ and they generate $\mathcal{F}$, together with $\mathcal{A}$. Therefore, according to Doplicher/Roberts [8,Lemma 3.2.,p.292] $\mathcal{G}$ is compact w.r.t. the pointwise norm topology. □

6.8.PROPOSITION. *Let $\mathcal{F}_0 \ni F = \sum_{\rho,j} F_{\rho j}\Phi_{\rho j}$, where the system of orthonormal bases $\{\Phi_{\rho j}\}^{d(\rho)}_{j=1}$ is fixed. Then*

$$\|F_{\rho j}\| \leq d(\rho)\|F\|_*, \quad j = 1, 2, ...d(\rho),\ \rho \in \hat{\mathcal{T}}, \tag{25}$$

i.e. $F_{\rho j}$ depends continuously on F.

Proof. We have

$$\frac{1}{d(\rho)}\|F_{\rho j}\|^2 = \frac{1}{d(\rho)}\|F_{\rho j}F^*_{\rho j}\| \leq \|\sum_{\alpha,j} d(\alpha)^{-1}F_{\alpha j}F^*_{\alpha j}\| = \|\langle F, F\rangle_{\mathcal{A}}\|^2 = |F|^2_{\mathcal{A}} \leq \|F\|^2_*.$$

This proves (25). □

6.9.COROLLARY. *The $\mathcal{A}$-scalar product $\langle F_1, F_2\rangle_{\mathcal{A}}$ on $\mathcal{F}_0$ is continuous w.r.t. the C^*-norm $\|\cdot\|_*$.*

Proof. According to Lemma 6.1., we have $\langle F_1, F_2\rangle_{\mathcal{A}} = (F_1F_2^*)_\iota$. So it is sufficient to prove that F_ι depends continuously on F. But this assertion is a special case of Proposition 6.8. □

Corollary 6.9. implies that the $\mathcal{A}$-scalar product has a continuous extension to $\mathcal{F}$ w.r.t. $\|\cdot\|_*$.

Obviously, on $\mathcal{F}_0$ we can define projections Π_ρ, $\rho \in \hat{\mathcal{T}}$, by the definition

$$\mathcal{F}_0 \ni F = \sum_{\sigma,j} A_{\sigma j}\Phi_{\sigma j} \to \Pi_\rho F := \sum_{j=1}^{d(\rho)} A_{\rho j}\Phi_{\rho j} \in \mathrm{spa}\{\mathcal{A}\mathcal{H}_\rho\}.$$

Note that this definition does not depend on the special orthonormal basis in $\mathcal{H}_\rho$.

$\Pi_\rho F$ depends continuously on F w.r.t. $\|\cdot\|_*$, in particular we get

$$\|\Pi_\rho F\| \leq d(\rho)^2\|F\|_*, \quad F \in \mathcal{F}_0.$$

Therefore, Π_ρ can be continuously extended on $\mathcal{F}$ as a projection and we obtain the relations

$$1 \leq \|\Pi_\rho\|_* \leq d(\rho)^2,$$

$$\Pi_\rho(A_1FA_2) = A_1\Pi_\rho(F)A_2, \quad f \in \mathcal{F},\ A_1, A_2 \in \mathcal{A},$$

$$\Pi_\rho\Pi_\sigma = \Pi_\sigma\Pi_\rho = \delta_{\rho\sigma}\Pi_\rho,$$

for a fixed system of representers $\rho, \sigma \in \hat{\mathcal{T}}$.

6.10.COROLLARY. *The relative commutant $\mathcal{A}' \cap \mathcal{F}$ is trivial, $\mathcal{A}' \cap \mathcal{F} = \mathbf{C}1$.*

Proof. Let $F \in \mathcal{F}$ and $AF = FA$ for all $A \in \mathcal{A}$. Then

$$A \cdot \Pi_\rho F = \Pi_\rho(AF) = \Pi_\rho(FA) = \Pi_\rho F \cdot A, \quad \rho \in \hat{\mathcal{T}}.$$

This means $\Pi_\rho F \in \mathcal{A}' \cap \mathcal{F}_0$, i.e. $\Pi_\rho F = \lambda_\rho 1$. hence for $\rho \neq \iota$ one has $\lambda_\rho = 0$ and this implies $F = \lambda_\iota 1$. □

6.11.LEMMA. *The relation*

$$\Pi_\rho\mathcal{F} = \mathrm{spa}\,\{\mathcal{A}\mathcal{H}_\rho\}, \quad \rho \in \hat{\mathcal{T}},$$

is valid.

Proof. Obviously, $\Pi_\rho\mathcal{F}_0 = \mathrm{spa}\,\mathcal{A}\mathcal{H}_\rho$. Since $\Pi_\rho\mathcal{F} = \mathrm{clo}_{\|\cdot\|_*}\Pi_\rho\mathcal{F}_0$ it is sufficient to prove that $\mathrm{spa}\,\mathcal{A}\mathcal{H}_\rho$ is closed w.r.t. $\|\cdot\|_*$. Let $F_n = \sum_{j=1}^{d(\rho)} A_{\rho j}^{(n)}\Phi_{\rho j} \in \mathrm{spa}\,\mathcal{A}\mathcal{H}_\rho$ and $\|F_n - F_m\|_* \to 0$, $n, m \to \infty$. The limit is denoted by F_∞. Then from $|F_n - F_m|_{\mathcal{A}}^2 \leq \|F_n - F_m\|_*^2$ we get

$$\|\langle F_n - F_m, F_n - F_m\rangle_{\mathcal{A}}\| = \|\sum_{j=1}^{d(\rho)}(A_{\rho j}^{(n)} - A_{\rho j}^{(m)})(A_{\rho j}^{(n)} - A_{\rho j}^{(m)})^*\|$$

$$\geq \|(A_{\rho j}^{(n)} - A_{\rho j}^{(m)})(A_{\rho j}^{(n)} - A_{\rho j}^{(m)})^*\| = \|A_{\rho j}^{(n)} - A_{\rho j}^{(m)}\|^2 \to 0$$

hence $F_\infty = \sum_{j=1}^{d(\rho)} A_{\rho j}^\infty\Phi_{\rho j}$ and $F_\infty \in \mathrm{spa}\,\mathcal{A}\mathcal{H}_\rho$. □

6.12.LEMMA. *The projection Π_ρ commutes with the action of $\mathcal{G}$, i.e.*

$$\Pi_\rho(gF) = g\Pi_\rho F, \quad F \in \mathcal{F}, g \in \mathcal{G}. \tag{26}$$

Proof. It is sufficient to prove (26) for all elements $F \in \mathcal{F}_0$ because (26) then follows using extension by continuity. But since $\mathcal{H}_\rho$ is invariant w.r.t. $\mathcal{G}$ and $gA = A$ for all $A \in \mathcal{A}$, we have $g\sum_{\rho,j} A_{\rho j}\Phi_{\rho j} = \sum_{\rho,j} A_{\rho j} g\Phi_{\rho j}$, hence $g(\Pi_\rho F) = \sum_{j=1}^{d(\rho)} A_{\rho j} g\Phi_{\rho j} = \Pi_\rho(gF)$ follows. □

6.13.PROPOSITION. *The fixed point algebra of $\mathcal{G}$ coincides with $\mathcal{A}$.*

Proof. First we note that $U(\mathcal{K})$ is a subgroup of $\mathcal{G}$, $U(\mathcal{K}) \subseteq \mathcal{G}$. This is obvious because

$$\langle U(k)F_1, U(k)F_2\rangle_{\mathcal{A}} = \langle F_1, F_2\rangle_{\mathcal{A}}, \quad k \in \mathcal{K},\ F_1, F_2 \in \mathcal{F}.$$

Since $\mathcal{K}$ acts irreducibly on $\mathcal{H}_\rho$, $\rho \in \hat{\mathcal{T}}$, and $\mathcal{H}_\rho$ is invariant w.r.t. $\mathcal{G}$ it follows that $\mathcal{H}_\rho$ is irreducible w.r.t. $\mathcal{G}$, too. Now, according to Lemma 6.12, from $gF = F$ for all $g \in \mathcal{G}$ we obtain $g\Pi_\rho F = \Pi_\rho F$, i.e. the Π_ρ-components of F are also invariant. So we have to prove $\Pi_\rho F = 0$ for all $\rho \neq \iota$. Now $\Pi_\rho F = \sum_{j=1}^{d(\rho)} A_{\rho j}\Phi_{\rho j}$ and $g(\Pi_\rho F) = \sum_{j,j'=1}^{d(\rho)} A_{\rho j} u_{j'j}^{(\rho)}(g)\Phi_{\rho j'}$ hence $A_{\rho j} = \sum_{j'=1}^{d(\rho)} u_{jj'}^{(\rho)}(g) A_{\rho j'}$ follows. Integrating this over $\mathcal{G}$ we obtain $A_{\rho j} = 0$. □

6.14.COROLLARY. *The groups $U(\mathcal{K})$ and $\mathcal{G}$ coincide.*

Proof. First one has to prove that $\operatorname{spec}\mathcal{G} = \hat{\mathcal{G}}$, i.e. all irreducible representations of $\mathcal{G}$ occur. This is obtained by considering the scalar functions $H_1^* g(H_2)$, $H_1, H_2 \in \mathcal{H}_\alpha$, $\alpha \in \mathcal{T}$. Using the Stone-Weierstra approximation theorem one gets that these functions span a dense set in $C(\mathcal{G})$ hence in $L^2(\mathcal{G})$. Then one can conclude that $U(\mathcal{K}) = \mathcal{G}$ (cf. Doplicher/Roberts [8, 3.3.Theorem, p.293]). □

We note that also a direct calculation shows that the representations induced by $\mathcal{G}$ on $\mathcal{H}_\alpha$ and $\mathcal{H}_{\overline{\alpha}}$, $\alpha \in \mathcal{T}$, are mutually conjugated. Namely we have $R_\alpha = \sum_{j=1}^{d(\alpha)} \Phi_{\overline{\alpha}j}\Phi_{\alpha j}$ hence $R_\alpha = \sum_{j=1}^{d(\alpha)} (g\Phi_{\overline{\alpha}j})(g\Phi_{\alpha j})$ and one obtains immediately

$$u_{j'j}^{(\overline{\alpha})}(g) = \overline{u_{j'j}^{(\alpha)}(g)}, \quad g \in \mathcal{G}.$$

Finally we note that the objects $\alpha \in \mathcal{T}$ turn out to be the restrictions to $\mathcal{A}$ of the so-called *canonical endomorphisms* w.r.t. $\mathcal{H}_\alpha$,

$$\alpha(X) = \sum_{j=1}^{d(\alpha)} \Phi_{\alpha j} X \Phi_{\alpha j}^*, \quad X \in \mathcal{A},\ \alpha \in \mathcal{T}.$$

7 Acknowledgment

It is a pleasure to thank S.Doplicher for a discussion on the subject at Oberwolfach in February 1994. I also thank Fatma Kärsten who contributed by some useful remarks concerning this approach.

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