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A MODIFIED DEFECT RELATION FOR HOLOMORPHIC CURVES

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1. Introduction and main results.

By a holomorphic curve, we mean a holomorphic mapping

$$x : V \longrightarrow \boldsymbol{P}_n ,$$

where V is an open Riemann surface and $\boldsymbol{P}_n$ is the n-dimensional complex projective space. In 1927, R. Nevanlinna [3] created a new theory concerning the distribution of values of a holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_1$. Nevanlinna's main result is that f assumes almost all values in $\boldsymbol{P}_1$ "equally often", and those values that f fails to assume often enough have total "defect" at most 2. H. Cartan [2] generalized this "defect relation" to holomorphic curves $x : \boldsymbol{C} \to \boldsymbol{P}_n$ counting how often x takes values in hyperplanes. L. Ahlfors [1] later extended Cartan's result to holomorphic curves $x : V \to \boldsymbol{P}_n$, which he cast in a geometric form. H. Wu [5] reorganized Ahlfors' theory in a modern fashion. We freely use the symboles, notations and terminologies from H. Wu [5] except for special declaration.

The purpose of this paper is to modify the Second Main Theorem for holomorphic curves, and furthermore, simplify the defect relation. Let τ be a harmonic exhaustion on V and $\sigma = \tau + \sqrt{-1}\,\rho$ be the special coordinate function. By a theorem of Gunning and Narasimhan [5, p. 102], there is a holomorphic function γ on V whose differential vanishes nowhere. Thus in every sufficiently small open subset of V, the restriction of γ to it is a coordinate function. Define

$$H(r) = \frac{1}{2\pi}\int_{\partial V[t]} \log\left|\frac{d\sigma}{d\gamma}\right| *d\tau \Big|_{r_0}^{r} ,$$

$$T_k^0(r) = T_k(r) + N_k(r) ,$$

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$$\varepsilon_k^q=\begin{cases}1, & \text{if } 0\leqq q\leqq k\\ \dfrac{(n-q)(k+1)}{(n-k)(q+1)}, & \text{if } k\leqq q\leqq n-1,\end{cases}$$

Then we obtain

THEOREM 1. *Let* $x: V\to \boldsymbol{P}_n$ *be a nondegenerate holomorphic curve and* V *admits a harmonic exhaustion, then for* $k=0, \cdots, n-1$

$$E(r)+H(r)+S_k(r)=N_{k-1}(r)-2N_k(r)+N_{k+1}(r) \tag{1}$$

$$T_{k-1}^0-2T_k^0+T_{k+1}^0=H+\mu(T) \tag{2}$$

and for $k=1, \cdots, n$

$$N_k(r)=(k+1)N_0(r)+\frac{k(k+1)}{2}(E(r)+H(r))+\sum_{j=0}^{k-1}(k-j)S_j(r). \tag{3}$$

THEOREM 2. *Let* $x: V\to \boldsymbol{P}_n$ *be a nondegenerate holomorphic curve and* V *admits a harmonic exhaustion. Let* $\{A^q\}$ *be a finite system of q-dimensional projective subspaces of* $\boldsymbol{P}_n$ *in general position. Then the generalized compensating terms* $m_k(A^q)=m_k(r, A^q)$ *satisfy the following inequality*

$$\sum_{A^q} m_k(A^q)=\varepsilon_k^q\binom{n}{q}\left(\frac{n+1}{k+1}T_k^0-N_n+\frac{1}{2}(n+1)(n-k)H\right)+\mu(T^2). \tag{4}$$

We also have the equality

$$\begin{aligned}&\frac{n+1}{k+1}T_k^0(r)-N_n(r)+\frac{1}{2}(n+1)(n-k)H(r)\\ &\quad=\frac{n+1}{k+1}T_k(r)-Q_k(r)-\frac{1}{2}(n+1)(n-k)E(r).\end{aligned} \tag{5}$$

where

$$Q_k(r)=\frac{n-k}{k+1}\sum_{j=0}^{k-1}(j+1)S_j(r)+\sum_{j=k}^{n-1}(n-j)S_j(r).$$

Remark. If $\tilde{x}=(x_0, \cdots, x_n): V\to \boldsymbol{C}^{n+1}$ is a reduced representation of x, then

$$N_n(r)=\int_{r_0}^{r} n(t, W=0)dt, \tag{6}$$

where $W=W(x_0, \cdots, x_n)$ is the Wronskian determinant of x_j $(j=0, \cdots, n)$ and

$$n(t, W=0)=\text{sum of the orders of zeroes of } W \text{ in } V[t].$$

Thus if $V=\boldsymbol{C}$ and $k=0$, (4) is just the Cartan's Second Main Theorem [2], [4].

I learned about differential geometry and complex analysis from H. Wu and Y. T. Siu, whom I wish to thank for sharing their insights with me.

2. Proof of Theorem 1.

Given a holomorphic curve $x: V \to \boldsymbol{P}_n$, with a reduced representation $\tilde{x} = (x_0, \cdots, x_n): V \to \boldsymbol{C}^{n+1}$. According to H. Wu [5] the quantity X_z^k is defined as follows: fix a coordinate neighborhood U in V and a coordinate function z on U,

$$X_z^k = \tilde{x} \wedge \tilde{x}^{(1)} \wedge \cdots \wedge \tilde{x}^{(k)}, \qquad k=0, \cdots, n,$$

where $X_z^0 = \tilde{x}^{(0)} = \tilde{x}$ and

$$\tilde{x}^{(i)} = \left(\frac{d^i x_0}{dz^i}, \cdots, \frac{d^i x_n}{dz^i}\right),$$

Then the following results are well-known [5]

(7) $$X_\gamma^k = \left(\frac{d\sigma}{d\gamma}\right)^{k(k+1)/2} X_\sigma^k \qquad [5, \text{ p. } 69]$$

(8) $$T_k(r) = \frac{1}{2\pi}\int_{\partial V[t]} \log |X_\gamma^k| * d\tau \Big|_{r_0}^{r} - N_k(r) \qquad [5, \text{ p. } 104]$$

(9) $$E(r) + S_k(r) + T_{k-1}(r) - 2T_k(r) + T_{k+1}(r)$$
$$= \frac{1}{2\pi}\int_{\partial V[t]} \log \frac{|X_\sigma^{k-1}|\,|X_\sigma^{k+1}|}{|X_\sigma^k|^2} * d\tau \Big|_{r_0}^{r} \qquad [5, \text{ p. } 130]$$

(10) $$E + S_k + T_{k-1} - 2T_k + T_{k+1} = \mu(T). \qquad [5, \text{ p. } 132]$$

where (7) holds in $V - V[r(\tau)] - \{\text{critical points of } \tau\}$. Since $r_0 \geqq r(\tau)$ and an integration always ignores finite point sets (the critical points of τ are all isolated), by (7) and (8), we have .

(11) $$\frac{1}{2\pi}\int_{\partial V[t]} \log |X_\sigma^k| * d\tau \Big|_{r_0}^{r} = T_k(r) + N_k(r) - \frac{1}{2}k(k+1)H(r).$$

Consequently, (9) and (11) imply (1).

Note that $N_{-1}(r) = 0$. So upon (1) summing over k from 0 to $j-1$, we have:

(12) $$j(E(r) + H(r)) + \sum_{i=0}^{j-1} S_i(r) = N_j(r) - N_{j-1}(r) - N_0(r).$$

Upon summing over j from 1 to k, we finally have (3). (1) and (10) imply (2). q. e. d.

3. Preliminary lemmas.

To prove Theorem 2 we need some lemmas.

LEMMA 1 [5, p. 131]. (i) *If $\psi_1 \leqq \psi$ off a compact set and $\varphi \leqq \varphi_1$ off a compact* *set, then* $\phi=\mu(\varphi)$ *implies* $\phi_1=\mu(\varphi_1)$.

(ii) *If* $\phi=\mu(\varphi)$, *then* $\phi+O(1)=\mu(\varphi)$.

(iii) *If* C *is a positive constant and* $\phi=\mu(\varphi)$, *then*

$$C\phi=\mu(\varphi).$$

(iv) *If* $\phi=\mu(\varphi)$ *and* ϕ_1 *is positive off a compact set, then*

$$\phi-\phi_1=\mu(\varphi).$$

(v) *Suppose* $\phi=\mu(\varphi)$ *and* $\phi_1=\mu(\varphi)$. *then* $\phi+\phi_1=\mu(\varphi)$.

Remark. We say $\phi=\mu(\varphi)$ for two continuous functions φ and ϕ if and only if

$$\int_{r_0}^{r} ds \int_{r_0}^{s} \exp\{K\phi(t)\} dt < C\varphi(r)+C' \qquad [5, \text{ p. } 131]$$

for some positive constants K, C and C'.

LEMMA 2. *For* $k=0, \cdots, n-2$,

$$(k+1)T^0_{k+1}=(k+2)T^0_k+\frac{1}{2}(k+1)(k+2)H+\mu(T) \tag{13}$$

and for $k=1, \cdots, n-1$,

$$(n-k)T^0_{k-1}=(n-k+1)T^0_k+\frac{1}{2}(n-k)(n-k+1)H-N_n+\mu(T). \tag{14}$$

Proof. By (2) and Lemma 1 (iii), we have

$$(k+1)(T^0_{k-1}-2T^0_k+T^0_{k+1}-H)=\mu(T) \tag{15}$$

and

$$(n-k)(T^0_{k-1}-2T^0_k+T^0_{k+1}-H)=\mu(T). \tag{16}$$

Upon (15) and (16) summing over k from 0 to k and k to $n-1$ respectively, and using Lemma 1 (v), we get (13) and (14). q.e.d.

COROLLARY 1. *If* $j\geqq k$, *then*

$$(k+1)T^0_j=(j+1)T^0_k+\frac{1}{2}(j-k)(k+1)(j+1)H+\mu(T). \tag{17}$$

If $j\leqq k$, *then*

$$(n-k)T^0_j=(n-j)T^0_k+\frac{1}{2}(k-j)(n-k)(n-j)H-(k-j)N_n+\mu(T). \tag{18}$$

Proof. Straightforward induction from the lemma.

COROLLARY 2. *If*

$$H_k = \limsup_{r\to\infty} \frac{H(r)}{T_k^0(r)} < +\infty ,$$

where we assume that V *has an infinite harmonic exhaustion and that* x *is nondegenerate, then there exists a positive constant* c *such that*

$$\| T_k^0(r) \leqq T^0(r) \leqq c T_k^0(r) , \tag{19}$$

where $T^0(r) = \max\{T_0^0(r), \cdots, T_{n-1}^0(r)\}$, *and the sign* "$\|$" *in front of an inequality means that the inequality is only valid in* $[0, \infty) - I$ *with* $\int_I d \log t < \infty$.

Proof. We know that $\psi = \mu(\varphi)$ implies

$$\| \psi(r) < \lambda \log (C\varphi(r) + C') \tag{20}$$

for a constant $\lambda > 1$ ([5], (4.62)). Hence (17) and (18) imply

$$\|(k+1)T_j^0(r) < (j+1)T_k^0(r) + \frac{1}{2}(j-k)(k+1)(j+1)H(r)$$
$$+ \lambda \log (CT(r) + C') \qquad \text{if } j \geqq k$$

and

$$\|(n-k)T_j^0(r) < (n-j)T_k^0(r) + \frac{1}{2}(k-j)(n-k)(n-j)H(r)$$
$$+ \lambda \log (CT(r) + C') \qquad \text{if } j \leqq k .$$

Obviously, they together imply that for some positive constants c_1 and c_2.

$$\| T^0(r) < c_1 T_k^0(r) + c_2 \log (CT^0(r) + C') \cdot$$

Because $T_k(r) \to \infty$ as $r \to \infty$, so $T^0(r) \to \infty$ as $r \to \infty$. Thus for sufficiently large r,

$$c_2 \log (CT^0(r) + C') > \frac{1}{2} T^0(r) .$$

Combining with the above inequalities, we obtain (19). q.e.d.

REMARK. If x is nondegenerate and V has an infinite harmonic exhaustion and

$$\lambda_k = \limsup_{r\to\infty} \frac{-E(r)}{T_k(r)} < +\infty .$$

we also have

$$\| T_k(r) \leqq T(r) \leqq c T_k(r) . \qquad [5, \text{ p. } 140] \tag{21}$$

LEMMA 3. *If y_j are indeterminates over the ring $\boldsymbol{Z}$ and if $y_j=0$ for $j>n$, then we have the algebraic identity*

$$D_q(k,\, l\,;\, y)\equiv -\sum_{j=k}^{n-1}\sum_{i=0}^{l} P_q(j-i,\, l-i)(y_{j-i-1}-2y_{j-i}+y_{j-i+1}) \tag{22}$$

$$=P_q(k+1,\, l+1)y_k-\left(P_q(n+1,\, l+1)-\binom{n}{l+1}\binom{-1}{q-l}\right)y_n$$

$$-\sum_{i=k-l-1}^{k}\binom{i}{l-k+i+1}\binom{n-i-1}{q+k-l-i}y_i-\sum_{i=k+1}^{n}\binom{i}{l}\binom{n-i-1}{q-l-1}y_i\,,$$

where $0\leqq l\leqq \min(k,\, q)$. If $l=q$, then

$$D_q(k,\, q\,;\, y)=\binom{n+1}{q+1}y_k-\binom{n}{q}y_n-\sum_{i=k-q-1}^{k}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}y_i\,. \tag{23}$$

By definition,

$$P_q(k,\, l)=\binom{n+1}{q+1}-\sum_{j\geqq 0}\binom{k+1}{l+j+1}\binom{n-k}{q-l-j}, \qquad [5, \text{ p. } 182]$$

where $\binom{\alpha}{\beta}$ is defined for all integers by the binomial series

$$(1+x)^\alpha=\sum_{\beta=-\infty}^{+\infty}\binom{\alpha}{\beta}x^\beta\,.$$

Proof. We often use the following identities:

$$\binom{\alpha}{\beta}+\binom{\alpha}{\beta-1}=\binom{\alpha+1}{\beta}\,;\ \binom{\alpha}{\beta}=0\,, \qquad \text{if } \beta<0$$

$$\sum_{i+j=\beta}\binom{\alpha+1}{i}\binom{n-\alpha}{j}=\binom{n+1}{\beta} \qquad [5, \text{ p. } 194]$$

and

$$\sum_{j=q}^{q+r}\binom{p+q+r-j}{p}\binom{j}{q}=\binom{p+q+r+1}{r} \qquad [5, \text{ p. } 198] \tag{24}$$

which directly imply

$$P_q(j-i,\, l-i)=\begin{cases} 0 & \text{if } i\geqq l+1 \\ \binom{n+1}{q+1} & \text{if } i<l-q \end{cases} \tag{25}$$

and

$$P_q(k+1,\, l+1)-P_q(k,\, l)=\binom{k+1}{l+1}\binom{n-k-1}{q-l}. \qquad [5, \text{ p. } 195] \tag{26}$$

By (25), we can write

(27) $$D_q(k, l; y)=D'_q(k, l; y)+D''_q(k, l; y),$$

where

$$D'_q(k, l; y)=-\sum_{j=k}^{n-1}\sum_{i=-\infty}^{+\infty}P_q(j-i, l-i)(y_{j-i-1}-2y_{j-i}+y_{j-i+1})$$

and

$$D''_q(k, l; y)=\sum_{j=k}^{n-1}\sum_{i=-\infty}^{-1}P_q(j-i, l-i)(y_{j-i-1}-2y_{j-i}+y_{j-i+1}).$$

Obviously,

$$D''_q(k, l; y)=\sum_{j=k}^{n-1}(P_q(j+1, l+1)y_j-P_q(j, l)y_{j+1})$$

$$+\sum_{j=k}^{n-1}\sum_{i=j+1}^{+\infty}(P_q(i+1, l-j+1+i)-2P_q(i, l-j+i)+P_q(i-1, l-j+i-1))y_i$$

Change order summing, we obtain

$$D''_q(k, l; y)=P_q(k+1, l+1)y_k-P_q(n+1, l+1)y_n+\sum_{i=k+1}^{n}(a_i+b_i)y_i,$$

where

$$a_i=P_q(i+1, l+1)-P_q(i-1, l),$$

$$b_i=\sum_{j=k}^{i-1}(P_q(i+1, l-j+i+1)-2P_q(i, l-j+i)+P_q(i-1, l-j+i-1)).$$

By definition,

$$P_q(i, l)-P_q(i-1, l)=\sum_{j\ge 0}\left\{\binom{i}{l+j+1}\binom{n-i+1}{q-l-j}-\binom{i+1}{l+j+1}\binom{n-i}{q-l-j}\right\}$$

$$=\sum_{j\ge 0}\left\{\binom{i}{l+j+1}\left[\binom{n-i}{q-l-j}+\binom{n-i}{q-l-j-1}\right]-\left[\binom{i}{l+j+1}+\binom{i}{l+j}\right]\binom{n-i}{q-l-j}\right\}$$

$$=\sum_{j\ge 0}\left\{\binom{i}{l+j+1}\binom{n-i}{q-l-j-1}-\binom{i}{l+j}\binom{n-i}{q-l-j}\right\}$$

$$=-\binom{i}{l}\binom{n-i}{q-l},$$

which and (26) imply

$$a_i=P_q(i+1, l+1)-P_q(i, l)+P_q(i, l)-P_q(i-1, l)$$

$$=\binom{i+1}{l+1}\binom{n-i-1}{q-l}-\binom{i}{l}\binom{n-i}{q-l}$$

$$=\left[\binom{i}{l+1}+\binom{i}{l}\right]\binom{n-i-1}{q-l}-\binom{i}{l}\binom{n-i}{q-i}$$

$$=\binom{i}{l+1}\binom{n-i-1}{q-l}-\binom{i}{l}\binom{n-i-1}{q-l-1}.$$

By (26), we have

$$b_i=\sum_{j=k}^{i-1}\left\{\binom{i+1}{l-j+i+1}\binom{n-i-1}{q-l+j-i}-\binom{i}{l-j+i}\binom{n-i}{q-l+j-i+1}\right\}$$

$$=\sum_{j=k}^{i-1}\left\{\left[\binom{i}{l-j+i+1}+\binom{i}{l-j+i}\right]\binom{n-i-1}{q-l+j-i}\right.$$

$$\left.-\binom{i}{l-j+i}\left[\binom{n-i-1}{q-l+j-i+1}+\binom{n-i-1}{q-l+j-i}\right]\right\}$$

$$=\sum_{j=k}^{i-1}\left\{\binom{i}{l-j+i+1}\binom{n-i-1}{q-l+j-i}-\binom{i}{l-j+i}\binom{n-i-1}{q-l+j-i+1}\right\}$$

$$=\binom{i}{l-k+i+1}\binom{n-i-1}{q-l+k-i}-\binom{i}{l+1}\binom{n-i-1}{q-l}.$$

Hence we finally obtain

$$D_q''(k,\ l\ ;\ y)=P_q(k+1,\ l+1)y_k-P_q(n+1,\ l+1)y_n$$
$$+\sum_{i=k+1}^{n}\left\{\binom{i}{l-k+i+1}\binom{n-i-1}{q-l+k-i}-\binom{i}{l}\binom{n-i-1}{q-l-1}\right\}y_i. \tag{28}$$

In similar fashion, we have

$$D_q'(k,\ l\ ;\ y)=-\sum_{j=k}^{n-1}\sum_{i=-\infty}^{+\infty}(P_q(i+1,\ l-j+i+1)$$
$$-2P_q(i,\ l-j+1)+P_q(i-1,\ l-j+i-1))y_i \tag{29}$$

$$=-\sum_{i=-\infty}^{+\infty}\sum_{j=k}^{n-1}\left\{\binom{i+1}{l-j+i+1}\binom{n-i-1}{q-l+j-i}-\binom{i}{l-j+i}\binom{n-i}{q-l+j-i+1}\right\}y_i$$

$$=-\sum_{i=-\infty}^{+\infty}\sum_{j=k}^{n-1}\left\{\left[\binom{i}{l-j+i+1}+\binom{i}{l-j+i}\right]\binom{n-i-1}{q-l+j-i}\right.$$

$$\left.-\binom{i}{l-j+i}\left[\binom{n-i-1}{q-l+j-i+1}+\binom{n-i-1}{q-l+j-i}\right]\right\}y_i$$

$$=-\sum_{i=-\infty}^{+\infty}\sum_{j=k}^{n-1}\left\{\binom{i}{l-j+i+1}\binom{n-i-1}{q-l+j-i}-\binom{i}{l-j+i}\binom{n-i-1}{q-l+j-i+1}\right\}y_i$$

$$=-\sum_{i=-\infty}^{+\infty}\left\{\binom{i}{l-k+i+1}\binom{n-i-1}{q-l+k-i}-\binom{i}{l-n+i+1}\binom{n-i-1}{q-l+n-i}\right\}y_i$$

$$=-\sum_{i=k-l-1}^{n}\binom{i}{l-k+i+1}\binom{n-i-1}{q-l+k-i}y_i+\binom{n}{l+1}\binom{-1}{q-l}y_n,$$

because

$$y_i=0 \qquad \text{if } i>n\,,$$

$$\binom{i}{l-k+i+1}=0 \qquad \text{if } i<k-l-1\,,$$

$$\binom{n-i-1}{q-l+n-i}=0 \qquad \text{if } i\leqq n-1\,.$$

Thus (27), (28) and (29) imply (22). q.e.d.

COROLLARY.

$$C_k^q\equiv\sum_{j=k}^{n-1}\sum_{i=0}^{q}P_q(j-i,\,q-i)=\frac{n-k}{2}\left\{(n+1)\binom{n}{q}-(k-q)\binom{n+1}{q}\right\}. \tag{30}$$

Proof. In Lemma 3 we let

$$y_j=\begin{cases} j & \text{if } j\leqq n \\ 0 & \text{if } j>n\,, \end{cases}$$

then $D_q(k,\,q\,;\,y)=0$, so we have

$$\sum_{i=k-q-1}^{k}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}i=\binom{n+1}{q+1}k-\binom{n}{q}n \tag{31}$$

In Lemma 3 we take

$$y_j=\begin{cases} j(j+1) & \text{if } j\leqq n \\ 0 & \text{if } j>n. \end{cases}$$

Then

$$C_k^q=-\frac{1}{2}D_q(k,\,q\,;\,y)=\frac{n(n+1)}{2}\binom{n}{q}-\frac{k(k+1)}{2}\binom{n+1}{q+1}+I\,, \tag{32}$$

where

$$\begin{aligned} I&=\frac{1}{2}\sum_{i=k-q-1}^{k}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}i(i+1) \\ &=\frac{k-q}{2}\sum_{i=k-q-1}^{k}\binom{i+1}{q-k+i+1}\binom{n-i-1}{k-i}i \\ &=\frac{k-q}{2}\left\{\sum_{i=k-q-1}^{k}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}i+\sum_{i=k-q}^{k}\binom{i}{q-k+i}\binom{n-i-1}{k-i}i\right\}. \end{aligned}$$

By (31), we have

$$I=\frac{k-q}{2}\left\{k\binom{n+1}{q+1}-n\binom{n}{q}+k\binom{n+1}{q}-n\binom{n}{q-1}\right\}. \tag{33}$$

Thus (33) and (32) imply (30). q.e.d.

4. Proof of Theorem 2.

We have the following inequalities [5]

$$\sum_{A^q} m_k(A^q) = -\sum_{j=k}^{n-1} \sum_{i=0}^{q} P_q(j-i,\, q-i)(E+S_{j-i}+T_{j-i-1} \tag{34}$$

$$-2T_{j-i}+T_{j-i+1})+\mu(T^2), \qquad \text{if } 0\leqq q\leqq k \qquad [5,\ \text{p. } 193]$$

and

$$\sum_{A^q} m_k(A^q) = -\sum_{j=0}^{k} \sum_{i=0}^{n-q-1} P_{n-q-1}(n-j-i-1,\, n-q-i-1)(E+S_{j+i} \tag{35}$$

$$+T_{j+i-1}-2T_{j+i}+T_{j+i+1})+\mu(T^2), \qquad \text{if } q\geqq k. \quad [5,\ \text{p. } 201]$$

Firstly, let us deduce (4) for the case $0\leqq q\leqq k$. By (1) and Lemma 3, we have

$$\sum_{A^q} m_k(A^q) = D_q(k,\, q\,;\, T^0)+C_k^q H+\mu(T^2) \tag{36}$$

$$=\binom{n+1}{q+1}T_k^0-\binom{n}{q}N_n+\sum_{i=k-q-1}^{k}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}(-T_i^0)$$

$$+C_k^q H+\mu(T^2).$$

By Corollary 1 of Lemma 2 and (24), the sum of the right hand side of the above identity equals

$$-\sum_{i=k-q-1}^{k}\frac{i+1}{k+1}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}T_k^0$$

$$+\frac{1}{2}\sum_{i=k-q-1}^{k}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}(k-i)(i+1)H+\mu(T)$$

$$=-\frac{k-q}{k+1}\binom{n+1}{q+1}T_k^0+\frac{1}{2}(n-k)(k-q)\binom{n+1}{q}H+\mu(T),$$

where

$$\sum_{i=k-q-1}^{k}\frac{i+1}{k+1}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}$$

$$=\frac{k-q}{k+1}\sum_{i=k-q-1}^{k}\binom{i+1}{q-k+i+1}\binom{n-i-1}{k-i}$$

$$=\frac{k-q}{k+1}\sum_{j=k-q}^{k+1}\binom{n-j}{n-k-1}\binom{j}{k-q}=\frac{k-q}{k+1}\binom{n+1}{q+1} \qquad \text{(by (24))}$$

and

$$\sum_{i=k-q-1}^{k}\binom{i}{q-k+i+1}\binom{n-i-1}{k-i}(k-i)(i+1)$$

$$=(n-k)(k-q)\sum_{i=k-q-1}^{k}\binom{i+1}{q-k+i+1}\binom{n-i-1}{k-i-1}$$

$$=(n-k)(k-q)\sum_{j=k-q}^{k+1}\binom{n-j}{n-k}\binom{j}{k-q}$$

$$=(n-k)(k-q)\left\{\binom{n+1}{q}+\binom{k+1}{k-q}\binom{n-k-1}{n-k}\right\} \qquad \text{(by (24))}$$

$$=(n-k)(k-q)\binom{n+1}{q}.$$

Hence (30), (36) and Lemma 1 imply

$$\sum_{A^q} m_k(A^q)=\binom{n}{q}\left(\frac{n+1}{k+1}T_k^0-N_n+\frac{1}{2}(n+1)(n-k)H\right)+\mu(T^2). \tag{37}$$

Next, we deduce (4) for the case $q\geqq k$. Take $y_j=T^0_{n-1-j}$ in Lemma 3. Then (23) and (35) imply

$$\begin{aligned}\sum_{A^q} m_k(A^q)&=D_{n-q-1}(n-k-1,\ n-q-1\,;\ y)+C^{n-q-1}_{n-k-1}H+\mu(T^2)\\&=\binom{n+1}{q+1}T_k^0-\sum_{j=k}^{n-q+k}\binom{n-1-j}{q-k-1}\binom{j}{k}T_j^0+C^{n-q-1}_{n-k-1}H+\mu(T^2),\end{aligned} \tag{38}$$

$(y_n=T^0_{-1}=0)$. By Corollary 1 of Lemma 2 and (24), the sum of the right hand side of the above equals

$$-\sum_{j=k}^{n-q+k}\frac{n-j}{n-k}\binom{n-1-j}{q-k-1}\binom{j}{k}T_k^0+\frac{1}{2}\sum_{j=k}^{n-q+k}(j-k)(n-j)\binom{n-1-j}{q-k-1}\binom{j}{k}H$$

$$-\sum_{j=k}^{n-q+k}\frac{j-k}{n-k}\binom{n-1-j}{q-k-1}\binom{j}{k}N_n+\mu(T)$$

$$=-\frac{q-k}{n-k}\binom{n+1}{q+1}T_k^0+\frac{1}{2}(k+1)(q-k)\binom{n+1}{q+2}H-\frac{k+1}{n-k}\binom{n}{q+1}N_n+\mu(T),$$

where

$$\sum_{j=k}^{n-q+k}\frac{n-j}{n-k}\binom{n-1-j}{q-k-1}\binom{j}{k}$$

$$=\frac{q-k}{n-k}\sum_{j=k}^{n-q+k}\binom{n-j}{q-k}\binom{j}{k}=\frac{q-k}{n-k}\binom{n+1}{q+1}, \qquad \text{(by (24))}$$

$$\sum_{j=k}^{n-q+k}(j-k)(n-j)\binom{n-1-j}{q-k-1}\binom{j}{k}$$

$$=(k+1)(q-k)\sum_{j=k}^{n-q+k}\binom{n-j}{q-k}\binom{j}{k+1}$$

$$=(k+1)(q-k)\binom{n+1}{q+2} \qquad \text{(by (24))}$$

and

$$\sum_{j=k}^{n-q+k}\frac{j-k}{n-k}\binom{n-1-j}{q-k-1}\binom{j}{k}$$

$$=\frac{k+1}{n-k}\sum_{j=k+1}^{n-q+k}\binom{n-1-j}{q-k-1}\binom{j}{k+1}$$

$$=\frac{k+1}{n-k}\binom{n}{n-q-1}=\frac{k+1}{n-k}\binom{n}{q+1}, \qquad \text{(by (24))}$$

Hence (30), (38) and Lemma 1 imply

$$\text{(39)} \qquad \sum_{A^q} m_k(A^q)=\frac{n-q}{n-k}\binom{n+1}{q+1}T_k^0-\frac{k+1}{n-k}\binom{n}{q+1}N_n+\frac{(k+1)(n+1)}{2}\binom{n}{q+1}H+\mu(T^2).$$

Now (4) follows from (37) and (39).

Finally, by (3), we have

$$\text{(49)} \qquad \frac{n+1}{k+1}N_k(r)-N_n(r)+\frac{(n+1)(n-k)}{2}H(r)=-Q_k(r)-\frac{(n+1)(n-k)}{2}E(r)$$

which implies (5). q.e.d.

5. Discussion.

In this section, we assume that V has an infinite harmonic exhaustion and the conditions in Theorem 2 hold. Define

$$\theta_k=\limsup_{r\to\infty}\frac{Q_k(r)}{T_k(r)} \quad \text{and} \quad \Theta_k=\limsup_{r\to\infty}\frac{N_n(r)}{T_k^0(r)}.$$

For each q-dimensional projective subspace A^q of $\boldsymbol{P}_n$, we define the defects of A^q to be:

$$\delta_k(A^q)=\liminf_{r\to\infty}\frac{m_k(r, A^q)}{T_k(r)} \quad \text{and} \quad \Delta_k(A^q)=\liminf_{r\to\infty}\frac{m_k(r, A^q)}{T_k^0(r)}.$$

Clearly $0\leqq\Delta_k(A^q)\leqq\delta_k(A^q)\leqq 1$. If $H_k<+\infty$, by Theorem 2, (20) and Corollary 2 of Lemma 2 we have

$$\text{(41)} \qquad \sum_{A^q}\Delta_k(A^q)\leqq\varepsilon_k^q\binom{n}{q}\left\{\frac{n+1}{k+1}-\Theta_k+\frac{(n+1)(n-k)}{2}H_k\right\}.$$

If $\chi_k < +\infty$, by Theorem 2, (20) and (21) we have

$$\sum_{A^q} \delta_k(A^q) \leqq \varepsilon_k^q \binom{n}{q} \left\{ \frac{n+1}{k+1} - \theta_k + \frac{(n+1)(n-k)}{2} \chi_k \right\}. \tag{42}$$

Let $Z(\tilde{x})$ be the set of zero points of $|\tilde{x}|$. Let $\nu_p(f)$ denote the order of zero at p of a function f on V. Clearly, we see

$$\nu_p(|X_\gamma^1|) \geqq 2\nu_p(|\tilde{x}|) - 1, \qquad \text{if } p \in Z(\tilde{x}).$$

Define

$$s_p = \begin{cases} \text{the stationary index of } x \text{ at } p, & \text{if } p \text{ is a critical point of } x \\ 0 & \text{otherwise} \end{cases}$$

and

$$I_p = \nu_p(|X_\gamma^1|) - 2\nu_p(|\tilde{x}|) - s_p.$$

We can prove that

$$I_p = 0, \qquad \text{if } p \in V - Z(\tilde{x}).$$

Define

$$i(t) = \sum_{p \in V[t]} I_p \quad \text{and} \quad I(r) = \int_{r_0}^{r} i(t)dt.$$

Then

$$I(r) = N_1(r) - 2N_0(r) - S_0(r).$$

Hence (1) implies

$$E(r) + H(r) = I(r). \tag{43}$$

If $V = \boldsymbol{C}$ or $\boldsymbol{C} - \{0\}$, we can choose $\tilde{x}$ such that $Z(\tilde{x}) = \emptyset$, so

$$H(r) = -E(r).$$

If x is transcendental and $Z(\tilde{x})$ is a finite set, then

$$\lim_{r \to \infty} \frac{I(r)}{T_0(r)} = 0.$$

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