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A modified-Leray-α subgrid scale model of turbulence

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Abstract

Inspired by the remarkable performance of the Leray-α (and the Navier–Stokes alpha (NS-α), also known as the viscous Camassa–Holm) subgrid scale model of turbulence as a closure model to Reynolds averaged equations (RANS) for flows in turbulent channels and pipes, we introduce in this paper another subgrid scale model of turbulence, the modified Leray-α (ML-α) subgrid scale model of turbulence. The application of the ML-α to infinite channels and pipes gives, due to symmetry, similar reduced equations as Leray-α and NS-α. As a result the reduced ML-α model in infinite channels and pipes is equally impressive as a closure model to RANS equations as NS-α and all the other alpha subgrid scale models of turbulence (Leray-α and Clark-α). Motivated by this, we present an analytical study of the ML-α model in this paper. Specifically, we will show the global well-posedness of the ML-α equation and establish an upper bound for the dimension of its global attractor. Similarly to the analytical study of the NS-α and Leray-α subgrid scale models of turbulence we show that the ML-α model will follow the usual $k^{-5/3}$ Kolmogorov power law for the energy spectrum for wavenumbers in the inertial range that are smaller than $1/\alpha$ and then have a steeper power law for wavenumbers greater than $1/\alpha$ (where $\alpha > 0$ is the length scale associated with the width of the filter). This result essentially shows that there is some sort of parametrization of the large wavenumbers (larger than $1/\alpha$) in terms of the smaller wavenumbers. Therefore, the ML-α

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model can provide us another computationally sound analytical subgrid large eddy simulation model of turbulence.

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1. Introduction

In recent years analytic subgrid scale models of turbulence have been extensively studied (see, e.g., [5–7, 9, 3, 15, 16]). Of particular interest to us in this paper is the so-called Leray-α model of turbulence:

$$\begin{aligned} &\partial_t v - \nu\Delta v + (u\cdot\nabla)v = -\nabla p + f,\\ &\nabla\cdot u = \nabla\cdot v = 0,\\ &v = u - \alpha^2\Delta u,\\ &u(x,0) = u^{\mathrm{in}}(x). \end{aligned} \tag{1}$$

Formally, the above system is the Navier–Stokes equations system when $\alpha = 0$, i.e. $u = v$. In order to study the question of existence of solutions to the Navier-Stokes equation (NSE), Leray considered in his pioneering work [23] a general regularization form of the Navier-Stokes equation in which the relationship between u and v in (1) is given by $u = \phi_\alpha * v$, where ϕ_α is an arbitrary smoothing kernel such that u converges to v, in some sense, as $\alpha \to 0^+$. In the particular case of system (1), the kernel ϕ_α is nothing other than the Green's function associated with the Helmholtz operator $(I - \alpha^2\Delta)$. For this very reason, system (1) is called the Leray-α model. For abstract mathematical study, one can use different smoothing kernels which could lead to a more general smoothing operator between u and v (see, e.g., [25]). However, motivated by the remarkable performance of the Navier-Stokes-α (also known as the viscous Camassa–Holm equations (VCHE) or the Lagrangian-averaged Navier-Stokes-α (LANS-α)) as a closure model of turbulence in infinite channels and pipes (see [5–7]) the Helmholtz relation between the unknown functions u and v becomes the natural choice. This is because this Helmholtz relation between the unknown function u and v leads to a simple system of differential equations which can be solved explicitly when the Navier-Stokes-α (NS-α) system is applied to infinite channels and pipes. Most importantly, these explicit solutions give excellent agreement with experimental data for a wide range of huge Reynolds numbers, up to about 17×10^6. This Helmholtz relation between u and v can be traced back from the derivation of the Euler-α model employing the Hamilton variational principle [21]. In [11], however, another approach connecting Lagrangian and Eulerian formulations for the Euler and Navier–Stokes equations was introduced. This exact connection between Lagrangian and Eulerian formulations gives another perspective—a numerical approximation point of view—for looking at the relation between the Navier–Stokes equations and the NS-α (LANS-α or VCHE) and Leray-α models.

Viewed as a subgrid scale model of turbulence, system (1) has been studied analytically in [9] and computationally in [18, 19, 22]. In particular, it was stressed in [9] that by using this model as a closure model in turbulent channels and pipes one obtains the same reduced system of equations as those produced by the Navier-Stokes-α (NS-α) model, whose solutions give, as we have already remarked above, excellent agreement with empirical data for a wide range of large Reynolds numbers [5–7]. It is also worth noting that the numerical study of the LANS-α subgrid scale turbulence model in [24] shows that this model, indeed, captures most of the large-scale features of a turbulent flow, in particular, those scales of motion larger than the lengthscale α, while the scales of motion smaller than α follow a faster decay of energy when compared with the energy of the NSE making it a more computable analytical subgrid large eddy simulation model of turbulence. It was this very remarkably successful comparison with experimental data and the manifestation of the expected results of the NS-α computations in comparison with the direct numerical simulation (DNS) of NSE which stoke the interest of the turbulence community in the alpha subgrid scale models of turbulence. Inspired by the above Leray-α model (1) we consider the following modified Leray-α model:

$$\begin{aligned}
&\partial_t v - \nu\Delta v + (v\cdot\nabla)u = -\nabla p + f,\\
&\nabla\cdot u = \nabla\cdot v = 0,\\
&v = u - \alpha^2\Delta u,\\
&u(x,0) = u^{\mathrm{in}}(x),
\end{aligned}\tag{2}$$

which we will call, in short, the ML-α model. To obtain the ML-α model we replaced the nonlinear term $(u\cdot\nabla)v$ in the Leray-α model (1) by $(v\cdot\nabla)u$. Interestingly enough, as we will demonstrate in section 3, when considering the ML-α as a closure model of the Reynolds averaged equations in turbulent channels and pipes, one arrives at the same system (up to the modified pressure) of reduced equations as those obtained from the NS-α (LANS-α or VCHE) [5–7], the Clark-α [3] and the Leray-α model [9]. (Also, as in [3, 5–7, 9] we use the Helmholtz operator in the definition of the variable v: $v = u-\alpha^2\Delta u$.) Hence, the ML-α model should, in principle, enjoy the same success story as the other alpha subgrid scale models of turbulence. This is the main purpose of introducing this model and consequently this analytical study. A computational investigation studying the performance of the ML-α model in comparison with other subgrid scale models of turbulence is the subject of future research.

We start by introducing some preliminary background and *a priori* estimates in section 2. In section 3 we show that the reductions of the ML-α model in channels and pipes are the same (up to modified pressure) as those of the NS-α in channels and pipes, respectively. In sections 4 and 5 we show the global well-posedness of the ML-α subgrid scale model of turbulence and establish estimates for the dimension of its global attractor. It is worth mentioning that from the *a priori* estimates established in section 4, one can extract subsequences of solutions which converge as $\alpha\to 0^+$ (in the appropriate sense) to a weak solution of the Navier-Stokes equations on any time interval $[0,T]$. This can be done along the lines of VCHE or NS-α in [15], or as has been observed for the Leray-α model in [9]. Sections 5 and 6 contain a discussion on the number of degrees of freedom and energy spectra of the ML-α model.

2. Functional setting and preliminaries

Let $\Omega = [0, 2\pi L]^3$. The ML-α subgrid scale turbulence model (2) of viscous incompressible flows, subject to periodic boundary condition, with basic domain Ω, is written in the expanded form:

$$\begin{aligned}
&\partial_t(u-\alpha^2\Delta u) - \nu\Delta(u-\alpha^2\Delta u) + ((u-\alpha^2\Delta u)\cdot\nabla)u = -\nabla p + f,\\
&\nabla\cdot u = 0,\\
&u(x,0) = u^{\mathrm{in}}(x),
\end{aligned}\tag{3}$$

where u represents the unknown 'filtered' fluid velocity vector, p is the unknown 'filtered' pressure scalar, $\nu > 0$ is the constant kinematic viscosity and $\alpha > 0$ is a lengthscale parameter which represents the width of the filter. The function f is a given body forcing assumed, for the simplicity of our presentation, to be time independent and u^{in} is the given initial velocity.

Next, we introduce some preliminary background material following the usual notation used in the context of the mathematical theory of Navier-Stokes equations (NSEs) (see, e.g., [12, 30, 31]).

(i) We denote by L^p and H^m the usual Lebesgue and Sobolev spaces. We denote by $|\cdot|$ and $(\cdot,\cdot)$ the L^2-norm and L^2-inner product and by $\|\cdot\|$ and $((\cdot,\cdot)) = (\nabla\cdot,\nabla\cdot)$ the H^1-norm and H^1-inner product, respectively.

(ii) Let $\mathcal{F}$ be the set of all vector trigonometric polynomials with periodic domain Ω. We then set

$$\mathcal{V} = \left\{\phi \in \mathcal{F} : \nabla\cdot\phi = 0 \text{ and } \int_\Omega \phi(x)\,\mathrm{d}x = 0\right\}.$$

We set H and V to be the closures of $\mathcal{V}$ in L^2 and H^1, respectively.

(iii) The orthogonal projection of $(L^2)^3$ onto H is denoted by $P_\sigma : (L^2)^3 \to H$ and is called the Helmholtz–Leray projection, and we denote by $A = -P_\sigma\Delta$ the Stokes operator subject to periodic boundary condition with domain $D(A) = (H^2(\Omega))^3 \cap V$. We note that in the space-periodic case,

$$Au = -P_\sigma\Delta u = -\Delta u, \qquad \text{for all } u \in D(A).$$

The operator A^{-1} is a self-adjoint positive definite compact operator from H into H (cf [12, 30]). We denote by $0 < L^{-2} = \lambda_1 \leqslant \lambda_2 \leqslant \ldots\lambda_j\ldots$ the eigenvalues of A, repeated according to their multiplicities. It is well known that in three dimensions the eigenvalues of the operator A satisfy the Weyl type formula (see, e.g., [2, 12, 28, 31]); namely, there exists a dimensionless constant $c_0 > 0$ such that

$$\frac{j^{2/3}}{c_0} \leqslant \frac{\lambda_j}{\lambda_1} \leqslant c_0 j^{2/3}, \qquad \text{for } j = 1, 2, \ldots. \tag{4}$$

(iv) We recall the following three-dimensional interpolation and Sobolev inequalities (see, e.g., [1]):

$$\|\phi\|_{L^3} \leqslant c\|\phi\|_{L^2}^{1/2}\|\phi\|_{H^1}^{1/2} \quad \text{and} \quad \|\phi\|_{L^6} \leqslant c\|\phi\|_{H^1}, \qquad \text{for every } \phi \in H^1(\Omega). \tag{5}$$

Also, recall Agmon's inequality [2] (see also [12]):

$$\|\phi\|_{L^\infty} \leqslant c\|\phi\|_{H^1}^{1/2}\|\phi\|_{H^2}^{1/2}, \qquad \text{for every } \phi \in H^2(\Omega). \tag{6}$$

Hereafter, c will denote a generic dimensionless constant.

(v) For $w_1, w_2 \in \mathcal{V}$, we define the bilinear form

$$B(w_1, w_2) = P_\sigma((w_1\cdot\nabla)w_2). \tag{7}$$

In the following lemma, we will list certain relevant inequalities and properties of B.

Lemma 1. *The bilinear form B defined in (7) satisfies the following.*

(i) B can be extended as a continuous map $B : V \times V \to V'$, where V' is the dual space of V. In particular, for every $w_1, w_2, w_3 \in V$, the bilinear form B satisfies the following inequalities:

$$|\langle B(w_1, w_2), w_3\rangle_{V'}| \leqslant c|w_1|^{1/2}\|w_1\|^{1/2}\|w_2\|\|w_3\|. \tag{8}$$

$$\langle B(w_1, w_2), w_3\rangle_{V'} = -\langle B(w_1, w_3), w_2\rangle_{V'}, \tag{9}$$

and, in particular,

$$\langle B(w_1, w_2), w_2\rangle_{V'} = 0. \tag{10}$$

(ii) Further, we have

$$|\langle B(w_1, w_2), w_3\rangle_{D(A)'}| \leqslant c|w_1|\|w_2\|\|w_3\|^{1/2}|Aw_3|^{1/2} \tag{11}$$

for every $w_1 \in H$, $w_2 \in V$ and $w_3 \in D(A)$.

Proof. The proof of (i) can be found in, e.g., [12, 30]. The proof of (ii) can be found, e.g., in [15]. □

Following the above functional notation we observe that $v = u + \alpha^2 Au$ and the system (2) (or (3)) can be written as

$$\begin{aligned} &\frac{\mathrm{d}v}{\mathrm{d}t} + \nu Av + B(v, u) = f, \\ &v = u + \alpha^2 Au, \\ &u(0) = u^{\mathrm{in}}. \end{aligned} \tag{12}$$

Without loss of generality, we assume $P_\sigma f = f$ since we can modify the pressure to include the gradient part of f.

We also observe, thanks to the Poincaré inequality, that

$$|v| \leqslant (\lambda_1^{-1} + \alpha^2)|Au| = (L^2 + \alpha^2)|Au|. \tag{13}$$

Definition 2 (regular solution). *Let $f \in H$, $u(0) = u^{\mathrm{in}} \in V$ and let $T > 0$. A function $u \in C([0, T); V) \cap L^2([0, T); D(A))$ with $(\mathrm{d}u/\mathrm{d}t) \in L^2([0, T); H)$ is said to be a regular solution of (3) (or (12)) on the interval $[0, T)$ if it satisfies*

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(u + \alpha^2 Au), w \right\rangle_{D(A)'} + \nu \left\langle A(u + \alpha^2 Au), w \right\rangle_{D(A)'} + \left\langle B(u + \alpha^2 Au, u), w \right\rangle_{D(A)'} = (f, w) \tag{14}$$

for every $w \in D(A)$ and almost every $t \in [0, T)$, and where (14) is understood in the following sense: for almost every t_0, $t \in [0, T)$ we have

$$\begin{aligned} &(u(t) + \alpha^2 Au(t), w) - (u(t_0) + \alpha^2 Au(t_0), w) + \nu \int_{t_0}^{t} (u(s) + \alpha^2 Au(s), Aw)\mathrm{d}s \\ &\qquad + \int_{t_0}^{t} \left\langle B(u(s) + \alpha^2 Au(s), u(s)), w \right\rangle_{D(A)'} \mathrm{d}s = \int_{t_0}^{t} (f, w)\mathrm{d}s. \end{aligned} \tag{15}$$

3. The modified-Leray-α as a turbulence closure model

It is well accepted that the Navier-Stokes equations (NSE) are the governing equations of the dynamics of viscous incompressible fluid flows. As of now, our current scientific methods or tools are unable to solve NSE analytically or to perform reliable direct numerical simulation of NSE due to the large range of scales of motion that need to be resolved when the Reynolds number is high. For our practical needs, we attempt to look for averaged quantities, i.e. flow variables such as velocity and pressure are long time-averaged. However, these time-averaged governing equations (called Reynolds averaged Navier-Stokes (RANS) equations) are not closed (i.e. we have fewer equations than unknowns). Turbulence models are attempting to close this system by modelling some of the unknown quantities in terms of the other known ones. As it stands, no single turbulence model is valid for general types of flows.

In this section, we will consider the ML-α model as a closure model to the RANS equations for flows in turbulent channels and pipes. We will show that the reduction of the system of equations in (3) or (2) in the infinite channels and pipes are the same (up to modified pressure) as the system of equations obtained in the case of NS-α (VCHE), [5–7]), Leray-α [9] and Clark-α [10]. Hence, the general solution to the reduced ML-α system of equations in channels and pipes will be the same (up to modified pressure) as the general solution of the NS-α in pipe and channel symmetry as obtained in [5–7]. As a result, we will observe an excellent agreement between the experimental empirical data and the solutions of the reduced ML-α in channels and pipes when we identify these solutions with the solutions of RANS equations (i.e. use it as a closure model) under the corresponding symmetry conditions, sharing the same success stories of the NS-α model (VCHE) and other alpha models.

To be more specific, let us begin by recalling RANS equations in channels and pipes (see, e.g., [26]). For a given function $\phi(x,t)$ we denote

$$\langle\phi\rangle(x) = \bar{\phi}(x) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \phi(x,t)\mathrm{d}t, \tag{16}$$

assuming that such a limit exists (see, e.g., [17] for the generalization of the notion of the limit to make sense of infinite time averages.) The long (infinite) time average of the NSE, i.e. the stationary RANS equations, is given by

$$\begin{aligned}
&(\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{u}} = \nu\Delta\bar{\boldsymbol{u}} - \boldsymbol{\nabla}\bar{p} - \overline{(\mathbf{u}-\bar{\boldsymbol{u}})\cdot\boldsymbol{\nabla}(\mathbf{u}-\bar{\boldsymbol{u}})},\\
&\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0.
\end{aligned} \tag{17}$$

Observe that the system above is not closed since we cannot express it in terms of $\bar{\boldsymbol{u}}$ alone.

3.1. The RANS equations for turbulent channel flows

Based on experimental observations of turbulent Poiseuille flows in infinite channels (see, e.g., [26, 32]), the mean velocity in (17) for turbulent channel flows admits the form $\bar{\boldsymbol{u}} = \left[\bar{U}(z), 0, 0\right]^T$, where $\bar{U}(z) = \bar{U}(-z)$, with mean pressure $\bar{p} = \bar{P}(x,y,z)$. Hence, the RANS system (17) under such symmetry reduces to

$$\begin{aligned}
&-\nu\bar{U}'' + \partial_z\langle wu\rangle = -\partial_x\bar{P},\\
&\partial_z\langle wv\rangle = -\partial_y\bar{P},\\
&\partial_z\left\langle w^2\right\rangle = -\partial_z\bar{P},
\end{aligned} \tag{18}$$

where the prime ($'$) denotes the derivative in the z direction and $(u,v,w)^T = \boldsymbol{u}-\bar{\boldsymbol{u}}$ is the fluctuation in the velocity in the infinite channel $\{(x,y,z) \in \mathbb{R}, -d \leqslant z \leqslant d\}$. Moreover, experimental observations (see, e.g., [26, 32]) indicate that the Reynolds stresses $\langle wu\rangle$, $\langle wv\rangle$ and $\left\langle w^2\right\rangle$ are also functions of the variable z alone. At the boundary we have the conditions $\bar{U}(\pm d) = 0$ (no-slip) and $\nu\bar{U}'(\pm d) = \mp\tau_0$, where τ_0 is the boundary shear stress. Thus, using the boundary conditions $\langle wu\rangle(\pm d) = \langle wv\rangle(\pm d) = 0$, the Reynolds equations imply that $\langle wv\rangle = 0$ and $\bar{P} = P_0 - \tau_0 x/d - \left\langle w^2\right\rangle(z)$, with integration constant P_0.

3.2. The reduced ML-α for channel flows

For the ML-α system of equations, under the channel symmetry, we denote by $\boldsymbol{U}$ the velocity u in (2) and we seek its steady-state solutions in the form $\boldsymbol{U} = [U(z), 0, 0]^T$, with even reflection symmetry condition $U(z) = U(-z)$ and boundary condition $U(\pm d) = 0$. Under these conditions, the steady ML-α reduces to

$$\begin{aligned}
&-\nu V'' = -\nu U'' + \nu\alpha^2 U'''' = -\partial_x p,\\
&0 = -\partial_y p,\\
&0 = -\partial_z p,
\end{aligned} \tag{19}$$

where $V = U - \alpha^2 U$ and p is a pressure function. Note here that we need additional boundary conditions to determine V. Such boundary conditions are not available based on physical considerations. However, in this case, and under the symmetry of the channel, the missing boundary conditions appear as free parameters that will be determined through a tuning process with empirical data.

3.3. Identifying ML-α with RANS—the channel case

Following the idea of [5–7] we identify the systems (18) and (19) with each other, which is the essence of our closure assumption. We compare (18) and (19), and as a result, we identify the various counterparts as

$$\begin{aligned}
&\bar{U} = U,\\
&\partial_z \langle wu \rangle = \nu\alpha^2 U'''' + p_1,\\
&\partial_z \langle wv \rangle = 0,\\
&\nabla(\bar{P} + \langle w^2 \rangle) = \nabla(p - p_1 x),
\end{aligned} \tag{20}$$

for some constant p_1. This identification gives

$$\begin{aligned}
&\langle wv \rangle = 0,\\
&-\langle wu \rangle (z) = -p_1 z - \nu\alpha^2 U'''
\end{aligned} \tag{21}$$

and leaves $\langle w^2 \rangle$ undetermined up to an arbitrary function of z. The identification in (20) is exactly the same (up to modified pressure and possibly $\langle w^2 \rangle$) identification that was derived when identifying the NS-α model (VCHE) with the RANS equations in the channel symmetry in [5–7]. The same identification holds true in the case of the Leray-α model [9] and the Clark-α model in [3]. Therefore, the general solution of ML-α and NS-α will be identical (up to a modified pressure) and, in particular, the mean flows in both cases will have the same functions. As a result, the performance of the ML-α model as a closure model to RANS equations for channel flows will be the same as the performance of NS-α which gave an excellent match with empirical data as has been reported in detail in [5–7].

A similar result applies to turbulent pipe flows.

3.4. The RANS equations for turbulent pipe flows

We consider a cylindrical pipe oriented along the x axis of radius d. Here we used d to denote radius (instead of diameter) to be consistent with the lateral distance d from the axis of symmetry that we used for the infinite channel and we will denote the coordinate system as (r, θ, x) to be consistent with [5–7]. Based on experimental observations of turbulent Hagen–Poisueille flows, the average of the velocity in the infinite pipe depends only on r and the mean velocity in (17) for turbulent pipe flows in cylindrical coordinates admits the form $\bar{\boldsymbol{u}} = [0, 0, \bar{U}(r)]^T$, with mean pressure $\bar{p} = \bar{P}(r, \theta, x)$. Hence, the RANS system reduces to

$$\begin{aligned}
&\partial_r \langle u^2 \rangle = -\partial_r \bar{P},\\
&0 = -\frac{1}{r}\partial_\theta \bar{P},\\
&-\nu\left(\bar{U}'' + \frac{1}{r}\bar{U}'\right) + \partial_r \langle uw \rangle = -\partial_x \bar{P},
\end{aligned} \tag{22}$$

where the prime ($'$) denotes the derivative with respect to r and $(u, v, w)^T = \boldsymbol{u} - \bar{\boldsymbol{u}}$ is the fluctuation in the velocity in the infinite pipe $\{(r, \theta, x) \mid 0 \leqslant r \leqslant d,\ 0 \leqslant \theta \leqslant 2\pi,\ x \in \mathbb{R}\}$. At the boundary we have the no-slip boundary condition $\bar{U}(d) = 0$ and $\nu\bar{U}'(d) = \tau_0$, where τ_0 is the boundary shear stress.

3.5. *The reduced ML-α in an infinite pipe*

Under the pipe symmetry, we denote by $\boldsymbol{U}$ the velocity u in ML-α (2) and we seek its steady-state solutions in cylindrical coordinates in the form $\boldsymbol{U} = [0, 0, U(r)]^T$ subject to boundary condition $U(d) = 0$. Under this condition, the steady ML-α reduces to

$$\begin{aligned}
&0 = -\partial_r p,\\
&0 = -\frac{1}{r}\partial_\theta p,\\
&-\nu\left(V'' + \frac{1}{r}V'\right) = -\nu(U'' + \frac{1}{r}U') + \nu\alpha^2\left[\frac{1}{r}(U'' + \frac{1}{r}U')' + \left(U'' + \frac{1}{r}U'\right)''\right] = -\partial_x p,
\end{aligned} \tag{23}$$

where $V = U - \alpha^2(U'' + (1/r)U')$, p is a pressure function and $(')$ denotes the derivative with respect to r.

3.6. *Identifying ML-α with RANS—the pipe case*

Here again we follow the idea in [5–7] and identify the system (22) and (23). Indeed, by identifying the various counterparts we have

$$\begin{aligned}
&\bar{U} = U,\\
&\partial_r \left\langle u^2\right\rangle = 0,\\
&\partial_r \langle uw\rangle = \nu\alpha^2\left[\frac{1}{r}\left(U'' + \frac{1}{r}U'\right)' + \left(U'' + \frac{1}{r}U'\right)''\right].
\end{aligned} \tag{24}$$

This is exactly the same identification (up to modified pressure) with the RANS equations as in the case of NS-α (VCHE) (see in [6] (5.1)) under the pipe symmetry. We conclude that similarly to the behaviour of the general solution of the reduced NS-α equation in the pipe, the general solution of the reduced ML-α equation in the pipe will have an excellent match with the empirical mean velocity in the pipe for a wide range of huge Reynolds numbers as has been reported in [5–7].

4. Existence and uniqueness

In this section we will prove the global well-posedness of the system in (3). The estimates and steps presented here are formal. One can justify them in a rigorous fashion by, for the instance, establishing them first for the finite dimensional Galerkin approximation scheme and then passing to the limit using the appropriate Aubin compactness theorems (see, e.g., [12, 15, 25, 30, 31]). In this section, we fix $T > 0$ to be arbitrarily large.

4.1. *H^1 estimates*

We take the inner product of (12) with u and use (10) to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|u|^2 + \alpha^2\|u\|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) = (f, u). \tag{25}$$

Note that by the Cauchy–Schwarz inequality we have

$$|(f, u)| \leqslant \begin{cases} |A^{-1}f||Au|,\\ |A^{-1/2}f|\|u\| \end{cases} \tag{26}$$

and by Young's inequality we have

$$|(f,u)| \leqslant \begin{cases} \dfrac{|A^{-1}f|^2}{2\nu\alpha^2} + \dfrac{\nu}{2}\alpha^2|Au|^2, \\ \dfrac{|A^{-1/2}f|^2}{2\nu} + \dfrac{\nu}{2}\|u\|^2. \end{cases} \tag{27}$$

We let $K_1 = \min\{|A^{-1}f|^2/\nu\alpha^2, |A^{-1/2}f|^2/\nu\}$, from the above inequalities we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(|u|^2 + \alpha^2\|u\|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) \leqslant K_1. \tag{28}$$

We then apply Poincaré's inequality to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(|u|^2 + \alpha^2\|u\|^2) + \nu\lambda_1(|u|^2 + \alpha^2\|u\|^2) \leqslant K_1. \tag{29}$$

Applying Gronwall's inequality we obtain

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \leqslant \mathrm{e}^{-\nu\lambda_1 t}(|u(0)|^2 + \alpha^2\|u(0)\|^2) + \frac{K_1}{\nu\lambda_1}(1 - \mathrm{e}^{-\nu\lambda_1 t}), \tag{30}$$

that is,

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \leqslant k_1 := |u(0)|^2 + \alpha^2\|u(0)\|^2 + \frac{K_1}{\nu\lambda_1}. \tag{31}$$

Thus, $u \in L^\infty([0,T],V)$.

4.2. H^2 estimates

Integrating (28) over the interval $(t, t+r)$ for $r > 0$, we obtain

$$\nu\int_t^{t+r}(\|u(s)\|^2 + \alpha^2|Au(s)|^2)\mathrm{d}s \leqslant rK_1 + |u(t)^2| + \alpha^2\|u(t)\|^2 \leqslant rK_1 + k_1. \tag{32}$$

This implies that $u \in L^2([0,T], D(A))$. Now let us take the inner product of (12) with Au; we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \alpha^2|Au|^2) + \nu(|Au|^2 + \alpha^2|A^{3/2}u|^2) + (B(v,u), Au) = (f, Au). \tag{33}$$

Note that

$$|(f, Au)| \leqslant \begin{cases} |A^{-1/2}f||A^{3/2}u|, \\ |f||Au|. \end{cases} \tag{34}$$

Again by Young's inequality we have

$$|(f, Au)| \leqslant \begin{cases} \dfrac{|A^{-1/2}f|^2}{\nu\alpha^2} + \dfrac{\nu}{4}\alpha^2|A^{3/2}u|^2, \\ \dfrac{|f|^2}{\nu} + \dfrac{\nu}{4}|Au|^2. \end{cases} \tag{35}$$

We denote by $K_2 = \min\{|A^{-1/2}f|^2/\nu\alpha^2, |f|^2/\nu\}$. Then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \alpha^2|Au|^2) + \frac{3\nu}{4}(|Au|^2 + \alpha^2|A^{3/2}u|^2) \leqslant K_2 + |\langle B(v, Au), u\rangle_{D(A)'}|. \tag{36}$$

Using (8), Young's inequality, (5) and (13) we obtain

$$\begin{aligned} |\langle B(v, Au), u\rangle_{D(A)'}| &\leqslant c|v|^{1/2}\|v\|^{1/2}|A^{3/2}u|\|u\| \\ &\leqslant c(\lambda_1^{-1} + \alpha^2)|Au|^{1/2}|A^{3/2}u|^{3/2}\|u\| \\ &\leqslant c(\lambda_1^{-1} + \alpha^2)^4\frac{\|u\|^4|Au|^2}{(\nu\alpha^2)^3} + \frac{3\nu\alpha^2}{4}|A^{3/2}u|^2. \end{aligned} \tag{37}$$

Using the above estimates and (36) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2+\alpha^2|Au|^2) \leqslant K_2 + c(\lambda_1^{-1}+\alpha^2)^4\frac{\|u\|^4|Au|^2}{(\nu\alpha^2)^3}. \tag{38}$$

We integrate the above equation over the interval (s, t) and use (31) and (32) to obtain

$$\|u(t)\|^2+\alpha^2|Au(t)|^2 \leqslant \|u(s)\|^2+\alpha^2|Au(s)|^2+2(t-s)K_2$$
$$+(\lambda_1^{-1}+\alpha^2)^4\frac{2ck_1^2}{(\nu\alpha^2)^4\alpha^4}[(t-s)K_1+k_1]. \tag{39}$$

Now, we integrate with respect to s over the interval $(0, t)$ and use (32)

$$t(\|u(t)\|^2+\alpha^2|Au(t)|^2) \leqslant \frac{1}{\nu}(tK_1+k_1)+t^2K_2+(\lambda_1^{-1}+\alpha^2)^4\frac{2ck_1^2}{(\nu\alpha^2)^4\alpha^4}\left[\frac{t^2K_1}{2}+tk_1\right], \tag{40}$$

for all $t \geqslant 0$.

For $t \geqslant 1/\nu\lambda_1$ we integrate (39) with respect to s over the interval $(t-1/\nu\lambda_1, t)$:

$$\frac{1}{\nu\lambda_1}(\|u(t)\|^2+\alpha^2|Au(t)|^2) \leqslant \frac{1}{\nu}\left(\frac{1}{\nu\lambda_1}K_1+k_1\right)+\left(\frac{1}{\nu\lambda_1}\right)^2K_2$$
$$+(\lambda_1^{-1}+\alpha^2)^4\frac{2ck_1^2}{(\nu\alpha^2)^4\alpha^4}\left[\left(\frac{1}{\nu\lambda_1}\right)^2\frac{K_1}{2}+\frac{k_1}{\nu\lambda_1}\right]. \tag{41}$$

Thus, from (40) and (41) we conclude that

$$\|u(t)\|^2+\alpha^2|Au(t)|^2 \leqslant k_2(t) \tag{42}$$

for all $t > 0$. We note that $k_2(t)$ enjoys the following properties.

(i) $k_2(t)$ is finite for all $t > 0$.
(ii) If $u^{\mathrm{in}} \in V$, but $u^{\mathrm{in}} \notin D(A)$, then the $\lim_{t\to 0^+} k_2(t) = \infty$.
(iii) $\limsup_{t\to\infty} k_2(t) < \infty$.

From (39) it is clear that if $u^{\mathrm{in}} \in D(A)$ then $u \in L^\infty([0,T]; D(A))$. On the other hand, if $u^{\mathrm{in}} \in V$, but $u^{\mathrm{in}} \notin D(A)$, we conclude from the above that $u \in L^\infty_{loc}((0,T], D(A)) \cap L^2([0,T], D(A))$.

Theorem 3 (global existence and uniqueness). *Let $f \in H$ and $u^{\mathrm{in}} \in V$. Then for any $T > 0$, (3) has a unique regular solution u in $[0,T)$. Furthermore, this solution depends continuously on the initial data as a map from V to $C([0,T], V)$.*

Proof. One can establish the global existence of regular solution of the ML-α system by applying a standard Galerkin approximation procedure together with the *a priori* estimates established above and then apply Aubin's compactness theorem to pass to the limit.

Uniqueness of regular solution. Next we will show the continuous dependence of regular solutions on the initial data and, in particular, we will show the uniqueness of regular solutions.

Let u and $\bar{u}$ be any two solutions of (3) on the interval $[0,T]$, with initial values $u(0) = u^{\mathrm{in}} \in V$ and $\bar{u}(0) = \bar{u}^{\mathrm{in}} \in V$, respectively. Let us denote by $v = (u+\alpha^2 Au)$, $\bar{v} = (\bar{u}+\alpha^2 A\bar{u})$, $\delta u = u - \bar{u}$ and $\delta v = v - \bar{v}$. Then from (3) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta v + \nu A\delta v + B(\delta v, u) + B(\bar{v}, \delta u) = 0. \tag{43}$$

By taking the $D(A)'$ action of (43) with δu,

$$\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\delta v, \delta u\right\rangle_{D(A)'} + \nu(\|\delta u\|^2+\alpha^2|A\delta u|^2) + \langle B(\delta v, u), \delta u\rangle_{D(A)'} + \langle B(\bar{v}, \delta u), \delta u\rangle_{D(A)'} = 0, \tag{44}$$

and by applying a lemma of Lions–Magenes concerning the derivative of functions with values in Banach space (cf chapter III, p 169, [30]) and by an analogue of (10) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\delta u|^2+\alpha^2\|\delta u\|^2)+\nu(\|\delta u\|^2+\alpha^2\|A\delta u\|^2)+\langle B(\delta v, u), \delta u\rangle_{D(A)'}=0. \tag{45}$$

Now we use (11) and (13) to reach

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\delta u|^2+\alpha^2\|\delta u\|^2)+\nu(\|\delta u\|^2+\alpha^2\|A\delta u\|^2)\leqslant c\,|\delta v|\|\delta u\|^{1/2}|A\delta u|^{1/2}\|u\|$$
$$\leqslant c(\lambda_1^{-1}+\alpha^2)^4\,|A\delta u|^{3/2}\|\delta u\|^{1/2}\|u\|. \tag{46}$$

We then apply Young's inequality to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\delta u|^2+\alpha^2\|\delta u\|^2)+\nu(\|\delta u\|^2+\alpha^2\|A\delta u\|^2)\leqslant\frac{C(\lambda_1^{-1}+\alpha^2)^4}{(\nu\alpha)^3}\,\|\delta u\|^2\|u\|^4+\frac{\nu\alpha^2}{2}|A\delta u|^2, \tag{47}$$

where $C=(27/8)c$. By Gronwall inequality, we obtain

$$(|\delta u(t)|^2+\alpha^2\|\delta u(t)\|^2)\leqslant(|\delta u(0)|^2+\alpha^2\|\delta u(0)\|^2)\exp\left(\int_0^t\frac{C(\lambda_1^{-1}+\alpha^2)^4\|u\|^4}{(\nu\alpha^2)^3\alpha^2}\right). \tag{48}$$

Since $u\in L^\infty([0,T];V)$, thanks to (31), we conclude the continuous dependence of the solutions on initial data, as a map from V into $C([0,T],V)$, for every bounded interval $[0,T]$. In particular, we also conclude the uniqueness of the solution to (3). □

As mentioned earlier, the above *a priori* estimates will enable us to obtain uniform estimates on $\mathrm{d}u/\mathrm{d}t$, in the appropriate norms, which are independent of α. Following similar arguments as those presented in [15], one can extract subsequences which converge to weak solutions of the 3D Navier Stokes equations as $\alpha\to 0^+$.

5. Global attractors, their dimension and connection to small scales

In this section we will show the existence of global attractor $\mathcal{A}\subset V$ for system (3). Moreover, we will show that $\mathcal{A}$ has finite Hausdorff and fractal dimensions. To estimate the dimension of the attractor, we recall the following lemmas (see [12, 31] and [15], respectively).

Lemma 4 (the Lieb–Thirring inequality). *Let $\{\psi_j\}_{j=1}^N$ be an orthonormal set of functions in $(H)^k=\underbrace{H\oplus H\ldots\oplus H}_{k\text{-times}}$. Then there exists a constant C_{LT}, which depends on k but is independent of N, such that*

$$\int_\Omega\left(\sum_{j=1}^N\psi_j(x)\cdot\psi_j(x)\right)^{5/3}\mathrm{d}x\leqslant C_{\mathrm{LT}}\sum_{j=1}^N\int_\Omega(\nabla\psi_j(x):\nabla\psi_j(x))\mathrm{d}x. \tag{49}$$

Lemma 5. *Let $\{\phi_j\}_{j=1}^N\in V$ be an orthonormal set of functions with respect to the inner product $[\cdot,\cdot]$:*

$$[\phi_i,\phi_j]=(\phi_i,\phi_j)+\alpha^2((\phi_i,\phi_j))=\delta_{ij}.$$

Let $\psi_j(x)=(\phi_j(x),\alpha(\partial\phi_j(x)/\partial x_1),\alpha(\partial\phi_j(x)/\partial x_2),\alpha(\partial\phi_j(x)/\partial x_3))$ and $\phi^2(x)=\sum_{j=1}^N(\phi_j(x)\cdot\phi_j(x))$. Then there exists a constant C_F, which is independent of N, such that

$$\|\phi\|_{L^\infty}^2\leqslant\frac{C_F}{\alpha^2}\left(\sum_{j=1}^N\int_\Omega(\nabla\psi_j(x):\nabla\psi_j(x))\mathrm{d}x\right)^{1/2}. \tag{50}$$

From the existence and uniqueness properties of the solutions to (3), we get a semi-group of solution operators, denoted as $\{S(t)\}_{t\geqslant 0}$, which associates, with each $u^{\text{in}} \in V$, the semi-flow for time $t \geqslant 0 : S(t)u^{\text{in}} = u(\cdot, t)$.

Theorem 6. *There is a compact global attractor $\mathcal{A} \subset V$ for system (3). Moreover, we have an upper bound for the Hausdorff and fractal dimension of the attractor $\mathcal{A}$*

$$d_H(\mathcal{A}) \leqslant d_F(\mathcal{A}) \leqslant c\, G^{3/2}\left(\frac{1}{\gamma\lambda_1\alpha^2}\right)^{3/4}, \tag{51}$$

where $G = |f|/\nu^2\lambda_1^{3/4}$ is the Grashoff number and $1/\gamma = \min\{1, 1/(\alpha^2\lambda_1)\}$.

Proof. The compactness of the semigroup $\{S(t)\}_{t\geqslant 0}$ and the existence of bounded absorbing ball $\mathcal{B}$ guarantee the existence of the nonempty compact global attractor $\mathcal{A}$ (see, e.g., [12, 20, 27, 29, 31]). First, let us show that there is an absorbing ball in V and $D(A)$. By (30), we have

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \leqslant \mathrm{e}^{-\nu\lambda_1 t}(|u(0)|^2 + \alpha^2\|u(0)\|^2) + \frac{K_1}{\nu\lambda_1}(1 - \mathrm{e}^{-\nu\lambda_1 t}). \tag{52}$$

When t is large enough such that $\mathrm{e}^{-\nu\lambda_1 t}(|u(0)|^2 + \alpha^2\|u(0)\|^2) \leqslant K_1/\nu\lambda_1$, then we have

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \leqslant 2\frac{K_1}{\nu\lambda_1}, \tag{53}$$

where we recall $K_1 = \min\{|A^{-1}f|^2/\nu\alpha^2, |A^{-1/2}f|^2/\nu\}$. In particular,

$$\limsup_{t\to\infty}\,(|u(t)|^2 + \alpha^2\|u(t)\|^2) \leqslant 2\frac{K_1}{\nu\lambda_1} =: R_V. \tag{54}$$

Therefore, system (3) has the ball $\mathcal{B}_V(0)$ in V of radius R_V as an absorbing ball in V.

Next, we would like to show that there is an absorbing ball $\mathcal{B}_{D(A)}(0)$ in $D(A)$. By (41) and (42) we conclude that

$$\limsup_{t\to\infty}\,(\|u(t)\|^2 + \alpha^2|Au(t)|^2) \leqslant \limsup_{t\to\infty} k_2(t) =: R_{D(A)} < \infty, \tag{55}$$

and therefore we have the ball $\mathcal{B}_{D(A)}(0)$ in $D(A)$ with radius $R_{D(A)}$ as an absorbing ball in $D(A)$.

By Rellich's lemma we have that $S(t) : V \to \mathcal{D}(A) \subset\subset V$ is a compact semigroup from V to itself.

Since $S(t)\mathcal{B}_V(0) \subset \mathcal{B}_V(0)$, it follows that for each $s > 0$ the set $C_s := \overline{\cup_{t\geqslant s} S(t)\mathcal{B}_V(0)}^V$ is nonempty and compact in V. By monotonicity of C_s for $s > 0$ and by the finite intersection property of compact sets, we see that

$$\mathcal{A} = \bigcap_{s>0}\overline{\bigcup_{t\geqslant s} S(t)\mathcal{B}_V(0)}^V \subset V \tag{56}$$

is a nonempty compact set in V and indeed is the unique global attractor in V. To estimate the Hausdorff and fractal dimensions of the global attractor we will use the trace formula (see, e.g., [12, 31]).

We start by following similar techniques as in [15]. We linearize the ML-α model about a regular solution $u(t)$ (or $v(t) = u(t) + \alpha^2 Au(t)$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta v + \nu A\delta v + B(\delta v, u) + B(v, \delta u) = 0, \tag{57}$$

where $\delta v = \delta u + \alpha^2 A\delta u$. Therefore, δu evolves according to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta u + \nu A\delta u + (I + \alpha^2 A)^{-1}[B(\delta u + \alpha^2 A\delta u, u) + B(u + \alpha^2 Au, \delta u)] = 0, \tag{58}$$

which we write symbolically as

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta u + T(t)\delta u = 0, \tag{59}$$

where $T(t)\psi = \nu A\psi + (I+\alpha^2 A)^{-1}[B((I+\alpha^2 A)\psi, u) + B(v, \psi)]$. Let $\delta u_i(0)$, $j = 1, \ldots, N$ be a set of linearly independent vectors in V and let $\delta u_j(t)$ be the corresponding solutions of (58) with initial value $\delta u_j(0)$ for $j = 1, \ldots, N$. Let

$$\mathcal{T}_N(t) = \mathrm{Trace}(P_N(t) \circ T(t) \circ P_N(t)), \tag{60}$$

where $P_N(t)$ is the orthogonal projection of V onto the span $\{\delta v_1(t), \delta v_2(t), \ldots, \delta v_N(t)\}$. Let $\{\phi_j\}_{j=1,\ldots,N}$ be an orthonormal basis, with respect to the inner product $[\cdot,\cdot] = (\cdot,\cdot) + \alpha^2((\cdot,\cdot))$ of the space $P_N V$. From (60) we have

$$\begin{aligned}
\mathcal{T}_N(t) &= \sum_{j=1}^{N} [T(t)\phi_j(\cdot,t), \phi_j(\cdot,t)] \\
&= \sum_{j=1}^{N} \nu[A\phi_j, \phi_j] + [(I+\alpha^2 A)^{-1} B((I+\alpha^2 A)\phi_j, u), \phi_j] + [(I+\alpha^2 A)^{-1} B(v, \phi_j), \phi_j] \\
&= \nu \sum_{j=1}^{N} [A\phi_j, \phi_j] + \sum_{j=1}^{N} (B((I+\alpha^2 A)\phi_j, u), \phi_j) + \sum_{j=1}^{N} (B(v, \phi_j), \phi_j) \\
&= \nu \sum_{j=1}^{N} [A\phi_j, \phi_j] + \sum_{j=1}^{N} (B(\phi_j, u), \phi_j) + \alpha^2 \sum_{j=1}^{N} (B(A\phi_j, u), \phi_j).
\end{aligned} \tag{61}$$

By the definition of the inner product $[\cdot,\cdot]$, we have

$$\begin{aligned}
\sum_{j=1}^{N} [A\phi_j, \phi_j] &= \sum_{j=1}^{N} (A\phi_j, \phi_j) + \alpha^2 \sum_{j=1}^{N} (A\phi_j, A\phi_j) \\
&= \sum_{j=1}^{N} \int_\Omega (\nabla\psi_j(x,t) : \nabla\psi_j(x,t))\mathrm{d}x =: Q_N(t),
\end{aligned} \tag{62}$$

where

$$\psi_j = \left(\phi_j,\ \alpha\frac{\partial}{\partial x_1}\phi_j,\ \alpha\frac{\partial}{\partial x_2}\phi_j,\ \alpha\frac{\partial}{\partial x_3}\phi_j\right)^T. \tag{63}$$

Also note that

$$(\psi_j, \psi_k) = \delta_{jk}. \tag{64}$$

Setting

$$\mathcal{R}_N^{(1)}(t) = \sum_{j=1}^{N} (B(\phi_j, u), \phi_j), \qquad \mathcal{R}_N^{(2)}(t) = \sum_{j=1}^{N} (B(A\phi_j, u), \phi_j),$$

we have

$$\mathcal{T}_N(t) = \nu Q_N(t) + \mathcal{R}_N^{(1)}(t) + \alpha^2 \mathcal{R}_N^{(2)}(t). \tag{65}$$

We denote by $\psi^2 := \sum_{j=1}^N \psi_j \cdot \psi_j$. For $\mathcal{R}_N^{(1)}(t)$ we have

$$
\begin{aligned}
|\mathcal{R}_N^{(1)}(t)| &\leqslant \sum_{j=1}^N |(B(\phi_j, u), \phi_j)| \leqslant \int_\Omega \sum_{j=1}^N |(\phi_j \cdot \nabla)u\, \phi_j| \mathrm{d}x \leqslant \int_\Omega \sum_{j=1}^N \phi_j^2\, |\nabla u| \mathrm{d}x \\
&\leqslant \int_\Omega \psi^2 |\nabla u| \mathrm{d}x \leqslant \|\psi^2\|_{L^{6/5}} \|\nabla u\|_{L^6} \leqslant c\, \|\psi^2\|_{L^{6/5}} |Au| \\
&\leqslant c \left(\int_\Omega \psi^2 \mathrm{d}x \right)^{7/12} \left(\int_\Omega (\psi^2)^{5/3} \mathrm{d}x \right)^{1/4} |Au| \\
&\leqslant c\, N^{7/12} Q_N(t)^{1/4} |Au(t)| \leqslant c \frac{N^{7/9}}{\nu^{1/3}} |Au|^{4/3} + \frac{\nu}{4} Q_N,
\end{aligned} \tag{66}
$$

where we used (5), (64), lemma 4 and Hölder's and Young's inequalities.

Next, integrating by parts we obtain

$$
\begin{aligned}
\mathcal{R}_N^{(2)}(t) &= \sum_{j=1}^N (B(A\phi_j, u), \phi_j) = -\sum_{j=1}^N \int_\Omega (\Delta\phi_j \cdot \nabla u)\phi_j\, \mathrm{d}x \\
&= \sum_{j=1}^N \int_\Omega (\nabla\phi_j \cdot \nabla u)\nabla\phi_j\, \mathrm{d}x + \sum_{j=1}^N \int_\Omega (\nabla\phi_j \cdot \nabla\nabla u)\phi_j\, \mathrm{d}x =: \mathcal{I}_N^{(1)}(t) + \mathcal{I}_N^{(2)}(t).
\end{aligned}
$$

For the first term we have

$$
\alpha^2 \mathcal{I}_N^{(1)}(t) \leqslant \alpha^2 \sum_{j=1}^N \int_\Omega |\nabla\phi_j(x)|^2 |\nabla u(x)| \mathrm{d}x \leqslant \int_\Omega \psi^2(x) |\nabla u(x)| \mathrm{d}x. \tag{67}
$$

This term has been taken care of in (66) and is bounded by the right-hand side of (66).

For the second term using lemma 5 we have

$$
\begin{aligned}
\alpha^2 \mathcal{I}_N^{(2)}(t) &\leqslant \alpha^2 \int_\Omega \left(\sum_{j=1}^N (\nabla\phi_j(x) : \nabla\phi_j(x)) \right)^{1/2} \left(\sum_{j=1}^N \phi_j^2(x) \right)^{1/2} |\nabla\nabla u(x)| \mathrm{d}x \\
&\leqslant \alpha \int_\Omega \psi(x)\phi(x) |\nabla\nabla u(x)| \mathrm{d}x \leqslant \alpha \|\phi\|_{L^\infty} \int_\Omega |\nabla\nabla u(x)| \psi(x) \mathrm{d}x \\
&\leqslant c_F^{1/2} Q_N^{1/4}(t) \int_\Omega |\nabla\nabla u(x)| \psi(x) \mathrm{d}x \leqslant c Q_N^{1/4}(t) |Au| N^{1/2},
\end{aligned} \tag{68}
$$

which for $N \geqslant 1$ is again bounded by the right-hand side of (66).

Combining the estimates so obtained above we finally find

$$
\mathcal{T}_N(t) \geqslant \frac{\nu}{2} Q_N(t) - c\nu^{-1/3} N^{7/9} |Au|^{4/3}. \tag{69}
$$

By the asymptotic behaviour of the eigenvalues of the operator A (see (4)) and (64) we get

$$
Q_N(t) = \sum_{j=1}^N \|\psi_j\|^2 \geqslant \sum_{j=1}^N \lambda_j \geqslant c_0 \lambda_1 N^{5/3}. \tag{70}
$$

Now, by the trace formula (see, e.g., [12, 31] and the references therein), if N is large enough so that

$$
\liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathcal{T}_N(t)\, \mathrm{d}t > 0 \tag{71}
$$

then N is an upper bound for the Hausdorff [12, 31] and fractal [4] dimensions of the global attractor.

Thus, by (69) and (70) it is sufficient to require N to be large enough such that

$$\nu^{4/3}\lambda_1 N^{8/9} > c \sup_{u^{\text{in}}\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(t)|^{4/3}\,\mathrm{d}t. \tag{72}$$

On the other hand, using Hölder's inequality we get from (32)

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(t)|^{4/3}\mathrm{d}t &\leqslant \left(\limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(t)|^{2}\mathrm{d}t\right)^{2/3}\\ &\leqslant \left(\limsup_{T\to\infty} \frac{1}{T\alpha^2}\int_0^T (\|u\|^2+\alpha^2|Au(t)|^{2})\mathrm{d}t\right)^{2/3} \leqslant \left(\frac{K_1}{\nu\alpha^2}\right)^{2/3}, \end{aligned}\tag{73}$$

since by the Poincaré inequality

$$K_1 \leqslant \min\left\{\frac{|f|^2}{\nu\lambda_1}, \frac{|f|^2}{\nu\lambda_1^2\alpha^2}\right\} = \frac{|f|^2}{\nu\lambda_1\gamma} = G^2\,\frac{\nu^3\lambda_1^{1/2}}{\gamma}.$$

From this and (72) we deduce that

$$d_H(\mathcal{A}) \leqslant d_F(\mathcal{A}) \leqslant c\, G^{3/2}\left(\frac{1}{\gamma\lambda_1\alpha^2}\right)^{3/4}. \tag{74}$$

□

We now interpret the estimate for the attractor dimension in terms of the mean rate of dissipation of energy in turbulent flows and by following [15] we define the corresponding mean rate of dissipation of ‘energy’ for the ML-α model (see (25)) as

$$\bar{\epsilon} = L^{-3}\nu \sup_{u^{\text{in}}\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T (\|u(s)\|^2+\alpha^2|Au(s)|^2)\mathrm{d}s. \tag{75}$$

Thus, and in analogy with the Kolmogorov dissipation length in the classical theory of turbulence, we set the dissipation length scale for the ML-α model as

$$l_d = \left(\frac{\nu^3}{\bar{\epsilon}}\right)^{1/4}. \tag{76}$$

Identifying the dimension of the global attractor with the number of degrees of freedom, we will show that the number of degrees of freedom for the ML-α model is bounded from above by a quantity which scales like $(L/\alpha)^{3/2}(L/l_d)^3$. This is straightforward.

In fact, in view of (75) we can write (73) as follows:

$$\begin{aligned}\sup_{u^{\text{in}}\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(t)|^{4/3}\mathrm{d}t &\leqslant \sup_{u^{\text{in}}\in\mathcal{A}}\left(\limsup_{T\to\infty} \frac{1}{T\alpha^2}\int_0^T (\|u\|^2+\alpha^2|Au(t)|^{2})\mathrm{d}t\right)^{2/3}\\ &\leqslant \left(\frac{\bar{\epsilon}L^3}{\nu\alpha^2}\right)^{2/3}. \end{aligned}\tag{77}$$

Using this in (72) and recalling (76) we obtain the following estimate for the dimension of the global attractor and hence the upper bound on the number of degrees of freedom in the ML-α model:

$$d_H(\mathcal{A}) \leqslant d_F(\mathcal{A}) \leqslant c\left(\frac{L}{\alpha}\right)^{3/2}\left(\frac{L}{l_d}\right)^{3}. \tag{78}$$

Remark 1. We observe that this estimate for the dimension of the global attractor (the number of degrees of freedom) is similar to the estimates obtained for NS-α (VCHE) in [15] and for Clark-α in [3]. All these estimates are proportional to $(L/l_d)^3$, which agree with the heuristic physical estimates suggested by the classical theory of turbulence about the number of degrees of freedom in three-dimensional turbulent flows. On the other hand, we would also like to point out that the bounds for the dimension of the global attractor of the Leray-α model, i.e. system (1), which was established in [9], are proportional to $(L/l_d)^{12/7}$. It is worth stressing that even though the exponent $12/7$ in the Leray-α model is remarkably smaller than the exponent 3 in the dimension of the global attractor of the ML-α model, the quantities involved in the definition of the viscous dissipation length-scales, l_d, for the Leray-α and ML-α models are different: while the former is based on the time average of the H^3-norm of u, the latter is based on the average of the corresponding H^2-norm. Furthermore, the equations governing the dynamics of the solutions to these models are also different, which, in turn, affect the corresponding dissipation lengthscales l_d.

6. Energy spectra

Following similar arguments to those presented in [14] and [16] (see also [3, 9, 17]) we will study in this section the energy spectra of the ML-α model. We obtain similar results for the decay of the energy spectrum for the filtered velocity u as in NS-α [16] and Clark-α [3]. In particular, we observe that there are two different power laws for the energy cascade. For wave numbers $k \ll 1/\alpha$, we obtain the usual $k^{-5/3}$ Kolmogorov power law; however, for $k \gg 1/\alpha$ we obtain a steeper power law, evidence that the ML-α is a good candidate for a subgrid scale model of turbulence. We are still unable to determine from theory the exact value of n in (82). We hope that a numerical study of this model will shed some light on the exact value of n, a subject for future research.

We will use the following notation:

$$
\begin{aligned}
& b(u, v, w) = (B(u, v), w), \\
& \hat{u}_k = \frac{1}{(2\pi L)^3} \int_\Omega u(x) \mathrm{e}^{-\mathrm{i}k\cdot x}\, \mathrm{d}x, \\
& \hat{v}_k = \frac{1}{(2\pi L)^3} \int_\Omega v(x) \mathrm{e}^{-\mathrm{i}k\cdot x}\, \mathrm{d}x, \\
& u_k = \sum_{k \leqslant |j| < 2k} \hat{u}_j \mathrm{e}^{\mathrm{i}j\cdot x}, \\
& v_k = \sum_{k \leqslant |j| < 2k} \hat{v}_j \mathrm{e}^{\mathrm{i}j\cdot x}, \\
& u_k^< = \sum_{j<k} u_j, \qquad v_k^< = \sum_{j<k} v_j, \\
& u_k^> = \sum_{2k \leqslant j} u_j, \qquad v_k^> = \sum_{2k \leqslant j} v_j.
\end{aligned}
$$

We split the flow into three parts according to the three lengthscale ranges. Assume $k_f < k$, where k_f is the largest wavenumber involved in the forcing term. Thus,

$$
\begin{aligned}
& u = u_k^< + u_k + u_k^>, \\
& v = v_k^< + v_k + v_k^>.
\end{aligned}
$$

The energy balance equation for the ML-α model for an eddy of size k^{-1} is given by

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v_k, u_k) + \nu(-\Delta v_k, u_k) = T_k - T_{2k}, \tag{79}$$

where

$$T_k := -b(v_k^<, u_k^<, u_k) + b(v_k + v_k^>, u_k + u_k^>, u_k^<), \tag{80}$$

T_k represents the net amount of energy per unit time that is transferred into wavenumbers larger than or equal to k, T_{2k} represents the net amount of energy per unit time that is transferred into wavenumbers larger than or equal to $2k$. Thus, $T_k - T_{2k}$ represents the net amount of energy per unit time that is transferred into wavenumbers between $[k, 2k)$.

Taking an ensemble average (long time average) of (79) we get

$$\nu \left\langle (-\Delta v_k, u_k) \right\rangle = \langle T_k \rangle - \langle T_{2k} \rangle \,. \tag{81}$$

We define the energy spectrum

$$E_\alpha(k) = (1 + \alpha^2 |k|^2) \sum_{|j|=k} |\hat{u}_j|^2;$$

then we can rewrite the time-averaged energy transfer equation (81) as

$$\nu k^3 E_\alpha(k) \sim \nu \int_k^{2k} k^2 E_\alpha(k)\,\mathrm{d}k \sim \langle T_k \rangle - \langle T_{2k} \rangle \,.$$

Thus as long as $\nu k^3 E_\alpha(k) << \langle T_k \rangle$ (that is, $\langle T_{2k} \rangle \approx \langle T_k \rangle$, there is no leakage of energy due to dissipation), the wavenumber k belongs to the inertial range. Similar to the other alpha subgrid scale models, it is not known what is the correct averaged velocity of an eddy of length size k^{-1}. That is, we do not know *a priori* in these models the exact eddy turnover time of an eddy of size k^{-1}. As we will see below, we have a few candidates for such an averaged velocity. Namely,

$$U_k^0 = \left\langle \frac{1}{L^3} \int_\Omega |v_k|^2 \mathrm{d}x \right\rangle^{1/2} \sim \left(\int_k^{2k} (1 + \alpha^2 k^2) E_\alpha(k) \right)^{1/2} \sim \left(k(1 + \alpha^2 k^2) E_\alpha(k) \right)^{1/2},$$

$$U_k^1 = \left\langle \frac{1}{L^3} \int_\Omega u_k \cdot v_k \mathrm{d}x \right\rangle^{1/2} \sim \left(\int_k^{2k} E_\alpha(k) \right)^{1/2} \sim (k E_\alpha(k))^{1/2},$$

$$U_k^2 = \left\langle \frac{1}{L^3} \int_\Omega |u_k|^2 \mathrm{d}x \right\rangle^{1/2} \sim \left(\int_k^{2k} \frac{E_\alpha(k)}{(1 + \alpha^2 k^2)} \right)^{1/2} \sim \left(\frac{k E_\alpha(k)}{1 + \alpha^2 k^2} \right)^{1/2},$$

that is,

$$U_k^n = \frac{(k E_\alpha(k))^{1/2}}{(1 + \alpha^2 k^2)^{(n-1)/2}} \qquad (n = 0, 1, 2). \tag{82}$$

In the inertial range, the Kraichnan energy cascade mechanism states that the corresponding turnover time of eddies of spatial size $1/k$ with given average velocity as above is about

$$\tau_k^n := \frac{1}{k U_k^n} = \frac{(1 + \alpha^2 k^2)^{(n-1)/2}}{k^{3/2} (E_\alpha(k))^{1/2}} \qquad (n = 0, 1, 2).$$

Therefore, the energy dissipation rate ϵ is

$$\epsilon \sim \frac{1}{\tau_k^n} \int_k^{2k} E_\alpha(k)\,\mathrm{d}k \sim \frac{k^{5/2} \, (E_\alpha(k))^{3/2}}{(1 + \alpha^2 k^2)^{(n-1)/2}}, \tag{83}$$

and hence

$$E_\alpha(k) \sim \frac{\epsilon^{2/3} (1 + \alpha^2 k^2)^{(n-1)/3}}{k^{5/3}}.$$

Note that the translational kinetic energy spectrum of the variable u is given by

$$E^u(k) \equiv \frac{E_\alpha(k)}{1+\alpha^2k^2} \sim \begin{cases} \dfrac{\epsilon_\alpha^{2/3}}{k^{5/3}}, & \text{when } k\alpha \ll 1\,, \\ \dfrac{\epsilon_\alpha^{2/3}}{\alpha^{2(4-n)/3}k^{(13-2n)/3}}, & \text{when } k\alpha \gg 1\,. \end{cases}$$

Therefore, depending on the appropriate average velocity of an eddy of size k^{-1} for the ML-α model, we would get the corresponding energy spectra which has a much faster decaying power law $k^{(2n-13)/3}$ ($n = 0, 1, 2$) than the usual Kolmogorov $k^{-5/3}$ power law, in the subrange $k\alpha \gg 1$. This signifies that the ML-α model, like the other alpha models, is a good candidate for the subgrid scale model of turbulence.

7. Concluding remarks

In this paper we propose using a new analytical subgrid scale turbulence model which yields the same closure ansatz as that of the NS-α subgrid scale turbulence model in infinite channels and pipes. A good assessment of the success of this new subgrid scale model was obtained by comparing its general solution with the general solution of the NS-α model in infinite channels and pipes which gave a remarkable match to the empirical data for a wide range of huge Reynolds numbers. In addition we prove the global well-posedness of this new model and show that it has a finite dimensional global attractor. Our explicit estimates for the dimension of the global attractor, in terms of the relevant physical parameters, are compatible with the suggested physical heuristic arguments for the number of degrees of freedom in turbulent flows. Furthermore, the steeper behaviour of the slope of the energy spectrum for larger wavenumbers within the inertial range, in comparison with the usual $k^{-5/3}$ Kolmogorov energy spectrum, signifies that there is less energy in the higher wavenumbers as is expected from any good subgrid scale model of turbulence.

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