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Research Article

A Modified New Two-Parameter Estimator in a Linear Regression Model

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The literature has shown that ordinary least squares estimator (OLSE) is not best when the explanatory variables are related, that is, when multicollinearity is present. This estimator becomes unstable and gives a misleading conclusion. In this study, a modified new two-parameter estimator based on prior information for the vector of parameters is proposed to circumvent the problem of multicollinearity. This new estimator includes the special cases of the ordinary least squares estimator (OLSE), the ridge estimator (RRE), the Liu estimator (LE), the modified ridge estimator (MRE), and the modified Liu estimator (MLE). Furthermore, the superiority of the new estimator over OLSE, RRE, LE, MRE, MLE, and the two-parameter estimator proposed by Ozkale and Kaciranlar (2007) was obtained by using the mean squared error matrix criterion. In conclusion, a numerical example and a simulation study were conducted to illustrate the theoretical results.

1. Introduction

The general linear regression model in matrix form is defined as

$$y = X\beta + \varepsilon, \tag{1}$$

where y is a $n \times 1$ vector of the dependent variable, X is a known $n \times p$ full-rank matrix of explanatory variables, β is a $p \times 1$ vector of regression coefficients, and ε is $n \times 1$ vector of disturbance such that $E(\varepsilon) = 0$ and $\mathrm{Cov}(\varepsilon) = \sigma^2 I$. The ordinary least squares estimator (OLSE) of β in model (1) is defined as

$$\widehat{\beta}_{\mathrm{OLS}} = (X'X)^{-1}X'y. \tag{2}$$

According to the Gauss–Markov theorem, the OLS estimator is considered best, linear, and unbiased, possessing minimum variance in the class of all linear unbiased estimators. However, different studies have shown that the OLS estimator is not best when the explanatory variables are related, that is, when multicollinearity is present [1]. This estimator becomes unstable and gives a misleading conclusion. Many biased estimators have been proposed as an alternative to OLSE to circumvent this problem. These include Stein estimator [2], principal components estimator [3], ridge estimator (RRE) estimator [1], contraction estimator [4], modified ridge regression estimator (MRRE) [5], and Liu estimator [6].

Hoerl and Kennard [1] proposed a ridge estimator (RRE)

$$\widehat{\beta}_{\mathrm{RRE}}(k) = (X'X + kI)^{-1}X'X\widehat{\beta}_{\mathrm{OLS}}y = T_k\widehat{\beta}_{\mathrm{OLS}}, \quad k > 0, \tag{3}$$

where $T_k = (X'X + kI)^{-1}X'X$. $\widehat{\beta}_{\mathrm{RRE}}(k)$ was obtained by augmenting the equation $0 = k^{1/2}\beta + \varepsilon'$ to the original equation (1) and then applying the OLS estimator. Mayer and Willke [4] defined the contraction estimator

$$\widehat{\beta}(\rho) = (1+\rho)^{-1}\widehat{\beta}, \quad \rho > 0. \tag{4}$$

Liu [6] combined the Stein estimator with a ridge estimator to combat the problem of multicollinearity. $\widehat{\beta}_{\mathrm{LE}}(d)$

was obtained by augmenting the equation $d\widehat{\beta} = \beta + \varepsilon'$ to the original equation (1) and then applying OLS. This is defined as follows:

$$\widehat{\beta}_{\mathrm{LE}}(d) = \left(X'X + I\right)^{-1}\left(X'y + dI\right)\widehat{\beta}_{\mathrm{OLS}} = T_d\widehat{\beta}_{\mathrm{OLS}}, \quad 0 < d < 1, \tag{5}$$

where $T_d = (X'X + I)^{-1}(X'y + dI)$.

Swindel [5] modified the ridge estimator (MRRE) by adding a prior information. The estimator is defined as follows:

$$\widehat{\beta}_{\mathrm{MRRE}}(k, b) = \left(X'X + kI\right)^{-1}\left(X'y + kb\right), \tag{6}$$

where b represent the prior information on β. MRRE tends to b as k tends to infinity. Also, MRRE returns the estimates of the OLS estimator when $k = 0$.

Based on prior information, Li and Yang [7] proposed a modified Liu estimator (MLE):

$$\widehat{\beta}_{\mathrm{MLE}}(d, b) = \left(X'X + I\right)^{-1}\left[\left(X'X + dI\right)\widehat{\beta}_{\mathrm{OLS}} + (1-d)b\right]. \tag{7}$$

MLE includes OLS and Liu as special cases. In recent times, different researchers have suggested the use of two-parameter estimators to handle multicollinearity. Ozkale and Kaciranlar [8] proposed the two-parameter estimator (TPE), which is defined as

$$\begin{aligned}\widehat{\beta}_{\mathrm{TPE}}(k, d) &= \left(X'X + kI\right)^{-1}\left(X'Y + kd\widehat{\beta}_{\mathrm{OLS}}\right) \\ &= \left(X'X + kI\right)^{-1}\left(X'X + kd\right)\widehat{\beta}_{\mathrm{OLS}} = T_{kd}\widehat{\beta},\end{aligned} \tag{8}$$

where $k > 0, 0 < d < 1$. TPE includes OLS, RRE, LE, and the contraction estimators as special cases.

The primary focus of this study is to provide an alternative method in a linear regression model to circumvent the problem of multicollinearity. A modified two-parameter (MTP) estimator is proposed based on prior information and is compared with OLS, LE, RRE, MRRE, MLE, and TPE, respectively, using the mean squared error matrix (MSEM) criterion. The article is structured as follows: We introduce the new estimator in Section 2. In Section 3, we discuss the superiority of the new estimator. Section 4 consists of a numerical example and a simulation study. Concluding remarks are provided in Section 5.

2. Modified Two-Parameter Estimator

Let $T_k = (X'X + kI)^{-1}X'X = I - k(X'X + kI)^{-1}$, and MRRE in equation (6) can be re-expressed as

$$\begin{aligned}\widehat{\beta}_{\mathrm{MRRE}}(k, b) &= \left(X'X + kI\right)^{-1}X'y + k\left(X'X + kI\right)^{-1}b \\ &= \left(X'X + kI\right)^{-1}X'X\widehat{\beta}_{\mathrm{OLS}} + k\left(X'X + kI\right)^{-1}b \\ &= T_k\widehat{\beta}_{\mathrm{OLS}} + \left(I - T_k\right)b.\end{aligned} \tag{9}$$

Similarly, $T_d = (X'X + I)^{-1}(X'X + dI)$, and then the modified Liu estimator in equation (7) can be written as

$$\begin{aligned}\widehat{\beta}_{\mathrm{MLE}}(d, b) &= T_d\widehat{\beta}_{\mathrm{OLS}} + \left(I - T_d\right)b \\ &= \left(X'X + I\right)^{-1}\left[\left(X'X + dI\right)\widehat{\beta}_{\mathrm{OLS}} + (1-d)b\right].\end{aligned} \tag{10}$$

MRRE and MLE are the convex combination of the prior information b and the OLS estimator. From equation (8), $T_{kd} = (X'X + kI)^{-1}(X'X + kdI) = I - k(1-d)(X'X + kI)^{-1}$; therefore, the modified two-parameter based on the prior information can be defined as follows:

$$\begin{aligned}\widehat{\beta}_{\mathrm{MTPE}}(k, d, b) &= T_{kd}\widehat{\beta}_{\mathrm{OLS}} - \left(I - T_{kd}\right)b \\ &= \left(X'X + kI\right)^{-1}\left(X'X + kdI\right)\widehat{\beta}_{\mathrm{OLS}} \\ &\quad + \left(I - \left(X'X + kI\right)^{-1}\left(X'X + kdI\right)\right)b \\ &= \left(X'X + kI\right)^{-1}\left(X'X + kdI\right)\widehat{\beta}_{\mathrm{OLS}} \\ &\quad + \left(k(1-d)\left(X'X + kI\right)^{-1}\right)b \\ &= \left(X'X + kI\right)^{-1}\left[\left(X'X + kdI\right)\widehat{\beta}_{\mathrm{OLS}} + k(1-d)b\right].\end{aligned} \tag{11}$$

Also, MTPE is a convex combination of the prior information and OLSE. It includes the special cases of OLSE, RRE, MRE, LE, and MLE. The following cases are possible:

$\widehat{\beta}_{\mathrm{MTPE}}(k, 1, b_0) = \widehat{\beta}_{\mathrm{MTPE}}(0, d, b_0) = \widehat{\beta}_{\mathrm{OLS}}$; ordinary least squares estimator
$\widehat{\beta}_{\mathrm{MTPE}}(1, d, 0) = \widehat{\beta}_{\mathrm{LE}}(d)$; Liu estimator
$\widehat{\beta}_{\mathrm{MTPE}}(k, 0, b_0) = \widehat{\beta}_{\mathrm{MRRE}}(k, b_0)$; modified ridge estimator
$\widehat{\beta}_{\mathrm{MTPE}}(k, 0, 0) = \widehat{\beta}_{\mathrm{RRE}}(k)$; ridge estimator
$\widehat{\beta}_{\mathrm{MTPE}}(1, d, b_0) = \widehat{\beta}_{\mathrm{MLE}}(d, b_0)$; modified Liu estimator

Suppose there exist an orthogonal matrix T such that $T'X'XT = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$, where λ_i is the ith eigenvalue of $X'X$. Λ and T are the matrices of eigenvalues and eigenvectors of $X'X$, respectively. Substituting $Z = XQ$, $\alpha = Q'\beta$ in model (1), then the equivalent model can be rewritten as

$$y = Z\alpha + \varepsilon. \tag{12}$$

The following representations of the estimators are as follows:

$$\begin{aligned}\widehat{\alpha}_{\mathrm{OLS}} &= \Lambda^{-1}Z'Y, \\ \widehat{\alpha}_{\mathrm{LE}}(d) &= (\Lambda + I)^{-1}(\Lambda + dI)\Lambda^{-1}Z'y, \\ \widehat{\alpha}_{\mathrm{RRE}}(k) &= (\Lambda + kI)^{-1}Z'y, \\ \widehat{\alpha}_{\mathrm{MRRE}}(k, b) &= (\Lambda + kI)^{-1}(Z'y + kb), \\ \widehat{\alpha}_{\mathrm{MLE}}(d, b) &= (\Lambda + I)^{-1}\left[(\Lambda + dI)\Lambda^{-1}Z'y + (1-d)b\right], \\ \widehat{\alpha}_{\mathrm{TPE}}(k, d) &= (\Lambda + kI)^{-1}(\Lambda + kdI)\Lambda^{-1}Z'y, \\ \widehat{\alpha}_{\mathrm{MTPE}}(k, d, b) &= (\Lambda + kI)^{-1}\left[(\Lambda + kdI)\Lambda^{-1}Z'y + k(1-d)b\right].\end{aligned} \tag{13}$$

The following notations and lemmas are needful to prove the statistical property of $\widehat{\beta}_{\mathrm{MTPE}}(k,d,b_0)$.

Lemma 1. *Let M be an $n\times n$ positive definite matrix, that is, $M>0$, and α be some vector, then $M-\alpha\alpha'\geq 0$ if and only if $\alpha' M^{-1}\alpha\leq 1$* [9].

Lemma 2. *Let $\widehat{\beta}_i = A_i y, i=1,2$ be two linear estimators of β. Suppose that $D=Cov(\widehat{\beta}_1)-Cov(\widehat{\beta}_2)>0$, where $Cov(\widehat{\beta}_i), i=1,2$ denotes the covariance matrix of $\widehat{\beta}_i$ and $b_i = Bias(\widehat{\beta}_i)=(A_iX-I)\beta, i=1,2$. Consequently,*

$$\begin{aligned}\Delta\left(\widehat{\beta}_1-\widehat{\beta}_2\right)&=\mathrm{MSEM}\left(\widehat{\beta}_1\right)-\mathrm{MSEM}\left(\widehat{\beta}_2\right)\\&=\sigma^2 D+b_1b_1'-b_2b_2'>0,\end{aligned}\tag{14}$$

if and only if $b_2'[\sigma^2 D+b_1b_1']^{-1}b_2<1$, where $MSEM(\widehat{\beta}_i)=Cov(\widehat{\beta}_i)+b_ib_i'$ [10].

3. Establishing Superiority of Modified Two-Parameter Estimator Using MSEM Criterion

In this section, MTPE is compared with the following estimators: OLS, RRE, LE, MRRE, MLE, and TPE.

3.1. Comparison between the MTPE and OLS Using MSEM Criterion. From the representation $\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)=(\Lambda+kI)^{-1}[(\Lambda+kdI)\Lambda^{-1}Z'y+k(1-d)b]$, the bias vector and covariance matrix of MTPE are obtained as follows:

$$\begin{aligned}E\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)&=E\left((\Lambda+kI)^{-1}\left[(\Lambda+kdI)\Lambda^{-1}Z'y\right.\right.\\&\quad\left.\left.+k(1-d)b\right]\right)\\&=(\Lambda+kI)^{-1}\left[(\Lambda+kdI)\Lambda^{-1}Z'Z\alpha\right.\\&\quad\left.+k(1-d)b\right]\\&=(\Lambda+kI)^{-1}\left[(\Lambda+kdI)\alpha+k(1-d)b\right]\\&=(\Lambda+kI)^{-1}(\Lambda+kdI)\alpha\\&\quad+(\Lambda+kI)^{-1}k(1-d)b,\end{aligned}\tag{15}$$

where $E(y)=Z\alpha$.

Recall that $k(1-d)=(\Lambda+kI)-(\Lambda+kdI)$ and let $B_{k,d,b}=(\Lambda+kI)^{-1}(\Lambda+kdI)$. Therefore,

$$\begin{aligned}E\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)&=B_{k,d,b}\alpha+\left(I-B_{k,d,b}\right)b,\\\mathrm{bias}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)&=B_{k,d,b}\alpha+\left(I-B_{k,d,b}\right)b-\alpha,\end{aligned}\tag{16}$$

$$\mathrm{Cov}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)=\sigma^2B_{k,d,b}\Lambda^{-1}B_{k,d,b}'.\tag{17}$$

Hence,

$$\begin{aligned}\mathrm{MSEM}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)&=\sigma^2B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\\&\quad+\left(B_{k,d,b}-I\right)(\alpha-b)(\alpha-b)'\left(B_{k,d,b}-I\right)'.\end{aligned}\tag{18}$$

From the representation, $\widehat{\alpha}=\Lambda^{-1}Z'Y$, the MSEM of OLS is given as

$$\mathrm{MSEM}\left(\widehat{\alpha}_{\mathrm{OLS}}\right)=\sigma^2\Lambda^{-1}.\tag{19}$$

Comparing (18) and (19),

$$\begin{aligned}&\mathrm{MSEM}\left(\widehat{\alpha}_{\mathrm{OLS}}\right)-\mathrm{MSEM}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)\\&\quad=\sigma^2\left(\Lambda^{-1}-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)\\&\qquad+\left(B_{k,d,b}-I\right)(\alpha-b)(\alpha-b)'\left(B_{k,d,b}-I\right)'.\end{aligned}\tag{20}$$

Let $k>0$ and $0<d<1$. Thus, the following theorem holds.

Theorem 3. *Consider two biased competing homogenous linear estimators $\widehat{\alpha}_{OLS}$ and $\widehat{\alpha}_{MTPE}(k,d,b)$. If $k>0$ and $0<d<1$, the estimator $\widehat{\alpha}_{MTPE(k,d,b)}$ is superior to estimator $\widehat{\alpha}$ using the MSEM criterion, that is, $MSEM(\widehat{\alpha}_{OLS})-MSEM(\widehat{\alpha}_{MTPE}(k,d,b))>0$ if and only if*

$$\begin{aligned}&(\alpha-b)'\left(B_{k,d,b}-I\right)'\left[\sigma^2\left(\Lambda^{-1}-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)\right]^{-1}\\&\quad\cdot\left(B_{k,d,b}-I\right)(\alpha-b)<1.\end{aligned}\tag{21}$$

Proof. Using (17) and (19), the following was obtained:

$$\begin{aligned}\mathrm{Cov}(\widehat{\alpha})-\mathrm{Cov}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)&=\sigma^2\left(\Lambda^{-1}-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)\\&=\sigma^2\mathrm{diag}\left\{\frac{1}{\lambda_i}-\frac{(\lambda_i+kd)^2}{\lambda_i(\lambda_i+k)^2}\right\}_{i=1}^{p}.\end{aligned}\tag{22}$$

$\Lambda^{-1}-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'$ will be positive definite (pd) if and only if $(\lambda_i+k)^2-(\lambda_i+kd)^2>0$ or $(\lambda_i+k)-(\lambda_i+kd)>0$. It was observed that $(\lambda_i+k)-(\lambda_i+kd)=k(1-d)>0$ for $0<d<1$ and $k>0$. Therefore, $\Lambda^{-1}-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'$ is pd. By Lemma 2, the proof is completed.

3.2. Comparison between the MTPE and RRE Using MSEM Criterion. From the representation, $\widehat{\alpha}_{\mathrm{RRE}}(k)=(\Lambda+kI)^{-1}Z'y$, the bias vector and covariance matrix of RRE is given as follows:

$$\begin{aligned}\mathrm{bias}\left(\widehat{\alpha}_{\mathrm{RRE}}(k)\right)&=-k(\Lambda+kI)^{-1}\alpha,\\\mathrm{Cov}\left(\widehat{\alpha}_{\mathrm{RRE}}(k)\right)&=\sigma^2(\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1}.\end{aligned}\tag{23}$$

Hence,

$$\mathrm{MSEM}\left(\widehat{\alpha}_{\mathrm{RRE}}(k)\right)=\sigma^2B_k\Lambda B_k'+k^2B_k\alpha\alpha'B_k',\tag{24}$$

where $B_k=(\Lambda+kI)^{-1}$. The difference between $\widehat{\alpha}_{\mathrm{RRE}}(k)$ and $\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)$ in the MSEM sense is as follows:

$$\begin{aligned}&\mathrm{MSEM}\left(\widehat{\alpha}_{\mathrm{RRE}}(k)\right)-\mathrm{MSEM}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)\\&\quad=\sigma^2\left(B_k\Lambda B_k'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)+k^2B_k\alpha\alpha'B_k\\&\qquad-\left(B_{k,d,b}-I\right)(\alpha-b)(\alpha-b)'\left(B_{k,d,b}-I\right)'.\end{aligned}\tag{25}$$

Let $k>0$ and $0<d<1$. Thus, the following theorem holds.

Theorem 4. *Consider two biased competing homogenous linear estimators* $\widehat{\alpha}_{RRE}(k)$ *and* $\widehat{\alpha}_{MTPE}(k,d,b)$. *If* $k>0$ *and* $0<d<1$, *the estimator* $\widehat{\alpha}_{MTPE(k,d,b)}$ *is superior to estimator* $\widehat{\alpha}_{RRE}(k)$ *using the MSEM criterion, that is,* $MSEM(\widehat{\alpha}_{RRE}(k)) - MSEM(\widehat{\alpha}_{MTPE}(k,d,b)) > 0$ *if and only if*

$$(\alpha-b)'\left(B_{k,d,b}-I\right)'\left[\sigma^2\left(B_k\Lambda B_k'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)+k^2B_k\alpha\alpha' B_k\right]^{-1}\left(B_{k,d,b}-I\right)(\alpha-b)<1. \quad (26)$$

3.3. Comparison between the MTPE and LE Using MSEM Criterion. From the representation, $\widehat{\alpha}_{\mathrm{LE}}(d)=(\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}Z'y$, the bias vector and covariance matrix of RRE are provided as follows:

$$\operatorname{bias}\left(\widehat{\alpha}_{LE}(d)\right)=(B_d-I)\alpha, \quad (27)$$

$$\operatorname{Cov}\left(\widehat{\alpha}_{\mathrm{LE}}(d)\right)=\sigma^2B_d\Lambda^{-1}B_d'. \quad (28)$$

Hence,

$$\operatorname{MSEM}\left(\widehat{\alpha}_{\mathrm{LE}}(d)\right)=\sigma^2B_d\Lambda^{-1}B_d'+(B_d-I)\alpha\alpha'(B_d-I)', \quad (29)$$

where $B_d=(\Lambda+I)^{-1}(\Lambda+dI)$. Considering the difference between (18) and (29),

$$\operatorname{MSEM}\left(\widehat{\alpha}_{\mathrm{LE}}(d)\right)-\operatorname{MSEM}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)=\sigma^2(D)+b_1b_1'-b_2b_2', \quad (30)$$

where $D=B_d\Lambda^{-1}B_d'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'$, $b_1=(B_d-I)\alpha$, and $b_2=(B_{k,d,b}-I)(\alpha-b)$.

Theorem 5. *Consider two biased competing homogenous linear estimators* $\widehat{\alpha}_{LE}(d)$ *and* $\widehat{\alpha}_{MTPE}(k,d,b)$. *If* $k>0$ *and* $0<d<1$, *the estimator* $\widehat{\alpha}_{MTPE(k,d,b)}$ *is superior to estimator* $\widehat{\alpha}_{LE}(d)$ *using the MSEM criterion, that is,* $MSEM(\widehat{\alpha}_{LE}(d))-MSEM(\widehat{\alpha}_{MTPE}(k,d,b))>0$ *if and only if*

$$(\alpha-b)'\left(B_{k,d,b}-I\right)'\left[\sigma^2\left(B_d\Lambda^{-1}B_d'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)+(B_d-I)\alpha\alpha'(B_d-I)'\right]^{-1}\left(B_{k,d,b}-I\right)(\alpha-b)<1. \quad (31)$$

Proof. Using (17) and (28), the following was obtained:

By computation,

$$\sigma^2(D)=\sigma^2\left(B_d\Lambda^{-1}B_d'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)=\sigma^2\operatorname{diag}\left\{\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{(\lambda_i+kd)^2}{\lambda_i(\lambda_i+k)^2}\right\}. \quad (32)$$

$B_d\Lambda^{-1}B_d'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'$ will be positive definite (pd) if and only if, $(\lambda_i+d)^2(\lambda_i+k)^2-(\lambda_i+1)^2(\lambda_i+kd)^2>0$. For $0<d<1$ and $k>1$, it was observed that $(\lambda_i+d)^2(\lambda_i+k)^2-(\lambda_i+1)^2(\lambda_i+kd)^2>0$. Therefore, $B_d\Lambda^{-1}B_d'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'$ is pd. By Lemma 2, the proof is completed.

3.4. Comparison between the MTPE and MRRE Using MSEM Criterion. From the representation, $\widehat{\alpha}_{\mathrm{MRRE}}(k,b)=(\Lambda+kI)^{-1}(Z'y+kb)$, the bias vector and covariance matrix of MRRE are provided as follows:

$$\operatorname{bias}\left(\widehat{\alpha}_{\mathrm{MRRE}}(k,b)\right)=\left(B_k-I\right)(\alpha-b), \quad (33)$$

$$\operatorname{Cov}\left(\widehat{\alpha}_{\mathrm{MRRE}}(k,b)\right)=\sigma^2B_k\Lambda^{-1}B_k'. \quad (34)$$

Hence,

$$\operatorname{MSEM}\left(\widehat{\alpha}_{\mathrm{MRRE}}(k,b)\right)=\sigma^2B_k\Lambda^{-1}B_k'+\left(B_k-I\right)(\alpha-b)(\alpha-b)'\left(B_k-I\right)', \quad (35)$$

where $B_k=(\Lambda+kI)^{-1}$. Considering the difference between (18) and (35),

$$\begin{aligned}\operatorname{MSEM}\left(\widehat{\alpha}_{\mathrm{MRRE}}(k)\right)-\operatorname{MSEM}\left(\widehat{\alpha}_{\mathrm{MTPE}}(k,d,b)\right)&=\sigma^2\left(B_k\Lambda^{-1}B_k'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)\\&+\left(B_k-I\right)(\alpha-b)(\alpha-b)'\left(B_k-I\right)'\\&-\left(B_{k,d,b}-I\right)(\alpha-b)(\alpha-b)'\left(B_{k,d,b}-I\right)'.\end{aligned} \quad (36)$$

Theorem 6. *Consider two biased competing homogenous linear estimators* $\widehat{\alpha}_{MRRE}(k,b)$ *and* $\widehat{\alpha}_{MTPE}(k,d,b)$. *If* $k>0$ *and* $0<d<1$, *the estimator* $\widehat{\alpha}_{MTPE(k,d,b)}$ *is superior to the estimator* $\widehat{\alpha}_{MRRE}(k,b)$ *using the MSEM criterion, that is, MSEM* $(\widehat{\alpha}_{MRRE}(k,b))-MSEM(\widehat{\alpha}_{MTPE}(k,d,b))>0$ *if and only if*

$$\left(B_{k,d,b}-I\right)'(\alpha-b)'\left[\sigma^2\left(B_k\Lambda^{-1}B_k'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)+\left(B_k-I\right)(\alpha-b)(\alpha-b)'\left(B_k-I\right)'\right]^{-1}\left(B_{k,d,b}-I\right)(\alpha-b)<1. \quad (37)$$

Proof. Using (17) and (34), the following was obtained:

$$\sigma^2\left(B_k\Lambda^{-1}B_k'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'\right)=\sigma^2\operatorname{diag}\left\{\frac{\lambda_i}{(\lambda_i+k)^2}-\frac{(\lambda_i+kd)^2}{\lambda_i(\lambda_i+k)^2}\right\} \quad (38)$$

Evidently, for $0<d<1$ and $k>0$, $B_k\Lambda^{-1}B_k'-B_{k,d,b}\Lambda^{-1}B_{k,d,b}'$ will be positive definite (pd). By Lemma 2, the proof is completed.

3.5. Comparison between the MTPE and MLE Using MSEM Criterion. From the representation, $\widehat{\alpha}_{\mathrm{MLE}}(d,b)=(\Lambda+I)^{-1}[(\Lambda+dI)\Lambda^{-1}Z'y+(1-d)b]$, the bias vector and covariance matrix of MLE are provided as follows:

$$\text{bias}\left(\hat{\alpha}_{\text{MLE}}(d)\right) = \left(B_d - I\right)(\alpha - b), \tag{39}$$

$$\text{Cov}\left(\hat{\alpha}_{\text{MLE}}(d)\right) = \sigma^2 B_d \Lambda^{-1} B_d'. \tag{40}$$

Hence,

$$\text{MSEM}\left(\hat{\alpha}_{\text{MLE}}(d)\right) = \sigma^2 B_d \Lambda^{-1} B_d' + \left(B_d - I\right)(\alpha - b)(\alpha - b)'\left(B_d - I\right)', \tag{41}$$

where $B_d = (\Lambda + I)^{-1}(\Lambda + dI)$. The mean square error difference between (18) and (41) is given as

$$\begin{aligned}\Delta_1 &= \text{MSEM}\left(\hat{\alpha}_{\text{MLE}}(d)\right) - \text{MSEM}\left(\hat{\alpha}_{\text{MTPE}}(k, d, b)\right) \\ &= \sigma^2 (D) + b_1 b_1' - b_2 b_2',\end{aligned} \tag{42}$$

where $D = B_d \Lambda^{-1} B_d' - B_{k,d,b} \Lambda^{-1} B_{k,d,b}'$, $b_1 = (B_d - I)(\alpha - b)$, $b_2 = (B_{k,d,b} - I)(\alpha - b)$.

Theorem 7. *Consider two biased competing homogenous linear estimators $\hat{\alpha}_{MLE}(d, b)$ and $\hat{\alpha}_{MTPE}(k, d, b)$. If $k > 0$ and $0 < d < 1$, the estimator $\hat{\alpha}_{MTPE(k,d,b)}$ is superior to the estimator $\hat{\alpha}_{MLE}(d, b)$ using the MSEM criterion, that is, MSEM $(\hat{\alpha}_{MLE}(d, b))$ – MSEM $(\hat{\alpha}_{MTPE}(k, d, b)) > 0$ if and only if $b_2'[\sigma^2(D) + b_1 b_1']^{-1} b_2 < 1$, where $D = B_d \Lambda^{-1} B_d' - B_{k,d,b} \Lambda^{-1} B_{k,d,b}'$, $b_1 = (B_d - I)(\alpha - b)$, and $b_2 = (B_{k,d,b} - I)(\alpha - b)$.*

Proof. Using (17) and (40), the following was obtained:

By computation,

$$D = B_d \Lambda^{-1} B_d' - B_{k,d,b} \Lambda^{-1} B_{k,d,b}' = Q \,\text{diag}\left(\tau_1, \ldots, \tau_p\right) Q'. \tag{43}$$

By computation,

$$\begin{aligned}\sigma^2 (D) &= \sigma^2 \left(B_d \Lambda^{-1} B_d' - B_{k,d,b} \Lambda^{-1} B_{k,d,b}'\right) \\ &= \sigma^2 \,\text{diag}\left(\frac{\left(\lambda_i + d\right)^2}{\lambda_i \left(\lambda_i + 1\right)^2} - \frac{\left(\lambda_i + kd\right)^2}{\lambda_i \left(\lambda_i + k\right)^2}\right).\end{aligned} \tag{44}$$

$\sigma^2 (B_d \Lambda^{-1} B_d' - B_{k,d,b} \Lambda^{-1} B_{k,d,b}')$ will be positive definite if and only if $(\lambda_i + d)^2 (\lambda_i + k)^2 - (\lambda_i + kd)^2 (\lambda_i + 1)^2 > 0$.

3.6. Comparison between the MTPE and TPE Using MSEM Criterion. From the representation $\hat{\alpha}_{\text{TPE}}(k, d) = (\Lambda + kI)^{-1}(\Lambda + kdI)\Lambda^{-1} Z' y$, the bias vector and covariance matrix of TPE are provided as follows:

$$\text{bias}\left(\hat{\alpha}_{\text{TPE}}(k, d)\right) = \left(B_{k,d,b}\alpha - I\right)\alpha, \tag{45}$$

$$\text{Cov}\left(\hat{\alpha}_{\text{TPE}}(k, d)\right) = \sigma^2 B_{k,d,b} \Lambda^{-1} B_{k,d,b}'. \tag{46}$$

Hence,

$$\begin{aligned}&\text{MSEM}\left(\hat{\alpha}_{\text{TPE}}(k, d)\right) \\ &= \sigma^2 B_{k,d,b} \Lambda^{-1} B_{k,d,b}' + \left(B_{k,d,b} - I\right)\alpha\alpha'\left(B_{k,d,b} - I\right)'.\end{aligned} \tag{47}$$

Considering the matrix difference between (18) and (47)

$$\begin{aligned}\Delta_2 &= \text{MSEM}\left(\hat{\alpha}_{\text{TPE}}(k, d)\right) - \text{MSEM}\left(\hat{\alpha}_{\text{MTPE}}(k, d, b)\right) \\ &= \left(B_{k,d,b} - I\right)\left[\alpha\alpha' - (\alpha - b)(\alpha - b)'\right]\left(B_{k,d,b} - I\right)'.\end{aligned} \tag{48}$$

Obviously, $\Delta_2 \geq 0$ if and only if $\alpha\alpha' - (\alpha - b)(\alpha - b)' \geq 0$; thus, the following results hold.

Theorem 8. *The modified two-parameter estimator $\hat{\alpha}_{MTPE}(k, d, b)$ is superior to the two-parameter estimator $\hat{\alpha}_{TPE}(k, d)$ in the MSEM sense if and only if $\alpha\alpha' - (\alpha - b)(\alpha - b)' \geq 0$.*

4. Selection of Bias Parameters

Selecting an appropriate parameter is crucial in this study. The use of the Ridge estimator largely depends on the ridge parameter, k. Several methods for estimating this ridge parameter have been proposed. This includes Hoerl and Kennard [1], Kibria [11], Muniz and Kibria [12], Aslam [13], Dorugade [14], Kibria and Banik [15], Lukman and Ayinde [16], Lukman et al. [17], and others. For the purpose of practical application of this new estimator, the optimum values of k and d are obtained. In order to obtain an optimum value of k, we assume the value of d is fixed.

Recall from equation (18),

$$\begin{aligned}\Delta &= \text{MSEM}\left(\hat{\alpha}_{\text{MTPE}}(k, d, b)\right) = \sigma^2 B_{k,d,b} \Lambda^{-1} B_{k,d,b}' \\ &\quad + \left(B_{k,d,b} - I\right)(\alpha - b)(\alpha - b)'\left(B_{k,d,b} - I\right)' \\ &= \sigma^2 \sum_{i=1}^{p} \frac{\left(\lambda_i + kd\right)^2}{\lambda_i \left(\lambda_i + k\right)^2} + k^2 (d - 1)^2 \sum_{i=1}^{p} \frac{\left(\alpha_i - b\right)^2}{\left(\lambda_i + k\right)^2}.\end{aligned} \tag{49}$$

Differentiating equation (49) with respect to k gives the following result:

$$\begin{aligned}\frac{\partial \Delta}{\partial k} &= -2\sigma^2 \sum_{i=1}^{p} \frac{\lambda_i \left((1 - d)\left(\lambda_i + kd\right)\right)}{\lambda_i \left(\lambda_i + k\right)^3} \\ &\quad + 2k (d - 1)^2 \sum_{i=1}^{p} \frac{\lambda_i \left(\alpha_i - b\right)^2}{\left(\lambda_i + k\right)^3}.\end{aligned} \tag{50}$$

Let $(\partial \Delta / \partial k) = 0$, the value of k is as follows:

$$k = \frac{\sigma^2 \lambda_i}{\lambda_i \left(\alpha_i - b\right)^2 - d\left(\lambda_i \left(\alpha_i - b\right)^2 + \sigma^2\right)}, \tag{51}$$

σ^2 and α_i are replaced by their unbiased estimators $\hat{\sigma}^2$ and $\hat{\alpha}_i$. The harmonic mean version is defined as

$$\hat{k}_{\text{HMP}} = \frac{p}{\sum_{i=1}^{p} 1/\hat{k}}, \tag{52}$$

where $\hat{k} = (\hat{\sigma}^2 \lambda)_i / (\lambda_i (\hat{\alpha}_i - b)^2 - \hat{d}(\lambda_i (\hat{\alpha}_i - b)^2 + \hat{\sigma}^2))$.

Recall that $\hat{\alpha}_{\text{MTPE}}(k, 0, 0) = \hat{\alpha}_{\text{RRE}}(k)$, considering this special case implies that $\hat{k}$ in equation (51) will become

$$\widehat{k} = \frac{\widehat{\sigma}^2}{\widehat{\alpha}_i^2}, \tag{53}$$

which is the estimated value of k introduced by Hoerl and Kennard [1]. Hoerl et al. [18] defined the harmonic version of the ridge parameter, k, as follows:

$$\widehat{k}_{\text{HKB}} = \frac{p\widehat{\sigma}^2}{\sum_i^p \widehat{\alpha}_i^2}. \tag{54}$$

The optimum value of d is obtained by differentiating equation (49) with respect to d with fixed k. The result is as follows:

$$\frac{\partial \Delta}{\partial d} = 2k\sigma^2 \sum_{i=1}^{p} \frac{(\lambda_i + kd)}{\lambda_i (\lambda_i + k)^2} + 2k^2 (d-1) \sum_{i=1}^{p} \frac{(\alpha_i - b)^2}{(\lambda_i + k)^2}. \tag{55}$$

Let $(\partial \Delta / \partial d) = 0$, the value of d is as follows:

$$d_{\text{MTPE}} = \frac{\sum_{i=1}^{p} \left[\left(k\lambda_i (\alpha - b)^2 \right) - (\sigma^2 \lambda_i) \right]}{\sum_{i=1}^{p} \left(\sigma^2 k + k\lambda_i (\alpha - b)^2 \right)}. \tag{56}$$

σ^2 and α_i are replaced by their unbiased estimators $\widehat{\sigma}^2$ and $\widehat{\alpha}_i$. Recall that $\widehat{\alpha}_{\text{MTPE}}(1, d, 0) = \widehat{\alpha}_{\text{LE}}(k)$, considering this special case implies that d in equation (54) will become

$$d_{\text{liu}} = \frac{\sum_{i=1}^{p} \left(\alpha_i^2 - \widehat{\sigma}^2 \right) \lambda_i}{\sum_{i=1}^{p} \left(\sigma^2 + \lambda_i \alpha_i^2 \right)}. \tag{57}$$

Equation (57) is the same as the optimum value of d proposed by Liu [6], which is defined as follows:

$$d_{\text{opt}} = \frac{\sum_{i=1}^{p} \left(\left(\alpha_i^2 - \widehat{\sigma}^2 \right) / (\lambda_i + 1)^2 \right)}{\sum_{i=1}^{p} \left((\sigma^2 + \lambda_i \alpha_i^2) / \lambda_i (\lambda_i + 1)^2 \right)}. \tag{58}$$

Theorem 9. *If*

$$\widehat{d} < \min \left(\frac{\lambda_i (\alpha_i - b)^2}{\lambda_i (\alpha_i - b)^2 + \sigma^2} \right), \tag{59}$$

for all i, then $\widehat{k}$ are always positive.

Proof. The values of k in (51) are always positive if $((\sigma^2 \lambda_i) / (\lambda_i (\alpha_i - b)^2 - d(\lambda_i (\alpha_i - b)^2 + \sigma^2))) > 0$. Since $\sigma^2 \lambda_i > 0, \lambda_i (\alpha_i - b)^2 - d(\lambda_i (\alpha_i - b)^2 + \sigma^2)$ must be positive for all i, it is observed that $d < ((\lambda_i (\alpha_i - b)^2) / (\lambda_i (\alpha_i - b)^2 + \sigma^2))$ for all i. This inequality depends on the unknown parameters σ^2 and α_i which is replaced by their unbiased estimators $\widehat{\sigma}^2$ and $\widehat{\alpha}_i$.

The selection of the estimator of the parameters d and k in $\widehat{\alpha}_{\text{MTPE}}(k, d, b)$ can be obtained iteratively as follows:

Step 1: calculate $\widehat{d}$ from (59).

Step 2: estimate $\widehat{k}_{\text{HKB}}$ by using $\widehat{d}$ in step 1.

Step 3: estimate $\widehat{d}_{\text{MTPE}}$ from (56) by using the estimator $\widehat{k}_{\text{HKB}}$ in step 2.

Step 4: if $\widehat{d}_{\text{MTPE}}$ is negative use $\widehat{d}_{\text{MTPE}} = \widehat{d}$, $\widehat{d}_{\text{MTPE}}$ can take negative value. However, $\widehat{d}$ takes value between 0 and 1.

5. Numerical Example and Monte-Carlo Simulation

Hussain dataset which was originally adopted by Eledum and Zahri [19] is used in this study to illustrate the performance of the new estimator. The dataset was also adopted in the study of Lukman et al. [20]. This is provided in Table 1. The regression model is defined as follows:

$$y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon_i, \tag{60}$$

where y_i represents the product value in the manufacturing sector, X_1 the values of the imported intermediate commodities, X_2 imported capital commodities, X_3 represents the value of imported raw materials. The variance inflation factors are $\text{VIF}_1 = 128.29$, $\text{VIF}_2 = 103.43$, and $\text{VIF}_3 = 70.87$. $\lambda_4 = 105.419$ and the condition number of $X'X$ is approximately 5660049. The variance inflation factor and the condition number both indicate the presence of severe multicollinearity.

The prior information of $b = 0.95\,\widehat{\beta}$ as used in the study of Li and Yang [7] is adopted. The estimated mean square values of the estimators OLSE, RRE, LE, MRRE, MLE, TPE, and MTPE are provided in Table 2. The values of k and d were computed using the estimators of k and d proposed in this study. k and d in equations (52) and (56) are obtained to be 1036.427 and 0.0043, respectively. From both tables, OLSE has the least performance among all the estimators. It was observed from Table 2 that the modified estimators (MLE, MRRE, and MTPE) outperform their counterparts. However, the proposed estimator MTPE outperforms other estimators.

Also, we conducted a Monte-Carlo simulation study to examine the performances of the estimators further. The simulation procedure used by Lukman and Ayinde [16] was also used to generate the explanatory variables in this study. This is given as

$$X_{ij} = \left(1 - \gamma^2\right)^{1/2} z_{ij} + \gamma z_{ip}, \quad i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, p, \tag{61}$$

where z_{ij} is independent standard normal distribution with mean zero and unit variance, γ^2 is the correlation between any two explanatory variables, and p is the number of explanatory variables. The values of γ were taken as 0.85, 0.9, and 0.99, respectively. In this study, the number of explanatory variable (p) was taken to be four.

The dependent variable is generated as follows:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i, \tag{62}$$

where $\varepsilon_i \sim (0, \sigma^2)$. The parameter values were chosen such that $\beta'\beta = 1$ which is a common restriction in simulation studies of this type [16]. The values of β are taken to be $\beta_1 = 0.8$, $\beta_2 = 0.1$, and $\beta_3 = 0.6$. Sample sizes 50 and 100 were used. Three different values of σ (0.01, 0.1, and 1) were also used. The experiment is replicated 5000 times. The estimated MSE is calculated as

Table 1

y	X_1	X_2	X_3
115.20	38.10	30.40	7.00
134.30	39.20	32.40	12.50
151.00	36.30	31.40	2.30
169.00	31.10	28.40	3.60
170.80	40.00	31.40	7.00
187.50	55.00	37.00	6.00
205.20	55.00	50.00	4.00
235.70	47.00	42.00	8.00
257.70	47.00	28.10	8.70
276.70	50.00	44.70	4.50
327.00	69.00	50.00	8.50
353.80	85.00	61.40	39.20
419.50	88.00	76.10	17.70
489.00	91.00	88.70	32.90
594.90	285.00	203.30	121.00
807.60	448.00	615.90	133.90
1014.00	324.00	562.10	82.50
1208.00	281.00	716.00	99.30
1380.00	349.00	771.30	103.90
1518.00	508.40	807.40	87.70
1763.00	533.20	1222.00	217.10
1914.00	592.80	1188.00	184.90
2338.00	726.40	1478.00	227.90
2275.00	706.30	1434.00	221.40
2562.00	796.70	1630.00	250.50
2750.00	856.00	1759.00	269.50
3000.00	934.90	1930.00	294.90
2859.00	890.30	1833.00	280.60
3794.00	1185.00	2472.00	375.20
4848.00	1696.00	3581.00	539.50
4048.00	1458.00	3065.00	463.00

Table 2: Estimated regression coefficients and mean square error of estimators.

Estimates	Estimators						
	OLSE	RRE	LE	MRRE	MLE	TPE	MTPE
$\widehat{\beta}_0$	208.87	207.12	191.96	198.54	208.87	207.13	195.81
$\widehat{\beta}_1$	−1.314	−1.314	−1.314	−1.314	−1.314	−1.314	−1.314
$\widehat{\beta}_2$	1.515	1.515	1.515	1.514	1.515	1.515	1.514
$\widehat{\beta}_3$	−2.017	−2.017	−2.006	−2.006	−2.017	−2.017	−2.004
MSE	1850.48	1822.65	1849.39	109.08	1564.06	1822.76	108.37

$$\text{MSE}(\widehat{\beta}) = \frac{1}{5000} \sum_{j=1}^{5000} \left(\widehat{\beta}_{ij} - \beta_i\right)' \left(\widehat{\beta}_{ij} - \beta_i\right), \qquad (63)$$

where $\widehat{\beta}_{ij}$ denotes the estimate of the ith parameter in the jth replication and β_i is the true parameter values. The estimated MSEs of the estimators for different values of n, p, σ, and γ are shown in Tables 3–6. The results from the simulation study show that the estimated MSE increases as the level of error variance increases. We observed that as the degree of multicollinearity (ρ) increases, the estimated MSEs also increase. Also, RRE, MRRE, LE, MLE, TPE, and MTPE have smaller MSE than the OLS estimator. The proposed estimator MTPE outperforms other estimators depending on the choice of prior information. The results of the simulation study support the real-life analysis in this paper.

6. Conclusions

In this article, we proposed a modified two-parameter estimator to overcome the multicollinearity problem in a linear regression model. Also, we established the superiority of this new estimator over other existing estimators in terms of matrix mean squared error criterion. This new estimator is considered to include the ordinary least squares estimator (OLSE), the ridge estimator (RRE), the Liu estimator (LE), the modified ridge estimator (MRE),

Table 3: Estimated MSE values of the OLSE, RRE, MRRE, LE, and MLE when $n = 50$.

					$\rho = 0.85$				
Estimators		$k = 0.01$			$k = 0.05$			$k = 0.1$	
	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1$
OLS	$3.46E-05$	0.003874	0.333639	$3.46E-05$	0.003874	0.333639	$3.46E-05$	0.003874	0.333639
RRE	$2.52E-05$	0.003185	0.311938	$3.14E-05$	0.003145	0.319639	$3.08E-05$	0.002959	0.297564
MRRE	$1.57E-05$	$1.98E-03$	$1.87E-01$	$1.95E-05$	$1.95E-03$	$1.99E-01$	$1.91E-05$	$1.84E-03$	$2.37E-01$
					$\rho = 0.9$				
OLS	$7.76E-05$	0.007438	0.719922	$7.76E-05$	0.007438	0.719922	$7.76E-05$	0.007438	0.719922
RRE	$6.79E-05$	0.006505	0.629549	$6.26E-05$	0.006321	0.629699	$5.89E-05$	0.00608	0.579048
MRRE	$4.22E-05$	$4.04E-03$	$3.78E-01$	$3.89E-05$	$3.93E-03$	$3.91E-01$	$3.66E-05$	$3.78E-03$	$4.61E-01$
					$\rho = 0.99$				
OLS	0.004043	0.408836	40.89173	0.004043	0.408836	40.89173	0.004043	0.408836	40.89173
RRE	0.002534	0.26822	24.79705	0.001099	0.109383	10.50059	0.00064	0.062891	6.359499
MRRE	0.001571	0.166296	15.37417	0.000681	0.067817	6.510364	0.000397	0.038992	3.942889
					$\rho = 0.85$				
Estimators		$d = 0.01$			$d = 0.05$			$d = 0.1$	
	0.01	0.1	1	0.01	0.1	1	0.01	0.1	1
OLS	$3.46E-05$	0.003874	0.333639	$3.46E-05$	0.003874	0.333639	$3.46E-05$	0.003874	0.333639
LE	$3.15E-05$	$2.02E-03$	$1.92E-01$	$2.97E-05$	$2.00E-03$	$2.11E-01$	$3.03E-05$	$2.18E-03$	$1.96E-01$
MLE	$1.96E-05$	$1.25E-03$	$1.20E-01$	$1.85E-05$	$1.25E-03$	$1.31E-01$	$1.88E-05$	$1.35E-03$	$1.22E-01$
					$\rho = 0.9$				
OLS	$7.76E-05$	0.007438	0.719922	$7.76E-05$	0.007438	0.719922	$7.76E-05$	0.007438	0.719922
LE	$4.75E-05$	$3.31E-03$	$3.30E-01$	$4.66E-05$	$3.43E-03$	$3.23E-01$	$4.25E-05$	$3.39E-03$	$3.43E-01$
MLE	$2.95E-05$	$2.05E-03$	$1.98E-01$	$2.89E-05$	$2.13E-03$	$2.01E-01$	$2.64E-05$	$2.11E-03$	$2.73E-01$
					$\rho = 0.99$				
OLS	0.004043	0.408836	40.89173	0.004043	0.408836	40.89173	0.004043	0.408836	40.89173
LE	$4.45E-05$	$3.00E-03$	$2.79E-01$	$6.72E-05$	$5.84E-03$	$5.25E-01$	$1.00E-04$	$8.96E-03$	$9.42E-01$
MLE	$2.76E-05$	$1.86E-03$	$1.68E-01$	$4.17E-05$	$3.63E-03$	$3.26E-01$	$6.22E-05$	$5.57E-03$	$7.50E-01$

Table 4: Estimated MSE values of the OLSE, TPE, and MTPE when $n = 50$.

Rho			0.85			0.9			0.99		
d	k	Sigma	OLSE	TPE	MTPE	OLSE	TPE	MTPE	OLSE	TPE	MTPE
		0.01	$3.46E-05$	$2E-05$	$9.13E-06$	$7.76E-05$	$4.14E-05$	$1.9E-05$	0.004043	0.00212	0.000969
	0.01	0.1	0.003874	0.001977	0.000181	0.007438	0.003827	0.00035	0.408836	0.22438	0.020514
		1	0.333639	0.188943	0.001727	0.719922	0.382846	0.0035	40.89173	20.74408	0.189653
		0.01	$3.46E-05$	$1.99E-05$	$9.09E-06$	$7.76E-05$	$3.78E-05$	$1.7E-05$	0.004043	0.000919	0.00042
0.01	0.05	0.1	0.003874	0.00189	0.000173	0.007438	0.003928	0.00036	0.408836	0.091505	0.008366
		1	0.333639	0.190314	0.00174	0.719922	0.384937	0.00352	40.89173	8.784307	0.080311
		0.01	$3.46E-05$	$1.86E-05$	$8.52E-06$	$7.76E-05$	$3.73E-05$	$1.7E-05$	0.004043	0.000535	0.000245
	0.1	0.1	0.003874	0.002014	0.000184	0.007438	0.003826	0.00035	0.408836	0.052611	0.00481
		1	0.333639	0.196492	0.001796	0.719922	0.367537	0.00336	40.89173	5.320063	0.048639
		0.01	$3.46E-05$	$1.89E-05$	$8.64E-06$	$7.76E-05$	$4.08E-05$	$1.9E-05$	0.004043	0.002114	0.000966
	0.01	0.1	0.003874	0.001943	0.000178	0.007438	0.003912	0.00036	0.408836	0.224209	0.020498
		1	0.333639	0.205892	0.001882	0.719922	0.368826	0.00337	40.89173	20.64197	0.18872
		0.01	$3.46E-05$	$1.98E-05$	$9.03E-06$	$7.76E-05$	$3.78E-05$	$1.7E-05$	0.004043	0.000961	0.000439
0.05	0.05	0.1	0.003874	0.001967	0.00018	0.007438	0.003769	0.00034	0.408836	0.09861	0.009015
		1	0.333639	0.186682	0.001707	0.719922	0.397967	0.00364	40.89173	9.495418	0.086812
		0.01	$3.46E-05$	$1.97E-05$	$9.02E-06$	$7.76E-05$	$3.96E-05$	$1.8E-05$	0.004043	0.000585	0.000267
	0.1	0.1	0.003874	0.001951	0.000178	0.007438	0.003655	0.00033	0.408836	0.060326	0.005515
		1	0.333639	0.205296	0.001877	0.719922	0.353004	0.00323	40.89173	5.839301	0.053386
		0.01	$3.46E-05$	$1.84E-05$	$8.4E-06$	$7.76E-05$	$3.69E-05$	$1.7E-05$	0.004043	0.002045	0.000935
	0.01	0.1	0.003874	0.001917	0.000175	0.007438	0.003819	0.00035	0.408836	0.209985	0.019198
		1	0.333639	0.191645	0.001752	0.719922	0.385686	0.00353	40.89173	22.09732	0.202026
		0.01	$3.46E-05$	$1.95E-05$	$8.91E-06$	$7.76E-05$	$3.84E-05$	$1.8E-05$	0.004043	0.001	0.000457
0.1	0.05	0.1	0.003874	0.001859	0.00017	0.007438	0.003834	0.00035	0.408836	0.102129	0.009337
		1	0.333639	0.192989	0.001764	0.719922	0.365375	0.00334	40.89173	10.09752	0.092317
		0.01	$3.46E-05$	$2.01E-05$	$9.19E-06$	$7.76E-05$	$3.56E-05$	$1.6E-05$	0.004043	0.000656	0.0003
	0.1	0.1	0.003874	0.002062	0.000189	0.007438	0.003699	0.00034	0.408836	0.058421	0.005341
		1	0.333639	0.18755	0.001715	0.719922	0.366358	0.00335	40.89173	6.172316	0.056431

Table 5: Estimated MSE values of the OLSE, RRE, MRRE, LE, and MLE when $n = 100$.

					$\rho = 0.85$				
		$k = 0.01$			$k = 0.05$			$k = 0.1$	
	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 0.01$	$\sigma = 0.1$	$\sigma = 1$
OLS	0.00578	0.54722	0.57255	0.00578	0.547223	0.57255	0.005784	0.547223	0.572553
RRE	1.77E − 05	0.00176	0.16786	1.77E − 05	0.001679	0.16908	1.66E − 05	0.001789	0.174567
MRRE	1.63E − 05	0.0017	0.07039	7.42E − 06	0.000254	0.07131	8.04E − 06	0.000848	0.066346
					$\rho = 0.9$				
OLS	9.19887	8.5034	8.50041	9.19887	8.5034	8.50041	9.198873	8.5034	8.500411
RRE	3.68E − 05	0.0034	0.34013	3.36E − 05	0.00349	0.34198	3.31E − 05	0.003399	0.326526
MRRE	2.76E − 05	2.55E − 03	2.55E − 01	2.52E − 05	2.62E − 03	2.56E − 01	2.48E − 05	2.55E − 03	2.45E − 01
					$\rho = 0.99$				
OLS	1.80265	171.595	169.891	1.80265	171.595	169.891	1.802645	171.595	169.8914
RRE	0.00188	0.19934	18.4294	0.00082	0.081294	7.80411	0.000476	0.046741	4.726424
MRRE	0.00117	0.12359	11.4262	0.00051	0.050402	4.83855	0.000295	0.028979	2.930383
					$\rho = 0.85$				
Sigma		$d = 0.01$			$d = 0.05$			$d = 0.1$	
	0.01	0.1	1	0.01	0.1	1	0.01	0.1	1
OLSE	0.00578	0.54722	0.57255	0.00578	0.547223	0.57255	0.005784	0.547223	0.572553
LE	2.34E − 05	1.50E − 03	1.43E − 01	2.21E − 05	1.49E − 03	1.57E − 01	2.25E − 05	1.62E − 03	1.46E − 01
MLE	1.45E − 05	9.32E − 04	8.88E − 02	1.37E − 05	9.25E − 04	9.75E − 02	1.40E − 05	1.01E − 03	9.07E − 02
					$\rho = 0.9$				
OLSE	9.19887	8.5034	8.50041	9.19887	8.5034	8.50041	9.198873	8.5034	8.500411
LE	3.53E − 05	2.46E − 03	2.45E − 01	3.46E − 05	2.55E − 03	2.40E − 01	3.16E − 05	2.52E − 03	2.55E − 01
MLE	2.19E − 05	1.53E − 03	1.47E − 01	2.15E − 05	1.58E − 03	1.49E − 01	1.96E − 05	1.57E − 03	2.03E − 01
					$\rho = 0.99$				
OLSE	1.80265	171.595	169.891	1.80265	171.595	169.891	1.802645	171.595	169.8914
LE	3.31E − 05	2.23E − 03	2.08E − 01	4.99E − 05	4.34E − 03	3.90E − 01	7.44E − 05	6.66E − 03	7.00E − 01
MLE	2.05E − 05	1.39E − 03	1.25E − 01	3.10E − 05	2.70E − 03	2.42E − 01	4.62E − 05	4.14E − 03	5.57E − 01

Table 6: Estimated MSE values of the OLSE, TPE, and MTPE when $n = 100$.

d	k	σ		$\rho = 0.85$			$\rho = 0.9$			$\rho = 0.99$	
			OLSE	TPE	MTPE	OLSE	TPE	MTPE	OLSE	TPE	MTPE
		0.01	5.78E − 03	1.77E − 05	8.11E − 06	9.20E + 00	3.68E − 05	1.68E − 05	1.80E + 00	1.88E − 03	8.61E − 04
	0.01	0.1	5.47E − 01	1.76E − 03	1.61E − 04	8.50E + 00	3.40E − 03	3.11E − 04	1.72E + 02	1.99E − 01	1.82E − 02
		1	5.73E − 01	1.68E − 01	1.54E − 03	8.50E + 00	3.40E − 01	3.11E − 03	1.70E + 02	1.84E + 01	1.68E − 01
		0.01	5.78E − 03	1.77E − 05	8.08E − E − 06	9.20E + 00	3.36E − 05	1.54E − 05	1.80E + 00	8.16E − 04	3.73E − 04
0.01	0.05	0.1	5.47E − 01	1.68E − 03	1.53E − 04	8.50E + 00	3.49E − 03	3.19E − 04	1.72E + 02	8.13E − 02	7.43E − 03
		1	5.73E − 01	1.69E − 01	1.55E − 03	8.50E + 00	3.42E − 01	3.13E − 03	1.70E + 02	7.80E + 00	7.13E − 02
		0.01	5.78E − 03	1.66E − 05	7.57E − 06	9.20E + 00	3.31E − 05	1.51E − 05	1.80E + 00	4.76E − 04	2.17E − 04
	0.1	0.1	5.47E − 01	1.79E − 03	1.64E − 04	8.50E + 00	3.40E − 03	3.11E − 04	1.72E + 02	4.67E − 02	4.27E − 03
		1	5.73E − 01	1.75E − 01	1.60E − 03	8.50E + 00	3.27E − 01	2.99E − 03	1.70E + 02	4.73E + 00	4.32E − 02
		0.01	5.78E − 03	1.68E − 05	7.68E − 06	9.20E + 00	3.63E − 05	1.66E − 05	1.80E + 00	1.88E − 03	8.58E − 04
	0.01	0.1	5.47E − 01	1.73E − 03	1.58E − 04	8.50E + 00	3.48E − 03	3.18E − 04	1.72E + 02	1.99E − 01	1.82E − 02
		1	5.73E − 01	1.83E − 01	1.67E − 03	8.50E + 00	3.28E − 01	3.00E − 03	1.70E + 02	1.83E + 01	1.68E − 01
		0.01	5.78E − 03	1.76E − 05	8.03E − 06	9.20E + 00	3.36E − 05	1.54E − 05	1.80E + 00	8.54E − 04	3.90E − 04
0.05	0.05	0.1	5.47E − 01	1.75E − 03	1.60E − 04	8.50E + 00	3.35E − 03	3.06E − 04	1.72E + 02	8.76E − 02	8.01E − 03
		1	5.73E − 01	1.66E − 01	1.52E − 03	8.50E + 00	3.54E − 01	3.23E − 03	1.70E + 02	8.44E + 00	7.71E − 02
		0.01	5.78E − 03	1.75E − 05	8.01E − 06	9.20E + 00	3.52E − 05	1.61E − 05	1.80E + 00	5.20E − 04	2.38E − 04
	0.1	0.1	5.47E − 01	1.73E − 03	1.58E − 04	8.50E + 00	3.25E − 03	2.97E − 04	1.72E + 02	5.36E − 02	4.90E − 03
		1	5.73E − 01	1.82E − 01	1.67E − 03	8.50E + 00	3.14E − 01	2.87E − 03	1.70E + 02	5.19E + 00	4.74E − 02
		0.01	5.78E − 03	1.63E − 05	7.46E − 06	9.20E + 00	3.28E − 05	1.50E − 05	1.80E + 00	1.82E − 03	8.30E − 04
	0.01	0.1	5.47E − 01	1.70E − 03	1.56E − 04	8.50E + 00	3.39E − 03	3.10E − 04	1.72E + 02	1.87E − 01	1.71E − 02
		1	5.73E − 01	1.70E − 01	1.56E − 03	8.50E + 00	3.43E − 01	3.13E − 03	1.70E + 02	1.96E + 01	1.79E − 01
		0.01	5.78E − 03	1.73E − 05	7.91E − 06	9.20E + 00	3.41E − 05	1.56E − 05	1.80E + 00	8.88E − 04	4.06E − 04
0.1	0.05	0.1	5.47E − 01	1.65E − 03	1.51E − 04	8.50E + 00	3.41E − 03	3.11E − 04	1.72E + 02	9.07E − 02	8.30E − 03
		1	5.73E − 01	1.71E − 01	1.57E − 03	8.50E + 00	3.25E − 01	2.97E − 03	1.70E + 02	8.97E + 00	8.20E − 02
		0.01	5.78E − 03	1.79E − 05	8.16E − 06	9.20E + 00	3.16E − 05	1.44E − 05	1.80E + 00	5.83E − 04	2.66E − 04
	0.1	0.1	5.47E − 01	1.83E − 03	1.68E − 04	8.50E + 00	3.29E − 03	3.00E − 04	1.72E + 02	5.19E − 02	4.75E − 03
		1	5.73E − 01	1.67E − 01	1.52E − 03	8.50E + 00	3.25E − 01	2.98E − 03	1.70E + 02	5.48E + 00	5.01E − 02

and the modified Liu estimator (MLE) as special cases. Finally, a numerical example and a simulation study were conducted to illustrate the theoretical results. Results show that the performance of the proposed estimator (MTPE) is superior to others.

Data Availability

The data used to support the findings of this study are included in Table 1.

Disclosure

This manuscript is accepted for a poster session, available in the following link: http://www.isi2019.org/wp-content/uploads/2019/03/CPS-list-by-CPS-POSTER-by-CPS_Title-no-1-March-2019.pdf.

Conflicts of Interest

There are no conflicts of interest regarding the publication of this paper.

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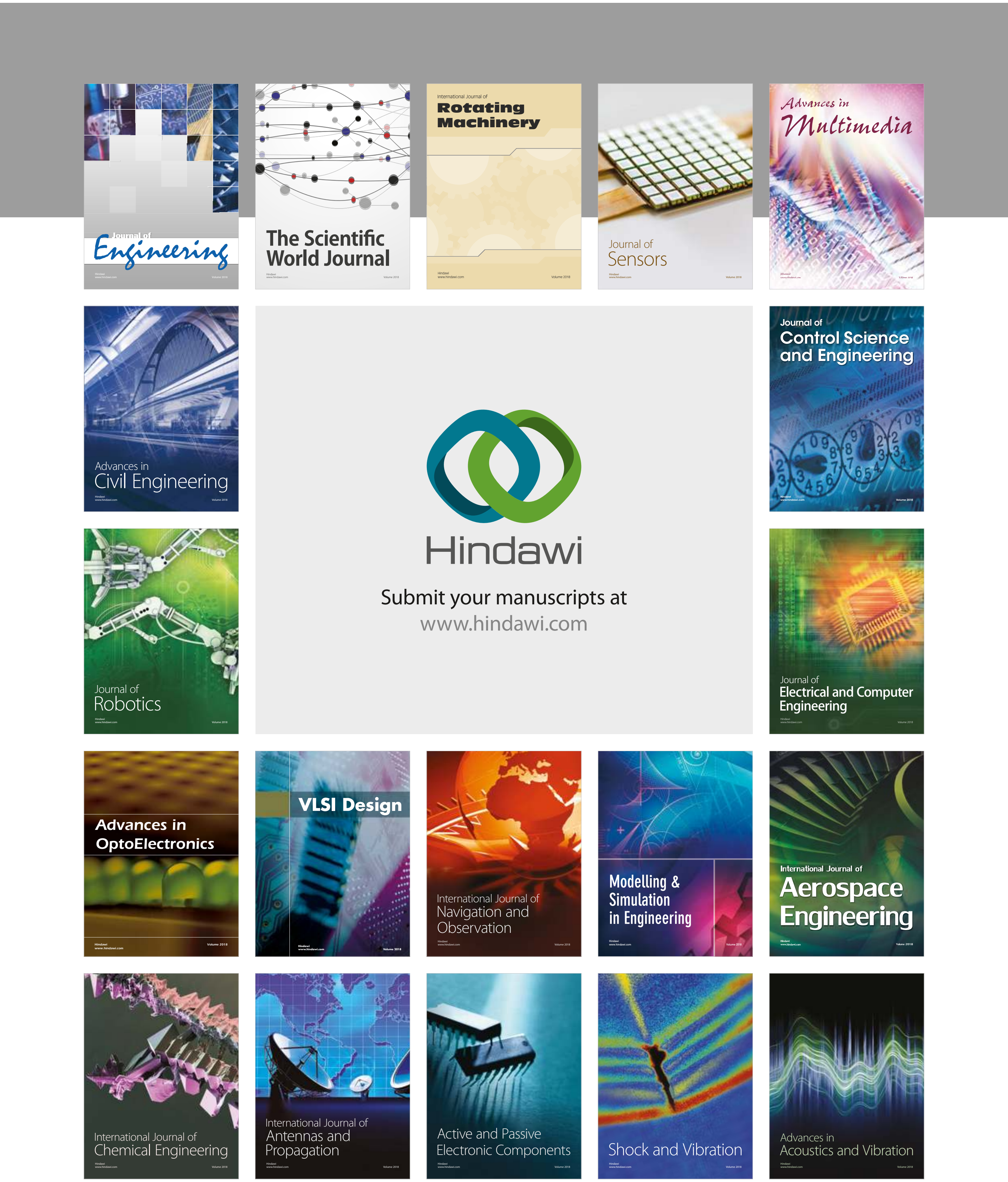

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