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A MODIFIED SHRINKING PROJECTION METHODS FOR NUMERICAL RECKONING FIXED POINTS OF G-NONEXPANSIVE MAPPINGS IN HILBERT SPACES WITH GRAPHS

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Abstract. In this paper, we introduce four new iterative schemes by modifying the shrinking projection method with Ishikawa iteration and S-iteration. The strong convergence theorems are given for obtaining a common fixed point of two G-nonexpansive mappings in a Hilbert space with a directed graph. We also give some numerical experiments for supporting our main theorems and compare convergence rate between them.

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1. INTRODUCTION

Let C be a nonempty, closed and convex subset of a normed space X. A mapping $T : C \to C$ is said to be

1. *contraction* if there exists $\alpha \in (0,1)$ such that $\|Tx - Ty\| \leq \alpha \|x - y\|$ for all $x, y \in C$;

2. *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

The first important result on fixed points for contractive-type mapping was the well known Banach's contraction principle appeared in explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution for an integral equation [4]. Since this date, many articles studied and considered fixed point theorems and the existence of fixed points of a single-valued nonlinear mapping (see, for examples [2, 6, 22]).

In 1953, Mann [11] introduced the famous iteration procedure as follows:

$$x_1 \in C$$
$$x_{n+1} = \delta_n x_n + (1 - \delta_n) T x_n,$$

for all $n \in \mathbb{N}$ where $\{\delta_n\} \subset [0,1]$ and $\mathbb{N}$ the set of all positive integers. This iteration is used to obtain weak convergence theorem (see for example [16, 18]).

In 1974, Ishikawa [8] generalized the Mann's iterative algorithm by introduce the following iteration:

$$\begin{aligned} x_0 &\in C \\ x_{n+1} &= \delta_n x_n + (1-\delta_n)Ty_n, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \end{aligned}$$

for all $n \in \mathbb{N}$ where $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1].

In 2007, Agarwal et al. [1] introduced and studied the S-iteration process for a class of nearly asymptotically nonexpansive mappings in Banach spaces and this scheme has a better convergence rate than Ishikawa iteration for a class of contractions in metric spaces.

In 2008, Takahashi et al. [23] just involved one closed convex set for a family of nonexpansive mappings $\{T_n\}$ and obtaining another modification of the Mann's iteration method:

$$\begin{aligned} u_0 &\in H, u_1 = P_{C_1}x_0 \text{ with } C_1 = C, \\ y_n &= \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} &= P_{C_{n+1}}x_0. \end{aligned}$$

They proved that if $\alpha_n \leq a$ for all $n \geq 1$ and for some $0 < a < 1$, then the sequence $\{u_n\}$ converges strongly to $P_{Fix(T)}x_0$.

In 2008, by combination of the concepts in fixed point and graph theory, Jachymski [9] generalized the Banach's contraction principle in a complete metric space endowed with a directed graph. Many papers dealt with this point for existence of fixed points of monotone nonexpansive, G-nonexpansive and G-contraction mappings on a hyperbolic metric, Banach and Hilbert spaces endowed with graph and directed graph. Also these articles discussed Browders convergence theorem for G -nonexpansive mapping in a Hilbert space with a directed graph, weak and strong convergence of the Ishikawa iteration for G-nonexpansive mappings (see for example [3, 13, 24, 25]).

Motivated by the work of [1, 25], Suparatulatorn et al. [20] studied the following modified S-iteration process:

$$\begin{aligned} x_0 &\in C, \\ y_n &= (1-\sigma_n)x_n + \sigma_n S_1 x_n, \\ x_{n+1} &= (1-\delta_n)S_1 x_n + (1-\delta_n)S_2 y_n, \ n \geq 0, \end{aligned}$$

where $\{\delta_n\}$ and $\{\sigma_n\}$ are sequences in (0, 1) and $S_1, S_2 : C \to C$ are G-nonexpansive mappings. Also they proved weak and strong convergence for approximating common fixed points of two G-nonexpansive mappings in a uniformly convex Banach space X endowed with a graph under this iteration.

Motivated and inspired by the above works, we introduce the four different iterative schemes by using the shrinking projection method for approximating a common fixed point of two G-nonexpansive mappings in Hilbert spaces. We then obtain strong convergence theorems. Finally, we discuss some important numerical results to illustrate the rate convergence of the four iterations.

2. Preliminaries and lemmas

In this section, we give some known definitions and lemmas which will be used in the later sections.

Let C be a nonempty subset of a real Banach space X. Let $\triangle$ denote the diagonal of the cartesian product $C \times C$, i.e., $\triangle = \{(x, x) : x \in \triangle\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \triangle$. We assume G has no parallel edge. So we can identify the graph G with the pair $(V(G), E(G))$. A mapping $S : G \to G$ is said to be

• $G-$contraction if S satisfies the conditions:

(i) S is edge-preserving, i.e.,

$$(x, y) \in E(G) \Rightarrow (Sx, Sy) \in E(G),$$

(ii) S decreases weights of edges of G, i.e., there exists $\delta \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \delta \|x - y\|.$$

• $G-$nonexpansive if S satisfies the condition (i) and

(iii) S non-increases weights of edges of G, i.e.,

$$(x, y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \|x - y\|.$$

Definition 1. The symbol G^{-1} is called the conversion of a graph G and it is a graph obtained from G by reversing the direction of edges as:

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Definition 2. The sequence $\{x_j\}_{j=0}^{N}$ of $N + 1$ vertices is called a path in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$), where $x_0 = x$, $x_N = y$ and $(x_j, x_{j+1}) \in E(G)$ for $j = 0, 1, ..., N - 1$.

Definition 3. If there is a path between any two vertices of the graph G, then a graph G is said to be connected.

Definition 4. If (x, y) and $(y, z) \in V(G)$, then $(x, z) \in V(G)$. This property is called the transitivity of a directed graph $G = (V(G), E(G))$ for all $x, y, z \in V(G)$.

Definition 5. Let $G = (V(G), E(G))$ be a directed graph. The set of edges $E(G)$ is said to be *convex* if for any $(x, y), (z, w) \in E(G)$ and for each $t \in (0,1)$, then $(tx + (1-t)z, ty + (1-t)w) \in E(G)$.

Definition 6. Let $x_0 \in V(G)$ and A subset of $V(G)$. We say that
(i) A is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in A$.
(ii) A dominates x_0 if for each $x \in A$, $(x, x_0) \in E(G)$.

Lemma 1 ([19]). *Let C be a nonempty, closed and convex subset of a Hilbert space H and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. Let $T : C \to C$ be a G-nonexpansive mapping and $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ for some $x \in C$. If, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$ and $\{x_n - Tx_n\} \to y$ for some $y \in H$. Then $(I - T)x = y$.*

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \le \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the *metric projection* of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \le 0$ holds for all $x \in H$ and $y \in C$; see [21].
We know that the following result.

Lemma 2. *Let H be a real Hilbert space. Then*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2,$$

for all $t \in [0,1]$ and $x, y \in H$.

Lemma 3 ([10]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in C : \ \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

Lemma 4 ([12]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection from H onto C. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x - y\|^2, \ \ \forall x \in H, \ \forall y \in C.$$

3. Main results

In this section, by using the shrinking projection method, we obtain four different strong convergence theorems for finding the same common fixed point of two G-nonexpasive mappings in real Hilbert spaces.

Theorem 1. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G=(V(G),E(G))$ be a directed graph such that $V(G)=C$ and $E(G)$ is convex. Let $S_1,S_2:C\to C$ be G-nonexpansive mappings such that $F:=F(S_1)\cap F(S_2)\neq\varnothing$, F is closed and $F(S_i)\times F(S_i)\subseteq E(G)$ for all $i=1,2$. Let $\{s_n\}$ be a sequence generated by*

$$\begin{aligned} s_1 &\in C, \text{ with } C_1=C,\\ y_n &= (1-\beta_n)s_n+\beta_n S_1 s_n,\\ z_n &= (1-\alpha_n)s_n+\alpha_n S_2 y_n,\\ C_{n+1} &= \{z\in C_n : \|z_n-z\|\le\|s_n-z\|\},\\ s_{n+1} &= P_{C_{n+1}}s_1,\ n\ge 1, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\},\{\beta_n\}\subset[0,1]$. Assume that the following conditions hold:
(i) $\{s_n\}$ dominates p for all $p\in F$ and if there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k}\rightharpoonup w\in C$, then $(s_{n_k},w)\in E(G)$;
(ii) $\liminf_{n\to\infty}\alpha_n>0$; (iii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.
Then the sequence $\{s_n\}$ converges strongly to $P_F s_1$.

Proof. We split the proof into five steps.
Step 1. Show that $P_{C_{n+1}}s_1$ is well-defined for each $s_1\in C$. As shown in Theorem 3.2 of Tiammee et al. [24], $F(S_i)$ is convex for all $i=1,2$. It follows from the assumption that F is closed and convex. Hence, $P_F s_1$ is well-defined. We see that $C_1=C$ is closed and convex. Assume that C_n is closed and convex. From the definition of C_{n+1} and Lemma 3, we get C_{n+1} is closed and convex. Let $p\in F$. Since $\{s_n\}$ dominates p and S_1 is edge-preserving, we have $(S_1 s_n,p)\in E(G)$. This implies that $(y_n,p)=((1-\beta_n)s_n+\beta_n S_1 s_n,p)\in E(G)$ by $E(G)$ is convex. Since S_2 is edge-preserving, we have

$$\begin{aligned} \|z_n-p\| &\le (1-\alpha_n)\|s_n-p\|+\alpha_n\|S_2 y_n-p\|\\ &\le (1-\alpha_n)\|s_n-p\|+\alpha_n((1-\beta_n)\|s_n-p\|+\beta_n\|S_1 s_n-p\|)\\ &\le \|s_n-p\|. \end{aligned} \tag{3.2}$$

We can conclude that $p\in C_{n+1}$. Thus $F\subset C_{n+1}$. This implies that $P_{C_{n+1}}s_1$ is well-defined.
Step 2. Show that $\lim_{n\to\infty}\|s_n-s_1\|$ exists. Since F is a nonempty, closed and convex subset of H, there exists a unique $v\in F$ such that $v=P_F s_1$. From $s_n=P_{C_n}s_1$ and $s_{n+1}\in C_n$, $\forall n\in\mathbb{N}$, we get

$$\|s_n-s_1\|\le\|s_{n+1}-s_1\|,\quad \forall n\in\mathbb{N}. \tag{3.3}$$

On the other hand, as $F\subset C_n$, we obtain

$$\|s_n-s_1\|\le\|v-s_1\|,\quad \forall n\in\mathbb{N}. \tag{3.4}$$

It follows from (3.3) and (3.4) that the sequence $\{s_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty}\|s_n-s_1\|$ exists.

Step 3. Show that $s_n \to w \in C$ as $n \to \infty$. For $m > n$, by the definition of C_n, we see that $s_m = P_{C_m} s_1 \in C_m \subset C_n$. From Lemma 4, we have

$$\|s_m - s_n\|^2 \le \|s_m - s_1\|^2 - \|s_n - s_1\|^2.$$

From Step 3, we obtain that $\{s_n\}$ is a Cauchy sequence. Hence, there exists $w \in C$ such that $s_n \to w$ as $n \to \infty$. In particular, we have

$$\lim_{n\to\infty} \|s_{n+1} - s_n\| = 0. \tag{3.5}$$

Step 4. Show that $w \in F$. Since $s_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - s_n\| \le \|z_n - s_{n+1}\| + \|s_{n+1} - s_n\| \le 2\|s_{n+1} - s_n\| \to 0 \tag{3.6}$$

as $n \to \infty$. Since $\liminf_{n\to\infty} \alpha_n > 0$ and (3.6), we have

$$\|S_2 y_n - s_n\| = \frac{1}{\alpha_n}\|z_n - s_n\| \to 0, \tag{3.7}$$

as $n \to \infty$. From $\{s_n\}$ dominates p for all $p \in F$ and Lemma 2, we get

$$\begin{aligned} \|z_n - p\|^2 &\le (1-\alpha_n)\|s_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2 \\ &\le (1-\alpha_n)\|s_n - p\|^2 + \alpha_n\big((1-\beta_n)\|s_n - p\|^2 \\ &\quad + \beta_n \|S_1 s_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 s_n - s_n\|^2\big) \\ &\le \|s_n - p\|^2 - \alpha_n(1-\beta_n)\beta_n \|S_1 s_n - s_n\|^2. \end{aligned} \tag{3.8}$$

This implies that

$$\alpha_n(1-\beta_n)\beta_n \|S_1 s_n - s_n\|^2 \le \|s_n - p\|^2 - \|z_n - p\|^2. \tag{3.9}$$

From our assumptions and (3.6), we have

$$\lim_{n\to\infty} \|S_1 s_n - s_n\| = 0. \tag{3.10}$$

This implies that

$$\lim_{n\to\infty} \|y_n - s_n\| = \lim_{n\to\infty} \beta_n \|S_1 s_n - s_n\| = 0. \tag{3.11}$$

It follows from (3.7) and (3.11) that

$$\|S_2 y_n - y_n\| \le \|S_2 y_n - s_n\| + \|s_n - y_n\| \to 0, \tag{3.12}$$

as $n \to \infty$. By Lemma 1, (3.10), (3.11) and (3.12), we have $w \in F$.
Step 5. Show that $w = v = P_F s_1$. Since $s_n = P_{C_n} s_1$, we have

$$\langle s_1 - s_n, s_n - p\rangle \ge 0, \quad \forall p \in C_n. \tag{3.13}$$

By taking the limit in (3.13), we obtain

$$\langle s_1 - w, w - p\rangle \ge 0, \quad \forall p \in C_n. \tag{3.14}$$

Since $F \subset C_n$, so $w = P_F s_1$. This completes the proof. □

Theorem 2. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $S_1, S_2 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \neq \varnothing$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1,2$. Let $\{t_n\}$ be a sequence generated by*

$$\begin{aligned} t_1 &\in C, \text{ with } C_1 = C, \\ y_n &= (1-\beta_n)t_n + \beta_n S_1 t_n, \\ z_n &= (1-\alpha_n)y_n + \alpha_n S_2 y_n, \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \leq \|t_n - z\|\}, \\ t_{n+1} &= P_{C_{n+1}} t_1, \ n \geq 1, \end{aligned} \tag{3.15}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Assume that the following conditions hold:
(i) $\{t_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \rightharpoonup w \in C$, then $(t_{n_k}, w) \in E(G)$;
(ii) $\liminf_{n\to\infty} \alpha_n > 0$; (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{t_n\}$ converges strongly to $P_F t_1$.

Proof. We set $t_n = s_n$, by the same proof of Step 1 in Theorem 1, we have $P_F t_1$ well-defined, C_{n+1} is closed convex for all $n \in \mathbb{N}$ and $(y_n, p) \in E(G)$ for each $p \in F$. Since S_1, S_2 are edge-preserving, we have

$$\begin{aligned} \|z_n - p\| &\leq (1-\alpha_n)\|y_n - p\| + \alpha_n \|S_2 y_n - p\| \\ &\leq (1-\beta_n)\|t_n - p\| + \beta_n \|S_1 t_n - p\| \leq \|t_n - p\|. \end{aligned}$$

We can conclude that $p \in C_{n+1}$. Thus $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} t_1$ is well-defined. By the same proof of Step 2-3 in Theorem 1, we obtain $t_n \to w \in C$ as $n \to \infty$. We next show that $w \in F$. Since $t_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - t_n\| \leq \|z_n - t_{n+1}\| + \|t_{n+1} - t_n\| \leq 2\|t_{n+1} - t_n\| \to 0 \tag{3.16}$$

as $n \to \infty$. Since $\{t_n\}$ dominates p for all $p \in F$, we get

$$\begin{aligned} \|z_n - p\|^2 &\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2 \\ &\leq (1-\beta_n)\|t_n - p\|^2 + \beta_n \|S_1 t_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 t_n - t_n\|^2 \\ &\leq \|t_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 t_n - t_n\|^2. \end{aligned} \tag{3.17}$$

This implies that

$$(1-\beta_n)\beta_n \|S_1 t_n - t_n\|^2 \leq \|t_n - p\|^2 - \|z_n - p\|^2. \tag{3.18}$$

From our assumption (ii) and (3.18), we have

$$\lim_{n\to\infty} \|S_1 t_n - t_n\| = 0. \tag{3.19}$$

This implies that

$$\lim_{n\to\infty} \|y_n - t_n\| = \lim_{n\to\infty} \beta_n \|S_1 t_n - t_n\| = 0. \tag{3.20}$$

It follows from (3.16) and (3.20) that

$$\|y_n - z_n\| \le \|y_n - t_n\| + \|t_n - z_n\| \to 0, \tag{3.21}$$

as $n \to \infty$. From $\liminf_{n\to\infty} \alpha_n > 0$ and (3.21), we have

$$\lim_{n\to\infty} \|S_2 y_n - y_n\| = \lim_{n\to\infty} \frac{1}{\alpha_n} \|z_n - y_n\| = 0. \tag{3.22}$$

By Lemma 1, (3.19), (3.20) and (3.22), we have $w \in F$. From Step 5 in Theorem 1, we obtain $w = P_F t_1$. This completes the proof. □

Theorem 3. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $S_1, S_2 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \neq \varnothing$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1,2$. Let $\{u_n\}$ be a sequence generated by*

$$\begin{aligned} u_1 &\in C, \text{ with } C_1 = C, \\ y_n &= (1-\beta_n)u_n + \beta_n S_1 u_n, \\ z_n &= (1-\alpha_n)S_1 u_n + \alpha_n S_2 y_n, \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} &= P_{C_{n+1}} u_1,\ n \ge 1, \end{aligned} \tag{3.23}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Assume that the following conditions hold:
(i) $\{u_n\}$ dominates p for all $p \in F$ and if there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup w \in C$, then $(u_{n_k}, w) \in E(G)$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{u_n\}$ converges strongly to $P_F u_1$.

Proof. We set $u_n = s_n$, by the same proof of Step 1 in Theorem 1, we have $P_F u_1$ well-defined, C_{n+1} is closed convex for all $n \in \mathbb{N}$ and $(y_n, p) \in E(G)$ for each $p \in F$. Since S_1, S_2 are edge-preserving, we have

$$\begin{aligned} \|z_n - p\| &\le (1-\alpha_n)\|S_1 u_n - p\| + \alpha_n \|S_2 y_n - p\| \\ &\le (1-\alpha_n)\|u_n - p\| + \alpha_n((1-\beta_n)\|u_n - p\| + \beta_n \|S_1 u_n - p\|) \\ &\le \|u_n - p\|. \end{aligned} \tag{3.24}$$

We can conclude that $p \in C_{n+1}$. Thus $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} u_1$ is well-defined. By the same proof of Step 2-3 in Theorem 1, we obtain $u_n \to w \in C$ as $n \to \infty$. We next show that $w \in F$. Since $u_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - u_n\| \le \|z_n - u_{n+1}\| + \|u_{n+1} - u_n\| \le 2\|u_{n+1} - u_n\| \to 0 \tag{3.25}$$

as $n \to \infty$. Since $\{u_n\}$ dominates p for all $p \in F$, we get

$$\|z_n - p\|^2 \le (1-\alpha_n)\|S_1 u_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2$$

$$\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\big((1-\beta_n)\|u_n - p\|^2 + \beta_n\|S_1u_n - p\|^2 - (1-\beta_n)\beta_n\|S_1u_n - u_n\|^2\big) \leq \|u_n - p\|^2 - \alpha_n(1-\beta_n)\beta_n\|S_1u_n - u_n\|^2. \tag{3.26}$$

This implies that

$$(1-\beta_n)\beta_n\|S_1u_n - u_n\|^2 \leq \|u_n - p\|^2 - \|z_n - p\|^2. \tag{3.27}$$

From our assumption (ii) and (3.27), we have

$$\lim_{n\to\infty}\|S_1u_n - u_n\| = 0. \tag{3.28}$$

This implies that

$$\lim_{n\to\infty}\|y_n - u_n\| = \lim_{n\to\infty}\beta_n\|S_1u_n - u_n\| = 0. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$\|S_1u_n - y_n\| \leq \|S_1u_n - u_n\| + \|u_n - y_n\| \to 0, \tag{3.30}$$

as $n \to \infty$. For $p \in F$, it follows from (3.24) that

$$\begin{aligned}\|z_n - p\|^2 &= (1-\alpha_n)\|S_1u_n - p\|^2 + \alpha_n\|S_2y_n - p\|^2 - (1-\alpha_n)\alpha_n\|S_1u_n - S_2y_n\|^2\\ &\leq (1-\alpha_n)\|u_n - p\|^2 + \alpha_n\|y_n - p\|^2 - (1-\alpha_n)\alpha_n\|S_1u_n - S_2y_n\|^2\\ &\leq \|u_n - p\|^2 - (1-\alpha_n)\alpha_n\|S_1u_n - S_2y_n\|^2.\end{aligned}$$

This implies that

$$(1-\alpha_n)\alpha_n\|S_1u_n - S_2y_n\|^2 \leq \|u_n - p\|^2 - \|z_n - p\|^2. \tag{3.31}$$

From the assumption (i) and (3.25), we have

$$\lim_{n\to\infty}\|S_1u_n - S_2y_n\| = 0. \tag{3.32}$$

It follows from (3.30) and (3.32) that

$$\|S_2y_n - y_n\| \leq \|S_2y_n - S_1u_n\| + \|S_1u_n - y_n\| \to 0, \tag{3.33}$$

as $n \to \infty$.

$$\lim_{n\to\infty}\|S_2y_n - y_n\| = \lim_{n\to\infty}\frac{1}{\alpha_n}\|z_n - y_n\| = 0. \tag{3.34}$$

By Lemma 1, (3.28), (3.29) and (3.34), we have $w \in F$. From Step 5 in Theorem 1, we obtain $w = P_Fu_1$. This completes the proof. □

Theorem 4. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Let $S_1, S_2 : C \to C$ be G-nonexpansive mappings such that $F := F(S_1) \cap F(S_2) \neq \varnothing$, F is closed and $F(S_i) \times F(S_i) \subseteq E(G)$ for all $i = 1,2$. Let $\{v_n\}$ be a sequence generated by*

$$v_1 \in C, \text{ with } C_1 = C,$$

$$\begin{aligned} y_n &= (1-\beta_n)v_n + \beta_n S_1 v_n, \\ z_n &= (1-\alpha_n)S_1 y_n + \alpha_n S_2 y_n, \\ C_{n+1} &= \{z \in C_n : \|z_n - z\| \le \|v_n - z\|\}, \\ v_{n+1} &= P_{C_{n+1}} v_1, \ n \ge 1, \end{aligned} \tag{3.35}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. *Assume that the following conditions hold:*
(i) (v_n, p), $(p, v_n) \in E(G)$ *for all* $p \in F$ *and* $n \in \mathbb{N}$ *and if there exists a subsequence* $\{v_{n_k}\}$ *of* $\{v_n\}$ *such that* $v_{n_k} \rightharpoonup w \in C$*, then* $(v_{n_k}, w) \in E(G)$*;*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$*;*
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{x_n\}$ *converges strongly to* $P_F v_1$.

Proof. We set $v_n = s_n$, by the same proof of Step 1 in Theorem 1, we have $P_F v_1$ well-defined, C_{n+1} is closed convex for all $n \in \mathbb{N}$ and $(y_n, p) \in E(G)$ for each $p \in F$. Since S_1, S_2 are edge-preserving, we have

$$\begin{aligned} \|z_n - p\| &\le (1-\alpha_n)\|S_1 y_n - p\| + \alpha_n \|S_2 y_n - p\| \\ &\le (1-\beta_n)\|v_n - p\| + \beta_n \|S_1 v_n - p\| \\ &\le \|v_n - p\|. \end{aligned} \tag{3.36}$$

We can conclude that $p \in C_{n+1}$. Thus $F \subset C_{n+1}$. This implies that $P_{C_{n+1}} v_1$ is well-defined. By the same proof of Step 2-3 in Theorem 1, we obtain $v_n \to w \in C$ as $n \to \infty$. We next show that $w \in F$. Since $v_{n+1} \in C_n$, it follows from (3.5) that

$$\|z_n - v_n\| \le \|z_n - v_{n+1}\| + \|v_{n+1} - v_n\| \le 2\|v_{n+1} - v_n\| \to 0 \tag{3.37}$$

as $n \to \infty$. Since $\{v_n\}$ dominates p for all $p \in F$, we get

$$\begin{aligned} \|z_n - p\|^2 &\le (1-\alpha_n)\|S_1 y_n - p\|^2 + \alpha_n \|S_2 y_n - p\|^2 \\ &\le (1-\beta_n)\|v_n - p\|^2 + \beta_n \|S_1 v_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 v_n - v_n\|^2) \\ &\le \|v_n - p\|^2 - (1-\beta_n)\beta_n \|S_1 v_n - v_n\|^2. \end{aligned} \tag{3.38}$$

This implies that

$$(1-\beta_n)\beta_n \|S_1 v_n - v_n\|^2 \le \|v_n - p\|^2 - \|z_n - p\|^2. \tag{3.39}$$

From our assumption (ii) and (3.39), we have

$$\lim_{n\to\infty} \|S_1 v_n - v_n\| = 0. \tag{3.40}$$

This implies that

$$\lim_{n\to\infty} \|y_n - v_n\| = \lim_{n\to\infty} \beta_n \|S_1 v_n - v_n\| = 0. \tag{3.41}$$

It follows from (3.37) and (3.41) that

$$\|y_n - z_n\| \le \|y_n - v_n\| + \|v_n - z_n\| \to 0, \tag{3.42}$$

as $n \to \infty$. We next show that $(v_n, y_n) \in E(G)$ for all $n \in \mathbb{N}$. Let $p \in F$. Since $(p, v_n) \in E(G)$ and S_1 is edge-preserving, so $(p, S_1 v_n) \in E(G)$ for all $n \in \mathbb{N}$. Then, $(p, y_n) = (p, (1-\beta_n)v_n + \beta_n S_1 v_n) \in E(G)$ by $E(G)$ is convex. Since G is transitive, $(v_n, y_n) \in E(G)$. This implies that

$$\|S_1 y_n - y_n\| \le \|S_1 y_n - S_1 v_n\| + \|S_1 v_n - v_n\| + \|v_n - y_n\| \\ \le 2\|y_n - S_1 v_n\| + \|S_1 v_n - v_n\|.$$

It follows from (3.40), (3.41) and (3.43) that

$$\lim_{n\to\infty} \|S_1 y_n - y_n\| = 0. \tag{3.43}$$

It follows from (3.42) and (3.43) that

$$\|S_1 y_n - z_n\| \le \|S_1 y_n - y_n\| + \|y_n - z_n\| \to 0, \tag{3.44}$$

as $n \to \infty$. From $\liminf_{n\to\infty} \alpha_n > 0$ and (3.44), we have

$$\lim_{n\to\infty} \|S_2 y_n - z_n\| = \lim_{n\to\infty} \frac{1}{\alpha_n}\|z_n - S_1 y_n\| = 0. \tag{3.45}$$

It follows from (3.42) and (3.46) that

$$\|S_2 y_n - y_n\| \le \|S_2 y_n - z_n\| + \|z_n - y_n\| \to 0, \tag{3.46}$$

as $n \to \infty$. By Lemma 1, (3.41), (3.43) and (3.46), we have $w \in F$. From Step 5 in Theorem 1, we obtain $w = P_F v_1$. This completes the proof. □

4. Convergence rate

In this section, we give examples and numerical results for supporting our main theorem. Moreover, we compare convergence rate of all iterations in Theorem 1-4. In 1976, Rhoades [17] gave the idea how to compare the rate of convergence between two iterative methods as follows:

Definition 7 ([17]). Let C be a nonempty closed convex subset of a Banach space X and $S : C \to C$ be be a mapping. Suppose $\{x_n\}$ and $\{y_n\}$ are two iterations which converge to a fixed point p of S. Then $\{x_n\}$ is said to converge faster than $\{y_n\}$ if

$$\|x_n - p\| \le \|y_n - p\|,$$

for all $n \ge 1$.

In 2011, Phuengrattana and Suantai [14] showed that the Ishikawa iteration converges faster than the Mann iteration for a class of continuous functions on the closed interval in a real line. In order to study, the order of convergence of a real sequence $\{x_n\}$ converging to p, we usually use the well-known terminology in numerical analysis, see [7], for example.

Definition 8 ([7])**.** Suppose $\{x_n\}$ is a sequence that converges to p, with $x_n \neq p$ for all n. If positive constants a and b exist with

$$\lim_{n\to\infty} \frac{|x_{n+1}-p|}{|x_n-p|^b} = a,$$

then $\{x_n\}$ converges to p of order a, with asymptotic error constant b. If $b = 1$ (and $a < 1$), the sequence is linearly convergent and if $b = 2$, the sequence is quadratically convergent.

In 2002, Berinde [5] employed above concept for comparing the rate of convergence between the two iterative methods as follows:

Definition 9 ([5])**.** Let $\{x_n\}$ and $\{y_n\}$ be two sequences of positive numbers that converge to p, q,respectively. Assume there exists

$$\lim_{n\to\infty} \frac{|x_n-p|}{|y_n-q|} = \ell.$$

(i) If $\ell = 0$, then it is said that the sequence $\{x_n\}$ converges to p faster than the sequence $\{y_n\}$ to q.
(ii) If $0 < \ell < \infty$, then we say that the sequence $\{x_n\}$ and $\{y_n\}$ have the same rate of convergence.

Definition 10 ([5, 15])**.** Let C be a nonempty closed convex subset of a Banach space X and $S : C \to C$ be a mapping. Suppose $\{x_n\}$ and $\{y_n\}$ are two iterations which converge to p fixed point q of S. We say that $\{x_n\}$ converges faster than $\{y_n\}$ to q if

$$\lim_{n\to\infty} \frac{\|x_n-p\|}{\|y_n-q\|} = 0.$$

We now give an example in Euclidian space $\mathbb{R}^3$ which shows numerical experiment for supporting our main results and comparing the rate of convergence of all iterations in Theorem 1-4.

Example 1. Let $H = \mathbb{R}^3$ and $C = [-2,0]^3$. Assume that $(x,y) \in E(G)$ if and only if $-1.5 \leq x_i, y_i \leq -0.5$ or $x = y$ for all $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in C$. Define a mappings $S_1, S_2 : C \to C$ by $S_1x = (\frac{\arcsin(x_1+1)}{2} - 1, \log(x_2+2) - 1, -1)$ and $S_2x = (-1,-1,\frac{\cot(x_3-\frac{\pi}{2}+1)}{2} - 1)$ for all $x = (x_1,x_2,x_3) \in C$. It is easy to check that S_1 and S_2 are G-nonexpansive such that $F(S_1) \cap F(S_2) = \{(-1,-1,-1)\}$. On the other hand, S_1 is not nonexpansive since for $x = (-2,-1.46,-1)$ and $y = (-1.49,-1.82,-1)$. This implies that $\|S_1(x)-S_1(y)\| > 0.70 > \|x-y\|$. Moreover, S_2 is not nonexpansive since for $x = (-1,-1,-1.55)$ and $y = (-1,-1,-1.97)$, we have $\|S_2(x)-S_2(y)\| > 0.42 > \|x-y\|$.

We provide a numerical test of a comparison of all iterations in Theorem 1-4 and choose $\alpha_n = \frac{n+1}{5n+3}$, $\beta_n = \frac{n+3}{10n+5}$. The stoping criterion is defined by $\|x_{n+1} - x_n\| < 10^{-7}$. The different choices of x_0 are given in Table 1.

TABLE 1. Comparison the methods in Theorem 1-4 of Example 1

Iterations	Choice 1: (-1.25,-0.9,-0.65) Iterations Number	CPU Time (sec)	Choice 2: (-1.45,-1.2,-0.7) Iterations Number	CPU Time (sec)
(3.1)	118	2.344643e-03	124	1.618338e-03
(3.15)	97	2.896767e-03	102	1.452308e-03
(3.23)	39	2.051223e-03	41	1.233451e-03
(3.35)	38	3.913172e-03	39	3.129209e-03

By computing, we obtain the sequences $\{x_n\}$ generated in Theorem 1-4 converge to (-1,-1,-1). We next show the following error plots of $\|x_{n+1} - x_n\|$.

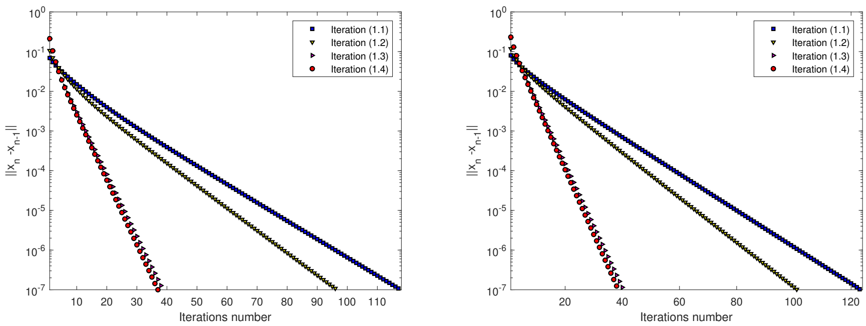

FIGURE 1. Error plots for sequences $\{x_n\}$ in Table 1 of choice 1 and choice 2, respectively.

We note that $p = (-1, -1, -1)$ is a common fixed point of S_1 and S_2. We compare the rate of convergence of $\{s_n\}$, $\{t_n\}$, $\{u_n\}$ and $\{v_n\}$ for Choice 1: $x_0 = (-1.25, -0.9, -0.65)$ and Choice 2: $x_0 = (-1.45, -1.2, -0.7)$.

TABLE 2. Comparison the rate of convergence of all iterations in Theorem 1-4 of Example 1 by choosing $x_0 = (-1.25, -0.9, -0.65)$

n	$\frac{\|v_n-p\|}{\|s_n-p\|}$	$\frac{\|v_n-p\|}{\|t_n-p\|}$	$\frac{\|v_n-p\|}{\|u_n-p\|}$	$\frac{\|u_n-p\|}{\|s_n-p\|}$	$\frac{\|u_n-p\|}{\|t_n-p\|}$	$\frac{\|t_n-p\|}{\|s_n-p\|}$
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
1	0.637538	0.979056	0.704657	0.651176	0.719731	0.904749
2	0.434675	0.952151	0.515409	0.456519	0.541310	0.843360
3	0.314266	0.924612	0.394249	0.339890	0.426395	0.797125
4	0.236078	0.899949	0.310695	0.262324	0.345236	0.759841
5	0.181039	0.878591	0.248470	0.206056	0.282805	0.728615
6	0.140193	0.859748	0.199756	0.163063	0.232342	0.701821
7	0.095220	0.842647	0.160677	0.113001	0.190681	0.592617
8	0.074299	0.826834	0.129054	0.089860	0.156083	0.575721
9	0.057963	0.811696	0.103409	0.071409	0.127399	0.560516

TABLE 3. Comparison the rate of convergence of all iterations in Theorem 1-4 of Example 1 by choosing $x_0 = (-1.45, -1.2, -0.7)$

n	$\frac{\|v_n-p\|}{\|s_n-p\|}$	$\frac{\|v_n-p\|}{\|t_n-p\|}$	$\frac{\|v_n-p\|}{\|u_n-p\|}$	$\frac{\|u_n-p\|}{\|s_n-p\|}$	$\frac{\|u_n-p\|}{\|t_n-p\|}$	$\frac{\|t_n-p\|}{\|s_n-p\|}$
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
1	0.713514	0.966823	0.768432	0.737998	0.794801	0.928531
2	0.527376	0.937880	0.598780	0.562306	0.638440	0.880750
3	0.397672	0.913037	0.471501	0.435549	0.516410	0.843417
4	0.302854	0.891493	0.372931	0.339716	0.418322	0.812092
5	0.231693	0.872302	0.295256	0.265611	0.338479	0.784719
6	0.177594	0.854743	0.233630	0.207774	0.273334	0.760149
7	0.136232	0.838398	0.184673	0.162490	0.220269	0.737690
8	0.104523	0.822910	0.145806	0.127016	0.177183	0.716861
9	0.080194	0.808107	0.114999	0.099237	0.142307	0.697345

Remark 1. From Figure 1, Table 1-3, it is shown that the iteration (3.35) has a good convergence speed and requires small number of iterations than the other three iterations for each of the choices.

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