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A MOMENT ESTIMATOR FOR THE INDEX OF AN EXTREME-VALUE DISTRIBUTION

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We extend Hill's well-known estimator for the index of a distribution function with regularly varying tail to an estimate for the index of an extreme-value distribution. Consistency and asymptotic normality are proved. The estimator is used for high quantile and endpoint estimation.

1. Introduction. Suppose one is given a sequence $X_1, X_2, \dots$ of i.i.d. observations from some distribution function F. Suppose for some constants $a_n > 0$ and b_n $(n = 1, 2, \dots)$ and some $\gamma \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left\{\frac{\max(X_1, X_2, \dots, X_n) - b_n}{a_n} \le x\right\} = G_\gamma(x), \tag{1.1}$$

for all x where $G_\gamma(x)$ is one of the extreme-value distributions

$$G_\gamma(x) = \exp\left(-(1+\gamma x)^{-1/\gamma}\right). \tag{1.2}$$

Here the index γ, is a real parameter [interpret $(1+\gamma x)^{-1/\gamma}$ as e^{-x} for $\gamma = 0$] and x is such that $1 + \gamma x > 0$. The question is how to estimate γ from a finite sample $X_1, X_2, \dots, X_n$.

In case one knows that $\gamma > 0$, one can use Hill's estimate [Hill (1975)] defined as

$$M_n^{(1)} := \frac{1}{k}\sum_{i=0}^{k-1} \log X_{(n-i,\,n)} - \log X_{(n-k,\,n)} \qquad (k < n), \tag{1.3}$$

where $X_{(1,n)} \le X_{(2,n)} \le \cdots \le X_{(n,n)}$ are the order statistics of $X_1, X_2, \dots, X_n$.

Mason (1982) proved weak consistency of $M_n^{(1)}$ for any sequence $k = k(n) \to \infty$, $k(n)/n \to 0$ $(n \to \infty)$ and Deheuvels, Haeusler and Mason (1988) proved strong consistency for any sequence $k(n)$ with $k(n)/\log\log n \to \infty$, $k(n)/n \to 0$ $(n \to \infty)$. It is well known that, under certain extra conditions,

$$\sqrt{k}\left(M_n^{(1)} - \gamma\right) \tag{1.4}$$

is asymptotically normal with mean 0 and variance γ^2 [see Davis and Resnick (1984), Csörgő and Mason (1985), Haeusler and Teugels (1985) and Goldie and Smith (1987)]. This leads to an asymptotic confidence interval for γ.

We now consider the estimation problem for general $\gamma \in \mathbb{R}$.

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Suppose $x^* = x^*(F) > 0$, where $x^*(F) := \sup\{x | F(x) < 1\}$ (this can be achieved by a simple shift), and define

$$M_n^{(2)} := \frac{1}{k} \sum_{i=0}^{k-1} \left(\log X_{(n-i,n)} - \log X_{(n-k,n)}\right)^2. \tag{1.5}$$

We shall prove (Section 2) that (1.1) implies that for $k = k(n) \to \infty$, $k(n)/n \to 0$ $(n \to \infty)$,

$$\lim_{n\to\infty} \hat{\gamma}_n = \gamma \quad \text{in probability}, \tag{1.6}$$

where

$$\hat{\gamma}_n := M_n^{(1)} + 1 - \frac{1}{2}\left\{1 - \frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}\right\}^{-1}. \tag{1.7}$$

Moreover, we shall prove that when $k(n)/(\log n)^\delta \to \infty$ $(n \to \infty)$ for some $\delta > 0$, then

$$\lim_{n\to\infty} \hat{\gamma}_n = \gamma \quad \text{a.s.} \tag{1.8}$$

We shall also give (Section 3) quite natural and general conditions under which the estimate is asymptotically normal so that an asymptotic confidence statement can be made. It seems that even when specialized to the Hill estimator, the result of Theorem 3.1 is the most general one obtained so far. In Sections 4 and 5 we use the moment estimator to obtain asymptotic confidence intervals for high quantiles of F and (in the case $\gamma < 0$) for $x^*(F)$. Section 6 contains some comments—in particular, the intuitive background of (1.7).

Somewhat related papers are Joe (1987) and Smith (1987).

Throughout the paper (except for part of Section 4), we assume

$$\lim_{n\to\infty} k(n) = \infty, \qquad \lim_{n\to\infty} k(n)/n = 0 \tag{1.9}$$

and familiarity with the theory of regularly varying functions and the function class Π [see, e.g., Geluk and de Haan (1987)].

2. Weak and strong consistency.

THEOREM 2.1. *If* (1.1) *holds*, $x^*(F) > 0$, $k(n)/n \to 0$ *and* $k(n) \to \infty$ $(n \to \infty)$, *then*

$$\lim_{n\to\infty} \hat{\gamma}_n = \gamma \quad \textit{in probability}. \tag{2.1}$$

If (1.1) *holds*, $x^*(F) > 0$, $k(n)/n \to 0$ *and* $k(n)/(\log n)^\delta \to \infty$ $(n \to \infty)$ *for some* $\delta > 0$, *then*

$$\lim_{n\to\infty} \hat{\gamma}_n = \gamma \quad \textit{a.s.} \tag{2.2}$$

For the proof we need some lemmas.

LEMMA 2.2. *Suppose* $U_1, U_2, \ldots$ *are i.i.d. random variables with a uniform* $[0,1]$ *distribution. Let* $\Gamma_n(t)$ *be the empirical distribution function based on*

$U_1, \dots, U_n$ $(n = 1, 2, \dots)$. *Then for* $0 < k(n) \le n$, $k(n)/(\log n)^\delta \to \infty$ *for some* $\delta > 0$ *and* $a < \delta/(2(1+\delta))$,

$$\lim_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1-a} \int_0^{k(n)/n} t^{-a-1}\{\Gamma_n(t) - t\}\, dt = 0 \quad a.s. \tag{2.3}$$

PROOF. For $a < 0$ we use a version of Theorem 2(iii) in Einmahl and Mason (1988), without monotonicity condition on $k(n)$ and $k(n)/n$. [It is easily seen that this weakening of the assumptions on $k(n)$ only entails an increase of the constant $2^{1/2}$ on the right.] We have

$$\begin{aligned}&\left|\left(\frac{n}{k(n)}\right)^{1+|a|} \int_0^{k(n)/n} t^{-1+|a|}\{\Gamma_n(t) - t\}\, dt\right| \\ &\le \left(\frac{n}{k(n)}\right)^{1+|a|} \sup_{0<t\le k(n)/n} |\Gamma_n(t) - t| \int_0^{k(n)/n} t^{-1+|a|}\, dt \\ &= \left\{|a|^{-1}\left(\frac{\log\log n}{k(n)}\right)^{1/2}\right\}\left[\left(\frac{n}{k(n)}\right)^{1/2}\left(\frac{n}{\log\log n}\right)^{1/2} \sup_{0<t\le k(n)/n} |\Gamma_n(t) - t|\right].\end{aligned}$$

Since the first factor tends to 0 and the second factor is a.s. bounded by the quoted theorem, we have proved (2.3) for $a < 0$.

For $0 \le a < \delta/(2(1+\delta))$ we use an appropriate version (similarly as before) of Theorem 1(ii) in Einmahl and Mason (1988). For $0 < \eta < \delta/(2(1+\delta)) - a$ and with $v = \frac{1}{2} - a - \eta$

$$\begin{aligned}&\left|\left(\frac{n}{k(n)}\right)^{1-a} \int_0^{k(n)/n} t^{-a-1}\{\Gamma_n(t) - t\}\, dt\right| \\ &\le \left(\frac{n}{k(n)}\right)^{1-a} \sup_{0<t\le k(n)/n} \frac{|\Gamma_n(t) - t|}{t^{1/2-v}} \cdot \int_0^{k(n)/n} t^{\eta-1}\, dt \\ &= \eta^{-1}\left(\frac{\log\log n}{k(n)}\right)^{1/2}\left[\left(\frac{n}{k(n)}\right)^{v}\left(\frac{n}{\log\log n}\right)^{1/2} \sup_{0<t\le k(n)/n} \frac{|\Gamma_n(t) - t|}{t^{1/2-v}}\right].\end{aligned}$$

Since the first factor tends to 0 and the second factor is bounded a.s. by the quoted theorem, we have proved (2.3) for $0 \le a < \delta/(2(1+\delta))$. □

LEMMA 2.3. *Let* $0 < k(n) \le n$ *and* $k(n)/(\log n)^\delta \to \infty$ $(n \to \infty)$ *for some* $\delta > 0$.

(i) *Suppose* $F(x) = x^\alpha$ $(0 < x < 1)$ *for some* $\alpha > 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{(i,n)}}{X_{(k(n)+1,n)}} = \frac{\alpha}{\alpha+1} \quad a.s. \tag{2.4}$$

(ii) *Suppose* $F(x) = 1 - x^{-\alpha}$ $(x > 1)$ *for some* $\alpha > 2(1+\delta)/\delta$. *Then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \frac{X_{(n-i,n)}}{X_{(n-k(n),n)}} = \frac{\alpha}{\alpha-1} \quad a.s. \tag{2.5}$$

PROOF. (i) Let F_n be the empirical distribution function based on $X_1, \dots, X_n$ from F. Lemma 2.2 implies, with $a = -1/\alpha$,

$$\lim_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1+1/\alpha} \int_0^{(k(n)/n)^{1/\alpha}} F_n(s)\, ds = \frac{1}{\alpha+1} \quad \text{a.s.} \tag{2.6}$$

Since [Wellner (1978)]

$$\lim_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1/\alpha} \cdot X_{(k(n)+1,n)} = 1 \quad \text{a.s.},$$

(2.6) implies

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{(i,n)}}{X_{(k(n)+1,n)}} \\
&= \limsup_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1/\alpha} \frac{1}{k(n)} \sum_{i=1}^{k(n)} X_{(i,n)} \\
&= \limsup_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1/\alpha} \cdot \frac{n}{k(n)} \int_0^{X_{(k(n),n)}} s\, dF_n(s) \\
&\le \limsup_{n\to\infty} \left[\left(\frac{n}{k(n)}\right)^{1/\alpha} X_{(k(n),n)} - \left(\frac{n}{k(n)}\right)^{1+1/\alpha} \int_0^{(k(n)(1-\varepsilon)/n)^{1/\alpha}} F_n(s)\, ds\right] \\
&= 1 - \liminf_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1+1/\alpha} \int_0^{(k(n)(1-\varepsilon)/n)^{1/\alpha}} F_n(s)\, ds \\
&= 1 - \frac{(1-\varepsilon)^{1+1/\alpha}}{(\alpha+1)} \quad \text{a.s.}
\end{aligned}$$

This with a similar lower bound gives the stated result.

(ii) Let F_n be the empirical distribution function based on $X_1, \dots, X_n$ from F. Lemma 2.2 implies, with $a = 1/\alpha$,

$$\lim_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1-1/\alpha} \int_{(n/k(n))^{1/\alpha}}^{\infty} \{1 - F_n(s)\}\, ds = \frac{1}{\alpha-1} \quad \text{a.s.} \tag{2.7}$$

Since [Wellner (1978)]

$$\lim_{n\to\infty} \left(\frac{k(n)}{n}\right)^{1/\alpha} \cdot X_{(n-k(n),n)} = 1 \quad \text{a.s.},$$

(2.7) implies

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \frac{X_{(n-i,n)}}{X_{(n-k(n),n)}}
&= \limsup_{n\to\infty} \left(\frac{n}{k(n)}\right)^{-1/\alpha} \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} X_{(n-i,n)} \\
&= \limsup_{n\to\infty} \left(\frac{n}{k(n)}\right)^{-1/\alpha} \cdot \frac{n}{k(n)} \cdot \int_{X_{(n-k(n)+1,n)}}^{\infty} s\,dF_n(s) \\
&= \limsup_{n\to\infty} \Bigg[\left(\frac{n}{k(n)}\right)^{-1/\alpha} X_{(n-k(n)+1,n)} \\
&\qquad + \left(\frac{n}{k(n)}\right)^{1-1/\alpha} \int_{X_{(n-k(n)+1,n)}}^{\infty} \{1 - F_n(s)\}\,ds\Bigg] \\
&\le 1 + \limsup_{n\to\infty} \left(\frac{n}{k(n)}\right)^{1-1/\alpha} \int_{(n(1-\varepsilon)/k(n))^{1/\alpha}}^{\infty} \{1 - F_n(s)\}\,ds \\
&= 1 + \frac{(1-\varepsilon)^{-1+1/\alpha}}{(\alpha - 1)} \quad \text{a.s.}
\end{aligned}$$

This with a similar lower bound gives the stated result. □

LEMMA 2.4. *Let* $0 < k(n) \le n$ *and* $k(n) \to \infty$ $(n \to \infty)$.
(i) *Suppose* $F(x) = x^{\alpha}$ $(0 < x < 1)$ *for some* $\alpha > 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{(i,n)}}{X_{(k(n)+1,n)}} = \frac{\alpha}{\alpha+1} \quad \textit{in probability}.$$

(ii) *Suppose* $F(x) = 1 - x^{-\alpha}$ $(x > 1)$ *for some* $\alpha > 1$. *Then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \frac{X_{(n-i,n)}}{X_{(n-k(n),n)}} = \frac{\alpha}{\alpha-1} \quad \textit{in probability}.$$

PROOF. (i) Note that

$$\left(X_{(1,n)}/X_{(k(n)+1,n)}, \dots, X_{(k(n),n)}/X_{(k(n)+1,n)}\right) \stackrel{d}{=} \left(Y_{(1,k(n))}, \dots, Y_{(k(n),k(n))}\right),$$

the order statistics from a sample $(Y_1, \dots, Y_{k(n)})$ from F. Hence

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{(i,n)}}{X_{(k(n)+1,n)}} \stackrel{d}{=} \frac{1}{k(n)} \sum_{i=1}^{k(n)} Y_i$$

and the law of large numbers applies. The proof of part (ii) is similar. □

LEMMA 2.5. *Suppose* (1.1) *holds and* $x^*(F) > 0$. *Let* $U = (1/(1-F))^{\leftarrow}$, *the arrow denoting the inverse function. Then, for some positive function* a,

$$\lim_{t\to\infty} \frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = \begin{cases} \log x, & \gamma \geq 0, \\ \dfrac{x^{\gamma}-1}{\gamma}, & \gamma < 0, \end{cases}$$

for all $x > 0$. *Moreover for each* $\varepsilon > 0$ *there exists* t_0 *such that, for* $t \geq t_0$ *and* $x \geq 1$, (i)

$$(2.8)\quad (1-\varepsilon)\frac{1-x^{-\varepsilon}}{\varepsilon} - \varepsilon < \frac{\log U(tx) - \log U(t)}{a(t)/U(t)} < (1+\varepsilon)\frac{x^{\varepsilon}-1}{\varepsilon} + \varepsilon,$$

provided $\gamma \geq 0$, *and* (ii)

$$(2.9)\quad 1-(1+\varepsilon)x^{\gamma+\varepsilon} < \frac{\log U(tx) - \log U(t)}{\log U(\infty) - \log U(t)} < 1-(1-\varepsilon)x^{\gamma-\varepsilon},$$

provided $\gamma < 0$.

PROOF. The statements follow from well-known inequalities for regularly varying functions ($\gamma < 0$) and Π-functions ($\gamma \geq 0$). Cf. Geluk and de Haan (1987), page 27. Note that we can take $a(t)/U(t) = \gamma$ for $\gamma > 0$ and $a(t)/U(t) = -\gamma\{\log U(\infty) - \log U(t)\}$ for $\gamma < 0$. □

PROOF OF THEOREM 2.1. We only give the proof of the strong consistency using Lemma 2.3. The proof of the weak consistency is similar, starting from Lemma 2.4 instead. Let $Y_1, Y_2, \ldots$ be i.i.d. with common distribution function $1 - 1/x$ ($x > 1$). Then $(X_1, X_2, \ldots) \overset{d}{=} (U(Y_1), U(Y_2), \ldots)$ and for all n also $(X_{(1,n)}, \ldots, X_{(n,n)}) \overset{d}{=} (U(Y_{(1,n)}), \ldots, U(Y_{(n,n)}))$. We work with the latter.

(i) Let $\gamma \geq 0$. Given $\varepsilon > 0$ for $r = 1, 2$ by Lemma 2.5(i) we have a.s. for sufficiently large n,

$$\frac{M_n^{(r)}}{\left\{a(Y_{(n-k(n),n)})/U(Y_{(n-k(n),n)})\right\}^r}$$

$$= \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \left\{\log U\left(\frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}} \cdot Y_{(n-k(n),n)}\right) - \log U(Y_{(n-k(n),n)})\right\}^r$$

$$\div \left\{a(Y_{(n-k(n),n)})/U(Y_{(n-k(n),n)})\right\}^r$$

$$< \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \left[\varepsilon + (1+\varepsilon)\frac{Y^{\varepsilon}_{(n-i,n)}/Y^{\varepsilon}_{(n-k(n),n)} - 1}{\varepsilon}\right]^r.$$

First suppose $r = 1$. Since $Y^{\varepsilon}_{(n-j,n)}$ is the $(n-j)$th order statistic from the distribution function $1 - 1/x^{1/\varepsilon}$ ($x > 1$), we can apply Lemma 2.4(ii) for $\varepsilon <$

$\delta/(2(1+\delta))$ and find

$$\limsup_{n\to\infty} \frac{M_n^{(1)}}{a(Y_{(n-k(n),n)})/U(Y_{(n-k(n),n)})} \le \varepsilon + (1+\varepsilon)\frac{\left\{\dfrac{\varepsilon^{-1}}{\varepsilon^{-1}-1}-1\right\}}{\varepsilon} \quad \text{a.s.}$$

This, together with a similar lower inequality, gives

$$\lim_{n\to\infty} \frac{M_n^{(1)}}{a(Y_{(n-k(n),n)})/U(Y_{(n-k(n),n)})} = 1 \quad \text{a.s.}$$

Next we note that the function a/U is slowly varying, hence

$$\lim_{n\to\infty} \frac{a\left(\dfrac{Y_{(n-k(n),n)}}{n/k(n)}\cdot\dfrac{n}{k(n)}\right)\Big/U\left(\dfrac{Y_{(n-k(n),n)}}{n/k(n)}\cdot\dfrac{n}{k(n)}\right)}{a\left(\dfrac{n}{k(n)}\right)\Big/U\left(\dfrac{n}{k(n)}\right)} = 1 \quad \text{a.s.}$$

The case $r=2$ is similar: One just works out the square and calculates the limits of all terms. It follows that for $r=1,2$,

$$\lim_{n\to\infty} \frac{M_n^{(r)}}{\left\{a\left(\dfrac{n}{k(n)}\right)\Big/U\left(\dfrac{n}{k(n)}\right)\right\}^r} = r! \quad \text{a.s.} \tag{2.10}$$

(ii) Let $\gamma<0$. Given $\varepsilon>0$ for $r=1,2$, we find as in part (i), now using Lemma 2.5(ii), that a.s. for sufficiently large n,

$$\frac{M_n^{(r)}}{\left\{\log U(\infty)-\log U(Y_{(n-k(n),n)})\right\}^r} < \frac{1}{k(n)}\sum_{i=0}^{k(n)-1}\left[1-(1-\varepsilon)\cdot\frac{Y_{(n-i,n)}^{\gamma-\varepsilon}}{Y_{(n-k(n),n)}^{\gamma-\varepsilon}}\right]^r.$$

First suppose $r=1$. Since $Y_{(n-i,n)}^{\gamma-\varepsilon}$ is the $(i+1)$st order statistic from the distribution function $x^{1/(-\gamma+\varepsilon)}$ $(0<x<1)$, we can apply Lemma 2.4(i) and find

$$\limsup_{n\to\infty} \frac{M_n^{(1)}}{\log U(\infty)-\log U(Y_{(n-k(n),n)})} \le 1-(1-\varepsilon)\frac{(\varepsilon-\gamma)^{-1}}{(\varepsilon-\gamma)^{-1}+1} \quad \text{a.s.}$$

This, together with a similar lower inequality, gives

$$\lim_{n\to\infty} \frac{M_n^{(1)}}{\log U(\infty)-\log U(Y_{(n-k(n),n)})} = \frac{-\gamma}{1-\gamma} \quad \text{a.s.}$$

Next note that the function $\log U(\infty)-\log U$ is regularly varying, hence

$$\lim_{n\to\infty} \frac{\log U(\infty)-\log U(\{Y_{(n-k(n),n)}k(n)/n\}\cdot[n/k(n)])}{\log U(\infty)-\log U(n/k(n))} = 1 \quad \text{a.s.}$$

The case $r=2$ is similar: One just works out the square and calculates the limits of all terms. It follows that for $r = 1, 2$ almost surely

$$\lim_{n\to\infty} \frac{M_n^{(r)}}{\{\log U(\infty) - \log U(n/k(n))\}^r} = \begin{cases} -\gamma/(1-\gamma), & r = 1, \\ 2\gamma^2/\{(1-\gamma)(1-2\gamma)\}, & r = 2. \end{cases} \tag{2.11}$$

(iii) Now (2.10) and (2.11) imply that for all real γ almost surely,

$$\lim_{n\to\infty} \frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}} = \begin{cases} 1/2, & \gamma \ge 0, \\ (1-2\gamma)/(2-2\gamma), & \gamma < 0 \end{cases} \tag{2.12}$$

and, since $\lim_{n\to\infty} a(n/k(n))/U(n/k(n)) = 0$ for $\gamma = 0$ and $\lim_{n\to\infty} \log U(\infty) - \log U(n/k(n)) = 0$ for $\gamma < 0$,

$$\lim_{n\to\infty} M_n^{(1)} = \max(0, \gamma) \quad \text{a.s.} \tag{2.13}$$

The result follows. □

3. Asymptotic normality.

THEOREM 3.1. *Suppose* (1.1) *holds and moreover, with* $U := (1/(1-F))^{\leftarrow}$:

(i) *For* $\gamma > 0$:

$$\pm\, t^{-\gamma} \cdot U(t) \in \Pi(b_1) \quad \textit{for some positive function } b_1. \tag{3.1}$$

(ii) *For* $\gamma = 0$: *There exist positive functions* b_2 *and* b_3 *such that*

$$\lim_{t\to\infty} \frac{\log U(tx) - \log U(t) - b_2(t)\log x}{b_3(t)} = \pm \frac{(\log x)^2}{2} \tag{3.2}$$

[*note that* $b_2(t) \sim a(t)/U(t)$, $t \to \infty$, *with* a *as defined in Lemma* 2.5].

(iii) *For* $\gamma < 0$:

$$\mp\, t^{-\gamma}\{U(\infty) - U(t)\} \in \Pi(b_4) \quad \textit{for some positive function } b_4. \tag{3.3}$$

Suppose also $\lim_{n\to\infty} k(n) = \infty$ *and*:

(iv) *For* $\gamma > 0$:

$$k(n) = o(n/g^{\leftarrow}(n)) \quad \textit{where } g(t) := t^{1-2\gamma}\{U(t)/b_1(t)\}^2. \tag{3.4}$$

(v) *For* $\gamma = 0$:

$$k(n) = o(n/g^{\leftarrow}(n)) \quad \textit{where } g(t) := t b_2^2(t)/b_3^2(t). \tag{3.5}$$

(vi) *For* $\gamma < 0$:

$$k(n) = o(n/g^{\leftarrow}(n)), \tag{3.6}$$

where $g(t) := t^{1-2\gamma}[\{\log U(\infty) - \log U(t)\}/b_4(t)]^2$.

Then

$$\sqrt{k(n)}\left(\frac{M_n^{(1)}}{f(\log X_{(n-k(n),n)})} - \rho_1(\gamma), \frac{M_n^{(2)}}{\{f(\log X_{(n-k(n),n)})\}^2} - \rho_2(\gamma)\right) \tag{3.7}$$

with $f(t) := a(1/\{1 - F(\exp t)\})/U(1/\{1 - F(\exp t)\})$ *has asymptotically a normal distribution* ($n \to \infty$) *with means zero and covariance matrix* (s_{ij}) *with*, *for* $\gamma \le 0$,

$$s_{11} = (1-\gamma)^{-2}(1-2\gamma)^{-1},$$
$$s_{12} = 4(1-\gamma)^{-2}(1-2\gamma)^{-1}(1-3\gamma)^{-1},$$
$$s_{22} = 4(5-11\gamma)(1-\gamma)^{-2}(1-2\gamma)^{-2}(1-3\gamma)(1-4\gamma),$$

and for $\gamma \ge 0$,

$$s_{11} = 1, \qquad s_{12} = 4, \qquad s_{22} = 20.$$

The functions ρ_1 *and* ρ_2 *are defined by*

$$\rho_1(\gamma) := \begin{cases} 1, & \gamma \ge 0, \\ 1/(1-\gamma), & \gamma < 0, \end{cases}$$

$$\rho_2(\gamma) := \begin{cases} 2, & \gamma \ge 0, \\ 2/\{(1-\gamma)(1-2\gamma)\}, & \gamma < 0. \end{cases}$$

REMARK. For $\gamma > 0$ the result specializes to $\sqrt{k(n)}\,(M_n^{(1)} - \gamma)$ is asymptotically $N(0, \gamma^2)$.

COROLLARY 3.2. *If the conditions of Theorem* 3.1 *are satisfied and if, moreover, in the case* $\gamma = 0$,

(3.8) $$k(n) = o(n/g_1^{\leftarrow}(n)) \quad \textit{where } g_1(t) := t\{U(t)/a(t)\}^2,$$

then

(3.9) $$\sqrt{k(n)}\,\{\hat{\gamma}_n - \gamma\}$$

has asymptotically a normal distribution with mean 0 *and variance*

(3.10) $$\begin{cases} 1+\gamma^2, & \gamma \ge 0, \\ (1-\gamma)^2(1-2\gamma)\left\{4 - 8\dfrac{1-2\gamma}{1-3\gamma} + \dfrac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right\}, & \gamma < 0. \end{cases}$$

REMARK. Neither (3.5) nor (3.8) implies the other.

EXAMPLE. The standard normal distribution satisfies (1.1) with $\gamma = 0$, $a(t) = \{U(t)\}^{-1}$ [note $a_n = a(n)$] and (3.2) with $b_2(t) = 1/\{U(t)\}^2 - 1/\{U(t)\}^4$, $b_3(t) = 2/\{U(t)\}^4$ and a minus sign. Because $U(t) \sim \sqrt{2\log t}$ $(t \to \infty)$, one finds that $g(t) \sim t(\log t)^2$ [cf. (3.5)] and $g_1(t) \sim 4t(\log t)^2$ [cf. (3.8)], $t \to \infty$, and hence the conclusion of Corollary 3.2 is true provided $k(n) = o((\log n)^2)$, $n \to \infty$.

Note that we found the same restriction on $\{k(n)\}$ for the asymptotic normality of Pickands' estimator [Dekkers and de Haan (1989)].

Before proving the theorem and its corollary, we formulate the conditions on U in terms of the distribution function F [for a proof see Dekkers and de Haan (1989), Section 3, where also some simpler alternative conditions and examples are given].

THEOREM 3.3. *The conditions* (i), (ii) *and* (iii) *of Theorem* 3.1 *imply* (1.1) *for the same* γ. *The conditions* (i), (ii) *and* (iii) *of Theorem* 3.1 *are equivalent to* (*respectively*): (i) *For* $\gamma > 0$:

$$\mp t^{1/\gamma}\{1 - F(t)\} \in \Pi. \tag{3.11}$$

(ii) *For* $\gamma = 0$: *There exists positive functions* f *and* α *with* $\lim_{t\uparrow x^*}\alpha(t) = 0$ *such that for* $x > 0$

$$\lim_{t\uparrow x^*}\frac{\dfrac{1 - F(\exp(t + xf(t)))}{1 - F(\exp(t))} - e^{-x}}{\alpha(t)} = \pm\frac{x^2}{2}e^{-x}. \tag{3.12}$$

(iii) *For* $\gamma < 0$:

$$\pm t^{-1/\gamma}\{1 - F(x^* - t^{-1})\} \in \Pi. \tag{3.13}$$

REMARK. For $\gamma > 0$ our second-order condition (3.11) is the same as the one used in Smith (1982).

REMARK. The conditions of Theorem 3.1 correspond to the conditions of Theorem 2.4 in Dekkers and de Haan (1989). A theorem similar to that of Theorem 3.1 can be given under the conditions of Theorem 2.5 of Dekkers and de Haan (1989).

For the proof of Theorem 3.1 we need the following lemmas.

LEMMA 3.4. *Let* $Y_{(1,n)} \le \cdots \le Y_{(n,n)}$ *be nth order statistics from the distribution function* $1 - x^{-1}$ $(x > 1)$. *Let* $0 < k(n) \le n$ *and* $k(n) \to \infty$ $(n \to \infty)$.
(i)

$$\sqrt{k(n)}\left(\frac{1}{k(n)}\sum_{i=0}^{k(n)-1}\log Y_{(n-i,n)} - \log Y_{(n-k(n),n)} - 1,\right.$$
$$\left.(20)^{-1/2}\left\{\frac{1}{k(n)}\sum_{i=0}^{k(n)-1}\left(\log Y_{(n-i,n)} - \log Y_{(n-k(n),n)}\right)^2 - 2\right\}\right) \tag{3.14}$$

is asymptotically normal $(n \to \infty)$ *with means* 0, *variances* 1 *and covariance* $2\sqrt{5}$.
(ii) *For* $\gamma < 0$

$$\sqrt{k(n)}\left(\frac{1}{k(n)}\sum_{i=0}^{k(n)-1}1 - \left(\frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}}\right)^{\gamma} + \frac{\gamma}{1-\gamma},\right.$$
$$\left.\frac{1}{k(n)}\sum_{i=0}^{k(n)-1}\left\{1 - \left(\frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}}\right)^{\gamma}\right\}^2 - \frac{2\gamma^2}{(1-\gamma)(1-2\gamma)}\right) \tag{3.15}$$

is asymptotically normal $(n \to \infty)$ *with means* 0, *variances* γ^2 *and* γ^4, *respec-*

tively, and covariance

$$\frac{2\gamma^3}{\sqrt{5}} \frac{\left(5 - 30\gamma + 40\gamma^2\right)^{1/2}}{\left(5 - 26\gamma + 33\gamma^2\right)^{1/2}}. \tag{3.16}$$

PROOF. We proceed as in the proof of Lemma 2.4.

(i) The random vector in (3.14) is equal in distribution to

$$\sqrt{k(n)}\left(\frac{1}{k(n)}\sum_{i=1}^{k(n)} Z_i - 1, (20)^{-1/2}\left\{\frac{1}{k(n)}\sum_{i=1}^{k(n)} Z_i^2 - 2\right\}\right),$$

where $Z_1, \dots, Z_n$ are i.i.d. from a standard exponential distribution. The statement of the lemma follows by applying the Cramér–Wold device and Liapounov's theorem [Chung (1974), page 200].

(ii) The random vector in (3.15) is equal in distribution to

$$\sqrt{k(n)}\left(\frac{1}{k(n)}\sum_{i=1}^{k(n)}(1 - R_i) + \frac{\gamma}{1-\gamma}, \frac{1}{k(n)}\sum_{i=1}^{k(n)}(1 - R_i)^2 - \frac{2\gamma^2}{(1-\gamma)(1-2\gamma)}\right),$$

where $R_1, R_2, \dots, R_n$ are i.i.d. from the distribution $x^{-1/\gamma}$ $(0 < x < 1)$. The statement of the lemma follows as before. □

LEMMA 3.5. *Suppose condition* (i), (ii) *or* (iii) *of Theorem* 3.1 *holds with the upper sign* (*i.e.*, + *for* $\gamma \ge 0$ *and* − *for* $\gamma < 0$). *For any* $\varepsilon > 0$ *there exists* t_0 *such that, for* $t \ge t_0$ *and* $x \ge 1$:

(i) *In the case* $\gamma > 0$:

$$\begin{aligned}(1-\varepsilon)\frac{1 - x^{-\varepsilon}}{\varepsilon} - \varepsilon &< \frac{\log U(tx) - \log U(t) - \gamma \log x}{t^\gamma b_1(t)/U(t)} \\ &< (1+\varepsilon)\frac{x^\varepsilon - 1}{\varepsilon} + \varepsilon.\end{aligned} \tag{3.17}$$

(ii) *In the case* $\gamma = 0$:

$$\begin{aligned}&\frac{(1-\varepsilon^2)(\log x)^2}{2} - 2\varepsilon \log x - \varepsilon \\ &\quad< \frac{\log U(tx) - \log U(t) - b_2(t)\log x}{b_3(t)} \\ &\quad< \frac{(1+\varepsilon)^2 x^\varepsilon (\log x)^2}{2} + 2\varepsilon \log x + \varepsilon.\end{aligned} \tag{3.18}$$

(iii) *In the case* $\gamma < 0$:

$$\begin{aligned}&(1-\varepsilon)x^\gamma \frac{1 - x^{-\varepsilon}}{\varepsilon} - \varepsilon x^\gamma \\ &\quad< \frac{\log U(tx) - \log U(t) - (1 - x^\gamma)\{\log U(\infty) - \log U(t)\}}{t^\gamma b_4(t)/U(\infty)} \\ &\quad< (1+\varepsilon)x^\gamma \cdot \frac{x^\varepsilon - 1}{\varepsilon} + \varepsilon x^\gamma.\end{aligned} \tag{3.19}$$

PROOF. (i)

$$\frac{\log U(tx) - \log U(t) - \gamma \log x}{t^\gamma b_1(t)/U(t)}$$
$$= \left\{\log\left(\frac{U(tx)}{x^\gamma U(t)}\right)\right\}\frac{U(t)}{t^\gamma b_1(t)}$$
$$\sim \left(\frac{U(tx)}{x^\gamma U(t)} - 1\right)\frac{U(t)}{t^\gamma b_1(t)} = \frac{(tx)^{-\gamma}U(t) - t^{-\gamma}U(t)}{b_1(t)}$$
$$\to \log x \qquad (t \to \infty) \quad \text{for all } x > 0,$$

i.e.,

$$\log U(t) - \gamma \log t \in \Pi\left(t^\gamma \cdot \frac{b_1(t)}{U(t)}\right).$$

Application of the well-known inequalities for Π-functions [Geluk and de Haan (1987), page 27] gives (3.17).

(ii) In the limit relation (3.2) we may choose [Omey and Willekens (1987)]

$$b_2(t) := CU(t) + b_3(t) := \log U(t) - \frac{1}{t}\int_0^t \log U(s)\,ds + b_3(t)$$

and CU satisfies

$$\lim_{t\to\infty} \frac{CU(tx) - CU(t)}{b_3(t)} = \log x, \tag{3.20}$$

for all $x > 0$, i.e., $CU \in \Pi(b_3)$. Moreover, $\log U(t) = CU(t) + \int_0^t CU(s)\,ds/s$, hence

$$\frac{\log U(tx) - \log U(t) - \{CU(t) + b_3(t)\}\log x}{b_3(t)}$$
$$= \frac{CU(tx) - CU(t)}{b_3(t)} + \int_1^x \frac{CU(st) - CU(t)}{b_3(t)}\frac{ds}{s} - \log x.$$

The well-known inequalities for Π-functions [Geluk and de Haan (1987), page 27] applied to CU then give (3.18).

(iii) $\log U(\infty) - \log U(t) = (U(\infty) - U(t))/U(\infty) + O((U(\infty) - U(t))^2)(t \to \infty)$, hence $-t^{-\gamma}\{U(\infty) - U(t)\} \in \Pi(b_4)$ implies $-t^{-\gamma}\{\log U(\infty) - \log U(t)\} \in \Pi(b_4/U(\infty))$. The inequalities for Π-functions yield for $t \geq t_0$ and $x \geq 1$

$$(1-\varepsilon)\frac{1 - x^{-\varepsilon}}{\varepsilon} - \varepsilon$$
$$< \frac{t^{-\gamma}\{\log U(\infty) - \log U(t)\} - (tx)^{-\gamma}\{\log U(\infty) - \log U(tx)\}}{b_4(t)/U(\infty)}$$
$$< (1+\varepsilon)\frac{x^\varepsilon - 1}{\varepsilon} + \varepsilon.$$

Rearranging gives (3.19). □

PROOF OF THEOREM 3.1. We shall give the proof for $\gamma = 0$ and a positive limit in (3.2). For other values of γ and the other choice of sign, the reasoning is similar. Let $Y_1, Y_2, \dots$ be i.i.d. with common distribution function $1 - 1/x$ $(x > 1)$. Then $(X_1, X_2, \dots) \stackrel{d}{=} (U(Y_1), U(Y_2), \dots)$ and for all n also, $(X_{(1,n)}, \dots, X_{(n,n)}) \stackrel{d}{=} (U(Y_{(1,n)}), \dots, U(Y_{(n,n)}))$. We work with the latter and proceed by providing bounds for the quantities concerned.

Since for $x \geq 1$ and $t \geq t_0$ by Lemma 3.5 and Lemma 2.5,

$$\left\{\frac{\log U(tx) - \log U(t)}{b_2(t)}\right\}^2$$
$$= (\log x)^2 + \left\{\frac{\log U(tx) - \log U(t)}{b_2(t)} - \log x\right\}$$
$$\times\left\{\frac{\log U(tx) - \log U(t)}{b_2(t)} + \log x\right\}$$
$$\leq (\log x)^2 + \frac{b_3(t)}{b_2(t)}\left\{(1+\varepsilon)^2 x^{\varepsilon}\frac{(\log x)^2}{2} + 2\varepsilon\log x + \varepsilon\right\}$$
$$\times\left\{(1+\varepsilon)\frac{x^{\varepsilon}-1}{\varepsilon} + \varepsilon + \log x\right\},$$

we have, after replacing t by $Y_{(n-k(n),n)}$ and xt by $Y_{(n-i,n)}$ and summing over i, eventually,

$$\sqrt{k(n)}\left\{\frac{M_n^{(2)}}{\{f(\log X_{(n-k(n),n)})\}^2} - 2\right\}$$

$$(3.21)\quad = \sqrt{k(n)}\left[\frac{1}{k(n)}\sum_{i=0}^{k(n)-1}\left\{\log U\left(Y_{(n-k(n),n)}\cdot\frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}}\right) - \log U\left(Y_{(n-k(n),n)}\right)\right\}^2 \div \left\{b_2\left(Y_{(n-k(n),n)}\right)\right\}^2 - 2\right]$$

$$\leq \sqrt{k(n)}\left[\frac{1}{k(n)}\sum_{i=0}^{k(n)-1}\left\{\log\frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}}\right\}^2 - 2\right]$$
$$+\left\{\sqrt{k(n)}\,\frac{b_3\left(Y_{(n-k(n),n)}\right)}{b_2\left(Y_{(n-k(n),n)}\right)}\right\}A_n,$$

where A_n is a linear combination of terms of the form

$$[1/k(n)] \sum_{i=0}^{k(n)-1} \left(Y_{(n-i,n)}/Y_{(n-k(n),n)}\right)^{\alpha_r}$$

where $\alpha_r < 1$ for every term. Hence $\lim_{n\to\infty} A_n$ exists in probability by Lemma 2.4. Further, since $b_3(t)/b_2(t)$ is slowly varying and $k(n)Y_{(n-k(n),n)}/n \to 1$ in probability [Smirnov (1949)], we have by (3.5)

$$\lim_{n\to\infty} \sqrt{k(n)} \cdot \frac{b_3\left(Y_{(n-k(n),n)}\right)}{b_2\left(Y_{(n-k(n),n)}\right)} = 0 \quad \text{in probability.}$$

A similar lower inequality is readily obtained. Now A_n is a linear combination of terms of the form

$$\frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \left\{\log \frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}}\right\}^{\alpha_r}$$

with $\alpha_r > 0$ for every term. Combining the results for the two bounds we get

$$\lim_{n\to\infty} \sqrt{k(n)} \left[\frac{M_n^{(2)}}{\left\{f\left(\log X_{(n-k(n),n)}\right)\right\}^2} - \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \left\{\log \frac{Y_{(n-i,n)}}{Y_{(n-k(n),n)}}\right\}^2 \right] = 0 \tag{3.22}$$

in probability. A similar statement for $M_n^{(1)}$ and Lemma 3.4 completes the proof. □

PROOF OF COROLLARY 3.2. Write (P, Q) for the limiting normal vector in (3.7). Then

$$\begin{aligned}
&\sqrt{k(n)} \left\{ \frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}} - \frac{\left(\rho_1(\gamma)\right)^2}{\rho_2(\gamma)} \right\} \\
&= \frac{\left(f\left(\log X_{(n-k(n),n)}\right)\right)^2}{\rho_2(\gamma) \cdot M_n^{(2)}} \left[\rho_2(\gamma)\sqrt{k(n)} \left\{ \frac{\left(M_n^{(1)}\right)^2}{\left(f\left(\log X_{(n-k(n),n)}\right)\right)^2} - \left(\rho_1(\gamma)\right)^2 \right\} \right. \\
&\qquad \left. - \left(\rho_1(\gamma)\right)^2 \sqrt{k(n)} \left\{ \frac{M_n^{(2)}}{\left(f\left(\log X_{(n-k(n),n)}\right)\right)^2} - \rho_2(\gamma) \right\} \right] \\
&\to 2\frac{\rho_1(\gamma)}{\rho_2(\gamma)} \cdot P - \frac{\left\{\rho_1(\gamma)\right\}^2}{\left\{\rho_2(\gamma)\right\}^2} \cdot Q
\end{aligned}$$

$(n \to \infty)$ in distribution. Hence

$$\sqrt{k(n)}\left[\left\{1-\frac{\frac{1}{2}}{1-\frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}}\right\}-\left\{1-\frac{\frac{1}{2}}{1-\frac{\left(\rho_1(\gamma)\right)^2}{\rho_2(\gamma)}}\right\}\right]$$

$$=\frac{\sqrt{k(n)}}{2}\frac{\left\{\frac{\left(\rho_1(\gamma)\right)^2}{\rho_2(\gamma)}-\frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}\right\}}{\left[\left\{1-\frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}\right\}\left\{1-\frac{\left(\rho_1(\gamma)\right)^2}{\rho_2(\gamma)}\right\}\right]}$$

$$\to \frac{\rho_1(\gamma)\left\{\frac{\rho_1(\gamma)Q}{2}-\rho_2(\gamma)P\right\}}{\left\{\rho_2(\gamma)-\{\rho_1(\gamma)\}^2\right\}^2}$$

$(n \to \infty)$ in distribution. Note that

$$1-\frac{\frac{1}{2}}{1-\frac{\left(\rho_1(\gamma)\right)^2}{\rho_2(\gamma)}}=\min(0,\gamma).$$

It remains to determine the asymptotic distribution of $\sqrt{k(n)}\,\{M_n^{(1)}-\max(0,\gamma)\}$. We claim that this expression tends to $P\cdot\max(0,\gamma)$ in distribution. For $\gamma>0$ this is correct. For $\gamma=0$ the extra condition of the corollary yields $\sqrt{k(n)}\,b_2(n/k(n)) \to 0\ (n\to\infty)$, hence $\sqrt{k(n)}\,f(\log X_{(n-k(n),\,n)}) \to 0$ and finally $\sqrt{k(n)}\,M_n^{(1)} \to 0\ (n\to\infty)$ in probability.

In a similar way we get $\sqrt{k(n)}\,M_n^{(1)} \to 0\ (n\to\infty)$ in probability for $\gamma<0$. The proof is complete. □

REMARK. It is clear from the proofs of Theorem 3.1 and Corollary 3.2 that if (i), (ii) or (iii) of Theorem 3.1 holds and if $k(n) \sim c\cdot n/g^{\leftarrow}(n)$ for some positive constant c $(n\to\infty)$, then $\sqrt{k(n)}\,\{\hat{\gamma}_n-\gamma\}$ has asymptotically a normal distribution with the same variance, but with mean $\pm\sqrt{c}$, where the sign corresponds with the sign in (3.1), (3.2) or (3.3) [i.e., in particular, $+\sqrt{c}$ corresponds with a + sign in (3.3)].

4. Quantile and endpoint estimation: Finite case. In Dekkers and de Haan (1989) we used differences of large order statistics as building blocks both for an estimator of γ (following J. Pickands III) and for estimating large quantiles. We shall now construct a similar estimate for a large quantile using sums of large order statistics.

The basic situation in this and the next section is the following. We have observed n independent drawings $X_1, X_2, \dots, X_n$ from a distribution function F satisfying (1.1). We want to find a level x_p (where p is a given number much less than 1) such that

$$F(x_p) = 1 - p. \tag{4.1}$$

With the function U as defined in Section 1, this means

$$x_p = U\left(\frac{1}{p}\right). \tag{4.2}$$

We propose to estimate x_p based on the observations $X_1, \dots, X_n$ as follows [cf. Dekkers and de Haan (1989)]:

$$\hat{x}_{p,n} := \frac{a_n^{\hat{\gamma}_n} - 1}{\hat{\gamma}_n} \cdot \frac{X_{(n-k,n)} M_n^{(1)}}{\rho_1(\hat{\gamma}_n)} + X_{(n-k,n)}, \tag{4.3}$$

with $\hat{\gamma}_n$ any consistent estimate of γ, $M_n^{(1)}$ and ρ_1 as defined before and

$$a_n := \frac{k}{n \cdot p}. \tag{4.4}$$

An asymptotic confidence interval for x_p can be constructed using the following result.

THEOREM 4.1. *Suppose $p = p_n \to 0$, $np_n \to c \in (0, \infty)$, $n \to \infty$. Let k [occurring in $X_{(n-k,n)}$ and for the definition of $M_n^{(1)}$, see (1.3)] be fixed, $k > c$. Then, provided (1.1) holds,*

$$\frac{\hat{x}_{p,n} - x_p}{X_{(n-k,n)} M_n^{(1)}} \tag{4.5}$$
$$\xrightarrow{d} \begin{cases} \left(\dfrac{\left(\frac{k}{c}\right)^{\gamma} - 1}{\gamma \rho_1(\gamma)} + \dfrac{1 - \left(\frac{1}{c} \cdot Q_k\right)^{\gamma}}{\gamma} \right) \Big/ \left\{ \dfrac{1}{k} \displaystyle\sum_{i=0}^{k-1} Z_i \right\}, & \gamma \ge 0, \\ \left(\dfrac{\left(\frac{k}{c}\right)^{\gamma} - 1}{\gamma \rho_1(\gamma)} + \dfrac{1 - \left(\frac{1}{c} \cdot Q_k\right)^{\gamma}}{\gamma} \right) \Big/ \left\{ \dfrac{1}{k} \displaystyle\sum_{i=0}^{k-1} \dfrac{\exp\left(\gamma \sum_{j=i}^{k-1} Z_j / j\right) - 1}{\gamma} \right\}, & \gamma < 0 \end{cases}$$

($n \to \infty$), with $Q_k, Z_0, Z_1, \dots, Z_{k-1}$ independent, Q_k gamma with k degrees of freedom, and Z_i, $i = 0, 1, \dots, k-1$, i.i.d. exponential.

REMARK. Note that the number of order statistics k used in the definition of $M_n^{(1)}$ remains bounded whereas, if for $\hat{\gamma}_n$ one uses (1.7), in order to get consistency for $\hat{\gamma}_n$ one needs to use an unbounded number k' of order statistics in its definitions.

The proof of Theorem 4.1 is based on the following lemma.

LEMMA 4.2 [cf. Beirlant and Teugels (1986)]. *Under the conditions and with the conventions of Theorem* 4.1,

$$\frac{M_n^{(1)}}{a(n)/U(n)} \xrightarrow{d} \begin{cases} \dfrac{1}{k}\sum_{i=0}^{k-1} Z_i, & \gamma \geq 0, \\ Q_k^{-\gamma}\dfrac{1}{k}\sum_{i=0}^{k-1} \dfrac{\exp\{\gamma\sum_{j=i}^{k-1} Z_j/j\} - 1}{\gamma}, & \gamma < 0. \end{cases} \tag{4.6}$$

REMARK. Note that for $\gamma \geq 0$ the limit law is of gamma type.

PROOF.

$$\begin{aligned}\frac{M_n^{(1)}}{a(n)/U(n)} &= \frac{1}{k}\sum_{i=0}^{k-1} \frac{\log X_{(n-i,n)} - \log X_{(n-k,n)}}{a(n)/U(n)} \\ &\stackrel{d}{=} \frac{1}{k}\sum_{i=0}^{k-1} \frac{\log\big(U\exp(E_{(n-i,n)} - E_{(n-k,n)} + E_{(n-k,n)})\big) - \log\big(U\exp E_{(n-k,n)}\big)}{a(\exp E_{(n-k,n)})/U(\exp E_{(n-k,n)})} \\ &\quad\times \frac{a(\exp E_{(n-k,n)}) \cdot U(n)}{a(n)\cdot U(\exp E_{(n-k,n)})}\end{aligned}$$

with $E_{(1,n)} \leq E_{(2,n)} \leq \cdots \leq E_{(n,n)}$ standard exponential order statistics. Now

$$E_{(n-i,n)} - E_{(n-k,n)} \stackrel{d}{=} \sum_{j=i}^{k-1} Z_j/j$$

for all n with $Z_1, Z_2, \dots, Z_n$ i.i.d. standard exponential by Rényi's representation for exponential order statistics and

$$E_{(k)} - \log n \xrightarrow{d} -\log Q_k \tag{4.7}$$

[Smirnov (1949)]. Using

$$\frac{\log U(tx) - \log U(t)}{a(t)/U(t)} \to \begin{cases} \log x, & \gamma \geq 0, \\ \dfrac{x^\gamma - 1}{\gamma}, & \gamma < 0 \end{cases}$$

and

$$\frac{U(t)}{U(tx)}\frac{a(tx)}{a(t)} \to \begin{cases} 1, & \gamma \geq 0, \\ x^\gamma, & \gamma < 0, \end{cases}$$

$t \to \infty$ for all $x > 0$, locally uniformly, we then get the result of the lemma. □

PROOF OF THEOREM 4.1.

$$\frac{\hat{x}_{p,n} - x_p}{X_{(n-k,n)} M_n^{(1)}} = \frac{a_n^{\hat{\gamma}_n} - 1}{\hat{\gamma}_n \rho_1(\hat{\gamma}_n)} + \left\{ \frac{X_{(n-k,n)} - U(n)}{a(n)} - \frac{U(n a_n/k) - U(n)}{a(n)} \right\} \times \frac{a(n)/U(n)}{M_n^{(1)}} \cdot \frac{U(n)}{X_{(n-k,n)}}.$$

Note that $(U(tx) - U(t))/a(t) \to (x^\gamma - 1)/\gamma$ $(t \to \infty)$ locally uniformly. An application of (4.7) and Lemma 4.2 above is now sufficient to complete the proof. □

In the case $\gamma < 0$, one can adapt the above reasoning for the boundary situation $p = 0$ to get a confidence interval for the upper endpoint $x^*(F) = U(\infty)$ of the distribution.

THEOREM 4.3. *Suppose* (1.1) *holds with* $\gamma < 0$. *Then* $x^* = x^*(F) := \sup\{x | F(x) < 1\}$ *is finite* (*and positive as assumed in Section* 1). *Set*

$$\hat{x}_n^* := X_{(n-k,n)} M_n^{(1)} \left(1 - \frac{1}{\hat{\gamma}_n}\right) + X_{(n-k,n)}. \tag{4.8}$$

Under the conditions of Theorem 4.1

$$\frac{\hat{x}_n^* - x^*}{X_{(n-k,n)} M_n^{(1)}} \xrightarrow{d} \left|\left(1 - \frac{1}{\gamma}\right) + \left\{\frac{1}{k}\sum_{i=0}^{k-1} \exp\left\{\gamma \sum_{j=i}^{k-1} \frac{Z_j}{j}\right\} - 1\right\}^{-1}.$$

PROOF.

$$\frac{\hat{x}_n^* - x^*}{X_{(n-k,n)} M_n^{(1)}} = 1 - \frac{1}{\hat{\gamma}_n} + \left\{ \frac{X_{(n-k,n)} - U(n)}{a(n)} - \frac{x^* - U(n)}{a(n)} \right\} \times \frac{U(n)}{X_{(n-k,n)}} \cdot \frac{a(n)/U(n)}{M_n^{(1)}}. \tag{4.9}$$

The rest of the proof is as before; note that $\{x^* - U(n)\}/a(n) \to -\gamma^{-1}$ $(n \to \infty)$. □

5. Endpoint and quantile estimation: Infinite case. We now consider estimating x_p again for the limiting situation $n \to \infty$ but allow the number of order statistics k involved in the definition of $X_{(n-k,n)}$ and $M_n^{(1)}$ to grow without bound. The following theorem enables one to construct a confidence interval for a quantile x_p when $p_n \to 0$, $np_n \to \infty$ $(n \to \infty)$.

THEOREM 5.1. *Suppose that* F *has a positive density* F' *so that* U' *exists. If* $U' \in RV_{\gamma-1}$ [*i.e.*, $F' \in RV_{-1/\gamma-1}$ *for* $\gamma > 0$, $1/F' \in \Gamma$ *for* $\gamma = 0$ *and*

$F'(x^* - 1/x) \in RV_{1/\gamma+1}$ *for* $\gamma < 0$], *then*

$$\sqrt{k(n)}\,\frac{X_{(n-k(n),\,n)} - U\left(\dfrac{1}{p_n}\right)}{X_{(n-k(n),\,n)}\cdot M_n^{(1)}} \tag{5.1}$$

is asymptotically normal with mean 0 *and variance* $\{1 - \min(0,\gamma)\}^2$, *provided* $p_n \to 0$, $np_n \to \infty$ $(n \to \infty)$ *and* $k(n) := [np_n]$.

PROOF. Since $\sqrt{k(n)}\,\{X_{(n-k(n),\,n)} - U(n/k(n))\}/(n/k(n))\cdot U'(n/k(n))$ is asymptotically standard normal [Dekkers and de Haan (1989), Lemma 3.1], also $X_{(n-k(n),\,n)} \sim U(n/k(n))$ $(n \to \infty)$ in probability.

Next note that, from the proof of Theorem 3.1 and $U' \in RV_{1/\gamma-1}$,

$$\frac{M_n^{(1)}}{\dfrac{n}{k(n)}U'\left(\dfrac{n}{k(n)}\right)\Big/U\left(\dfrac{n}{k(n)}\right)} \sim \frac{M_n^{(1)}}{Y_{(n-k(n),\,n)}\cdot U'\big(Y_{(n-k(n),\,n)}\big)/U\big(Y_{(n-k(n),\,n)}\big)}$$

$$\to \frac{1}{1-\min(0,\gamma)},$$

with $X_{(n-i,\,n)} \stackrel{d}{=} U(Y_{(n-i,\,n)})$, $i = 0, 1, \dots, n-1$, as before.

Finally, one checks that

$$\lim_{n\to\infty}\sqrt{k(n)}\,\frac{U\left(\dfrac{1}{p_n}\right) - U\left(\dfrac{n}{k(n)}\right)}{\dfrac{n}{k(n)}\cdot U'\left(\dfrac{n}{k(n)}\right)} = 0. \qquad \square$$

Next we consider the estimation of the endpoint of the distribution.

THEOREM 5.2. *Let* $k = k(n) \to \infty$ *and* $k(n)/n \to 0$ $(n \to \infty)$. *Suppose the conditions of Theorem* 3.1 *hold with* $\gamma < 0$. *Suppose moreover that U has a regularly varying derivative* U'. *Then, with* $\hat{x}_n^*$ *as defined in* (4.8),

$$\sqrt{k(n)}\cdot\frac{\hat{x}_n^* - x^*}{X_{(n-k(n),\,n)}M_n^{(1)}(1-\hat{\gamma}_n)} \tag{5.2}$$

is asymptotically normal $(n \to \infty)$ *with mean* 0 *and variance*

$$\frac{1}{\gamma^2}\left[\frac{1}{1-2\gamma} + \frac{1-2\gamma}{\gamma^2}\left\{4 - 8\frac{1-2\gamma}{1-3\gamma} + \frac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right\} - \frac{4}{1-3\gamma}\right]. \tag{5.3}$$

For the proof we need the following lemma.

LEMMA 5.3. *Suppose the conditions of Theorem 5.2 hold. Recall the function U from Lemma 2.5. The random vector*

$$(5.4)\quad \sqrt{k}\left(\frac{X_{(n-k,n)}M_n^{(1)}}{-\gamma\left\{U(\infty)-U\left(\frac{n}{k}\right)\right\}}-(1-\gamma)^{-1}, \hat{\gamma}_n-\gamma, \frac{X_{(n-k,n)}-U\left(\frac{n}{k}\right)}{-\gamma\left\{U(\infty)-U\left(\frac{n}{k}\right)\right\}}\right)$$

is asymptotically normal with means 0 *and covariance matrix* (s_{ij}) *with*

$$s_{11}=\frac{1+\gamma^2(1-2\gamma)}{(1-\gamma)^2(1-2\gamma)},\qquad s_{12}=-2+\frac{2(1-2\gamma)}{(1-3\gamma)},\qquad s_{13}=\frac{\gamma}{(1-\gamma)},$$

$$(5.5)\quad s_{22}=(1-\gamma)^2(1-2\gamma)\left[4-\frac{8(1-2\gamma)}{(1-3\gamma)}+\frac{(5-11\gamma)(1-2\gamma)}{(1-3\gamma)(1-4\gamma)}\right],$$

$$s_{23}=0,\qquad s_{33}=1.$$

PROOF. Note that (3.7) holds with

$$f(\log X_{(n-k,n)})=-\gamma\left\{U(\infty)-U\left(1/\left\{1-F(X_{(n-k,n)})\right\}\right)\right\}\div U\left(1/\left\{1-F(X_{(n-k,n)})\right\}\right).$$

We write the first component of (5.4) as

$$\sqrt{k}\left\{\frac{X_{(n-k,n)}M_n^{(1)}}{-\gamma\left\{U(\infty)-U\left(\frac{n}{k}\right)\right\}}-(1-\gamma)^{-1}\right\}$$

$$(5.6)\quad =\frac{U(\infty)-U(e^{E_{(n-k,n)}})}{U(\infty)-U\left(\frac{n}{k}\right)}\sqrt{k}\left\{\frac{M_n^{(1)}}{f(\log X_{(n-k,n)})}-(1-\gamma)^{-1}\right\}+\frac{\sqrt{k}}{1-\gamma}\left\{\frac{U(\infty)-U(e^{E_{(n-k,n)}})}{U(\infty)-U\left(\frac{n}{k}\right)}-1\right\},$$

with $E_{(1,n)}\le\cdots\le E_{(n,n)}$ standard exponential order statistics as before. Note that

(i) $U(e^{E_{(n-k,n)}})=X_{(n-k,n)}$;
(ii) $E_{(n-k,n)}-\log(n/k)\to 0$ in probability.

It is thus sufficient to consider the limit distribution of the random vector

$$(5.7)\quad \sqrt{k}\left(\frac{M_n^{(1)}}{f(\log X_{(n-k,n)})}-(1-\gamma)^{-1}, \hat{\gamma}_n-\gamma, \frac{X_{(n-k,n)}-U\left(\frac{n}{k}\right)}{U(\infty)-U\left(\frac{n}{k}\right)}\right).$$

The joint limit distribution of the first two components follows easily from the results of Section 3. It remains to prove that the third component is asymptotically standard normal and independent of the first two components. The asymptotic normality of the third component follows, e.g., from Lemma 3.1 of Dekkers and de Haan (1989).

If we rewrite all order statistics in terms of exponential order statistics $E_{(1,n)} \leq \cdots \leq E_{(n,n)}$ [as we did in Dekkers and de Haan (1989)], we see that by (3.22) the asymptotic distribution of $(M_n^{(1)}, M_n^{(2)})$ is totally determined by the asymptotic distribution of two functionals of $(E_{(n-k(n)+1,n)} - E_{(n-k(n),n)}, \ldots, E_{(n,n)} - E_{(n-k(n),n)})$ whereas the asymptotic distribution of the third component of (5.7) is totally determined by that of $E_{(n-k(n),n)}$ [cf. Dekkers and de Haan (1989), Lemma 3.1]. The asymptotic independence follows. □

PROOF OF THEOREM 5.2.

$$\sqrt{k}\,\frac{\hat{x}_n^* - x^*}{X_{(n-k,n)} M_n^{(1)}(1-\hat{\gamma}_n)}$$

$$= \sqrt{k}\left[-\frac{1}{\hat{\gamma}_n} + \frac{X_{(n-k,n)} - U(\infty)}{X_{(n-k,n)} M_n^{(1)}(1-\hat{\gamma}_n)}\right]$$

$$= \sqrt{k}\left\{-\frac{1}{\hat{\gamma}_n} + \frac{1}{\gamma}\right\} + \frac{\sqrt{k}}{(-\gamma)}\cdot\frac{X_{(n-k,n)} - U\left(\frac{n}{k}\right)}{U(\infty) - U\left(\frac{n}{k}\right)}\cdot\frac{(-\gamma)\left\{U(\infty) - U\left(\frac{n}{k}\right)\right\}}{X_{(n-k,n)} M_n^{(1)}(1-\hat{\gamma}_n)}$$

$$-\frac{1}{\gamma}\,\frac{-\gamma\left\{U(\infty) - U\left(\frac{n}{k}\right)\right\}}{X_{(n-k,n)} M_n^{(1)}}\sqrt{k}\left[\left\{\frac{X_{(n-k,n)} M_n^{(1)}}{-\gamma\left\{U(\infty) - U\left(\frac{n}{k}\right)\right\}} - (1-\gamma)^{-1}\right\}\right.$$

$$\left. - \left\{(1-\hat{\gamma}_n)^{-1} - (1-\gamma)^{-1}\right\}\right].$$

Application of Lemma 5.3 now gives the stated result. □

A somewhat related paper is Hall (1982).

6. Concluding remarks. We now provide an intuitive background for (1.7). It is well known that the convergence of the Hill estimator (1.3) for $\gamma > 0$ is the sample analogue of the following relation, which is necessary and sufficient for

(1.1) in the case $\gamma > 0$:

$$\gamma = \int_1^\infty u^{-1/\gamma} \frac{du}{u} \to \int_1^\infty \frac{1 - F(tu)}{1 - F(t)} \frac{du}{u} = \frac{\int_t^\infty (1 - F(u))(du/u)}{1 - F(t)} \tag{6.1}$$

$$= \frac{\int_t^\infty (\log x - \log t)\, dF(x)}{1 - F(t)} = E(\log X - \log t | X > t)$$

$(t \to \infty)$, where X is a r.v. with d.f. F. So the reason for using the log of the order statistics instead of the order statistics themselves is that otherwise the first integral may diverge. This forces us to use logarithms of order statistics instead of the order statistics themselves in the definition of $M_n^{(1)}$. That is not possible when the random variables are negative. In order to avoid this problem (which comes up only for $\gamma \le 0$) we have to impose the extra condition $x^*(F) > 0$. This does not cause any difficulty in applications. An analogue of (6.1) is known in the case $\gamma = 0$ [Balkema and de Haan (1974)]: (1.1) holds with $\gamma = 0$ if and only if

$$\lim_{t \uparrow x^*} \frac{E(\{X - t\}^2 | X > t)}{\{E(X - t | X > t)\}^2} = \frac{\int_0^\infty x^2\, d(1 - e^{-x})}{\{\int_0^\infty x\, d(1 - e^{-x})\}^2} = 2. \tag{6.2}$$

These two considerations led us to consider the quotient $M_n^{(2)}/\{M_n^{(1)}\}^2$. However, it is clear that this quotient does not discriminate sufficiently, since taking logarithms transforms r.v.'s in the domain of G_γ with $\gamma \ge 0$ into r.v.'s in the domain of G_0 [cf. (2.10)]. But by good luck $M_n^{(1)}$ itself also converges for any γ [see (2.11)] and discriminates the range of values of γ not covered by $M_n^{(2)}/\{M_n^{(1)}\}^2$.

In Dekkers and de Haan (1989) we discussed several other methods to estimate γ. A comparison of the different estimators both from a theoretical and from a practical point of view is the subject of further research.

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