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A MONOTONE APPROXIMATION FOR THE NONAUTONOMOUS SIZE-STRUCTURED POPULATION MODEL

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Abstract. In this paper we develop a monotone approximation method, based on an upper and lower solutions technique, for solving the nonautonomous size-structured model. Such a technique results in the existence and uniqueness of solutions for this equation. Furthermore, we establish a first-order convergence of the method and present a numerical example.

1. Introduction. In this paper we consider the following first-order nonlocal hyperbolic initial-boundary value problem that describes the dynamics of a size-structured population:

$$\begin{aligned} &u_t + (g(t,x)u)_x = -m(t,x)\,u, && 0 < t < T, \quad a < x < b, \\ &g(t,a)u(t,a) = C(t) + \int_a^b q(t,x)u(t,x)dx, && 0 < t < T, \\ &u(0,x) = u_0(x), && a \le x \le b. \end{aligned} \tag{1.1}$$

The above class of models was proposed by Sinko and Streifer [9] in analogy to the McKendrick-Von Foerster class of age-structured models. Here, the function $u(t,x)$ represents the density of individuals in the size class $[x, x+dx)$ at time t, i.e., the total number of individuals between sizes x_1 and x_2, $x_1 < x_2$, at time t is given by $\int_{x_1}^{x_2} u(t,x)dx$. The parameter $g(t,x)$ represents the time and size-dependent growth rate. The parameter $m(t,x)$ represents the mortality rate, and $q(t,x)$ represents the reproduction rate of an individual of size x at time t. The function $C(t)$ represents the inflow of a-size individuals from an external source.

The above problem has been widely investigated [1, 2, 6, 7]. In particular, several numerical methods for approximating the autonomous case of this equation have been

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developed. Two common approaches that have been used to study well-posedness and establish convergence of approximation methods for the autonomous case are the semigroup approach and the method of characteristics [1, 7]. The goal of this paper is to present a different approach for establishing well-posedness of the nonautonomous equation (1.1) as well as developing a monotone approximation scheme, which is computationally attractive, for solving this equation. To the best of our knowledge, comparison results and approximation methods based on monotone sequences have not been established for this equation.

To carry out our program, the following assumptions will be imposed on our parameters throughout the paper:

(A1) $g \in C^1([0,T] \times [a,b])$, $g > 0$ on $[0,T] \times [a,b)$, and $g(t,b) = 0$, $t \in [0,T]$;
(A2) $q \in C([0,T] \times [a,b])$ and $q \geq 0$ in $[0,T] \times [a,b]$;
(A3) $m \in C([0,T] \times [a,b])$ and $m \geq 0$ in $[0,T] \times [a,b]$;
(A4) $C(t) \in C([0,T])$, and $C(t) \geq 0$;
(A5) $u_0 \in C([a,b])$, $u_0 \geq 0$ and satisfies the following compatibility condition:

$$g(0,a)u_0(a) = C(0) + \int_a^b q(0,x)u_0(x)dx.$$

The plan of our paper is as follows. In Sec. 2 we establish a comparison result for (1.1). In Sec. 3 we construct monotone sequences of upper and lower solutions and show their linear convergence to the unique solution of (1.1). In Sec. 4 we present a numerical example illustrating the simplicity of applying this scheme to approximate the solution of our equation.

2. Comparison principle. For simplicity, let $D_T = (0,T) \times (a,b)$. We begin with the definition of upper and lower solutions of problem (1.1).

DEFINITION. *A function $u(t,x)$ is called an upper (a lower) solution of* (1.1) *on D_T if all the following hold:*

(i) $u \in C(D_T) \cap L^\infty(D_T)$;
(ii) $u(0,x) \geq (\leq)\ u_0(x)$ *in* $[a,b]$;
(iii) *For every $t \in (0,T)$ and every nonnegative $\xi(t,x) \in C^1(\overline{D_T})$,*

$$\begin{aligned}&\textstyle\int_a^b u(t,x)\xi(t,x)dx \\ &\quad \geq (\leq) \textstyle\int_a^b u(0,x)\xi(0,x)dx + \int_0^t \xi(\tau,a)\left(C(\tau) + \int_a^b q(\tau,x)u(\tau,x)dx\right)d\tau \\ &\qquad + \textstyle\int_0^t \int_a^b u(\tau,x)\left[\xi_\tau + g(\tau,x)\xi_x\right]dx\,d\tau \\ &\qquad\quad - \textstyle\int_0^t \int_a^b \xi(\tau,x)m(\tau,x)u(\tau,x)\,dx\,d\tau.\end{aligned}$$

A function $u(t,x)$ is said to be a solution to Eq. (1.1) if it satisfies the definition of both a lower solution and an upper solution to this equation. With the above definition we are ready to prove the following theorem.

THEOREM 2.1. Let u and v be an upper solution and a lower solution of (1.1), respectively. Then $u \geq v$ in $\overline{D_T}$.

Proof. Let $w(t,x) = v - u$. Then w satisfies

$$w(0,x) = v(0,x) - u(0,x) \leq 0 \quad \text{in } [a,b] \tag{2.1}$$

and

$$
\begin{aligned}
&\textstyle\int_a^b w(t,x)\xi(t,x)dx \\
&\qquad \le \textstyle\int_0^t \xi(\tau,a) \int_a^b q(\tau,x)w(\tau,x)\,dx\,d\tau \\
&\qquad\quad + \textstyle\int_0^t \int_a^b w[\xi_\tau + g(\tau,x)\xi_x]\,dx\,d\tau \\
&\qquad\quad - \textstyle\int_0^t \int_a^b \xi(\tau,x)\; m(\tau,x)\; w(\tau,x)\,dx\,d\tau.
\end{aligned} \tag{2.2}
$$

Letting $\zeta(t,x) = e^{-\gamma t}\xi(t,x)$, where γ (≥ 0) is chosen so that $\gamma - m \ge 0$ on D_T, then we have that

$$
\begin{aligned}
&e^{\gamma t} \textstyle\int_a^b w(t,x)\zeta(t,x)dx \\
&\qquad \le \textstyle\int_0^t e^{\gamma\tau}\zeta(\tau,a) \int_a^b q(\tau,x)w(\tau,x)\,dx\,d\tau \\
&\qquad\quad + \textstyle\int_0^t \int_a^b w(\tau,x)e^{\gamma\tau}[\zeta_\tau + g(\tau,x)\zeta_x]\,dx\,d\tau \\
&\qquad\quad + \textstyle\int_0^t \int_a^b e^{\gamma\tau}\zeta(\tau,x)\,(\gamma - m(\tau,x))\,w(\tau,x)\,dx\,d\tau.
\end{aligned} \tag{2.3}
$$

We now set up the following backward problem:

$$
\begin{aligned}
&\zeta_\tau + g\zeta_x = 0, && 0<\tau<t, && a<x<b, \\
&\zeta(\tau,b) = 0, && 0<\tau<t, && \\
&\zeta(t,x) = \chi(x), && && a \le x \le b.
\end{aligned} \tag{2.4}
$$

Here $\chi(x) \in C_0^\infty(a,b)$, $0 \le \chi \le 1$.

The existence of $\zeta \in C^1(\overline{D_T})$ follows from the fact that by the variable change $s = t-\tau$, (2.4) can be written into

$$
\begin{aligned}
&\zeta_s - g\zeta_x = 0, && 0<s<t, && a<x<b, \\
&\zeta(s,b) = 0, && 0<s<t, && \\
&\zeta(0,x) = \chi(x), && && a \le x \le b.
\end{aligned} \tag{2.5}
$$

Note that the initial and boundary values for ζ imply that $0 \le \zeta \le 1$ on D_T.

Substituting such a ζ in (2.3) yields

$$
\int_a^b w(t,x)\chi(x)dx \le L \int_0^t \int_a^b w(\tau,x)^+\,dx\,d\tau, \tag{2.6}
$$

where $L = \max\limits_{\overline{D_T}} [q(t,x) + \gamma - m(t,x)]$.

Since this inequality holds for every χ, we can choose a sequence $\{\chi_n\}$ on (a,b) converging to

$$
\chi = \begin{cases} 1 & \text{if } w(t,x) > 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Consequently, we find that

$$
\int_a^b w(t,x)^+ dx \le L \int_0^t \int_a^b w(\tau,x)^+ dx\,d\tau,
$$

which by Gronwall's inequality leads to

$$
\int_a^b w(t,x)^+ dx = 0.
$$

Thus, the conclusion follows.

REMARK 2.2. *If we assume that* $u, v \in L^\infty(D_T)$, *then we obtain that* $u \ge v$ *a.e. on* D_T.

3. Monotone approximation scheme. We start this section by constructing monotone sequences of upper and lower solutions. To this end, let $\alpha^0(t,x)$ and $\beta^0(t,x)$ be a lower solution and an upper solution of (1.1), respectively. We now define our two sequences $\{\alpha^k\}_{k=1}^{\infty}$ and $\{\beta^k\}_{k=1}^{\infty}$ as follows:

$$\tag{3.1} \begin{aligned} &\alpha_t^k + (g(t,x)\alpha^k)_x = -m(t,x)\ \alpha^k, && 0<t<T, && a<x<b,\\ &g(t,a)\alpha^k(t,a) = \widehat{R}^{k-1}(t), && 0<t<T, &&\\ &\alpha^k(0,x) = u_0(x), && && a\le x\le b, \end{aligned}$$

where $\widehat{R}^{k-1}(t) \equiv C(t) + \int_a^b q(t,x)\alpha^{k-1}(t,x)dx$, and

$$\tag{3.2} \begin{aligned} &\beta_t^k + (g(t,x)\beta^k)_x = -m(t,x)\ \beta^k, && 0<t<T, && a<x<b,\\ &g(t,a)\beta^k(t,a) = \widehat{B}^{k-1}(t), && 0<t<T, &&\\ &\beta^k(0,x) = u_0(x), && && a\le x\le b, \end{aligned}$$

where $\widehat{B}^{k-1}(t) \equiv C(t) + \int_a^b q(t,x)\beta^{k-1}(t,x)dx$.

The existence of α^k and β^k follows from the fact that $\widehat{R}^{k-1}$ and $\widehat{B}^{k-1}$ are given functions; hence these systems can be solved using the method of characteristics [3, 4, 5, 10]. In fact, the equations for the characteristic curves are given by

$$\tag{3.3} \begin{cases} \dfrac{d}{ds}t(s) = 1,\\[2ex] \dfrac{d}{ds}x(s) = g(t(s),x(s)). \end{cases}$$

Along a characteristic curve $(t(s),x(s))$, the solution α^k to Eq. (3.1) satisfies the following ordinary differential equation:

$$\tag{3.4} \frac{d}{ds}\alpha^k(s) = -(g_x(t(s),x(s)) + m(t(s),x(s)))\alpha^k(s).$$

Under assumption (A1), Eq. (3.3) has a unique solution for any initial point $(t(s_0),x(s_0))$. If we parametrize the characteristic curves with the variable t, then a characteristic curve passing through $(\widehat{t},\widehat{x})$ is given by $(t, X(t;\widehat{t},\widehat{x}))$, where X satisfies

$$\frac{d}{dt}X(t;\widehat{t},\widehat{x}) = g(t, X(t;\widehat{t},\widehat{x}))$$

and $X(\widehat{t};\widehat{t},\widehat{x}) = \widehat{x}$. By (A1) the function X is a strictly increasing function, and therefore a unique inverse function $\Gamma(x;\widehat{t},\widehat{x})$ exists. Hence if we define $G(x) = \Gamma(x;0,a)$, then $(G(x),x)$ represents the characteristic curve passing through $(0,a)$ and this curve divides the (t,x)-plane into two parts. For any point (t,x) with $t\le G(x)$, the solution $\alpha^k(t,x)$ is determined through the initial condition by

(3.5)

$$\alpha^k(t,x) = u_0(X(0;t,x))\exp\left\{-\int_0^t (g_x(s,X(s;t,x)) + m(s,X(s;t,x)))ds\right\}.$$

On the other hand, if $t > G(x)$ then the solution is determined via the boundary condition by

(3.6)

$$\alpha^k(t,x) = R^{k-1}(\Gamma(a;t,x)) \exp\left\{ -\int_{\Gamma(a;t,x)}^{t} (g_x(s,X(s;t,x)) + m(s,X(s;t,x)))ds \right\},$$

where $R^{k-1}(t) = \dfrac{\widehat{R}^{k-1}(t)}{g(t,a)}$. A similar representation can be obtained for the solution β^k to Eq. (3.2) by replacing R^{k-1} with B^{k-1}, where $B^{k-1} = \dfrac{\hat{B}^{k-1}}{g(t,a)}$.

Our next task is to show that the sequences $\{\alpha^k\}_{k=0}^{\infty}$ and $\{\beta^k\}_{k=0}^{\infty}$ are monotone. To this end, let $w = \alpha^0 - \alpha^1$. Then w satisfies (2.1)-(2.2). Hence, by the comparison result, $w(t,x) \leq 0$, which implies that $\alpha^0 \leq \alpha^1$. Similarly, we can show that $\beta^0 \geq \beta^1$, $\alpha^1 \leq \beta^0$, and $\alpha^0 \leq \beta^1$. From this, it easily follows that α^1 and β^1 are a lower solution and an upper solution, respectively. Hence, $\alpha^1 \leq \beta^1$.

Proceeding in a similar manner we can show that $\alpha^k \leq \alpha^{k+1} \leq \beta^{k+1} \leq \beta^k$ and that α^{k+1} and β^{k+1} are also a lower solution and an upper solution of (1.1), respectively. Hence by induction, we obtain two monotone sequences that satisfy

$$\alpha^0 \leq \alpha^1 \leq \cdots \leq \alpha^k \leq \beta^k \leq \cdots \leq \beta^1 \leq \beta^0 \qquad \text{in } \overline{D}_T$$

for each $k = 0, 1, 2, \ldots$. We remark that from the monotonicity of the sequences $\{\alpha^k\}_{k=0}^{\infty}$ and $\{\beta^k\}_{k=0}^{\infty}$ there exist functions α and β such that $\alpha^k \to \alpha$ and $\beta^k \to \beta$ pointwise in $\overline{D}_T$.

Having established the monotonicity of our sequences, we now prove the following convergence result.

THEOREM 3.1. Suppose that $\alpha^0(t,x)$ and $\beta^0(t,x)$ are a lower solution and an upper solution of (1.1), respectively. Then, the sequences $\{\alpha^k\}_{k=0}^{\infty}$ and $\{\beta^k\}_{k=0}^{\infty}$ converge uniformly to the unique solution $u(t,x)$ of Eq. (1.1) on D_T. Moreover, the order of convergence is linear.

Proof. We first prove the convergence result for the sequence $\{\alpha^k\}_{k=0}^{\infty}$. To this end, using standard arguments (see, [8] p. 189), the solution representation for α^k given in (3.5)-(3.6), the fact that $\alpha^0 \leq \alpha^k \leq \beta^0$, and the monotonicity of the sequence $\{\alpha^k\}$, we have that along the characteristic curves passing through the points $(0, x_0)$, the solution

$$\alpha^k(t, X(t;0,x_0)) = u_0(x_0) \exp\left\{ -\int_0^t (g_x(s,X(s;0,x_0)) + m(s,X(s;0,x_0)))ds \right\}$$

coincides with

$$\alpha(t, X(t;0,x_0) = u_0(x_0) \exp\left\{ -\int_0^t (g_x(s,X(s;0,x_0)) + m(s,X(s;0,x_0)))ds \right\}$$

on $0 \leq t \leq T$. Similarly, since $R^k(t)$ is monotone and uniformly bounded on the interval $0 \leq t \leq T$, along the characteristic curves passing through (t_0, a), the solution

$$\alpha^k(t, X(t;t_0,a)) = R^{k-1}(t_0) \exp\left\{ -\int_{t_0}^t (g_x(s,X(s;t_0,a)) + m(s,X(s;t_0,a)))ds \right\}$$

converges to

$$\alpha(t, X(t; t_0, a) = R(t_0) \exp\left\{ - \int_{t_0}^{t} (g_x(s, X(s; t_0, a)) + m(s, X(s; t_0, a)))ds \right\}$$

uniformly and monotonically on $t_0 \le t \le T$, where

$$R(t) = \frac{1}{g(t,a)} \left(C(t) + \int_a^b q(t,x)\alpha(t,x)dx \right).$$

Consequently, we can define

(3.7)

$$\underline{u}(t,x) = \begin{cases} u_0(X(0;t,x)) \exp\left\{ - \int_0^t (g_x(s, X(s;t,x)) + m(s, X(s;t,x)))ds \right\}, & t \le G(x), \\ R(\Gamma(a;t,x)) \exp\left\{ - \int_{\Gamma(a;t,x)}^t (g_x(s, X(s;t,x)) + m(s, X(s;t,x)))ds \right\}, & t > G(x). \end{cases}$$

Now using the compatibility condition (A5) imposed on u_0, the continuity of the limit function $\underline{u}(t,x)$ is easily established. Furthermore, we can show that $\underline{u}(t,x)$ is a solution of Eq. (1.1). In a similar manner, we can obtain a continuous limit function $\overline{u}$ from the sequence $\{\beta^k\}_{k=0}^{\infty}$. Using the comparison result, we have that $\underline{u} = u = \overline{u}$. This proves the existence of a solution. The uniqueness follows immediately from Theorem 2.1. In particular, if u and v are two solutions of Eq. (1.1), then from Theorem 2.1 we obtain that $v \le u$ and $u \le v$. As for the linear convergence, subtracting the solution representation of α^k given in (3.5)-(3.6) from that of the limit function $u(t,x)$ given in (3.7), we get

$$|\alpha^k(t,x) - u(t,x)| \le \begin{cases} 0, & t \le G(x), \\ M|R^{k-1}(\Gamma(a;t,x)) - R(\Gamma(a;t,x))|, & t > G(x), \end{cases}$$

where $M = \sup_{\overline{D}_T} \exp\left\{ - \int_{\Gamma(a;t,x)}^t (g_x(s, X(s;t,x)) + m(s, X(s;t,x)))ds \right\}$. Hence taking the supremum over all $t \in [0,T]$ and $x \in [a,b]$, we get

$$\sup_{\overline{D}_T} |\alpha^k(t,x) - u(t,x)| \le M||q||_\infty (b-a) \sup_{\overline{D}_T} |\alpha^{k-1}(t,x) - u(t,x)|.$$

4. Numerical results. For our numerical example, we have chosen the following initial condition and parameter functions

$$u_0 = \cos 4\pi x + 2, \;\; g(t,x) = 1 - x, \;\; C = 0, \;\; q(t,x) = 3xe^{-t}, \;\; \text{and} \quad m(t,x) = 2xe^{0.5t}.$$

We set $\alpha^0(t,x) = u_0(x)e^{-15t}$. It can easily be verified that $\alpha^0(t,x)$ generates a lower solution for our example problem. To solve Eq. (1.1), we use a first-order discretization of the solution representation for the approximation scheme given in (3.5)-(3.6) with $k = 0, 1, \ldots, 10$. In this example, the interval $[a,b] = [0,1]$ and the final time $T = 1$. In Figure 1 we present the final iteration $\alpha^{10}(t,x)$. We note that $\sup_{\overline{D}_T} |\alpha^{10}(t,x) - \alpha^9(t,x)| \approx 10^{-12}$, which indicates the rapid convergence of this method.

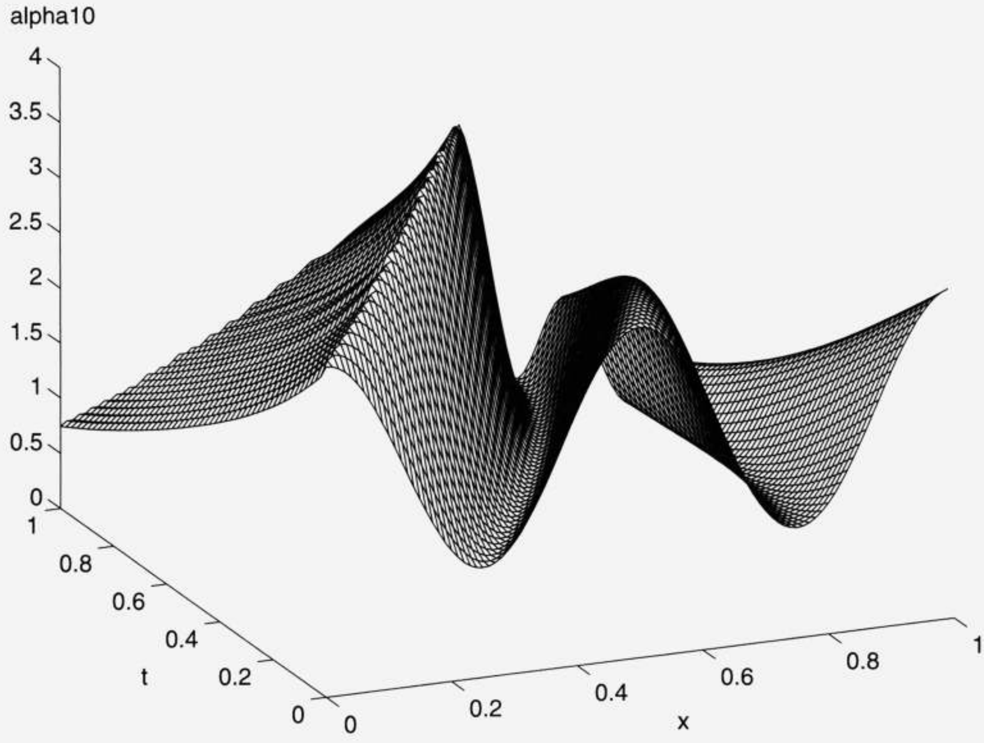

FIG. 1. The approximate solution $\alpha^{10}(t, x)$.

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