Filomat 34:5 (2020), 1487–1497
https://doi.org/10.2298/FIL2005487C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Monotone Bregan Projection Algorithm for Fixed Point and Equilibrium Problems in a Reflexive Banach Space

Sun Young Cho[a]

[a]*Department of Liberal Arts, Gyeongnam National University of Science and Technology, Jinju-Si, Gyeongsangnam-do, Korea*

Abstract. In this paper, a monotone Bregan projection algorithm is investigated for solving equilibrium problems and common fixed point problems of a family of closed multi-valued Bregman quasi-strict pseudocontractions. Strong convergence is guaranteed in the framework of reflexive Banach spaces.

1. Introduction-Preliminaries

Fixed Point Theory is a fascinating key component of nonlinear functional analysis. It has a large number of theoretical and real world applications in many fields, for example, machine learning, differential equations, game theory, economics, transportation, and control theory; see [2, 13, 20]. During the last decade, many convergence theorems for various convex optimization problems were established in infinite dimensional real Hilbert spaces through fixed point methods; see [8–11, 18, 19, 21, 28, 29] and the references therein. In the Banach setting, the approximation of fixed points via hybrid techniques is important, however, there are few results since the duality mapping is not easy to calculated in Banach spaces. In this paper, we are concerned with an equilibrium problem via a fixed method in the Banach setting.

Let E be a real reflexive Banach space with the norm $\|\cdot\|$ and let E^* be the dual space of E. Let $f : E \to (-\infty, +\infty]$ be a convex, proper and lower semi-continuous function. In this paper, we denote the domain of f by dom$f$, i.e., dom$f := \{x \in E : f(x) < +\infty\}$. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let any $x \in$ int dom$f$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^\circ(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

Recall that the function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int dom$f$; Gâteaux differentiable at x if the limit $f^\circ(x, y)$ exists for any y; uniformly Fréchet differentiable on a subset C of E if the limit $f^\circ(x, y)$ is attained uniformly for $x \in C$ and $\|y\| = 1$; Fréchet differentiable at x if the limit $f^\circ(x, y)$ is attained uniformly in $\|y\| = 1$. For function f, the following facts are known. (i) If f is Gâteaux differentiable at x, then $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x; (ii) If a continuous convex function $f \to \mathbb{R}$ is Gâteaux differentiable, ∇f is norm-to-weak* continuous; (iii) If f is Fréchet differentiable, ∇f is norm-to-norm continuous.

2010 *Mathematics Subject Classification*. Primary 47H05; Secondary 90C33
Keywords. Bregman projection, convergence analysis, equilibrium problem, fixed point, variational inequality.
Received: 04 May 2018; Revised: 08 January 2020; Accepted: 18 January 2020
Communicated by Adrian Petrusel
Email address: sycho@gntech.ac.kr (Sun Young Cho)

Let $x \in \text{int dom} f$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) - f(y) \leq \langle x^*, x - y\rangle, \quad \forall\, y \in E\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \quad \forall x^* \in E^*.$$

Recall that a function f is said to be (i) essentially stirctly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of dom∂f; (ii) essentially smooth if ∂f is both locally bounded and single-valued on its domain; (iii) Legendre, if it is both essentially smooth and essentially strictly convex.

In the framework of reflexive Banach spaces, we have the following facts: (i) f is essentially smooth if and only if f^* is essentially strictly convex; (ii) $(\partial f)^{-1} = \partial f^*$; (iii) f is Legendre if and only if f^* is Legendre; (iv) If f is Legendre, then ∇f is bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$=dom$\nabla f^*$=int dom$f^*$ and ran∇f^*=dom∇f=int dom$f$.

Let $f : E \to (-\infty, +\infty]$ be a G$\hat{a}$teaux differentiable function. The Bregman distance with respect to f is the function D_f : dom$f\times$int dom$f \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle\nabla f(x), y - x\rangle.$$

We remark here that the Bregman distance is not a distance in the usual sense.

Recall that bifunction $V_f : E \times E^* \to [0, \infty)$ associated with f is defined by

$$V_f(x, x^*) = f(x) + f^*(x^*) - \langle x, x^*\rangle, \quad \forall\, x \in E,\ x^* \in E^*.$$

Then V_f is nonnegative and satisfies $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$, $\forall x \in E, x^* \in E^*$. $D_f(\cdot,\cdot)$ has the following important property, called "three point identity". For any $x \in \text{dom} f$ and $y,\ z \in \text{int dom} f$,

$$\langle\nabla f(z) - \nabla f(y), x - y\rangle = D_f(x, y) + D_f(y, z) - D_f(x, z).$$

Let $f : E \to (-\infty, +\infty]$ be a convex and G$\hat{a}$teaux differentiable function and let $C \subset \text{dom} f$ be a nonempty, closed, and convex set. The Bregman projection $x \in \text{int dom} f$ onto C is the unique vector $P^f_C(x) \in C$ satisfying

$$D_f(P^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Letting $f(x) = \|x\|^2$, $\forall x \in E$, we find that the Bregman projection $P^f_C(x)$ is reduced the generalized projection $\Pi_C(x)$, defined by $\Pi_C(x) = \arg\min_{y\in C} \phi(y, x)$.

Let $B_r := \{z \in E : \|z\| \leq r\}$ and $S_E = \{x \in E : \|x\| = 1\}$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by

$$\rho_r(t) := \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}.$$

Let $f : E \to (-\infty, +\infty]$ be G$\hat{a}$teaux differentiable. The modulus of total convexity of f at $x \in \text{dom} f$ is the function $\nu_f(x,\cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in dom f, \|y - x\| = t\}.$$

The modulus of the total convexity of the function f on the set B is the function ν_f : int dom$f \times [0, +\infty) \to [0, +\infty]$ defined by $\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap dom f\}$.

Recall that a function f is said to be: (i) totally convex at x if $\nu_f(x, t) > 0$, whenever $t > 0$; (ii) totally convex if it is totally convex at any point $x \in \text{int dom} f$; (iii) totally convex on bounded sets if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$.

A function f is said to be: strongly coercive if $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$; sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Let C be a nonempty, closed, and convex subset of E. We use $CB(C)$ to denote the family of nonempty closed bounded subsets of C. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(C)$ defined by

$$H(A, B) = \max\{\sup_{y\in B} d(y, A), \sup_{x\in A} d(x, B), \}, \quad \forall A, B \in CB(C),$$

where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from point a to subset B. Let $T : C \to CB(C)$ be a multi-valued mapping. The fixed point set of T is denoted by $F(T) := \{p \in C : p = T(p)\}$. Recall that T is said to be multi-valued Bregman quasi-nonexpansive with respect to f if $F(T) \neq \emptyset$ and

$$D_f(p, u) \leq D_f(p, x), \quad \forall u \in Tx,\ x \in C,\ p \in F(T).$$

If $f(x) = \|x\|^2$ for all $x \in E$, it becomes a multi-valued quasi-ϕ-nonexpansive mapping, that is,

$$\phi(p, u) \leq \phi(p, x), \quad \forall u \in Tx,\ x \in C,\ p \in F(T).$$

Recall that T is said to be multi-valued Bregman quasi-strictly pseudo-contractive with respect to f if $F(T) \neq \emptyset$ and

$$D_f(p, u) \leq D_f(p, x) + kD_f(x, u), \quad \forall u \in Tx,\ x \in C,\ p \in F(T).$$

If $f(x) = \|x\|^2$ for all $x \in E$, it becomes a multi-valued quasi-ϕ-strictly pseudo-contractive mapping, that is,

$$\phi(p, u) \leq \phi(p, x) + k\phi(x, u), \quad \forall u \in Tx,\ x \in C,\ p \in F(T).$$

Let $g : C \times C \to \mathbb{R}$ be a bifunction. Recall that the equilibrium problem in the sense of Blum and Oettli [5] is find $\tilde{x}$ such that

$$g(\tilde{x}, y) \geq 0, \quad \forall y \in C. \tag{1}$$

In this paper, the set of solutions of the equilibrium problem is denoted by $EP(g)$. Equilibrium problem 1 provides us a a general and unified framework to study a wide class of problems arising in convex optimization problems; see [6, 12, 15, 16, 22] and the references.

In view of the generality and importance of equilibrium problems, fixed point algorithms have been extensively investigated for approximation solutions of problem (1); see [1, 7, 14, 23, 27, 33] and the references therein. It is known that Picard iterative method may fail to converge for nonexpansive-type mappings whose complementary mappings are monotone. Mann-type iterative method which is one of most popular iterative methods has recently attracted much attention in optimization and analysis communities. Mann-type iterative method is efficient for nonexpansive-type mappings, however, it is only weakly convergent in the framework of infinite dimensional spaces. To modify the Mann-type iterative method such that the strong convergence is guaranteed without compact assumptions, hybrid projection techniques were considered; see [26, 30, 34]. Unfortunately, the success achieved in using geometric properties in Hilbert spaces is not easy to carry over to the framework of Banach spaces. The main difficulty is that the normalized duality map appears in most Banach space inequalities This creates very serious technical difficulties in computation. Recently, attempts with the Bregman distance have been made to overcome these difficulties; see [17, 24, 25, 31, 32] and the references therein.

In this article, a monotone Bregan projection algorithm is investigated for solving equilibrium problems and common fixed point problems of a family of closed multi-valued Bregman quasi-strict pseudocontractions. Strong convergence is guaranteed in the framework of reflexive Banach spaces. Our algorithm is efficient for an infinite family of mappings, which is one of the highlights of this paper.

To study equilibrium problem (1), we impose the following restrictions on bifunction g.

(R-1) $g(x,x) \equiv 0, \forall a \in C$;

(R-2) $g(x,y) \geq \limsup_{t\downarrow 0} g(tz+(1-t)x,y), \forall x,y,z \in C$;

(R-3) $g(y,x)+g(x,y) \leq 0, \forall x,y \in C$;

(R-4) $y \mapsto g(x,y)$ is convex and weakly lower semi-continuous, $\forall x \in C$.

There are a lot of bifunction satisfying the above restrictions, for example, $g(x,y) = y - x$. For $r > 0$, the resolvent operator of bifunction g, $Res_r^g : E \to C$ is defined as follows:

$$Res_r^g(x) = \{z \in C : \langle y - z, \nabla f(z) - \nabla f(x)\rangle + rg(z,y) \geq 0, \quad \forall\, y \in C\}, \quad \forall x \in E. \tag{2}$$

Lemma 1.1. *[11] Let E be a reflexive Banach space and let C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (R-1)-(R-4) and let $Res_r^g : E \to C$ be resolvent defined by (2). Then the following statements hold:*

(a) Res_r^g is single-valued;

(b) $F(Res_r^g) = EP(g)$;

(c) $EP(g)$ is closed and convex;

(d) $D_f(p, Res_r^g x) + D_f(Res_r^g x, x) \leq D_f(p,x)$, $\forall\, p \in EP(g)$, $\forall\, x \in E$.

Lemma 1.2. *[4] Suppose that f is Gâteaux differentiable and totally convex on int dom$f$. Let $x \in$ int dom$f$ and let $C \subset$ int dom$f$ be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:*

(i) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(\hat{x}), \hat{x} - y\rangle \geq 0, \quad \forall\, y \in C,$$

(i) The vector $\hat{x}$ is the unique solution of the inequality

$$D_f(y,\hat{x}) + D_f(\hat{x},x) \leq D_f(y,x), \quad \forall\, y \in C,$$

(iii) The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f, i.e., $\hat{x} = P_C^f(x)$.

Lemma 1.3. *[3] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 1.4. *[3] Suppose $x \in E$ and $y \in$ int dom$f$. If f is essentially strictly convex, then $D_f(x,y) = 0 \Leftrightarrow x = y$. Function f is sequentially consistent if and only if f is totally convex on bounded sets.*

Lemma 1.5. *[3] Let $f : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of domf^*=E^* if and only if f is strongly coercive and uniformly convex on bounded subsets of E.*

Lemma 1.6. *Let $f : E \to R$ be a Legendre function which is uniformly Fréchet differentiable and bounded on subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T : C \to CB(C)$ be a multi-valued Bregman quasi-strictly pseudocontractive mapping with respect to f. Then, for any $x \in C$, $u \in Tx$, $p \in F(T)$ and $k \in [0,1)$ the following hold:*

$$D_f(x,u) \leq \frac{1}{1-k}\langle x - p, \nabla f(x) - \nabla f(u)\rangle.$$

Proof. Let $x \in C$, $u \in Tx$, $p \in F(T)$ and $k \in [0,1)$, by the definition of T, we have

$$D_f(p,u) \leq D_f(p,x) + kD_f(x,u).$$

This implies that

$$D_f(p,x) + D_f(x,u) + \langle p - x, \nabla f(x) - \nabla f(u)\rangle \leq D_f(p,x) + kD_f(x,u).$$

Hence, one has

$$D_f(x,u) \leq \frac{1}{1-k}\langle x - p, \nabla f(x) - \nabla f(u)\rangle.$$

This completes the proof. □

Lemma 1.7. *Let $f : E \to R$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T : C \to CB(C)$ be a multi-valued Bregman quasi-strictly pseudocontractive mapping with respect to f. Then $F(T)$ is a convex and closed set.*

Proof. Let $x, y \in F(T)$ and $p = tx + (1-t)y$ for $t \in (0,1)$. For all $w \in Tp$, one has

$$D_f(p,w) \leq \frac{1}{1-k}\langle p - y, \nabla f(p) - \nabla f(w)\rangle \tag{3}$$

and

$$D_f(p,w) \leq \frac{1}{1-k}\langle p - x, \nabla f(p) - \nabla f(w)\rangle \tag{4}$$

respectively. Multiplying (3) by $(1-t)$ and (4) by t, we have

$$D_f(p,w) \leq \frac{1}{1-k}\langle p - p, \nabla f(p) - \nabla f(w)\rangle,$$

which implies $D_f(p,w) = 0$. From Lemma 1.4, we have $p = w$, that is, $F(T)$ is convex.

Next, we show that $F(T)$ is closed. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $F(T)$ such that $x_n \to x^*$ as $n \to \infty$. We prove that $x^* \in F(T)$. In fact, for all $u \in Tx^*$, we have

$$D_f(x^*,u) \leq \frac{1}{1-\kappa}\langle x^* - x_n, \nabla f(x^*) - \nabla f(u)\rangle, \tag{5}$$

which implies $D_f(x^*,u) = 0$ by taking limit $n \to \infty$ in (5). Using Lemma 1.4 we obtain $x^* = u$, that is, $x^* \in F(T)$. So $F(T)$ is closed. This completes the proof. □

2. Main results

In this section, we state and prove our main theorem.

Theorem 2.1. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\prod$ be a index set. Let $T_i : C \to CB(C)$ be a closed and multi-valued Bregman quasi-strict pseudocontraction with fixed points. Let g_i be a bifunction with (R-1), (R-2), (R-3) and (R-4) for each $i \in \prod$. Assume that $\Omega := \cap_{i\in\prod}F(T_i) \bigcap \cap_{i\in\prod}EP(g_i) \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\prod} C_{1,i}, \\ x_1 = P^f_{C_1}(x_0), \\ y_{n,i} = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(z_{n,i})], \quad z_{n,i} \in T_i x_n, \\ r_{n,i} g_i(u_{n,i}, y) + \langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i})\rangle \geq 0, \quad \forall\ y \in C, \\ C_{n+1,i} = \{z \in C_{n,i} : D_f(z,u_{n,i}) \leq D_f(z,y_{n,i}) \leq D_f(z,x_n) + \frac{\kappa}{1-\kappa}\langle x_n - z, \nabla f(x_n) - \nabla f(z_{n,i})\rangle\}, \\ C_{n+1} = \cap_{i\in\prod} C_{n+i,i}, \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \end{cases} \tag{6}$$

where $\kappa \in [0,1)$, $\liminf_{n\to\infty} r_{n,i} > 0$, for $\forall i \in \prod$. Then $\{x_n\}$ converges strongly to $\widehat{p} = P^f_\Omega(x_1)$, where P^f_Ω is the Bregman projection from E onto Ω.

Proof. From Lemma 1.1 and Lemma 1.7, we see that $F(T)\cap EP(g)$ is convex and closed. Hence $P^f_{F(T)\cap EP(g)}(x_1)$ is well defined. Next, we prove that C_n is also convex and closed. It suffices to show that, for each fixed but arbitrary $i \in \prod$, $C_{n,i}$ is a convex and closed set. It is obvious that $C_{1,i} = C$ is convex and closed. We now let $C_{m,i}$ is a convex and closed set for some $m \geq 1$. Letting z_1 and z_2 be two arbitrary points in $C_{m+1,i}$, we find that $z_1, z_2 \in C_{m,i}$. Set $z_{1,2} = \lambda z_1 + (1-\lambda)z_2$, where λ is a real number in $(0,1)$. Since f is convex, we find that

$$D_f(z_{1,2}, u_{m,i}) \leq D_f(z_{1,2}, y_{m,i}) \leq D_f(z_{1,2}, x_m) + \frac{\kappa}{1-\kappa}\langle x_m - z_{1,2}, \nabla f(x_m) - \nabla f(z_{m,i})\rangle$$

In view of $z_{1,2} \in C_{n,i}$, we obtain that $C_{n,i} \in C_{m+1,i}$. This proves that $C_{m+1,i}$ is a convex and closed set. Hence, $C_{n,i}$ is also a convex and closed set. This implies that $\cap_{i\in\prod} C_{n,i}$ is convex and closed. So, $P^f_\Omega(x_0)$ is well defined.

Next, we show that $\Omega \subset C_n$. $\Omega \subset C_1 = C$ is obvious. Let $\Omega \subset C_{m,i}$. Note that $u_m = Res^g_{r_m} y_m$. For any $w \in \Omega \subset C_{m,i}$, we derive that

$$\begin{aligned}
D_f(w, u_m) &= D_f(w, \nabla f^*[\alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(z_m)]) \\
&= f(w) - \langle w, \alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(z_m)\rangle \\
&\quad + f^*(\alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(z_m)) \\
&\leq \alpha_m f(w) - \alpha_m\langle w, \nabla f(x_m)\rangle + \alpha_m f^*(x_m) \\
&\quad + (1-\alpha_m) f(w) - (1-\alpha_m)\langle w, \nabla f(z_m)\rangle + (1-\alpha_m) f^*(\nabla f(z_m)) \\
&= (1-\alpha_m) D_f(w, z_m) + \alpha_m D_f(w, x_m) \\
&\leq (1-\alpha_m)[D_f(w, x_m) + k D_f(x_m, z_m)] + \alpha_m D_f(w, x_m) \\
&\leq \frac{(1-\alpha_m)k}{1-k}\langle x_m - w, \nabla f(x_m) - \nabla f(z_m)\rangle + D_f(w, x_m) \\
&\leq \frac{k\langle x_m - w, \nabla f(x_m) - \nabla f(z_m)\rangle}{1-k} + D_f(w, x_m),
\end{aligned}$$

that is, $w \in C_{m+1,i}$. This proves that $\Omega \subset C_{n,i}$, which further implies that $\Omega \subset C_n = \cap_{i\in\prod} C_{n,i}$. Using Lemma 1.2 yields that

$$\langle y - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \leq 0, \quad \forall\, y \in C_n,$$

It follows from $\Omega \subset C_n$ that

$$\langle w - x_n, \nabla f(x_1) - \nabla f(x_n)\rangle \leq 0, \quad \forall\, w \in \Omega. \tag{7}$$

From Lemma 1.2, one has

$$D_f(x_n, x_1) = D_f(P^f_{C_n}(x_1), x_1) \leq D_f(w, x_1) - D_f(w, P^f_{C_n}(x_1)) \leq D_f(w, x_1),$$

for each $w \in \Omega$. Therefore, $\{D_f(x_n, x_1)\}$ is bounded. An application of Lemma 1.3 yields that $\{x_n\}$ is a bounded sequence. In view of the fact that $x_{n+1} = P^f_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$, and Since $x_n = P^f_{C_n}(x_1)$, one has $D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1)$. This implies that $\{D_f(x_n, x_1)\}$ is a nondecreasing sequence. Therefore $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Since $\{x_n\}$ is a bounded sequence and space E is a reflexive space, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup \widehat{p}$. Since C_n is closed and convex, we find that $\widehat{p} \in C_n$. On the other hand, one has

$$D_f(x_{n_j}, x_1) \leq D_f(\widehat{p}, x_1), \quad \forall\, n_j \in \mathbb{N} \cup \{0\}. \tag{8}$$

On the other hand, one has

$$\begin{aligned}\liminf_{j\to\infty} D_f(x_{n_j}, x_1) &= \liminf_{j\to\infty}\{f(x_{n_j}) - f(x_1) - \langle \nabla f(x_1), x_{n_j} - x_1\rangle\}\\ &\geq f(\widehat{p}) - f(x_1) - \langle \nabla f(x_1), \widehat{p} - x_1\rangle\\ &= D_f(\widehat{p}, x_1).\end{aligned} \tag{9}$$

It follows from (8) and (9) that

$$D_f(\widehat{p}, x_1) \leq \liminf_{j\to\infty} D_f(x_{n_j}, x_1) \leq \limsup_{j\to\infty} D_f(x_{n_j}, x_1) \leq D_f(\widehat{p}, x_1).$$

Hence, $\lim_{j\to\infty} D_f(x_{n_j}, x_1) = D_f(\widehat{p}, x_1)$. Employing Lemma 1.2, one obtains that $D_f(\widehat{p}, x_{n_j}) \leq D_f(\widehat{p}, x_1) - D_f(x_{n_j}, x_1)$. Hence, $\lim_{j\to\infty} D_f(\widehat{p}, x_{n_j}) = 0$. Using Lemma 1.4 that $\lim_{j\to\infty} x_{n_j} = \widehat{p}$. Since $\{D_f(x_n, x_0)\}$ is a convergent sequence, one obtains that

$$\lim_{n\to\infty} D_f(x_n, x_1) = D_f(\widehat{p}, x_1). \tag{10}$$

Using Lemma 1.2, one has

$$D_f(\widehat{p}, x_n) \leq D_f(\widehat{p}, x_1) - D_f(x_n, x_1). \tag{11}$$

Letting $n \to \infty$ in (11), one finds from Lemma 1.4 that

$$\lim_{n\to\infty} x_n = \widehat{p}. \tag{12}$$

On the other hand, one has

$$D_f(x_{n+1}, u_{n,i}) \leq D_f(x_{n+1}, y_{n,i}) \leq D_f(x_{n+1}, x_n) + \frac{\kappa}{1-\kappa}\langle x_n - x_{n+1}, \nabla f(x_n) - \nabla f(z_{n,i})\rangle,$$

which together with (12) implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, u_{n,i}) = \lim_{n\to\infty} D_f(x_{n+1}, y_{n,i}) = 0.$$

Since f is totally convex on bounded subsets of E, and sequentially consistent, one sees that

$$\lim_{n\to\infty} \|x_{n+1} - y_{n,i}\| = 0, \quad \lim_{n\to\infty} \|x_{n+1} - u_{n,i}\| = 0. \tag{13}$$

From (12) and (13), one obtains that

$$\lim_{n\to\infty} \|x_n - y_{n,i}\| = 0, \quad \lim_{n\to\infty} \|x_n - u_{n,i}\| = 0. \tag{14}$$

Since ∇f is uniformly continuous on each bounded subset of E, one has

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_{n,i})\| = 0. \tag{15}$$

It follows that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_{n,i})\| = \lim_{n\to\infty} \frac{1}{1-\alpha_n}\|\nabla f(x_n) - \nabla f(y_{n,i})\| = 0. \tag{16}$$

Using Lemma 1.6, we find from (16) that $\lim_{n\to\infty} \|x_n - z_{n,i}\| = 0$. Therefore $\lim_{n\to\infty} z_{n,i} = \lim_{n\to\infty} x_n = \widehat{p}$. In view of $z_{n,i} \in T_i x_n$, and from the closedness of T_i, it follows $\widehat{p} \in F(T_i)$. Hence, $\widehat{p} \in \cap_{i\in\prod} F(T_i)$.

Next, we prove $\widehat{p} \in \cap_{i\in\prod} EP(g_i)$. Since $\|u_{n,i} - y_{n,i}\| \le \|x_n - y_{n,i}\| + \|u_{n,i} - x_n\|$, we find from (14), one obtains that $\lim_{n\to\infty} \|u_{n,i} - y_{n,i}\| = 0$. Since ∇f is uniformly norm-to-norm continuous on bounded subsets of E, one has

$$\lim_{n\to\infty} \frac{\|\nabla f(u_{n,i}) - \nabla f(y_{n,i})\|}{r_{n,i}} = 0, \tag{17}$$

which together with $u_n = Res^g_{r_{n,i}} y_{n,i}$ implies that

$$r_{n,i} g_i(u_{n,i}, y) + \langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i})\rangle \ge 0, \quad \forall\, y \in C.$$

Hence, one has

$$\|y - u_{n,i}\| \frac{\|\nabla f(u_{n,i}) - \nabla f(y_{n,i})\|}{r_{n,i}} \ge \frac{\langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i})\rangle}{r_{n,i}} \ge g_i(y, u_{n,i}), \quad \forall y \in C.$$

Using (17), one sees that $g_i(y, \widehat{p}) \le 0$, $\forall y \in C$. For $t_i \in (0, 1)$ and $y \in C$, letting $y_{t_i} = t_i y + (1 - t_i)\widehat{p}$, we have $g_i(y_{t_i}, \widehat{p}) \le 0$. Hence

$$0 = g_i(y_{t_i}, y_{t_i}) \le (1 - t_i) g_i(y_{t_i}, p) + t_i g_i(y_{t_i}, y) \le t_i g_i(y_{t_i}, y).$$

Dividing by t_i, one has $g_i(y_t, y) \ge 0$, $\forall y \in C$. Letting $t \downarrow 0$, one finds that $g_i(\widehat{p}, y) \ge 0$, $\forall y \in C$. Hence $\widehat{p} \in \cap_{i\in\prod} EP(g_i)$. This proves that $\widehat{p} \in \Omega$.

Finally, we take $n \to \infty$ in (7) and obtain that

$$\langle w - \widehat{p}, \nabla f(x_1) - \nabla f(x_n)\rangle \le 0, \quad \forall\, w \in \Omega.$$

Using Lemma 1.2, one has $\widehat{p} = P^f_\Omega(x_1)$. This completes the proof. □

For the class of multi-valued Bregman quasi-nonexpansive mappings, we find the following result easily.

Corollary 2.2. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\prod$ be a index set. Let $T_i : C \to CB(C)$ be a closed and multi-valued Bregman quasi-nonexpansive mapping with fixed points. Let g_i be a bifunction with (R-1), (R-2), (R-3) and (R-4) for each $i \in \prod$. Assume that $\Omega := \cap_{i\in\prod} F(T_i) \bigcap \cap_{i\in\prod} EP(g_i) \ne \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\prod} C_{1,i}, \\ x_1 = P^f_{C_1}(x_0), \\ y_{n,i} = \nabla f^*[\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(z_{n,i})], \quad z_{n,i} \in T_i x_n, \\ r_{n,i} g_i(u_{n,i}, y) + \langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i})\rangle \ge 0, \quad \forall\, y \in C, \\ C_{n+1,i} = \{z \in C_{n,i} : D_f(z, u_{n,i}) \le D_f(z, y_{n,i}) \le D_f(z, x_n)\}, \\ C_{n+1} = \cap_{i\in\prod} C_{n+i,i}, \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \end{cases}$$

where $\liminf_{n\to\infty} r_{n,i} > 0$, for $\forall i \in \prod$. Then $\{x_n\}$ converges strongly to $\widehat{p} = P^f_\Omega(x_1)$, where P^f_Ω is the Bregman projection of E onto Ω.

If $f(x) = \|x\|^2$, $\forall x \in E$, then the class of multi-valued Bregman quasi-strict pseudo-contractions is reduced to the class of multi-valued quasi-ϕ-strict pseudo-contractions. We have the following result.

Corollary 2.3. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\prod$ be a index set. Let $T_i : C \to CB(C)$ be a closed and multi-valued Bregman quasi-strict pseudocontraction with fixed points. Let g_i be a bifunction with (R-1), (R-2), (R-3) and (R-4) for each $i \in \prod$. Assume that $\Omega := \cap_{i\in\prod} F(T_i) \bigcap \cap_{i\in\prod} EP(g_i) \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\prod} C_{1,i}, \\ x_1 = P^f_{C_1}(x_0), \\ y_{n,i} = J^{-1}[\alpha_n J(x_n) + (1-\alpha_n) J(z_{n,i})], \quad z_{n,i} \in T_i x_n, \\ r_{n,i} g_i(u_{n,i}, y) + \langle y - u_{n,i}, J(u_{n,i}) - J(y_{n,i})\rangle \geq 0, \quad \forall\, y \in C, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(z, u_{n,i}) \leq \phi(z, y_{n,i}) \leq \phi(z, x_n) + \frac{\kappa}{1-\kappa}\langle x_n - z, J(x_n) - J(z_{n,i})\rangle\}, \\ C_{n+1} = \cap_{i\in\prod} C_{n+i,i}, \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \end{cases}$$

where $\kappa \in [0,1)$, $\liminf_{n\to\infty} r_{n,i} > 0$, for $\forall i \in \prod$. Then $\{x_n\}$ converges strongly to $\widehat{p} = P^f_\Omega(x_1)$, where P^f_Ω is the generalized projection of E onto Ω.

Let E be a real Banach space and let E^* be the dual space of E. Let C be nonempty closed and convex subset of E and let $A : C \subseteq E \to E^*$ be a nonlinear mapping. The variational inequality problem for mapping A and its domain C is to find $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall\, y \in C. \tag{18}$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Recall that a mapping $A : C \to E^*$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall\, x,\ y \in C.$$

A mapping $A : C \to E^*$ is said to be γ-inverse strongly monotone if there exists $\gamma > o$ such that

$$\langle Ax - Ay, x - y\rangle \geq \gamma\|Ax - Ay\|^2, \quad \forall\, x, y \in C.$$

Lemma 2.4. *Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function and let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a continuous monotone mapping. For $s > 0$ and $x \in E$, define a mapping $Res^f_s : E \to C$ as follows: for all $x \in E$,*

$$Res^A_s := \{z \in C : \langle \nabla f(z) - \nabla f(x), y - z\rangle + s\langle Az, y - z\rangle \geq 0, \quad \forall y \in C\}$$

Then the following hold:
(1) Res^A_s is single-valued;
(2) $F(Res^A_s) = VI(C, A)$;
(3) $D_f(p, Res^A_s x) + D_f(Res^A_s x, x) \leq D_f(p, x)$, for $p \in F(Res^A_s)$;
(4) $VI(C, A)$ is closed and convex.

Based on above lemma and Theorem 2.1, the following result is not hard to derive.

Corollary 2.5. *Let E be a real reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\prod$ be a index set. Let g_i be a bifunction with (R-1), (R-2), (R-3) and (R-4) for each $i \in \prod$. Let $A_i : C \to E^*$ be a continuous monotone mapping with a mapping $Res^{A_i}_{s_i} : E \to C$ is defined by*

$$Res^{A_i}_{s_i} := \{z \in C : s_i\langle A_i z, y - z\rangle + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \quad \forall y \in C\}, \quad \forall x \in E$$

Assume that $\Omega := \cap_{i\in\prod} EP(g_i) \bigcap \cap_{i\in\prod} VI(C, A_i) \neq \emptyset$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\prod} C_{1,i}, \\ x_1 = P^f_{C_1}(x_0), \\ y_{n,i} = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res^{A_i}_{s_i} x_n)], \\ r_{n,i} g_i(u_{n,i}, y) + \langle y - u_{n,i}, \nabla f(u_{n,i}) - \nabla f(y_{n,i})\rangle \geq 0, \quad \forall\, y \in C, \\ C_{n+1,i} = \{z \in C_{n,i} : D_f(z, u_{n,i}) \leq D_f(z, y_{n,i}) \leq D_f(z, x_n)\}, \\ C_{n+1} = \cap_{i\in\prod} C_{n+i,i}, \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \end{cases}$$

where $\{s_i\}$ *is a sequence of positive real numbers,* $\liminf_{n\to\infty} r_{n,i} > 0$, *for* $\forall i \in \prod$. *Then* $\{x_n\}$ *converges strongly to* $\widehat{p} = P^f_\Omega(x_1)$, *where* P^f_Ω *is the Bregman projection of E onto* Ω.

Remark 2.6. *In this paper, we proposed a monotone Bregan projection algorithm for solving equilibrium problems and common fixed point problems of a family of closed multi-valued Bregman quasi-strict pseudocontractions. Our algorithm is strongly convergent wihtout any compact assumption. It deserve mentioning that our algorithm is valid for a family of uncountable many bifunctions and quasi-strict pseudocontractions in the framework of reflexive Banach spaces.*

References

[1] A. Abkar, M. Shekarbaigi, N. Aghamohammadi, *A new iterative algorithm for solving a system of generalized equilibrium problems,* J. Nonlinear Funct. Anal. 2018 (2018), Article ID 46.

[2] N.T. An, N.M. Nam, X. Qin, *Solving k-center problems involving sets based on optimization techniques,* J. Global Optim. 76 (2020), 189-209.

[3] H.H. Bauschke, J.M. Borwein, P.L. Combettes, *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces,* Commun. Contemp. Math. 3 (2001), 615-664.

[4] D. Butnariu, E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces,* Abstr. Appl. Anal. 2006 (2006), Art. ID 84919.

[5] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Student 63 (1994), 123-145.

[6] L.C. Ceng, Q.H. Ansari, J.C. Yao, *Some iterative methods for finding fixed points and for solving constrained convex minimization problems,* Nonlinear Aanl. 74 (2011), 5286-5302.

[7] L.C. Ceng, A. Petrusel, J.C. Yao, Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory 20 (2019), 113-134.

[8] S.S. Chang, C.F. Wen, J.C. Yao, *Zero point problem of accretive operators in Banach spaces,* Bull. Malaysian Math. Sci. Soc. 42 (2019), 105-118.

[9] S.S. Chang, C.F. Wen, J.C. Yao, *Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces,* Optimization 67 (2018), 1183-1196.

[10] B.A.B. Dehaish, *A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces,* J. Inequal. Appl. 2015 (2015), 51.

[11] B.A.B. Dehaish, *Weak and strong convergence of algorithms for the sum of two accretive operators with applications*, J. Nonlinear Convex Anal. 16 (2015), 1321-1336.

[12] X. Ju, S.A. Khan, *Well-posedness for bilevel vector equilibrium problems,* Appl. Set-Valued Anal. Optim. 1 (2019), 29-38.

[13] B.T. Kien, et al., *Second-order optimality conditions for multi-objective optimal control problems with mixed pointwise constraints and free right end points*, SIAM J. Control Optim. (2020), in press.

[14] L. Liu, X. Qin, *On the strong convergence of a projection-based algorithm in Hilbert spaces,* J. Appl. Anal. Comput. 10 (2020), 104-117.

[15] L.D. Muu, N.V. Quy, *DC-gap function and proximal methods for solving Nash-Cournot oligopolistic equilibrium models involving concave cost,* J. Appl. Numer. Optim. 1 (2019), 13-24.

[16] L.V. Nguyen, Q.H. Ansari, X. Qin, *Linear conditioning, weak sharpness and finite convergence for equilibrium problems,* J. Global Optim. 77 (2020), 405–424.

[17] E. Naraghirad, W. Takahashi, J.C. Yao, *Generalized retraction and fixed point theorems using Bregman functions in Banach spaces*, J. Nonlinear Convex Anal. 13 (2012), 141-156.

[18] F. U. Ogbuisi, Y. Shehu, *A projected subgradient-proximal method for split equality equilibrium problems of pseudomonotone bifunctions in Banach spaces*, J. Nonlinear Var. Anal. 3 (2019), 205–224.

[19] X. Qin, A. Petrusel, J.C. Yao, *CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces,* J. Nonlinear Convex Anal. 19 (2018), 157-165.

[20] X. Qin, N.T. An, *Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets,* Comput. Optim. Appl. 74 (2019), 821-850.

[21] X. Qin, J.C. Yao, *Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators,* J. Inequal. Appl. 2016 (2016), Article ID 232.

[22] X. Qin, J.C. Yao, *Projection splitting algorithms for nonself operators,* J. Nonlinear Convex Anal. 18 (2017), 925–935.

[23] X. Qin, L. Wang, J.C. Yao, *Inertial splitting method for maximal monotone mappings*, J. Nonlinear Convex Anal. 21 (2020), 2325–2333.

[24] S. Reich, S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces,* Numer. Funct. Anal. Optim. 31 (2010) 22-44.

[25] N. Shahzad, H. Zegeye, *Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings,* Fixed Point Theory Appl. 2014 (2014), Article ID 152.

[26] Y. Su, Z. Wang, H. Xu, *Strong convergence theorems for a common fixed point of two hemi-relatively nonexpansive mappings,* Nonlinear Anal. 71 (2009), 5616-5628.

[27] N. Shahzad, H. Zegeye, *Convergence theorems of common solutions for fixed point, variational inequality and equilibrium problems,* J. Nonlinear Var. Anal. 3 (2019), 189-203.

[28] W. Takahahsi, J.C. Yao, *The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. 20 (2019), 173-195.

[29] W. Takahashi, C.F. Wen, J.C. Yao, *The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space,* Fixed Point Theory 19 (2018), 407-419.

[30] W. Takahahsi, K. Zembayashi, *Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings,* Fixed Point Theory Appl. 2008 (2008), 528476.

[31] Z. M. Wang, *Strong convergence theorems for Bregman quasi-strict pseudo-contractions in reflexive Banach spaces with applications,* Fixed Point Theory Appl. 2015 (2015), Article ID 91.

[32] Z. M. Wang, A. R. Wei, *Some results on a finite family of Bregman quasi-strict pseudo-contractions,* J. Nonlinear Sci. Appl. 10 (2017) 975-989.

[33] X. Zhao, K.F. Ng, C. Li, J.C. Yao, *Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems*, Appl. Math. Optim. 78 (2018), 613-641.

[34] C. Zhang, J. Li, B. Liu, *Strong convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces,* Comput. Math. Appl. 61 (2011), 262-276.