*A Multi-Boundary Stefan Problem and the Disappearance of Phases**

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1. Introduction. This paper is concerned with a Stefan free boundary problem in which the free boundaries may intersect. Physically it can be thought of as describing a one-dimensional system consisting of a piece of ice immersed in water. Depending on the initial conditions, the piece of ice can melt or the water can freeze at each ice-water interface. In particular the ice can, at some finite time T, melt away entirely.

Mathematically, the problem is formulated as follows: Given the data φ_1, φ_2, φ_3, b_1 and b_2, find six functions $s_j = s_j(t)$, $j = 1, 2$, and $u_i = u_i(x, t)$, $i = 1, 2, 3, 4$, such that the 6-tuple $(s_1, s_2, u_1, u_2, u_3, u_4)$ satisfies

$$(1.1)\quad \begin{cases} \kappa_1 \dfrac{\partial^2 u_1}{\partial x^2} - \dfrac{\partial u_1}{\partial t} = 0, & -\infty < x < s_1(t), \quad 0 < t \leqq T, \\ u_1(x, 0) = \varphi_1(x) \geqq 0, & -\infty < x \leqq b_1, \quad s_1(0) = b_1, \\ u_1(s_1(t), t) = 0, & 0 \leqq t \leqq T, \end{cases}$$

$$(1.2)\quad \begin{cases} \kappa_2 \dfrac{\partial^2 u_2}{\partial x^2} - \dfrac{\partial u_2}{\partial t} = 0, & s_1(t) < x < s_2(t), \quad 0 < t < T, \\ u_2(x, 0) = \varphi_2(x) \leqq 0, & b_1 \leqq x \leqq b_2, \quad b_1 < b_2, \quad s_2(0) = b_2, \\ u_2(s_1(t), t) = u_2(s_2(t), t) = 0, & 0 \leqq t \leqq T, \end{cases}$$

$$(1.3)\quad \begin{cases} \kappa_1 \dfrac{\partial^2 u_3}{\partial x^2} - \dfrac{\partial u_3}{\partial t} = 0, & s_2(t) < x < \infty, \quad 0 < t \leqq T, \\ u_3(x, 0) = \varphi_3(x) \geqq 0, & b_2 \leqq x < \infty, \\ u_3(s_2(t), t) = 0, & 0 \leqq t \leqq T, \end{cases}$$

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$$
(1.4)\qquad \begin{cases} \dot{s}_1(t) = -k_1 \dfrac{\partial u_1}{\partial x}(s_1(t), t) + k_2 \dfrac{\partial u_2}{\partial x}(s_1(t), t), & 0 < t < T, \\[2ex] \dot{s}_2(t) = -k_1 \dfrac{\partial u_3}{\partial x}(s_2(t), t) + k_2 \dfrac{\partial u_2}{\partial x}(s_2(t), t), & 0 < t < T, \end{cases}
$$

and u_4 is defined and satisfies

$$
(1.5)\qquad \begin{cases} \kappa_1 \dfrac{\partial^2 u_4}{\partial x^2} - \dfrac{\partial u_4}{\partial t} = 0, \qquad -\infty < x < \infty, \qquad T < t, \\[2ex] u_4(x, T) = \begin{cases} u_1(x, T), & -\infty < x \leqq s_1(T) = s_2(T), \\ u_2(x, T), & s_2(T) \leqq x < \infty, \end{cases} \end{cases}
$$

only if T is finite, where T is such that $s_1(T) = s_2(T)$ and $s_1(t) < s_2(t), 0 \leqq t < T$. Here κ_1 and κ_2 denote the respective diffusivities of the two phases, and k_1 and k_2 denote the respective conductivities of the two phases.

In this paper we prove a theorem on the global existence and uniqueness of the solution $(s_1, s_2, u_1, u_2, u_3, u_4)$ and show that the free boundaries s_1 and s_2 depend continuously and monotonically on the data $\varphi_1, \varphi_2, \varphi_3, b_1$ and b_2. The main tool used in our analysis is the maximum principle, both in its strong form [10] and in the form of the parabolic version of Hopf's lemma [4]. As in a previous work of two of the authors [2], the constructive element in our approach is based on the idea of retarding the argument (*cf.* equation (3.12)) in the free boundary conditions in (1.4). We also investigate the behavior of $s_2(t) - s_1(t)$ at $t = T$ and relate the critical time T with the initial energy in the water-ice-water system.

There is an extensive literature on the one boundary problem. For example, see [1, 2, 3, 5, 6, 7, 8, 9]. However, the literature on the multi-boundary problem seems to be sparse. The only general treatment that we have found is that of Oleinik [11].

We require the following assumptions (A) on the Stefan data: φ_1, φ_2, and φ_3 are continuous except possibly for a finite number of bounded jumps; there exist positive constants $K_i, \eta_i, i = 1, 2$, such that

$$
(1.6)\qquad 0 \leqq \varphi_1(x) \leqq K_1(1 - \exp\{\kappa_1^{-1}\eta_1(x - b_1)\}), \qquad -\infty < x \leqq b_1,
$$

$$
(1.7)\qquad -\min K_2(1 - \exp\{-\kappa_2^{-1}\eta_2(x - b_1)\}), \\ K_2(1 - \exp\{\kappa_2^{-1}\eta_2(x - b_2)\}) \leqq \varphi_2(x) \leqq 0, \qquad b_1 \leqq x \leqq b_2,
$$

$$
(1.8)\qquad 0 \leqq \varphi_3(x) \leqq K_1(1 - \exp\{-\kappa_1^{-1}\eta_1(x - b_2)\}), \qquad b_2 \leqq x < \infty,
$$

and moreover, that

$$
(1.9)\qquad k_1\kappa_1^{-1}K_1 + k_2\kappa_2^{-1}K_2 < 1.
$$

Note that the $\eta_i, i = 1, 2$, which are fixed but of any size, describe the local Lipschitz behavior of the data $\varphi_i, i = 1, 2, 3$, at b_1 and b_2.

It would be desirable to remove the restriction (1.9) which is crucial for our existence proof. We emphasize, however, that the qualitative features of the physical problem (*cf.* sections 5, 6, 7) do not depend upon restriction (1.9). What is needed is a more sophisticated procedure for the global estimation of the derivatives $\dot{s}_1$ and $\dot{s}_2$. Our procedure says, in effect, that $\dot{s}_1$ and $\dot{s}_2$ can be estimated globally by linear techniques if the initial data is small enough.

If we were to consider the Stefan problem analogous to (1.1), $\cdots$, (1.5) for n boundaries, then with assumptions like (1.6), $\cdots$, (1.9), it will become clear to the reader that the method of demonstrating the existence of the solution of (1.1), $\cdots$, (1.5) is applicable to demonstrating the existence of the solution of the n-boundary Stefan problem. In the opinion of the authors, the assumption (1.9) is not satisfying enough mathematically to justify the morass of notation necessary to handle the disappearance of a multitude of phases. Consequently, we shall consider the two boundary problem (1.1), $\cdots$, (1.5) in detail and shall note from time to time the extensions to the n-boundary case.

By a solution (u_1, u_2, u_3, u_4) of (1.1), (1.2), (1.3), and (1.5) for given contionuous s_1 and s_2, we mean that

(1) $\partial^2 u_i/\partial x^2$, $\partial u_i/\partial t$, $i = 1, 2, 3, 4$, are continuous in their respective domains of definition.

(2) u_i, $i = 1, 2, 3, 4$, are continuous and bounded on the closure of their respective domains of definition except at points of discontinuity of the data φ_1, φ_2 and φ_3.

(3) u_i, $i = 1, 2, 3, 4$, satisfy respectively, (1.1), (1.2), (1.3), and (1.5).

It is well known that if s_1 and s_2 are Lipschitz continuous, then u_i, $i = 1, 2, 3, 4$, exist and are unique.

By a solution $(s_1, s_2, u_1, u_2, u_3, u_4)$ of (1.1), $\cdots$, (1.5), we mean that $s_1(0) = b_1$, $s_2(0) = b_2$, s_1 and s_2 are continuously differentiable for $0 < t < T$ and continuous for $0 \leqq t \leqq T$, that u_i, $i = 1, 2, 3, 4$, are the respective corresponding solutions of (1.1), (1.2), (1.3), and (1.5), and that $\partial u_1/\partial x(s_1(t), t)$, $\partial u_2/\partial x(s_1(t), t)$, $\partial u_2/\partial x(s_2(t), t)$, and $\partial u_3/\partial x(s_2(t), t)$ exist and satisfy (1.4).

2. Reformulation of the boundary conditions. Let $(s_1, s_2, u_1, u_2, u_3, u_4)$ be a solution of (1.1), $\cdots$, (1.5). Since

$$0 = \int_\sigma^t \int_{s_1(\tau)-N_1}^{s_1(\tau)} \left(\kappa_1 \frac{\partial^2 u_1}{\partial \xi^2} - \frac{\partial u_1}{\partial \tau}\right) d\xi \, d\tau \tag{2.1}$$

in the domain $\{s_1(\tau) - N_1 < \xi \leqq s_1(\tau), 0 < \sigma \leqq \tau \leqq t < T\}$, and

$$0 = \int_\sigma^t \int_{s_1(\tau)}^{s_3(\tau)} \left(\kappa_2 \frac{\partial^2 u_2}{\partial \xi^2} - \frac{\partial u_2}{\partial \tau}\right) d\xi \, d\tau \tag{2.2}$$

in the domain $\{s_1(\tau) \leqq \xi \leqq s_3(\tau), 0 < \sigma \leqq \tau \leqq t < T\}$, where $N_1 > 0$ and $s_3(\tau) = 2^{-1}(s_1(\tau) + s_2(\tau))$, it follows from (1.4) and integration by parts that

$$
\begin{aligned}
(2.3) \qquad s_1(t) = s_1(\sigma) &- k_1 \int_\sigma^t \frac{\partial u_1}{\partial \xi}(s_1(\tau) - N_1, \tau)\, d\tau \\
&+ k_2 \int_\sigma^t \frac{\partial u_2}{\partial \xi}(s_3(\tau), \tau)\, d\tau - k_1\kappa_1^{-1} \int_{s_1(t)-N_1}^{s_1(t)} u_1(\xi, t)\, d\xi \\
&- k_2\kappa_2^{-1} \int_{s_1(t)}^{s_3(t)} u_2(\xi, t)\, d\xi + k_1\kappa_1^{-1} \int_{s_1(\sigma)-N_1}^{s_1(\sigma)} u_1(\xi, \sigma)\, d\xi \\
&+ k_2\kappa_2^{-1} \int_{s_1(\sigma)}^{s_3(\sigma)} u_2(\xi, \sigma)\, d\xi - k_1\kappa_1^{-1} \int_\sigma^t u_1(s_1(\tau) - N_1, \tau)\, ds_1(\tau) \\
&+ k_2\kappa_2^{-1} \int_\sigma^t u_2(s_3(\tau), \tau)\, ds_3(\tau),
\end{aligned}
$$

where the last two integrals in (2.3) are Riemann–Stieltjes integrals. Likewise, it follows that

$$
\begin{aligned}
(2.4) \qquad s_2(t) = s_2(\sigma) &- k_1 \int_\sigma^t \frac{\partial u_3}{\partial \xi}(s_2(\tau) + N_2, \tau)\, d\tau \\
&+ k_2 \int_\sigma^t \frac{\partial u_2}{\partial \xi}(s_3(\tau), \tau)\, d\tau + k_1\kappa_1^{-1} \int_{s_2(t)}^{s_2(t)+N_2} u_3(\xi, t)\, d\xi \\
&+ k_2\kappa_2^{-1} \int_{s_3(t)}^{s_2(t)} u_2(\xi, t)\, d\xi - k_1\kappa_1^{-1} \int_{s_2(\sigma)}^{s_2(\sigma)+N_2} u_3(\xi, \sigma)\, d\xi \\
&- k_2\kappa_2^{-1} \int_{s_3(\sigma)}^{s_2(\sigma)} u_2(\xi, \sigma)\, d\xi - k_1\kappa_1^{-1} \int_\sigma^t u_3(s_2(\tau) + N_2, \tau)\, ds_2(\tau) \\
&+ k_2\kappa_2^{-1} \int_\sigma^t u_2(s_3(\tau), \tau)\, ds_3(\tau),
\end{aligned}
$$

where $N_2 > 0$ and the last two integrals in (2.4) are Riemann–Stieltjes integrals. Conversely, if $(s_1, s_2, u_1, u_2, u_3, u_4)$ satisfies (1.1), (1.2), (1.3), (1.5), (2.3), and (2.4) for all σ such that $0 < \sigma < T$ and if $\partial u_1/\partial x$, $\partial u_2/\partial x$, and $\partial u_3/\partial x$ exist and are continuous to the boundaries s_1 and s_2, then it follows from the analysis of the derivation of (2.3) and (2.4) that (1.4) is satisfied. Apparently, (2.3) and (2.4) are more general formulations of the boundary conditions in (1.4). However the appendix of [2] contains a proof of

Lemma 1. *Under assumption* (A), *let* (u_1, u_2, u_3) *denote the solution of* (1.1), (1.2), *and* (1.3) *and let* s_1 *and* s_2 *be Lipschitz continuous for* $0 \leqq t \leqq T$. *Then,* $\partial u_1/\partial x(s_1(t), t)$ *and* $\partial u_3/\partial x(s_2(t), t)$ *exist and are continuous for* $0 < t \leqq T$, *and* $\partial u_2/\partial x(s_1(t), t)$ *and* $\partial u_2/\partial x(s_2(t), t)$ *exist and are continuous for* $0 < t < T$.

Thus, within the class of Lipschitz continuous boundaries s_1 and s_2, the two formulations (1.1), $\cdots$, (1.5) and (1.1), (1.2), (1.3), (1.5), (2.3) and (2.4) for all σ such that $0 < \sigma < T$ are equivalent.

3. Existence. First, we need

Lemma 2. Under assumptions (A), *let* (u_1, u_2, u_3) *be a solution of* (1.1), (1.2), *and* (1.3), *where* s_1 *and* s_2 *are* C^1 *for* $0 < t < T$ *and*

$$\|\dot{s}_i\| = \sup_{0<t<T} |\dot{s}_i(t)| < \infty, \qquad i = 1, 2. \tag{3.1}$$

Then,

$$\left|\frac{\partial u_1}{\partial x}(s_1(t), t)\right| \leqq \kappa_1^{-1}K_1[\|\dot{s}_1\| + \eta_1], \qquad 0 < t \leqq T, \tag{3.2}$$

$$\left|\frac{\partial u_3}{\partial x}(s_2(t), t)\right| \leqq \kappa_1^{-1}K_1[\|\dot{s}_2\| + \eta_1], \qquad 0 < t \leqq T, \tag{3.3}$$

and

$$\left|\frac{\partial u_2}{\partial x}(s_i(t), t)\right| \leqq \kappa_2^{-1}K_2[\|\dot{s}_i\| + \eta_2], \qquad 0 < t < T, \qquad i = 1, 2. \tag{3.4}$$

Proof. Consider u_1. Set $\xi = x - s_1(t)$ and $U_1(\xi, t) = u_1(\xi + s_1(t), t)$. Then

$$LU_1 \equiv \kappa_1 \frac{\partial^2 U_1}{\partial \xi^2} + \dot{s}_1 \frac{\partial U_1}{\partial \xi} - \frac{\partial U}{\partial t} = 0, \qquad -\infty < \xi < 0, \qquad 0 < t < T,$$

$$\begin{aligned} U(\xi, 0) &= \varphi_1(\xi + b_1), \qquad -\infty < \xi < 0, \\ U(0, t) &= 0, \qquad 0 < t < T. \end{aligned} \tag{3.5}$$

Set

$$w(\xi, t) = K_1(1 - \exp\{\kappa_1^{-1}(\|\dot{s}_1\| + \eta_1)\xi\}). \tag{3.6}$$

Now, for $-\infty < \xi < 0, 0 < t < T$,

$$\begin{aligned} L(w - U_1) &= \kappa_1 \frac{\partial^2 w}{\partial \xi^2} + \dot{s}_1 \frac{\partial w}{\partial \xi} \\ &= -\kappa_1^{-1}K_1 \exp\{\kappa_1^{-1}(\|\dot{s}_1\| + \eta_1)\xi\}[(\|\dot{s}_1\| + \eta_1)^2 + (\|\dot{s}_1\| + \eta_1)\dot{s}_1] \\ &< -\kappa_1^{-1}K_1 \exp\{\kappa_1^{-1}(\|\dot{s}_1\| + \eta_1)\xi\}[\eta_1 \|\dot{s}_1\| + \eta_1^2] < 0. \end{aligned} \tag{3.7}$$

Also, for $t = 0, -\infty < \xi < 0$,

$$\begin{aligned} \varphi_1(\xi + b_1) = U_1(\xi, 0) &\leqq K_1(1 - \exp\{\kappa_1^{-1}\eta_1\xi\}) \\ &< K_1(1 - \exp\{\kappa_1^{-1}(\|\dot{s}_1\| + \eta_1)\xi\}) \\ &= w(\xi, 0), \end{aligned} \tag{3.8}$$

and

$$w(0, t) = U_1(0, t) = 0. \tag{3.9}$$

Hence, it follows from the maximum principle that

$$0 \leqq U_1(\xi, t) \leqq w(\xi, t). \tag{3.10}$$

Hence, from (3.9) and (3.10), it follows that

$$-\kappa_1^{-1}K_1[\|\dot{s}_1\| + \eta_1] \leqq \frac{\partial U_1}{\partial \xi}(0, t) \leqq 0 \tag{3.11}$$

which proves (3.2). The inequalities (3.3) and (3.4) follow from similar applications of the maximum principle. Q.E.D.

For each θ, $0 < \theta < (b_2 - b_1)/2$, we construct a family $(s_1^\theta, s_2^\theta, u_1^\theta, u_2^\theta, u_3^\theta, u_4^\theta)$ of approximations to the solution of (1.1), $\cdots$, (1.5) by retarding the argument

$$\begin{aligned}\dot{s}_1(t) &= -k_1 \frac{\partial u_1}{\partial x}(s_1(t-\theta), t-\theta) + k_2 \frac{\partial u_2}{\partial x}(s_1(t-\theta), t-\theta),\\ \dot{s}_2(t) &= -k_1 \frac{\partial u_3}{\partial x}(s_2(t-\theta), t-\theta) + k_2 \frac{\partial u_2}{\partial x}(s_2(t-\theta), t-\theta)\end{aligned} \tag{3.12}$$

in the free boundary conditions in (1.4). Let

$$\chi_1^\theta = \begin{cases} 1, & -\infty < x \leqq b_1 - \theta,\\ 0, & b_1 - \theta < x \leqq b_1, \end{cases}$$

$$\chi_2^\theta = \begin{cases} 0, & b_1 \leqq x \leqq b_1 + \theta,\\ 1, & b_1 + \theta \leqq x \leqq b_2 - \theta,\\ 0, & b_2 - \theta \leqq x \leqq b_2, \end{cases}$$

$$\chi_3^\theta = \begin{cases} 0, & b_2 \leqq x \leqq b_2 + \theta,\\ 1, & b_2 + \theta \leqq x < \infty, \end{cases}$$

and $\varphi_i^\theta = \chi_i^\theta \varphi_i$, $i = 1, 2, 3$. In the first interval $0 \leqq t \leqq \theta$, we set $s_1^\theta \equiv b_1$ and $s_2^\theta \equiv b_2$ and define $(u_1^\theta, u_2^\theta, u_3^\theta)$ to be the unique solution of (1.1), (1.2), and (1.3) in which φ_i, $i = 1, 2, 3$, s_j, $j = 1, 2$ have been replaced by φ_i^θ, $i = 1, 2, 3$, s_j^θ, $j = 1, 2$, respectively. Clearly, $\partial u_1^\theta/\partial x(b_1, t)$, $\partial u_2^\theta/\partial x(b_1, t)$, $\partial u_2^\theta/\partial x(b_2, t)$, and $\partial u_3^\theta/\partial x(b_2, t)$ exist and are continuous for $0 \leqq t \leqq \theta$. We proceed now by induction. Assume that $(s_1^\theta, s_2^\theta, u_1^\theta, u_2^\theta, u_3^\theta, u_4^\theta)$ has been constructed for $0 \leqq t \leqq n\theta$, that s_i^θ, $i = 1, 2$, are continuously differentiable, that $\partial u_1^\theta/\partial x(s_1^\theta(t), t)$, $\partial u_1^\theta/\partial x(s_2^\theta(t), t)$, $\partial u_2^\theta/\partial x(s_2^\theta(t), t)$, and $\partial u_3^\theta/\partial x(s_2^\theta(t), t)$ exist and are continuous and that

$$\begin{cases} s_1^\theta(t) = b_1 + \displaystyle\int_\theta^t \left\{-k_1 \frac{\partial u_1^\theta}{\partial x}(s_1^\theta(\tau-\theta), \tau-\theta) + k_2 \frac{\partial u_2^\theta}{\partial x}(s_1^\theta(\tau-\theta), \tau-\theta)\right\} d\tau,\\ s_2^\theta(t) = b_2 + \displaystyle\int_\theta^t \left\{-k_1 \frac{\partial u_3^\theta}{\partial x}(s_2^\theta(\tau-\theta), \tau-\theta) + k_2 \frac{\partial u_2^\theta}{\partial x}(s_2^\theta(\tau-\theta), \tau-\theta)\right\} d\tau. \end{cases} \tag{3.13}$$

In the next step $n\theta \leqq t \leqq (n+1)\theta$, we define s_i^θ, $i = 1, 2$, by (3.13) and solve (1.1), (1.2), and (1.3) for u_1^θ, u_2^θ, and u_3^θ for $n\theta \leqq t \leqq (n+1)\theta$. By the inductive hypothesis on $\partial u_1^\theta/\partial x$, $\partial u_2^\theta/\partial x$ and $\partial u_3^\theta/\partial x$ at the boundaries s_i^θ, $i = 1, 2$, it follows that s_1^θ and s_2^θ are continuously differentiable. Hence, by Lemma 1, $\partial u_1^\theta/\partial x(s_1^\theta(t), t)$, $\partial u_2^\theta/\partial x(s_1^\theta(t), t)$, $\partial u_2^\theta/\partial x(s_2^\theta(t), t)$, and $\partial u_3^\theta/\partial x(s_2^\theta(t), t)$ exist and are continuous for $n\theta \leqq t \leqq (n+1)\theta$. We continue in this way for all t or until we encounter a T_θ such that $s_1^\theta(T_\theta) = s_2^\theta(T_\theta)$ and $s_1^\theta(t) < s_2^\theta(t)$, $0 \leqq t < T_\theta$. If such a T_θ arises then we define

$$\text{(3.14)} \qquad \begin{cases} s_1^\theta(t) \equiv s_1^\theta(T_\theta), & T_\theta \leqq t < \infty, \\ s_2^\theta(t) \equiv s_2^\theta(T_\theta), & T_\theta \leqq t < \infty, \end{cases}$$

and solve (1.5) for u_4^θ, where in (1.5) T is replaced by T_θ, $u_1(x, T)$ is replaced by $u_1^\theta(x, T_\theta)$, and $u_3(x, T)$ is replaced by $u_3^\theta(x, T_\theta)$.

We summarize the results of the above construction in

Lemma 3. *For each θ, $0 < \theta < (b_2 - b_1)/2$, there exists a solution $(u_1^\theta, u_2^\theta, u_3^\theta)$ of (1.1), (1.2), and (1.3) in which $\varphi_i = \varphi_i^\theta$, $i = 1, 2, 3$, and $s_j = s_j^\theta$, $j = 1, 2$. The functions s_j^θ, $j = 1, 2$ are continuously differentiable for $\theta \leqq t < T_\theta$ and Lipschitz continuous for $0 \leqq t < \infty$. Moreover, the functions s_j^θ satisfy (3.12) and*

$$\text{(3.15)} \qquad |\dot{s}_j^\theta(t)| \leqq K_3, \qquad j = 1, 2, \qquad 0 \leqq t < T_\theta,$$

where

$$\text{(3.16)} \qquad K_3 = [1 - (k_1\kappa_1^{-1}K_1 + k_2\kappa_2^{-1}K_2)]^{-1}(k_1\kappa_1^{-1}K_1\eta_1 + k_2\kappa_2^{-1}K_2\eta_2).$$

Note that the Lipschitz constant for s_j^θ, $j = 1, 2$, is K_3 which is independent of θ.

Proof. The inequality (3.15) follows from Lemma 2, (3.12), and (1.9). The remainder has been done above.

Theorem 1. *Under the assumption* (A), *there exists a solution $(u_1, u_2, u_3, u_4, s_1, s_2)$ to the Stefan problem* (1.1), $\cdots$, (1.5). *The free boundaries s_1 and s_2 are continuously differentiable and satisfy*

$$|\dot{s}_j(t)| \leqq K_3 \quad \text{for} \quad 0 \leqq t < T,$$

where K_3 is defined by (3.16).

Proof. From (3.14) and (3.15), it follows that the functions $s_j^\theta(t)$ form an equicontinuous family, uniformly bounded on compact subsets. Pick a sequence of θ's tending to zero. By Ascoli–Arzela's theorem there is a subsequence, denote it by $s_j^\theta(t)$, that converges uniformly to a Lipschitz continuous function $s_j(t)$ on compact subsets of $0 \leqq t < \infty$. From the limit functions $s_1(t)$ and $s_2(t)$, define T as following (1.5). Next, solve (1.1), (1.2), (1.3), and (1.5) for (u_1, u_2, u_3, u_4) using the limit functions s_1 and s_2. Using (3.2), (3.3), (3.4), (3.15) and the maximum principle, it is easy to see that the corresponding subsequences u_1^θ, u_2^θ, u_3^θ, and u_4^θ converge subuniformly to u_1, u_2, u_3 and u_4, respectively.

Repeating the derivation of (2.3) for $0 < \theta < (b_2 - b_1)/2$, we find that

$$-k_1 \int_\sigma^t \frac{\partial u_1^\theta}{\partial \xi}(s_1^\theta(\tau), \tau)\, d\tau + k_2 \int_\sigma^t \frac{\partial u_2^\theta}{\partial \xi}(s_1^\theta(\tau), \tau)\, d\tau$$

$$= -k_1 \int_\sigma^t \frac{\partial u_1^\theta}{\partial \xi}(s_1^\theta(\tau) - N_1, \tau)\, d\tau + k_2 \int_\sigma^t \frac{\partial u_2^\theta}{\partial \xi}(s_3^\theta(\tau), \tau)\, d\tau$$

$$- k_1\kappa_1^{-1} \int_{s_1^\theta(t)-N_1}^{s_1^\theta(t)} u_1^\theta(\xi, t)\, d\xi - k_2\kappa_2^{-1} \int_{s_1^\theta(t)}^{s_3^\theta(t)} u_2^\theta(\xi, t)\, d\xi$$

$$+ k_1\kappa_1^{-1} \int_{s_1^\theta(\sigma)-N_1}^{s_1(\sigma)} u_1^\theta(\xi, \sigma)\, d\xi + k_2\kappa_2^{-1} \int_{s_1^\theta(\sigma)}^{s_3(\sigma)} u_2^\theta(\xi, \sigma)\, d\xi$$

$$- k_1\kappa_1^{-1} \int_\sigma^t u_1^\theta(s_1^\theta(\tau) - N_1, \tau)\, ds_1^\theta(\tau) + k_2\kappa_1^{-1} \int_\sigma^t u_2^\theta(s_3^\theta(\tau), \tau)\, ds_3^\theta(\tau).$$

From (3.12), it follows that

$$s_1^\theta(t + \theta) - s_1^\theta(\sigma + \theta) = -k_1 \int_\sigma^t \frac{\partial u_1^\theta}{\partial \xi}(s_1^\theta(\tau), \tau)\, d\tau + k_2 \int_\sigma^t \frac{\partial u_2^\theta}{\partial \xi}(s_1^\theta(\tau), \tau)\, d\tau.$$

Consequently, from the uniform convergence of the subsequences s_1^θ, s_2^θ, the subuniform convergence of the corresponding subsequences u_1^θ, u_2^θ, u_3^θ, and u_4^θ, the relations (3.2), (3.3), (3.4), and (3.15), and the maximum principle, it follows that the limit tuple $(s_1, s_2, u_1, u_2, u_3, u_4)$ satisfies (2.3). In a similar manner, it also satisfies (2.4). Hence, $(s_1, s_2, u_1, u_2, u_3, u_4)$ is a solution of the Stefan problem (1.1), (1.2), (1.3), (1.4), and (1.5).

4. Stability and uniqueness. Assume that the data $\varphi_1^i, \varphi_2^i, \varphi_3^i, b_1^i, b_2^i$, $i = 1, 2$, satisfy assumptions (A). Let $(s_1^i, s_2^i, u_1^i, u_2^i, u_3^i, u_4^i)$, $i = 1, 2$, denote corresponding solutions of the Stefan problem (1.1), $\cdots$, (1.5). Set

$$\Phi_i(x) = \begin{cases} \varphi_1^i(x), & -\infty < x \leqq b_1^i, \\ \varphi_2^i(x), & b_1^i \leqq x \leqq b_2^i, \\ \varphi_3^i(x), & b_2^i \leqq x < \infty, \qquad i = 1, 2. \end{cases} \tag{4.1}$$

Assume that Φ_i, $i = 1, 2$, are continuous and possess a continuous derivative with respect to x for all x except $x = b_1^i$ and $x = b_2^i$ and that there exists a constant K_4 such that

$$\left| \frac{d\Phi_i}{dx} \right| \leqq K_4, \qquad i = 1, 2, \qquad x \neq b_j^i. \tag{4.2}$$

Suppose that φ_1^i and φ_3^i, $i = 1, 2$ are such that

$$\int_{-\infty}^{\infty} \Phi_i(x)\, dx < \infty, \qquad i = 1, 2, \tag{4.3}$$

and that

$$\lim_{x\to-\infty} \frac{\partial u_1^i}{\partial x} = \lim_{x\to+\infty} \frac{\partial u_3^i}{\partial x} = 0 \tag{4.4}$$

uniformly for $0 \leqq t \leqq T^i$, $i = 1, 2$.

Theorem 2. *If* $|b_1^1 - b_1^2| \leqq 6^{-1}(b_2^1 - b_1^1)$, $|b_2^1 - b_2^2| \leqq 6^{-1}(b_2^1 - b_1^1)$, *and* $K_2 < \frac{1}{2}$, *then for* $0 \leqq t \leqq t_0 = \min((b_2^1 - b_1^1)/12K_3, T_1, T_2)$ *there exists a computable positive constant* $K_5 = K_5(t_0, K_1, K_2, K_3, K_4, \eta_1, \eta_2)$ *such that*

$$\begin{aligned} &|s_1^1(t) - s_1^2(t)| + |s_2^1(t) - s_2^2(t)| \\ &\quad \leqq K_5\Big\{|b_1^1 - b_1^2| + |b_2^1 - b_2^2| + \int_{-\infty}^{\infty} |\Phi_1(x) - \Phi_2(x)|\, dx \\ &\quad + \sup_{\max(b_1^1, b_1^2) \leqq x \leqq \min(b_2^1, b_2^2)} |\varphi_2^1(x) - \varphi_2^2(x)| \\ &\quad + \sup_{\max(b_1^1, b_1^2) \leqq x \leqq \min(b_2^1, b_2^2)} \left|\frac{d\varphi_2^1}{dx}(x) - \frac{d\varphi_2^2}{dx}(x)\right|\Big\}, \end{aligned} \tag{4.5}$$

where T_i, $i = 1, 2$, *are corresponding* T's *as defined following* (1.5).

Proof. Considering the relations (2.3) and (2.4) for $\sigma = 0$ and $N_1 = N_2 = \infty$, the proof, which we omit, is almost a repetition of the analysis given by two of the authors in [1] and summarized by two of the authors in [2].

Corollary. *Under the above assumptions, there can exist at most one solution* $(s_1, s_2, u_1, u_2, u_3, u_4)$ *to the Stefan problem* (1.1), $\cdots$, (1.5).

Remark. The appearance of the last two terms in (4.5), the hypothesis of differentiability of Φ_i, the hypothesis on K_2 and the conditions on b_1^i, b_2^i, $i = 1, 2$, resulted from the authors' method of estimating the differences

$$\int_0^t \frac{\partial u_2^1}{\partial \xi}(s_3^1(\tau), \tau)\, d\tau - \int_0^t \frac{\partial u_2^2}{\partial \xi}(s_3^2(\tau), \tau)\, d\tau$$

and

$$\int_0^t u_2^1(s_3^1(\tau), \tau)\, ds_3^1(\tau) - \int_0^t u_2^2(s_3^2(\tau), \tau)\, ds_3^2(\tau).$$

Note that uniqueness for the n-boundary Stefan problem has been demonstrated in a generalized sense by Oleinik [11].

5. Monotone dependence. Recalling (4.1), assume that Φ_i, $i = 1,2$ satisfy assumptions (A). Let $(s_1^i, s_2^i, u_1^i, u_2^i, u_3^i, u_4^i)$, $i = 1, 2$, denote the corresponding solution of the Stefan problem (1.1), $\cdots$, (1.5).

Theorem 3. *If* $b_1^1 \leqq b_1^2$, $b_2^2 \leqq b_2^1$, *and* $\Phi_1(x) \leqq \Phi_2(x)$, $-\infty < x < \infty$, *then*

$$s_1^1(t) \leqq s_1^2(t) \quad \text{and} \quad s_2^2(t) \leqq s_2^1(t) \tag{5.1}$$

for $0 \leqq t \leqq T_2$, *where* T_i, $i = 1, 2$ *are the corresponding* T's *as defined following*

(1.5). *Moreover,*

$$T_2 \leqq T_1 . \tag{5.2}$$

Proof. The proof, which is based upon the strong maximum principle and theorem 2, is analogous to that of theorem 6 [2].

6. The behavior of $s_2(t)-s_1(t)$ at $t = T$. Let $(s_1,s_2,u_1,u_2,u_3,u_4)$ denote a solution of (1.1), $\cdots$, (1.5).

Theorem 4. *If $\partial u_1/\partial x(s_1(t), t)$ and $\partial u_3/\partial x(s_2(t), t)$ are continuous for $0 < t \leqq T$, then there exist positive constants t_0 , q, and Q such that for $t_0 < t \leqq T$,*

$$q(T - t) \leqq s_2(t) - s_1(t) \leqq Q(T - t). \tag{6.1}$$

Proof. Considering (2.2) with t replaced by t_2 , σ replaced by t, and s_3 replaced by s_2 , it follows that

$$\kappa_2 \int_{t_1}^{t_2} \frac{\partial u_2}{\partial \xi} (s_2(\tau), \tau)\, d\tau - \kappa_2 \int_{t_1}^{t_2} \frac{\partial u_2}{\partial \xi} (s_1(\tau), \tau)\, d\tau = \int_{s_1(t_2)}^{s_2(t_2)} u_2(\xi, t_2)\, d\xi - \int_{s_1(t_1)}^{s_2(t_1)} u_2(\xi, t_1)\, d\xi. \tag{6.2}$$

By multiplying by $k_2\kappa_2^{-1}$ and adding the appropriate integrals of $\partial u_1/\partial x$ and $\partial u_3/\partial x$, we find that

$$\begin{aligned}[s_2(t_2) - s_1(t_2)] - [s_2(t_1) - s_1(t_1)] = &- k_1 \int_{t_1}^{t_2} \frac{\partial u_3}{\partial \xi} (s_2(\tau), \tau)\, d\tau + k_1 \int_{t_1}^{t_2} \frac{\partial u_1}{\partial \xi} (s_1(\tau), \tau)\, d\tau \\ &+ k_2\kappa_2^{-1} \int_{s_1(t_2)}^{s_2(t_2)} u_2(\xi, t_2)\, d\xi - k_2\kappa_2^{-1} \int_{s_1(t_1)}^{s_2(t_1)} u_2(\xi, t_1)\, d\xi.\end{aligned} \tag{6.3}$$

Consequently, setting $t_2 = T$ and $t_1 = t$ and multiplying (6.3) by -1, it follows that

$$\begin{aligned}s_2(t) - s_1(t) = k_1 \int_t^T \frac{\partial u_3}{\partial \xi} (s_2(\tau), \tau)\, d\tau \\ - k_1 \int_t^T \frac{\partial u_1}{\partial \xi} (s_1(\tau), \tau)\, d\tau + k_2\kappa_2^{-1} \int_{s_1(t)}^{s_2(t)} u_2(\xi, t)\, d\xi.\end{aligned} \tag{6.4}$$

By the strong maximum principle, it follows that $\partial u_3/\partial \xi(s_2(\tau), \tau) > 0$ and $\partial u_1/\partial \xi(s_1(\tau), \tau) < 0$, $0 < t_0 \leqq \tau \leqq T$. Hence, there exists a $q^* > 0$ such that

$$s_2(t) - s_1(t) \geqq 2k_1 q^*(T - t) + k_2\kappa_2^{-1} u_2(\xi^*, t)[s_2(t) - s_1(t)]. \tag{6.5}$$

Since $\lim_{t\uparrow T} u_2(\xi^*, t) = 0$, the left inequality of the result (6.1) is valid. The right inequality of (6.1) follows immediately from (6.4).

Remark. Under the assumptions (A) on the data φ_1 , φ_2 , φ_3 , b_1 and b_2 , the hypothesis of Theorem 4 is satisfied by the solution $(s_1, s_2, u_1, u_1, u_3, u_4)$ of the Stefan problem (1.1), $\cdots$, (1.5).

7. The relationship between the initial energy and T. Consider a solution $(s_1, s_2, u_1, u_2, u_3, u_4)$ of (1.1), $\cdots$, (1.5) for φ_1 and φ_3 which tend to zero sufficiently fast as $x \to -\infty$ and $x \to +\infty$ such that

$$(7.1) \qquad \int_{-\infty}^{b_1} \varphi_1(\xi)\, d\xi < \infty \quad \text{and} \quad \int_{b_2}^{\infty} \varphi_3(\xi)\, d\xi < \infty$$

and

$$(7.2) \qquad \lim_{x \to -\infty} \frac{\partial u_1}{\partial x}(x, t) = \lim_{x \to +\infty} \frac{\partial u_3}{\partial x}(x, t) = 0$$

uniformly for $0 < t \leqq T$. Now, let

$$(7.3) \qquad l(t) = s_2(t) - s_1(t)$$

and

$$(7.4) \qquad H(t) = k_1\kappa_1^{-1} \int_{-\infty}^{s_1(t)} u_1(\xi, t)\, d\xi + k_2\kappa_2^{-1} \int_{s_1(t)}^{s_2(t)} u_2(\xi, t)\, d\xi + k_1\kappa_1^{-1} \int_{s_2(t)}^{\infty} u_3(\xi, t)\, d\xi.$$

By considering the relations like (2.1) with $\sigma = 0$ and $N_1 = \infty$ and relations (6.2) and (6.3) with $t_2 = t$ and $t_1 = 0$, it follows that

$$(7.5) \qquad E(t) = E(0),$$

where

$$(7.6) \qquad E(t) = H(t) - l(t).$$

Note that (7.5) is simply a statement of the principle of conservation of heat energy.

Considering now the critical time of phase disappearance T, we prove the following theorem.

Theorem 5. *Suppose that*

$$(7.7) \qquad \lim_{t \to \infty} H(t) = 0.$$

If $E(0) > 0$, then T is finite. If $E(0) = 0$, then T is plus infinity and $\lim_{t\to\infty} l(t) = 0$. If $E(0) < 0$, then $\lim_{t\to\infty} l(t) = -E(0) > 0$ and there does not exist a finite T.

Proof. For $E(0) > 0$, it follows that $H(t) - l(t) = E(0) > 0$. Hence (7.7) implies that $l(t) < -\frac{1}{2}E(0) < 0$ for t sufficiently large. Thus, there must exist a $T < \infty$ such that $l(T) = 0$ and $l(t) > 0$ for $0 \leqq t < T$.

For $E(0) = 0$, it follows from (7.5) and (7.7) that $\lim_{t\to\infty} l(t) = 0$. Note that T must be infinite for if it is not, then the strong maximum principle would imply that $u_1 \equiv u_3 \equiv 0$ since

$$H(T) = k_1\kappa_1^{-1} \int_{-\infty}^{s_1(T)} u_1(\xi, T)\, d\xi + k_1\kappa_1^{-1} \int_{s_2(T)}^{\infty} u_3(\xi, T)\, d\xi = l(T) = 0. \tag{7.8}$$

For $E(0) < 0$, it follows that $\lim_{t\to\infty} l(t) = -E(0) > 0$. Note that there cannot exist a finite T such that $l(T) = 0$ since it would follow that $H(T) = E(0) < 0$ with $H(T)$ defined as in (7.8). Q.E.D.

The discussion of theorem 5 is really not complete until the hypothesis

$$\lim_{t\to\infty} H(t) = 0$$

has been related to φ_i, $i = 1, 2, 3$. By integrating the differential equations over the various regions, it follows from (1.1), $\cdots$, (1.5) and the maximum principle that $s_1(t)$ and $s_2(t)$ can be bounded in terms of $b_2 - b_1$ and integrals of the data φ_i, $i = 1, 2, 3$. Denote these bounds by σ_1 and σ_2; *i.e.*, for $0 \leqq t \leqq T$,

$$\sigma_1 \leqq s_1(t) < s_2(t) \leqq \sigma_2 . \tag{7.9}$$

Suppose now that $T = \infty$ and that φ_1, φ_2 and φ_3 are bounded and have compact support. Then it follows from the maximum principle and an elementary calculation that

$$\lim_{t\to\infty} H(t) = 0. \tag{7.10}$$

For example, in order to show that

$$\lim_{t\to\infty} \int_{s_2(t)}^{\infty} u_3(\xi, t)\, d\xi = 0 \tag{7.11}$$

it suffices to show that

$$\lim_{t\to\infty} \int_{\sigma_1}^{\infty} v_3(\xi, t)\, d\xi = 0, \tag{7.12}$$

where

$$\begin{aligned} &\kappa_1 \frac{\partial^2 v_3}{\partial x^2} - \frac{\partial v_3}{\partial t} = 0, \qquad \sigma_1 < x < \infty, \qquad 0 < t, \\ &v_3(x, 0) = \begin{cases} 0, & \sigma_1 \leqq x \leqq b_2 , \\ \varphi_3(x), & b_2 < x < \infty, \end{cases} \\ &v_3(\sigma_1, t) \equiv 0, \qquad 0 \leqq t. \end{aligned} \tag{7.13}$$

However, (7.12) follows from an elementary estimation of the representation of $v_3(x, t)$. Hence, we have shown

Lemma 4. *If $T = \infty$, and if φ_i, $i = 1, 2, 3$, are bounded and have compact support in $-\infty < x < \infty$, then*

$$\lim_{t \to \infty} H(t) = 0.$$

Remark. By setting the functions φ_1 and φ_3 equal to zero for sufficiently large $|x|$, it follows from theorem 3 that if $E(0) = +\infty$, then T is finite.

8. The physical significance of (1.9). Assume that in (1.1), ..., (1.5) we are considering a water-ice-water system for the gram-centimeter-second system of units. Then, the left hand sides of the equations in (1.4) must be multiplied by the heat of fusion times the density of water at 0°C. Consequently (1.9) should be replaced by essentially

$$c_1K_1 + c_2K_2 < \sim 80, \tag{8.1}$$

where c_1 and c_2 are the heat capacities of water and ice, respectively. Hence, (8.1) allows a temperature spread, $K_1 + K_2$, of about 80°C. Under normal pressures, such a temperature spread is certainly ample to cover the range of validity of the description of the water-ice-water system by (1.1), $\cdots$, (1.5).

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