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A MULTI-DIMENSIONAL BLOW-UP PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE IN $\mathbb{R}^N$

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Abstract. Let B be an N-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius R, and ∂B be its boundary. Also, let $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and let $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$, and 0 for $x \in \mathbb{R}^N \setminus B$. This article studies the following multi-dimensional semilinear parabolic problem with a concentrated nonlinear source on ∂B:

$$u_t - \triangle u = \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u) \text{ in } \mathbb{R}^N \times (0, T],$$
$$u(x,0) = \psi(x) \text{ for } x \in \mathbb{R}^N,\ u(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \le T,$$

where α and T are positive numbers, f and ψ are given functions such that $f(0) \ge 0$, $f(u)$ and $f'(u)$ are positive for $u > 0$, $f''(u) \ge 0$ for $u > 0$, and ψ is nontrivial on ∂B, nonnegative, and continuous such that $\psi \to 0$ as $|x| \to \infty$, $\int_{\mathbb{R}^N} \psi(x)\,dx < \infty$, and $\triangle\psi + \alpha(\partial\chi_B(x)/\partial\nu) f(\psi(x)) \ge 0$ in $\mathbb{R}^N$. It is shown that the problem has a unique nonnegative continuous solution before blowup occurs. We assume that $\psi(x) = M(0) > \psi(y)$ for $x \in \partial B$ and $y \notin \partial B$, where $M(t) = \sup_{x\in\mathbb{R}^N} u(x,t)$. It is proved that if u blows up in a finite time, then it blows up everywhere on ∂B. If, in addition, ψ is radially symmetric about the origin, then we show that if u blows up, then it blows up on ∂B only. Furthermore, if $f(u) \ge \kappa u^p$, where κ and p are positive constants such that $p > 1$, then it is proved that for any α, u always blows up in a finite time for $N \le 2$; for $N \ge 3$, it is shown that there exists a unique number α^* such that u exists globally for $\alpha \le \alpha^*$ and blows up in a finite time for $\alpha > \alpha^*$. A formula for computing α^* is given.

1. Introduction. Let $H = \partial/\partial t - \triangle$, T be a positive real number, $x = (x_1, x_2, \ldots, x_N)$ be a point in the N-dimensional Euclidean space $\mathbb{R}^N$, $\Omega = \mathbb{R}^N \times (0, T]$, B be an

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N-dimensional ball $\left\{x \in \mathbb{R}^N : \left|x - \bar{b}\right| < R\right\}$ centered at a given point $\bar{b}$ with a radius R, ∂B be the boundary of B, $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and

$$\chi_B(x) = \begin{cases} 1 \text{ for } x \in B, \\ 0 \text{ for } x \in \mathbb{R}^N \setminus B \end{cases}$$

be the characteristic function. Without loss of generality, let $\bar{b}$ be the origin. We would like to study the following multi-dimensional semilinear parabolic problem with a source on the surface of the ball:

$$\left.\begin{array}{c} Hu = \alpha \dfrac{\partial \chi_B(x)}{\partial \nu} f(u) \text{ in } \Omega, \\ u(x,0) = \psi(x) \text{ for } x \in \mathbb{R}^N,\ u(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \le T, \end{array}\right\} \tag{1.1}$$

where α is a positive constant, f and ψ are given functions such that $f(0) \ge 0$, $f(u)$ and $f'(u)$ are positive for $u > 0$, $f''(u) \ge 0$ for $u > 0$, and ψ is nontrivial on ∂B, nonnegative, and continuous such that $\psi \to 0$ as $|x| \to \infty$, $\int_{\mathbb{R}^N} \psi(x)\, dx < \infty$, and $\triangle\psi + \alpha\left(\partial\chi_B(x)/\partial\nu\right) f(\psi(x)) \ge 0$ in $\mathbb{R}^N$. We note that such a problem in a bounded domain, instead of $\mathbb{R}^N$, was studied by Chan and Tian ([2], [3]).

A solution u is said to blow up at the point (x, t_b) if there exists a sequence $\{(x_n, t_n)\}$ such that $u(x_n, t_n) \to \infty$ as $(x_n, t_n) \to (x, t_b)$.

In Section 2, we show that the nonlinear integral equation corresponding to the problem (1.1) has a unique nonnegative continuous solution u, which is a nondecreasing function of t. We then prove that u is the unique solution of the problem (1.1). Let $M(t)$ denote $\sup_{x \in \mathbb{R}^N} u(x,t)$. We assume that

$$\psi(x) = M(0) > \psi(y) \text{ for } x \in \partial B \text{ and } y \notin \partial B. \tag{1.2}$$

If t_b is finite, we show that u blows up everywhere on ∂B. If, in addition, ψ is radially symmetric about the origin, then we prove that if u blows up, then it blows up everywhere on ∂B only. Let κ and p be positive constants such that $p > 1$. If $f(u) \ge \kappa u^p$, then we prove, in Section 3, that for any α, u always blows up in a finite time if $N \le 2$. This behavior is completely different from that for $N \ge 3$. In Section 4, we show that for $N \ge 3$, there exists a unique number α^* such that u exists globally for $\alpha \le \alpha^*$ and blows up in a finite time for $\alpha > \alpha^*$. We also derive a formula for computing α^*. We note that whether a solution of the heat equation without a concentrated source in an unbounded domain blows up in a finite time was studied by Fujita [7], and Pinsky [8].

2. Existence, uniqueness, and blowup. To derive the integral equation from the problem (1.1), we use the adjoint operator $(-\partial/\partial t - \triangle)$ of H. Using Green's second identity, we obtain

$$u(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0)\,\psi(\xi)\, d\xi + \alpha \int_0^t \int_{\mathbb{R}^N} g(x,t;\xi,\tau)\, \frac{\partial \chi_B(\xi)}{\partial \nu} f(u(\xi,\tau))\, d\xi d\tau,$$

where

$$g(x,t;\xi,\tau) = \begin{cases} \dfrac{1}{\left[4\pi(t-\tau)\right]^{N/2}} \exp\left(-\dfrac{|x-\xi|^2}{4(t-\tau)}\right),\ t > \tau, \\ 0,\ t < \tau \end{cases} \tag{2.1}$$

is Green's function (cf. Stakgold [10, p. 198]) corresponding to the problem (1.1). Using integration by parts and the Divergence Theorem, we have

$$u(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0)\,\psi(\xi)\,d\xi + \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\,f(u(\xi,\tau))\,dS_\xi d\tau \tag{2.2}$$

(cf. Chan and Tragoonsirisak [4]). We note that $\int_{\mathbb{R}^N} \psi(x)\,dx < \infty$ is used to show that the first term on the right-hand side is continuous, and Lemma 2.1 of Chan and Tragoonsirisak [4] is used to show that the second term is continuous for $t > 0$.

To establish the next two results, we modify the techniques in proving Theorems 3 and 4 of Chan and Tian [2] for a blow-up problem in a bounded domain, and Theorems 2.1 and 2.2 of Chan and Tragoonsirisak [4] for a quenching problem in $\mathbb{R}^N$.

THEOREM 2.1. There exists some t_b such that for $0 \le t < t_b$, the integral equation (2.2) has a unique continuous nonnegative solution u, and u is a nondecreasing function of t. If t_b is finite, then u is unbounded in $[0, t_b)$.

Our next result shows that the solution of the integral equation (2.2) is the solution of the problem (1.1).

THEOREM 2.2. The problem (1.1) has a unique solution u for $0 \le t < t_b$.

Henceforth, we assume that (1.2) holds. Our next result shows that at t_b, u blows up everywhere on the surface of the ball.

THEOREM 2.3. If t_b is finite, then at t_b, u blows up everywhere on ∂B.

Proof. By Theorems 2.1 and 2.2, there exists some t_b such that for $0 \le t < t_b$, the problem (1.1) has a unique nonnegative continuous solution u, which is a nondecreasing function of t. Since $u(x,t)$ on $\partial B \times (0, t_b)$ is known, let us denote it by $\tilde{g}(x,t)$, and rewrite the problem (1.1) as two initial-boundary value problems:

$$\left.\begin{array}{c} Hu = 0 \text{ in } B \times (0, t_b), \\ u(x,0) = \psi(x) \text{ on } \bar{B},\ u(x,t) = \tilde{g}(x,t) \text{ on } \partial B \times (0, t_b); \end{array}\right\} \tag{2.3}$$

$$\left.\begin{array}{c} Hu = 0 \text{ in } \left(\mathbb{R}^N \backslash \bar{B}\right) \times (0, t_b), \\ u(x,0) = \psi(x) \text{ on } \mathbb{R}^N \setminus B,\ u(x,t) = \tilde{g}(x,t) \text{ on } \partial B \times (0, t_b), \\ u(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t < t_b. \end{array}\right\} \tag{2.4}$$

Let us consider the problem (2.3). From the strong maximum principle (cf. Friedman [6, p. 34]), u attains its maximum somewhere on ∂B for $t > 0$. For the problem (2.4), $u(x,t) \to 0$ as $|x| \to \infty$. Since u is a nondecreasing function of t, it follows from the Phragmén-Lindelöf Principle (cf. Protter and Weinberger [9, pp. 183-185]) that u attains its maximum somewhere on ∂B for $t > 0$. Thus for each given $\rho \in (0, t_b)$, u attains its maximum for $0 \le t \le \rho$ somewhere on $\partial B \times \{\rho\}$.

Suppose that there exists a smallest positive value of t, say t_1, and some $\bar{y} \notin \partial B$ such that $u(\bar{y}, t_1) = \min_{x \in \partial B} u(x, t_1)$. We claim that for $x \in \partial B, u(x, t_1) = u(\bar{y}, t_1)$. If this is not true, then there exists some $\bar{x} \in \partial B$ such that $u(\bar{x}, t_1) > \min_{x \in \partial B} u(x, t_1)$. Since u is continuous, there exists some point $(\tilde{y}, t_1)$ in a neighborhood of $(\bar{x}, t_1)$ such that $\tilde{y} \notin \partial B$ and $u(\tilde{y}, t_1) > \min_{x \in \partial B} u(x, t_1)$. This contradicts the definition of t_1. Thus, u attains its maximum at $(\bar{y}, t_1)$ for $0 \le t \le t_1$. If $\bar{y} \in B$, then it follows from the strong maximum principle and the continuity of u that $u \equiv u(\bar{y}, t_1)$ on $\bar{B} \times [0, t_1]$. This contradicts (1.2). If $\bar{y} \in (\mathbb{R}^N \setminus \bar{B})$, then let $\tilde{B}$ be an N-dimensional ball $\left\{x \in \mathbb{R}^N : |x| < \tilde{R}\right\}$ such that $\bar{y} \in \tilde{B}$. By the strong maximum principle and the continuity of u, $u \equiv u(\bar{y}, t_1)$ on $\left(\tilde{B} \backslash B\right) \times [0, t_1]$. Again, this contradicts (1.2). Thus for any $t > 0$,

$$u(x,t) > u(y,t) \text{ for any } x \in \partial B \text{ and any } y \notin \partial B. \tag{2.5}$$

We claim that for each $t > 0$, u attains the same value for $x \in \partial B$. If this is not true, then for some $t > 0$, there exists some $\tilde{x} \in \partial B$ such that $u(\tilde{x}, t) > \min_{x \in \partial B} u(x,t)$. By continuity, there exists some point $(\hat{y}, t)$ in a neighborhood of $(\tilde{x}, t)$ such that $\hat{y} \notin \partial B$ and $u(\hat{y}, t) > \min_{x \in \partial B} u(x,t)$. This contradicts (2.5). Hence for any $t > 0$,

$$u(x,t) = M(t) \text{ for } x \in \partial B,\ M(t) > u(y,t) \text{ for any } y \notin \partial B. \tag{2.6}$$

This implies that for each $t > 0$, u attains its absolute maximum on ∂B. Thus, if u blows up, then it blows up there. Since t_b is finite, it follows from Theorem 2.1 that u blows up everywhere on ∂B. □

Our next result shows that for the symmetric case, u blows up on ∂B only.

THEOREM 2.4. Under the additional assumption that ψ is radially symmetric about the origin, if t_b is finite, then at t_b, u blows up on ∂B only.

Proof. Let us construct a sequence $\{u_n\}$ in Ω by $u_0(x,t) = \psi(x)$, and for $n = 0, 1, 2, \ldots$,

$$H u_{n+1} = \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u_n) \text{ in } \Omega,$$
$$u_{n+1}(x,0) = \psi(x) \text{ for } x \in \mathbb{R}^N,\ u_{n+1}(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \le T.$$

From (2.2),

$$u_{n+1}(x,t) = \int_{\mathbb{R}^N} g(x,t;\xi,0)\,\psi(\xi)\,d\xi + \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\, f(u_n(\xi,\tau))\, dS_\xi d\tau. \tag{2.7}$$

We note that $\triangle\psi + \alpha(\partial\chi_B(x)/\partial\nu) f(\psi(x)) \ge 0$ in $\mathbb{R}^N$. Thus,

$$H(u_1 - u_0) \ge \alpha \frac{\partial \chi_B(x)}{\partial \nu}\left(f(u_0(x,t)) - f(\psi(x))\right) = 0 \text{ in } \Omega,$$
$$(u_1 - u_0)(x,0) = 0 \text{ for } x \in \mathbb{R}^N,\ (u_1 - u_0)(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \le T.$$

Since $g(x,t;\xi,\tau) > 0$ in $\left\{(x,t;\xi,\tau) : x \text{ and } \xi \text{ are in } \mathbb{R}^N, T \ge t > \tau \ge 0\right\}$, it follows from (2.2) that $u_1(x,t) \ge u_0(x,t)$ in Ω. Let us assume that for some positive integer j, $\psi \le u_1 \le u_2 \le u_3 \le \cdots \le u_{j-1} \le u_j$ in Ω. We have

$$H(u_{j+1} - u_j) = \alpha \frac{\partial \chi_B(x)}{\partial \nu}\left(f(u_j) - f(u_{j-1})\right) \text{ in } \Omega,$$
$$(u_{j+1} - u_j)(x,0) = 0 \text{ for } x \in \mathbb{R}^N,\ (u_{j+1} - u_j)(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \le T.$$

Since f is an increasing function and $u_j \ge u_{j-1}$, we have $f(u_j) - f(u_{j-1}) \ge 0$. It follows from (2.2) that $u_{j+1} \ge u_j$. By the principle of mathematical induction,

$$\psi \le u_1 \le u_2 \le \cdots \le u_{n-1} \le u_n \text{ in } \Omega.$$

Since u_n is an increasing sequence as n increases, it follows from the Monotone Convergence Theorem that we have (2.2) with $\lim_{n\to\infty} u_n(x,t) = u(x,t)$.

Since $\psi(x)$ is radially symmetric about the origin, namely $\psi(x) = \psi(|x|)$, it follows from (2.1) and the construction (2.7) that

$$\begin{aligned} u_1(x,t) &= \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-\xi|^2}{4t}\right) \psi(|\xi|)\, d\xi \\ &\quad + \alpha \int_0^t \frac{1}{[4\pi(t-\tau)]^{N/2}} \int_{\partial B} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right) f(\psi(|\xi|))\, dS_\xi d\tau \\ &= \frac{1}{(4\pi t)^{N/2}} \lim_{r\to\infty} \int_{B(0,r)} \exp\left(-\frac{|x-\xi|^2}{4t}\right) \psi(|\xi|)\, d\xi \\ &\quad + \alpha \int_0^t \frac{1}{[4\pi(t-\tau)]^{N/2}} \int_{\partial B} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right) f(\psi(|\xi|))\, dS_\xi d\tau, \end{aligned}$$

where $B(0,r)$ is the N-dimensional ball centered at the origin with a radius r. Thus, $u_1(x,t)$ is radially symmetric about the origin. We assume that for some positive integer j, $u_j(x,t)$ is radially symmetric about the origin, namely $u_j(x,t) = u_j(|x|,t)$. Then,

$$\begin{aligned} u_{j+1}(x,t) &= \frac{1}{(4\pi t)^{N/2}} \lim_{r\to\infty} \int_{B(0,r)} \exp\left(-\frac{|x-\xi|^2}{4t}\right) \psi(|\xi|)\, d\xi \\ &\quad + \alpha \int_0^t \frac{1}{[4\pi(t-\tau)]^{N/2}} \int_{\partial B} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right) f(u_j(|\xi|,\tau))\, dS_\xi d\tau \end{aligned}$$

is also radially symmetric about the origin. By the principle of mathematical induction, $u_n(x,t)$ is radially symmetric about the origin for $n = 0, 1, 2, \ldots$. Hence, $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ is radially symmetric about the origin.

From the problem (2.4), we have

$$\begin{gathered} u_t - \left(u_{rr} + \frac{N-1}{r} u_r\right) = 0 \text{ in } (R,\infty) \times (0,t_b), \\ u(r,0) = \psi(r) \text{ on } [R,\infty), \\ u(R,t) = M(t),\ u(r,t) \to 0 \text{ as } r \to \infty \text{ for } 0 < t < t_b. \end{gathered}$$

From Theorem 2.1, $u_t(x,t) \geq 0$ in $\left(\mathbb{R}^N \backslash \bar{B}\right) \times (0,t_b)$. Thus,

$$u_{rr} + \frac{N-1}{r} u_r = u_t \geq 0.$$

We note from (2.6) and the parabolic version of Hopf's lemma (cf. Friedman [6, p. 49]) that $u_r(R,t) < 0$ for $0 < t < t_b$. Hence for $0 < t < t_b$, $\lim_{r\to R^+} u_{rr}(r,t) \geq 0$ for $N \geq 1$. Therefore, if u blows up, then it blows up on ∂B only.

For the problem (2.3), it follows from Theorem 2.1 that $u_t(x,t) \geq 0$ in $B \times (0,t_b)$. By Corollary 2 of Friedman [6, p. 74], u is infinitely differentiable. Hence, $Hu_t = 0$ in $B \times (0,t_b)$. If $u_t = 0$ somewhere in $B \times (0,t_b)$, say at $t = t_2$, then it follows from the problem (2.3) and the strong maximum principle that $u_t \equiv 0$ in $B \times (0,t_2]$, and hence $u(x,t) = \psi(x)$ for $(x,t) \in B \times (0,t_2]$. By continuity, we have for $(x,t) \in \partial B \times [0,t_2]$, $u(x,t) = \psi(x) = M(0)$, which is bounded. Since the solution u is continuous on

$\partial B \times [0, t_b)$, there exists some $t_3\,(\geq t_2)$ such that $u_t > 0$ in $B \times [t_3, t_b)$. Because u is radially symmetric, we have

$$u_t - \left(u_{rr} + \frac{N-1}{r} u_r\right) = 0 \text{ in } (0, R) \times (0, t_b),$$
$$u(r, 0) = \psi(r) \text{ on } [0, R],$$
$$u_r(0, t) = 0,\, u(R, t) = M(t) \text{ for } 0 < t < t_b.$$

Thus,

$$u_{rr} + \frac{N-1}{r} u_r = u_t > 0 \tag{2.8}$$

in $B\times[t_3, t_b)$. Since $\lim_{r\to 0} u_{rr} + (N-1)\lim_{r\to 0}(u_r/r) = N u_{rr}(0,t)$, we have $u_{rr}(0,t) > 0$, implying that u is concave up near the origin $r = 0$. Because $u_r(0,t) = 0$ for $t_3 \leq t < t_b$, we have $u_r > 0$ near the origin for $t_3 \leq t < t_b$. We would like to show that $u(0,t)$ is bounded as t tends to t_b. Let us assume, on the contrary, that $u(0,t)$ tends to infinity as t tends to t_b. If $u_t(0,t)$ is bounded, say by a constant k_1, then

$$u(0,t) \leq u(0,0) + k_1 t \text{ for } 0 < t < t_b.$$

Because $u(0,0)$ is bounded, we have a contradiction. Thus, $u_t(0,t)$ tends to infinity as t tends to t_b. Since $u_t(0,t) = N u_{rr}(0,t)$, we have $u_{rr}(0,t)$ tending to infinity as t tends to t_b. Thus for $t_3 \leq t < t_b$, there are points in a neighborhood of the origin $r = 0$ with values larger than $u(0,t)$, and hence, u should blow up before t_b. This contradicts the definition of t_b. Hence, $u(0,t)$ is bounded as t tends to t_b. Next, we would like to show that the graph of u is concave up near ∂B. Since $u(r,t)$ tends to infinity as r tends to R and t tends to t_b, and u is a strictly increasing function of $t \in [t_3, t_b)$, we have for any given number M_1 sufficiently large, that there exists $\tilde{r}$ sufficiently close to R and some $\tilde{t}$ such that $u(r,t) > M_1$ for $r \in [\tilde{r}, R]$ and $t \in [\tilde{t}, t_b)$. We claim that for any given large number M_2, we can choose $\tilde{r}$ and $\tilde{t}$ such that $u_t(r,t) > M_2$ for $r \in [\tilde{r}, R]$ and $t \in [\tilde{t}, t_b)$. To prove this, let us assume that $u_t(r,t)$ is bounded, say by a constant M_2. Then, $u(r,t) \leq u(r,0) + M_2 t$. We note that for $M_1 > u(R,0) + M_2 t_b$, we have $\tilde{r}$ sufficiently close to R and some $\tilde{t}$ such that $u(r,t) > M_1$ for $r \in [\tilde{r}, R]$ and $t \in [\tilde{t}, t_b)$. Thus,

$$u(r,t) \leq u(r,0) + M_2 t \leq u(R,0) + M_2 t_b < M_1$$

for $r \in [\tilde{r}, R]$ and $t \in [\tilde{t}, t_b)$. We have a contradiction. Hence, $u_t(r,t)$ can be made as large as we please. By choosing r and t sufficiently close to R and t_b respectively, if $u_{rr}(r,t) \leq 0$, then it follows from (2.8) that $u_r(r,t)$ can be made as large as we please. This gives a contradiction to $u_{rr}(r,t) \leq 0$ since $u(r,t)$ can be made as large as we wish. Thus, u is concave up near ∂B. Because t_b is finite, it follows from Theorem 2.3 that u blows up on ∂B only. □

3. $\boldsymbol{N \leq 2}$**.** In the sequel, we assume that $f(u) \geq \kappa u^p$, where κ and p are positive constants such that $p > 1$. Let

$$I(x,t) = \int_{\partial B} g(x, t; \xi, 0)\, dS_\xi.$$

Lemma 3.1 of Chan and Tragoonsirisak [4] states that for $t \geq 1$ and any $x \in \bar{B}$,

$$(4\pi)^{-N/2} e^{-R^2} \omega_N R^{N-1} t^{-N/2} \leq I(x,t) \leq (4\pi)^{-N/2} \omega_N R^{N-1} t^{-N/2}, \tag{3.1}$$

where ω_N denotes the surface area of an N-dimensional unit sphere.

THEOREM 3.1. If $N \leq 2$, then for any α and any $\psi(x)$, the solution u of the problem (1.1) always blows up in a finite time.

Proof. Let

$$h(x) = \frac{e^{-|x|^2}}{\pi^{N/2}}.$$

We note that $h(x) > 0$, $h(x) \to 0$ as $|x| \to \infty$,

$$\begin{gathered}
\int_{\mathbb{R}^N} h(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{-x_1^2}}{\sqrt{\pi}} dx_1 \cdots \int_{-\infty}^{\infty} \frac{e^{-x_N^2}}{\sqrt{\pi}} dx_N = 1, \\
\int_{\partial B} h(x)\,dS_x = \frac{e^{-R^2}}{\pi^{N/2}} \int_{\partial B} dS_x = \frac{e^{-R^2}\omega_N R^{N-1}}{\pi^{N/2}}, \\
\int_{\bar{B}} h(x)\,dx < \int_{\mathbb{R}^N} h(x)\,dx = 1, \\
\triangle h = \frac{4e^{-|x|^2}|x|^2}{\pi^{N/2}} - 2Nh(x) \geq -2Nh(x).
\end{gathered} \tag{3.2}$$

Let

$$F(t) = \int_{\mathbb{R}^N} u(x,t)\,h(x)\,dx.$$

Since u is the solution of the problem (1.1), $F(t)$ may be regarded as a distribution. Thus,

$$\begin{aligned}
F'(t) &= \int_{\mathbb{R}^N} u_t(x,t)\,h(x)\,dx \\
&= \int_{\mathbb{R}^N} \left(\triangle u(x,t) + \alpha \frac{\partial \chi_B(x)}{\partial \nu} f(u(x,t)) \right) h(x)\,dx \\
&\geq \int_{\mathbb{R}^N} \triangle u(x,t)\,h(x)\,dx + \alpha\kappa \int_{\mathbb{R}^N} \frac{\partial \chi_B(x)}{\partial \nu} u^p(x,t)\,h(x)\,dx \\
&= \int_{\mathbb{R}^N} \triangle u(x,t)\,h(x)\,dx + \alpha\kappa \int_{\partial B} u^p(x,t)\,h(x)\,dS_x.
\end{aligned}$$

Using Green's second identity and (3.2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \triangle\, u(x,t)\, h(x)\, dx \\
&\quad = \lim_{\tilde{R}\to\infty} \int_{|x|<\tilde{R}} \triangle u(x,t)\, h(x)\, dx \\
&\quad = \lim_{\tilde{R}\to\infty} \int_{|x|<\tilde{R}} u(x,t) \triangle h(x)\, dx \\
&\quad = \int_{\mathbb{R}^N} u(x,t) \triangle h(x)\, dx \\
&\quad \geq -2N \int_{\mathbb{R}^N} u(x,t)\, h(x)\, dx \\
&\quad = -2NF(t)\,.
\end{aligned}$$

From (2.6),

$$F(t) \leq M(t) \int_{\mathbb{R}^N} h(x)\, dx = M(t)\,.$$

Thus,

$$\begin{aligned}
\int_{\partial B} u^p(x,t)\, h(x)\, dS_x &= M^p(t) \int_{\partial B} h(x)\, dS_x \\
&\geq F^p(t) \int_{\partial B} h(x)\, dS_x \\
&= \frac{e^{-R^2}\omega_N R^{N-1} F^p(t)}{\pi^{N/2}}.
\end{aligned}$$

Hence,

$$F'(t) + 2NF(t) \geq \frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{\pi^{N/2}} F^p(t)\,.$$

Solving this Bernoulli inequality, we obtain

$$F^{1-p}(t) \leq \frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{2N\pi^{N/2}} + Ce^{2N(p-1)t},$$

where C is to be determined. We can choose for $\tilde{t} \geq 0$,

$$C = \left(F^{1-p}\left(\tilde{t}\right) - \frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{2N\pi^{N/2}} \right) e^{2N(1-p)\tilde{t}}.$$

Thus for $t > \tilde{t} \geq 0$,

$$F^{p-1}(t) \geq \left[\frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{2N\pi^{N/2}} + \left(F^{1-p}\left(\tilde{t}\right) - \frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{2N\pi^{N/2}} \right) e^{2N(p-1)\left(t-\tilde{t}\right)} \right]^{-1}. \tag{3.3}$$

We would like to show that there exists $\tilde{t}$ such that

$$F^{1-p}\left(\tilde{t}\right) - \frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{2N\pi^{N/2}} < 0. \tag{3.4}$$

From (2.2) and (2.6),

$$\begin{aligned} u(x,t) &\geq \alpha\kappa \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\, u^p(\xi,\tau)\, dS_\xi d\tau \\ &= \alpha\kappa \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\, M^p(\tau)\, dS_\xi d\tau. \end{aligned}$$

For $t > 1$,

$$u(x,t) \geq \alpha\kappa \int_0^{t-1} M^p(\tau) \int_{\partial B} g(x,t;\xi,\tau)\, dS_\xi d\tau.$$

Since u is a nondecreasing function of t, we have $M^p(\tau) \geq M^p(0) > 0$. Thus,

$$\begin{aligned} u(x,t) &\geq \alpha\kappa M^p(0) \int_0^{t-1} \int_{\partial B} g(x,t;\xi,\tau)\, dS_\xi d\tau \\ &= \alpha\kappa M^p(0) \int_0^{t-1} I(x,t-\tau)\, d\tau \\ &= \alpha\kappa M^p(0) \int_1^t I(x,\theta)\, d\theta. \end{aligned}$$

Using (3.1), we have for any $x \in \bar{B}$,

$$\begin{aligned} u(x,t) &\geq \alpha\kappa M^p(0)\,(4\pi)^{-N/2} e^{-R^2} \omega_N R^{N-1} \int_1^t \theta^{-N/2} d\theta \\ &= \begin{cases} 2\alpha\kappa M^p(0)\,(4\pi)^{-N/2} e^{-R^2} \omega_N R^{N-1} \left(t^{1/2} - 1\right) \text{ if } N = 1, \\ \alpha\kappa M^p(0)\,(4\pi)^{-N/2} e^{-R^2} \omega_N R^{N-1} \ln t \text{ if } N = 2. \end{cases} \end{aligned}$$

Thus, there exists $\tilde{t}$ such that for $t \geq \tilde{t}$,

$$u(x,t) > \frac{\left(2N\pi^{N/2}\right)^{1/(p-1)}}{\alpha^{1/(p-1)} \left(\kappa e^{-R^2} \omega_N R^{N-1}\right)^{1/(p-1)} \left(\int_{\bar{B}} h(x)\, dx\right)}$$

for any $x \in \bar{B}$. Then,

$$\begin{aligned} F^{p-1}(\tilde{t}) &= \left(\int_{\mathbb{R}^N} u(x,\tilde{t})\, h(x)\, dx\right)^{p-1} \\ &\geq \left(\int_{\bar{B}} u(x,\tilde{t})\, h(x)\, dx\right)^{p-1} \\ &> \left[\frac{\left(2N\pi^{N/2}\right)^{1/(p-1)}}{\alpha^{1/(p-1)} \left(\kappa e^{-R^2} \omega_N R^{N-1}\right)^{1/(p-1)} \left(\int_{\bar{B}} h(x)\, dx\right)}\right]^{p-1} \left(\int_{\bar{B}} h(x)\, dx\right)^{p-1} \\ &= \frac{2N\pi^{N/2}}{\alpha\kappa e^{-R^2} \omega_N R^{N-1}}, \end{aligned}$$

which gives (3.4). From (3.3), there exists a finite time $t_b \left(> \tilde{t}\right)$ such that $\lim_{t \to t_b} F(t) = \infty$. Thus, $u(x,t)$ blows up in a finite time. □

4. $N \geq 3$. In this section, we show that the blow-up behavior for $N \geq 3$ is completely different from that for $N \leq 2$.

THEOREM 4.1. (i) For $N \geq 3$, if α is sufficiently small, then the solution u of the problem (1.1) exists globally.

(ii) For $N \geq 3$, if α is sufficiently large, then the solution u of the problem (1.1) blows up in a finite time.

Proof. (i) Since $f'(u) > 0$, we have $f(u) \leq f(2M(0))$ for $u(x,t) \leq 2M(0)$. Thus for $u(x,t) \leq 2M(0)$, it follows from (2.2) and $\int_{\mathbb{R}^N} g(x,t;\xi,0)\,d\xi = 1$ (cf. Evans [5, p. 46]) that

$$\begin{aligned} u(x,t) &\leq M(0) \int_{\mathbb{R}^N} g(x,t;\xi,0)\,d\xi + \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\, f(2M(0))\, dS_\xi d\tau \\ &= M(0) + \alpha f(2M(0)) \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\, dS_\xi d\tau. \end{aligned}$$

Let $\eta = (\xi_i - x_i)/(2\sqrt{t-\tau})$. Using $\int_{-\infty}^{\infty} e^{-\eta^2} d\eta = \sqrt{\pi}$, we have

$$\int_{\partial B} g(x,t;\xi,\tau)\, dS_\xi \leq \frac{1}{2\sqrt{\pi}\,(t-\tau)^{1/2}}.$$

For $0 < t \leq 1$, we have

$$\begin{aligned} u(x,t) &\leq M(0) + \frac{\alpha f(2M(0))}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{1/2}} \\ &\leq M(0) + \frac{\alpha f(2M(0))}{\sqrt{\pi}}. \end{aligned}$$

For $t > 1$, and for any $b \in \partial B$,

$$\begin{aligned} u(x,t) &\leq u(b,t) \\ &\leq M(0) + \alpha f(2M(0)) \left(\int_0^{t-1} \int_{\partial B} g(b,t;\xi,\tau)\, dS_\xi d\tau + \int_{t-1}^t \int_{\partial B} g(b,t;\xi,\tau)\, dS_\xi d\tau \right) \\ &\leq M(0) + \alpha f(2M(0)) \left(\int_0^{t-1} I(b,t-\tau)\, d\tau + \int_{t-1}^t \frac{d\tau}{2\sqrt{\pi}\,(t-\tau)^{1/2}} \right) \\ &= M(0) + \alpha f(2M(0)) \left(\int_1^t I(b,\theta)\, d\theta + \frac{1}{\sqrt{\pi}} \right) \\ &\leq M(0) + \alpha f(2M(0)) \left(\int_1^\infty I(b,\theta)\, d\theta + \frac{1}{\sqrt{\pi}} \right). \end{aligned} \tag{4.1}$$

Using (3.1), we have for $N \geq 3$,

$$\begin{aligned} \int_1^\infty I(b,\theta)\, d\theta &\leq (4\pi)^{-N/2} \omega_N R^{N-1} \int_1^\infty \theta^{-N/2} d\theta \\ &= \frac{(4\pi)^{-N/2} \omega_N R^{N-1}}{N/2 - 1} < \infty. \end{aligned}$$

Thus, we can choose $\alpha\,(> 0)$ sufficiently small such that the right-hand side of (4.1) is less than or equal to $2M(0)$. Hence, the solution u of the problem (1.1) exists globally.

(ii) Let $\tilde{t}=0$ in (3.3). We have

$$F^{p-1}(t)\geq\left[\frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{2N\pi^{N/2}}+\left(F^{1-p}(0)-\frac{\alpha\kappa e^{-R^2}\omega_N R^{N-1}}{2N\pi^{N/2}}\right)e^{2N(p-1)t}\right]^{-1}.$$

We note that $F(0)=\int_{\mathbb{R}^N}\psi(x)h(x)\,dx$. We would like to choose α sufficiently large such that $F^{1-p}(0)-\alpha\kappa e^{-R^2}\omega_N R^{N-1}/(2N\pi^{N/2})<0$. This can be accomplished by choosing

$$\alpha>\frac{2N\pi^{N/2}F^{1-p}(0)}{\kappa e^{-R^2}\omega_N R^{N-1}}.$$

Thus, there exists a finite time t_b such that $\lim_{t\to t_b}F(t)=\infty$ and hence $u(x,t)$ blows up in a finite time. □

Let k denote the positive constant $\int_{\mathbb{R}^N}\psi(\xi)\,d\xi$. Then,

$$\int_{\mathbb{R}^N}\exp\left(-\frac{|x-\xi|^2}{4t}\right)\psi(\xi)\,d\xi\leq\int_{\mathbb{R}^N}\psi(\xi)\,d\xi=k.$$

We have

$$\begin{aligned}\int_{\mathbb{R}^N}g(x,t;\xi,0)\psi(\xi)\,d\xi&=\frac{1}{(4\pi t)^{N/2}}\int_{\mathbb{R}^N}\exp\left(-\frac{|x-\xi|^2}{4t}\right)\psi(\xi)\,d\xi\\&\leq\frac{k}{(4\pi t)^{N/2}},\end{aligned}$$

which tends to 0 as $t\to\infty$. This shows that the initial data do not affect the solution as t tends to infinity. The fundamental solution (cf. Evans [5, pp. 22 and 615]) of the Laplace equation for $N\geq3$ is given by

$$G(x)=\frac{\Gamma\left(\frac{N}{2}+1\right)}{N(N-2)\pi^{N/2}}\frac{1}{|x|^{N-2}}.$$

The proof of the following result is the same as that of Theorem 4.2 of Chan and Tragoonsirisak [4].

THEOREM 4.2. If $u(x,t)\leq C$ for some positive constant C, then $u(x,t)$ converges from below to a solution $U(x)=\lim_{t\to\infty}u(x,t)$ of the nonlinear integral equation,

$$U(x)=\alpha\int_{\partial B}G(x-\xi)f(U(\xi))dS_\xi. \tag{4.2}$$

The next result shows that there exists a critical value for α.

THEOREM 4.3. For $N\geq3$, there exists a unique α^* such that u exists globally for $\alpha<\alpha^*$, and u blows up in a finite time for $\alpha>\alpha^*$.

Proof. To show that the larger the α, the larger the solution, let $\alpha>\beta$, and consider the sequence $\{v_n\}$ given by $v_0(x,t)=\psi(x)$, and for $n=0,1,2,\ldots$,

$$v_{n+1}(x,t)=\int_{\mathbb{R}^N}g(x,t;\xi,0)\psi(\xi)\,d\xi+\beta\int_0^t\int_{\partial B}g(x,t;\xi,\tau)f(v_n(\xi,\tau))\,dS_\xi d\tau.$$

Similar to the construction of the sequence $\{u_n\}$ in Ω in the proof of Theorem 2.4, we obtain

$$\begin{aligned} v(x,t) &= \lim_{n\to\infty} v_n(x,t) \\ &= \int_{\mathbb{R}^N} g(x,t;\xi,0)\,\psi(\xi)\,d\xi + \beta \int_0^t \int_{\partial B} g(x,t;\xi,\tau)\, f(v(\xi,\tau))\, dS_\xi d\tau. \end{aligned}$$

Since $u_n > v_n$ for $n = 1, 2, 3, \ldots$, we have $u \geq v$. Hence, the solution u is a nondecreasing function of α. It follows from Theorem 4.1 that there exists a unique α^* such that u exists globally for $\alpha < \alpha^*$ and u blows up in a finite time for $\alpha > \alpha^*$. □

We note that the critical value α^* is determined as the supremum of all positive values α for which a solution U of (4.2) exists. The proof of the next result (showing that the solution u exists globally when $\alpha = \alpha^*$) for the case $f(0) > 0$ is a modification of that for Theorem 7 of Chan and Jiang [1] for a degenerate one-dimensional problem in a bounded domain.

THEOREM 4.4. For $N \geq 3$,

$$\alpha^* = \frac{(N-2)\,\pi^{(N-3)/2}}{R\Gamma\left(\frac{N-1}{2}\right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi} \sup_{M(0)<s<\infty} \left(\frac{s}{f(s)}\right), \tag{4.3}$$

where for $N = 3$, $\prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi = 1$. Furthermore, u does not blow up in infinite time.

Proof. From (2.6), $U(x) = \lim_{t\to\infty} u(x,t)$ attains its maximum at $b \in \partial B$. From (4.2),

$$U(b) = \alpha \int_{\partial B} G(b-\xi)\, f(U(b))\, dS_\xi.$$

Thus,

$$\alpha = \left(\frac{1}{\int_{\partial B} G(b-\xi)\, dS_\xi}\right)\left(\frac{U(b)}{f(U(b))}\right),$$

and hence,

$$\alpha^* = \left(\frac{1}{\int_{\partial B} G(b-\xi)\, dS_\xi}\right) \sup_{M(0)<s<\infty}\left(\frac{s}{f(s)}\right).$$

From the proof of Theorem 4.5 of Chan and Tragoonsirisak [4],

$$\int_{\partial B} G(b-\xi)\, dS_\xi = \frac{R\Gamma\left(\frac{N-1}{2}\right) \prod_{i=1}^{N-3} \int_0^\pi \sin^i \varphi d\varphi}{(N-2)\,\pi^{(N-3)/2}}.$$

Thus, we have (4.3).

Let us consider the function $\varphi(s) = s/f(s)$.

CASE 1. If $f(0) = 0$, then we claim that $\varphi(s)$ is a decreasing function for $s > 0$. Since f is a convex function (cf. Stromberg [11, p. 199]) in $(0,\infty)$, we have for any $0 < s < s_2$,

$$f((1-t)s + ts_2) \leq (1-t)f(s) + tf(s_2),\ t \in [0,1].$$

Letting $s \to 0$, we have

$$f(ts_2) \leq tf(s_2).$$

Let $t = s_1/s_2$, where $0 < s_1 < s_2$. Then,

$$f(s_1) \leq \frac{s_1}{s_2} f(s_2),$$

which gives

$$\varphi(s_2) \leq \varphi(s_1),$$

implying that $\varphi(s)$ is a nonincreasing function of $s\,(> 0)$. It follows from (4.3) that

$$\alpha^* = \frac{(N-2)\,\pi^{(N-3)/2}}{R\Gamma\left(\frac{N-1}{2}\right)\prod_{i=1}^{N-3}\int_0^\pi \sin^i \varphi d\varphi}\left(\frac{M(0)}{f(M(0))}\right). \tag{4.4}$$

Case 2. If $f(0) > 0$, then $\varphi(s) > 0$ for $s > 0$, and $\varphi(0) = 0 = \lim_{s\to\infty}\varphi(s)$. We have $\varphi'(s) = (f(s) - sf'(s))/f^2(s)$. Therefore, a relative maximum or minimum occurs at $\tilde{s} \in (0,\infty)$, where $f(\tilde{s}) = \tilde{s}f'(\tilde{s})$. Since $\varphi''(\tilde{s}) = -\tilde{s}f''(\tilde{s})/f^2(\tilde{s}) < 0$, $\varphi(s)$ attains its absolute maximum when $\varphi(\tilde{s}) = 1/f'(\tilde{s})$. Thus, $\sup_{0<s<\infty}(s/f(s))$ occurs at $s = \tilde{s} \in (0,\infty)$. We note that the function $\varphi(s)$ is a strictly increasing function for $0 \leq s < \tilde{s}$, and a strictly decreasing function for $s > \tilde{s}$. Thus, if $M(0) < \tilde{s}$, then

$$\alpha^* = \frac{(N-2)\,\pi^{(N-3)/2}}{R\Gamma\left(\frac{N-1}{2}\right)\prod_{i=1}^{N-3}\int_0^\pi \sin^i \varphi d\varphi}\left(\frac{\tilde{s}}{f(\tilde{s})}\right). \tag{4.5}$$

If $M(0) \geq \tilde{s}$, then it follows from $\varphi(s)$ being a strictly decreasing function for $s > \tilde{s}$ that $\varphi(s)$ attains its supremum at $M(0)$. Thus,

$$\alpha^* = \frac{(N-2)\,\pi^{(N-3)/2}}{R\Gamma\left(\frac{N-1}{2}\right)\prod_{i=1}^{N-3}\int_0^\pi \sin^i \varphi d\varphi}\left(\frac{M(0)}{f(M(0))}\right). \tag{4.6}$$

From (4.4) to (4.6), α^* occurs at some finite positive value. Hence for $\alpha \leq \alpha^*$, u exists globally. Since u blows up in a finite time for $\alpha > \alpha^*$, u does not blow up in infinite time. □

For an illustration, we give below two examples on calculating α^* for some given functions f and some given initial data on the surface of the ball $M(0)$.

Example 4.5. Let $f(u) = u^p$. Since $f(0) = 0$, it follows from (4.4) that

$$\alpha^* = \frac{(N-2)\,\pi^{(N-3)/2}}{M^{p-1}(0)\,R\Gamma\left(\frac{N-1}{2}\right)\prod_{i=1}^{N-3}\int_0^\pi \sin^i \varphi d\varphi}.$$

EXAMPLE 4.6. Let $f(u) = (u+1)^p$. Since $f(0) > 0$, we have $\tilde{s} = 1/(p-1)$. From (4.5) and (4.6),

$$\alpha^* = \begin{cases} \dfrac{(p-1)^{p-1}(N-2)\pi^{(N-3)/2}}{p^p R\Gamma\left(\frac{N-1}{2}\right)\prod\limits_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi} & \text{if } M(0) < \dfrac{1}{p-1}, \\ \\ \dfrac{M(0)(N-2)\pi^{(N-3)/2}}{(M(0)+1)^p R\Gamma\left(\frac{N-1}{2}\right)\prod\limits_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi} & \text{if } M(0) \geq \dfrac{1}{p-1}. \end{cases}$$

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