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A MULTI-LEVEL CORRECTION SCHEME FOR EIGENVALUE PROBLEMS

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Abstract. In this paper, a type of multi-level correction scheme is proposed to solve eigenvalue problems by the finite element method. This type of multi-level correction method includes multi correction steps in a sequence of finite element spaces. In each correction step, we only need to solve a source problem on a finer finite element space and an eigenvalue problem on the coarsest finite element space. The accuracy of the eigenpair approximation can be improved after each correction step. This correction scheme can improve the efficiency of solving eigenvalue problems by the finite element method.

1. Introduction

The purpose of this paper is to propose a type of multi-level correction scheme based on the finite element discretization to solve eigenvalue problems. The two-grid method for solving eigenvalue problems has been proposed and analyzed by Xu and Zhou in [25]. The idea of the two-grid method comes from [23, 24] for nonsymmetric or indefinite problems and nonlinear elliptic equations. Since then, there have existed many numerical methods for solving eigenvalue problems based on the idea of the two-grid method (see, e.g., [1, 6, 10, 18, 19, 21]).

In this paper, we present a type of multi-level correction scheme to solve eigenvalue problems. With the proposed method, solving the eigenvalue problem will not be much more difficult than the solution of the corresponding source problem. Our method is some type of operator iterative method (see, e.g, [12, 25, 27]). The correction method for eigenvalue problems in this paper is based on a series of finite element spaces with different approximation properties which are related to the multi-level method (cf. [22]).

The standard Galerkin finite element method for eigenvalue problems has been extensively investigated, e.g., Babuška and Osborn [2, 3], Chatelin [5] and references cited therein. And many high efficient methods have also been proposed and analyzed for different types of eigenvalue problems (see, e.g., [6, 7, 11, 14, 15, 17–19, 21, 25]) based on the superconvergence theory (cf. [17–19, 21] and the "comosol guide"), the extrapolation technique (cf. [7, 11, 14, 15, 17]) and the two-grid method (cf. [6, 25]). Here we adopt some basic results in these papers to give the error

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estimates of the type of multi-level correction scheme that is introduced here. It will be shown that the convergence rate of the eigenpair approximations can be improved after each correction step.

The multi-level correction procedure can be described as follows: (1) solve the eigenvalue problem in the coarsest finite element space; (2) solve an additional source problem in an augmented space using the previously obtained eigenvalue multiplying its associated eigenfunction as the load vector; (3) solve the eigenvalue problem again on the finite element space that is constructed by combining the coarsest finite element space with the obtained eigenfunction approximation in step (2). Then go to step (2) for the next loop.

Similarly to [25], in order to describe our method clearly, we give the following simple Laplace eigenvalue problem to illustrate the main idea in this paper with the multi-grid implementation method (see Section 5).

Find(λ, u) such that

$$\left\{\begin{array}{rcll} -\Delta u &=& \lambda u, & \text{in } \Omega,\\ u &=& 0, & \text{on } \partial\Omega,\\ \int_\Omega u^2 d\Omega &=& 1, & \end{array}\right. \tag{1.1}$$

where $\Omega \subset \mathcal{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and Δ denotes the Laplace operator.

Let V_H denote the coarsest linear finite element space defined on the coarsest mesh $\mathcal{T}_H$. Additionally, we also need to construct a series of nested finite element spaces V_{h_2}, V_{h_3}, $\cdots$, V_{h_n} which are defined on the corresponding series of meshes $\mathcal{T}_{h_k}$ $(k = 2, 3, \cdots n)$ such that $V_H \subset V_{h_2} \subset \cdots \subset V_{h_n}$ (cf. [4, 8]). Our multi-level correction algorithm can be defined as follows (see Sections 3 and 4):

(1) Solve an eigenvalue problem in the coarsest space V_H:
Find $(\lambda_H, u_H) \in \mathcal{R} \times V_H$ such that $\|u_H\|_0 = 1$ and

$$\int_\Omega \nabla u_H \nabla v_H d\Omega \;=\; \lambda_H \int_\Omega u_H v_H d\Omega, \quad \forall v_H \in V_H.$$

(2) Set $h_1 = H$ and Do $k = 1, \cdots, n-2$:
- Solve the following auxiliary source problem:
Find $\widetilde{u}_{h_{k+1}} \in V_{h_{k+1}}$ such that

$$\int_\Omega \nabla \widetilde{u}_{h_{k+1}} \nabla v_{h_{k+1}} d\Omega \;=\; \lambda_{h_k} \int_\Omega u_{h_k} v_{h_{k+1}} d\Omega, \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}.$$

- Define a new finite element space $V_{H,h_{k+1}} = V_H + \text{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the following eigenvalue problem:
Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $\|u_{h_{k+1}}\|_0 = 1$ and

$$\int_\Omega \nabla u_{h_{k+1}} \nabla v_{H,h_{k+1}} d\Omega = \lambda_{h_{k+1}} \int_\Omega u_{h_{k+1}} v_{H,h_{k+1}} d\Omega, \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}.$$

end Do

(3) Solve the following auxiliary source problem:
Find $\widetilde{u}_{h_n} \in V_{h_n}$ such that

$$\int_\Omega \nabla u_{h_n} \nabla v_{h_n} d\Omega \;=\; \lambda_{h_{n-1}} \int_\Omega u_{h_{n-1}} v_{h_n} d\Omega, \quad \forall v_{h_n} \in V_{h_n}.$$

Then compute the Rayleigh quotient

$$\lambda_{h_n} = \frac{\|\nabla u_{h_n}\|_0^2}{\|u_{h_n}\|_0^2}.$$

If, for example, λ_H is the first eigenvalue of the problem at the first step and Ω is a convex domain, then we can establish the following results (see Sections 3 and 4 for details):

$$\|\nabla(u - u_{h_n})\|_0 = \mathcal{O}\Big(\sum_{k=1}^{n} h_k H^{n-k}\Big) \quad \text{and} \quad |\lambda_{h_n} - \lambda| = \mathcal{O}\Big(\sum_{k=1}^{n} h_k^2 H^{2(n-k)}\Big).$$

These two estimates means that we can obtain asymptotic optimal errors by taking $H = \sqrt[n]{h_n}$ and $h_k = H^k$ $(k = 1, \cdots, n-1)$. This result is different from the two-grid method [25] $(H = \sqrt{h_n})$ and the extended two-grid method [10] $(H = \sqrt[4]{h_n})$, since we can choose different n to control H under the condition of fixed h_n.

In this method, we replace solving the eigenvalue problem in the finest finite element space by solving a series of boundary value problems in a series of the nested finite element spaces and a series of eigenvalue problems in the coarsest finite element space. It is well known that there exists the multi-grid method that can solve boundary value problems efficiently. So this correction method can improve the efficiency of solving eigenvalue problems.

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for the eigenvalue problem. A type of one correction step is given in Section 3. In Section 4, we propose a type of multi-level correction algorithm to solve the eigenvalue problem by the finite element method. In Section 5, some numerical examples are presented to validate our theoretical analysis and some concluding remarks are given in the last section.

2. Discretization by finite element method

In this section, we introduce some notation and error estimates of the finite element approximation for eigenvalue problems. In this paper, the letter C (with or without subscripts) denotes a generic positive constant which may be different at different occurrences. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes.

Let $(V, \|\cdot\|)$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, respectively. Let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ be two symmetric bilinear forms on $V \times V$ satisfying

$$a(w,v) \lesssim \|w\|\|v\|, \quad \forall w \in V \text{ and } \forall v \in V, \tag{2.1}$$

$$\|w\|^2 \lesssim a(w,w), \quad \forall w \in V \tag{2.2}$$

and

$$0 < b(w,w), \quad \forall w \in V, \text{ and } w \neq 0. \tag{2.3}$$

From (2.1) and (2.2), we know that $\|\cdot\|_a := a(\cdot,\cdot)^{1/2}$ and $\|\cdot\|$ are two equivalent norms on V. We assume that the norm $\|\cdot\|$ is relatively compact with respect to the norm $\|\cdot\|_b := b(\cdot,\cdot)^{1/2}$ in the sense that any sequence which is bounded in $\|\cdot\|$, one can extract a subsequence which is Cauchy with respect to $\|\cdot\|_b$. We shall use $a(\cdot,\cdot)$ and $\|\cdot\|_a$ as the inner product and norm on V in the rest of this paper.

Set

$$W := \text{the completion of } V \text{ with respect to } \|\cdot\|_b.$$

Thus W is a Hilbert space with the inner product $b(\cdot,\cdot)$ and compactly imbedded in V. Construct a "negative space" by $V' =$ the dual of V with a norm $\|\cdot\|_{-a}$ given by

$$\|w\|_{-a} = \sup_{v\in V, \|v\|_a=1} b(w,v). \tag{2.4}$$

Then $W \subset V'$ compactly, and for $v \in V$, $b(w,v)$ has a continuous extension to $w \in V'$ such that $b(w,v)$ is continuous on V' by Hahn-Banach theorem (cf. [9]). We assume that $V_h \subset V$ is a family of finite-dimensional spaces that satisfy the following assumption:

For any $w \in V$,

$$\lim_{h\to 0} \inf_{v\in V_h} \|w-v\|_a = 0. \tag{2.5}$$

Let P_h be the finite element projection operator of V onto V_h defined by

$$a(w - P_h w, v) = 0, \quad \forall w \in V \text{ and } \forall v \in V_h. \tag{2.6}$$

Obviously,

$$\|P_h w\|_a \leq \|w\|_a, \quad \forall w \in V. \tag{2.7}$$

For any $w \in V$, by (2.5) we have

$$\|w - P_h w\|_a \;=\; o(1), \quad \text{as } h \to 0. \tag{2.8}$$

Define $\eta_a(h)$ as

$$\eta_a(h) = \sup_{f\in V, \|f\|_a=1} \inf_{v\in V_h} \|Tf - v\|_a, \tag{2.9}$$

where the operator $T : V' \mapsto V$ is defined as

$$a(Tf, v) \;=\; b(f,v), \quad \forall f \in V' \text{ and } \forall v \in V. \tag{2.10}$$

In order to derive the error estimate of eigenpair approximations in the negative norm $\|\cdot\|_{-a}$, we need the following negative norm error estimate of the finite element projection operator P_h.

Lemma 2.1 ([3, Lemma 3.3 and Lemma 3.4])**.**

$$\eta_a(h) \;=\; o(1), \quad \text{as } h \to 0 \tag{2.11}$$

and

$$\|w - P_h w\|_{-a} \;\lesssim\; \eta_a(h)\|w - P_h w\|_a, \quad \forall w \in V. \tag{2.12}$$

In our methodology description, we are concerned with the following general eigenvalue problem:

Find $(\lambda, u) \in \mathcal{R} \times V$ such that $b(u,u) = 1$ and

$$a(u,v) \;=\; \lambda b(u,v), \quad \forall v \in V. \tag{2.13}$$

For the eigenvalue λ, there exists the following Rayleigh quotient expression (see, e.g., [2, 3, 25])

$$\lambda = \frac{a(u,u)}{b(u,u)}. \tag{2.14}$$

From [3, 5], we know the eigenvalue problem (2.13) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity.

Now, let us define the finite element approximations of the problem (2.13). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we can construct a finite element space denoted by $V_h \subset V$. In order to do the multi-level correction method, we start the process on the original mesh $\mathcal{T}_H$ with the mesh size H and the original coarsest finite element space V_H defined on the mesh $\mathcal{T}_H$.

Then we define the approximation of eigenpair (λ, u) of (2.13) by the finite element method as:

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ such that $b(u_h, u_h) = 1$ and

$$a(u_h, v_h) = \lambda_h b(u_h, v_h), \qquad \forall v_h \in V_h. \tag{2.15}$$

From (2.15), the following Rayleigh quotient expression for λ_h holds (see, e.g., [2, 3, 25])

$$\lambda_h = \frac{a(u_h, u_h)}{b(u_h, u_h)}. \tag{2.16}$$

Similarly, we know from [3, 5] the eigenvalue problem (2.13) has eigenvalues

$$0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{k,h} \le \cdots \le \lambda_{N_h,h},$$

and the corresponding eigenfunctions

$$u_{1,h}, u_{2,h}, \cdots, u_{k,h}, \cdots, u_{N_h,h},$$

with $b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \le i, j \le N_h$ (N_h is the dimension of the finite element space V_h).

From the minimum-maximum principle (see, e.g., [2, 3]), the following upper bound result holds

$$\lambda_i \le \lambda_{i,h}, \quad i = 1, 2, \cdots, N_h.$$

Let $M(\lambda_i)$ denote the eigenfunction set corresponding to the eigenvalue λ_i which is defined by

$$\begin{aligned} M(\lambda_i) = \big\{ & w \in V : w \text{ is an eigenfunction of (2.13) corresponding to } \lambda_i \\ & \text{and } \|w\|_b = 1 \big\}. \end{aligned} \tag{2.17}$$

Then we define

$$\delta_h(\lambda_i) = \sup_{w\in M(\lambda_i)} \inf_{v\in V_h} \|w - v\|_a. \tag{2.18}$$

For the eigenpair approximations by the finite element method, there exist the following error estimates.

Proposition 2.1 ([2, Lemma 3.7, (3.29b)], [3, p. 699 and Section 9] and [5]). *(i) For any eigenfunction approximation $u_{i,h}$ of* (2.15) $(i = 1, 2, \cdots, N_h)$*, there is an eigenfunction u_i of* (2.13) *corresponding to λ_i such that $\|u_i\|_b = 1$ and*

$$\|u_i - u_{i,h}\|_a \quad \leq \quad C_i \delta_h(\lambda_i). \tag{2.19}$$

Furthermore,

$$\|u_i - u_{i,h}\|_{-a} \leq C_i \eta_a(h) \|u_i - u_{i,h}\|_a. \tag{2.20}$$

(ii) Assume $\lambda_{i,h}$ is the smallest eigenvalue approximation of the multiple eigenvalue λ_i and the corresponding eigenfunction approximation is $u_{i,h}$. Then there is an eigenfunction u_i such that $\|u_i\|_b = 1$ and

$$\|u_i - u_{i,h}\|_a \quad \leq \quad C_i \inf_{w \in M(\lambda_i)} \inf_{v \in V_h} \|w - v\|_a \tag{2.21}$$

and

$$\|u_i - u_{i,h}\|_{-a} \leq C_i \eta_a(h) \|u_i - u_{i,h}\|_a. \tag{2.22}$$

(iii) For each eigenvalue, we have

$$\lambda_i \leq \lambda_{i,h} \leq \lambda_i + C_i \delta_h^2(\lambda_i). \tag{2.23}$$

Here and hereafter, C_i is some constant depending on i but independent of the mesh size h.

3. One correction step

In this section, we present a type of correction step to improve the accuracy of the current eigenvalue and eigenfunction approximations. This correction method consists of solving an auxiliary source problem in the finer finite element space and an eigenvalue problem on the coarsest finite element space. For simplicity of notation, we set $(\lambda, u) = (\lambda_i, u_i)$ $(i = 1, 2, \cdots, k, \cdots)$ and $(\lambda_h, u_h) = (\lambda_{i,h}, u_{i,h})$ $(i = 1, 2, \cdots, N_h)$ to denote an eigenpair of problem (2.13) and (2.15), respectively.

To analyze our method, we introduce the error expansion of eigenvalue by the Rayleigh quotient formula which comes from [2, 3, 17, 18, 25].

Theorem 3.1. *Assume (λ, u) is the true solution of the eigenvalue problem* (2.13)*, $0 \neq \psi \in V$. Let us define*

$$\widehat{\lambda} = \frac{a(\psi, \psi)}{b(\psi, \psi)}. \tag{3.1}$$

Then we have

$$\widehat{\lambda} - \lambda \quad = \quad \frac{a(u - \psi, u - \psi)}{b(\psi, \psi)} - \lambda \frac{b(u - \psi, u - \psi)}{b(\psi, \psi)}. \tag{3.2}$$

Assume we have obtained an eigenpair approximation $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$. Now we introduce a type of correction step to improve the accuracy of the current eigenpair approximation (λ_{h_1}, u_{h_1}). Let $V_{h_2} \subset V$ be a finer finite element space such that $V_{h_1} \subset V_{h_2}$. Based on this finer finite element space, we define the following correction step.

Algorithm 3.1. *One Correction Step:*

(1) *Define the following auxiliary source problem:*
Find $\widetilde{u}_{h_2} \in V_{h_2}$ *such that*

$$a(\widetilde{u}_{h_2}, v_{h_2}) = \lambda_{h_1} b(u_{h_1}, v_{h_2}), \quad \forall v_{h_2} \in V_{h_2}. \tag{3.3}$$

Solve this equation to obtain a new eigenfunction approximation $\widetilde{u}_{h_2} \in V_{h_2}$.

(2) *Define a new finite element space* $V_{H,h_2} = V_H + \mathrm{span}\{\widetilde{u}_{h_2}\}$ *and solve the following eigenvalue problem:*
Find $(\lambda_{h_2}, u_{h_2}) \in \mathcal{R} \times V_{H,h_2}$ *such that* $b(u_{h_2}, u_{h_2}) = 1$ *and*

$$a(u_{h_2}, v_{H,h_2}) = \lambda_{h_2} b(u_{h_2}, v_{H,h_2}), \quad \forall v_{H,h_2} \in V_{H,h_2}. \tag{3.4}$$

The eigenpair approximation (λ_{h_2}, u_{h_2}) *obtained in Step (2) is the output of this algorithm which is denoted by* $(\lambda_{h_2}, u_{h_2}) = \mathit{Correction}(V_H, \lambda_{h_1}, u_{h_1}, V_{h_2})$.

Remark 3.2. If the concerned eigenvalue λ is multiple, we choose the smallest eigenvalue approximation of λ as λ_{h_2} and its associated eigenfunction approximation as u_{h_2} which has the biggest component in the direction of $\widetilde{u}_{h_2}$.

Remark 3.3 ([20]). We would like to consider the conditions of the eigenvalue problem (3.4). It is well known that the condition for eigenvalues of symmetric matrices is 1 (see [20, p. 71]). Since the condition for the eigenvector v corresponding to the eigenvalue λ of symmetric matrices is (see [20, p. 74])

$$\mathrm{cond}(v) = \frac{1}{\min_{\lambda_j \neq \lambda} |\lambda - \lambda_j|},$$

then the conditions of eigenvalue problem (3.4) in each correction step are almost the same as the eigenvalue problem defined on the coarsest finite element space. Furthermore, the obtained eigenpair approximations also give very good preliminary information for the next correction step. For example in the multi-space way of Example 6.1, the CPU time of eigenvalue solving is $0.063119s$ on the coarsest mesh and $0.014806s$ for the first correction when $H = 1/16$.

Theorem 3.4. *Assume there exists an exact eigenpair* (λ, u) *of* (2.13) *such that the current eigenpair approximation* $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$ *has the following error estimates:*

$$\|u - u_{h_1}\|_a \lesssim \varepsilon_{h_1}(\lambda), \tag{3.5}$$
$$\|u - u_{h_1}\|_{-a} \lesssim \eta_a(H)\|u - u_{h_1}\|_a, \tag{3.6}$$
$$|\lambda - \lambda_{h_1}| \lesssim \varepsilon_{h_1}^2(\lambda). \tag{3.7}$$

Then after one correction step, there exist an exact eigenpair $(\lambda, \widehat{u})$ *of* (2.13) *such that the resultant approximation* $(\lambda_{h_2}, u_{h_2}) \in \mathcal{R} \times V_{h_2}$ *has the following error estimates:*

$$\|\widehat{u} - u_{h_2}\|_a \lesssim \varepsilon_{h_2}(\lambda), \tag{3.8}$$
$$\|\widehat{u} - u_{h_2}\|_{-a} \lesssim \eta_a(H)\|u - u_{h_2}\|_a, \tag{3.9}$$
$$|\lambda - \lambda_{h_2}| \lesssim \varepsilon_{h_2}^2(\lambda), \tag{3.10}$$

where $\varepsilon_{h_2}(\lambda) := \eta_a(H)\varepsilon_{h_1}(\lambda) + \varepsilon_{h_1}^2(\lambda) + \delta_{h_2}(\lambda)$.

Proof. From problems (2.6), (2.13), and (3.3), and (3.5), (3.6), and (3.7), the following estimate holds:

$$\begin{aligned}\|\widetilde{u}_{h_2} - P_{h_2}u\|_a^2 &\lesssim a(\widetilde{u}_{h_2} - P_{h_2}u, \widetilde{u}_{h_2} - P_{h_2}u) = b(\lambda_{h_1}u_{h_1} - \lambda u, \widetilde{u}_{h_2} - P_{h_2}u)\\ &\lesssim \|\lambda_{h_1}u_{h_1} - \lambda u\|_{-a}\|\widetilde{u}_{h_2} - P_{h_2}u\|_a\\ &\lesssim (|\lambda_{h_1} - \lambda|\|u_{h_1}\|_{-a} + \lambda\|u_{h_1} - u\|_{-a})\|\widetilde{u}_{h_2} - P_{h_2}u\|_a\\ &\lesssim \big(\varepsilon_{h_1}^2(\lambda) + \eta_a(H)\varepsilon_{h_1}(\lambda)\big)\|\widetilde{u}_{h_2} - P_{h_2}u\|_a.\end{aligned}$$

Then we have

$$\|\widetilde{u}_{h_2} - P_{h_2}u\|_a \lesssim \varepsilon_{h_1}^2(\lambda) + \eta_a(H)\varepsilon_{h_1}(\lambda). \tag{3.11}$$

Combining (3.11) and the error estimate of finite element projection

$$\|u - P_{h_2}u\|_a \lesssim \delta_{h_2}(\lambda),$$

we have

$$\|\widetilde{u}_{h_2} - u\|_a \lesssim \varepsilon_{h_1}^2(\lambda) + \eta_a(H)\varepsilon_{h_1}(\lambda) + \delta_{h_2}(\lambda). \tag{3.12}$$

Now we come to estimate the error of the eigenpair solution (λ_{h_2}, u_{h_2}) of (3.4). Based on the error estimate theory of eigenvalue problem by finite element method (see, e.g., [2, 3]), (2.21)–(2.22) and Remark 3.2, there exists an eigenfunction $\widehat{u} \in M(\lambda)$ such that the following estimates hold (if λ is simple, we have $u = \widehat{u}$):

$$\|\widehat{u} - u_{h_2}\|_a \lesssim \inf_{w\in M(\lambda)} \inf_{v\in V_{H,h_2}} \|w - v\|_a \lesssim \|u - \widetilde{u}_{h_2}\|_a \tag{3.13}$$

and

$$\|\widehat{u} - u_{h_2}\|_{-a} \lesssim \widetilde{\eta}_a(H)\|u - u_{h_2}\|_a, \tag{3.14}$$

where

$$\widetilde{\eta}_a(H) = \sup_{f\in V, \|f\|_a=1} \inf_{v\in V_{H,h_2}} \|Tf - v\|_a \le \eta_a(H). \tag{3.15}$$

From (3.12), (3.13), (3.14), and (3.15), we can obtain (3.8) and (3.9). The estimate (3.10) can be derived by Theorem 3.1 and (3.8). □

Remark 3.5. The Algorithm 3.1 adopts the two-grid method to construct One Correction Step. In [10], the authors provide an extension of the two-grid method. We can also use this extended two-grid method to construct another One Correction Step by replacing the auxiliary source problem (3.3) by

$$a(\widetilde{u}_{h_2}, v_{h_2}) - \lambda_{h_1}b(\widetilde{u}_{h_2}, v_{h_2}) = \lambda_{h_1}b(u_{h_1}, v_{h_2}), \quad \forall v_{h_2} \in V_{h_2}. \tag{3.16}$$

Under the same conditions of Theorem 3.4 and the given eigenpair approximation (λ_{h_1}, u_{h_1}) has the error estimates (3.5), (3.6), and (3.7), the resultant approximation (λ_{h_2}, u_{h_2}) by the new correction step has the error estimates (3.8), (3.9), and (3.10) with $\varepsilon_{h_2}(\lambda) := \eta_a(H)\varepsilon_{h_1}^3(\lambda) + \delta_{h_2}(\lambda)$.

4. Multi-level correction scheme

In this section, we introduce a type of multi-level correction scheme based on the *One Correction Step* defined in Algorithm 3.1. This type of correction method can improve the convergence order after each correction step which is different from the two-grid methods in [10, 25].

Algorithm 4.1. *Multi-level Correction Scheme:*

(1) *Construct a coarse finite element space V_H and solve the following eigenvalue problem:*
Find $(\lambda_H, u_H) \in \mathcal{R} \times V_H$ such that $b(u_H, u_H) = 1$ and

$$a(u_H, v_H) = \lambda_H b(u_H, v_H), \quad \forall v_H \in V_H. \tag{4.1}$$

(2) *Set $h_1 = H$ and construct a series of nested finite element spaces $V_{h_2}, \cdots, V_{h_n}$ such that $\eta_a(H) \gtrsim \delta_{h_1}(\lambda) \geq \delta_{h_2}(\lambda) \geq \cdots \geq \delta_{h_n}(\lambda)$.*

(3) *Do $k = 1, \cdots, n-2$*
Obtain a new eigenpair approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$ by a correction step

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = Correction(V_H, \lambda_{h_k}, u_{h_k}, V_{h_{k+1}}). \tag{4.2}$$

end Do

(4) *Solve the following source problem:*
Find $u_{h_n} \in V_{h_n}$ such that

$$a(u_{h_n}, v_{h_n}) = \lambda_{h_{n-1}} b(v_{h_{n-1}}, v_{h_n}), \quad \forall v_{h_n} \in V_{h_n}. \tag{4.3}$$

Then compute the Rayleigh quotient of u_{h_n}:

$$\lambda_{h_n} = \frac{a(u_{h_n}, u_{h_n})}{b(u_{h_n}, u_{h_n})}. \tag{4.4}$$

Finally, we obtain an eigenpair approximation $(\lambda_{h_n}, u_{h_n}) \in \mathcal{R} \times V_{h_n}$.

Theorem 4.1. *After implementing Algorithm* 4.1, *there exists an eigenpair (λ, u) of* (2.13) *such that the resultant eigenpair approximation (λ_{h_n}, u_{h_n}) has the following error estimates:*

$$\|u_{h_n} - u\|_a \lesssim \varepsilon_{h_n}(\lambda), \tag{4.5}$$

$$|\lambda_{h_n} - \lambda| \lesssim \varepsilon_{h_n}^2(\lambda), \tag{4.6}$$

where $\varepsilon_{h_n}(\lambda) = \sum_{k=1}^{n} \eta_a(H)^{n-k}\delta_{h_k}(\lambda)$.

Proof. From $\eta_a(H) \gtrsim \delta_{h_1}(\lambda) \geq \delta_{h_2}(\lambda) \geq \cdots \geq \delta_{h_n}(\lambda)$ and Theorem 3.4, we have

$$\varepsilon_{h_{k+1}}(\lambda) \lesssim \eta_a(H)\varepsilon_{h_k}(\lambda) + \delta_{h_{k+1}}(\lambda), \quad \text{for } 1 \leq k \leq n-2. \tag{4.7}$$

Then by recursive relation, there exists an eigenfunction $u \in M(\lambda)$ of (2.13) such that the following estimate holds:

$$\begin{aligned} \|u_{h_{n-1}} - u\|_a &\lesssim \varepsilon_{h_{n-1}}(\lambda) \lesssim \eta_a(H)\varepsilon_{h_{n-2}}(\lambda) + \delta_{h_{n-1}}(\lambda) \\ &\lesssim \eta_a(H)^2\varepsilon_{h_{n-3}}(\lambda) + \eta_a(H)\delta_{h_{n-2}}(\lambda) + \delta_{h_{n-1}}(\lambda) \\ &\lesssim \sum_{k=1}^{n-1} \eta_a(H)^{n-k-1}\delta_{h_k}(\lambda) \end{aligned} \tag{4.8}$$

and

$$\|u_{h_{n-1}} - u\|_{-a} \lesssim \eta_a(H)\|u_{h_{n-1}} - u\|_a. \tag{4.9}$$

Based on the proof in Theorem 3.4 and (4.8)-(4.9), the final eigenfunction approximation u_{h_n} has the error estimate

$$\begin{aligned} \|u_{h_n} - u\|_a &\lesssim \varepsilon^2_{h_{n-1}}(\lambda) + \eta_a(H)\varepsilon_{h_{n-1}}(\lambda) + \delta_{h_n}(\lambda) \\ &\lesssim \sum_{k=1}^{n} \eta_a(H)^{n-k}\delta_{h_k}(\lambda). \end{aligned} \tag{4.10}$$

This is the estimate (4.5). From Theorem 3.1 and (4.10), we can obtain the estimate (4.6). □

Remark 4.2. Using the One Correction Step defined in Remark 3.5 and replacing the source problem (4.3) by

$$a(u_{h_n}, v_{h_n}) - \lambda_{h_{n-1}} b(u_{h_n}, v_{h_n}) = \lambda_{h_{n-1}} b(v_{h_{n-1}}, v_{h_n}), \quad \forall v_{h_n} \in V_{h_n}, \tag{4.11}$$

we can construct a new Multi-level Correction Scheme which has the error estimates (4.5) and (4.6) with $\varepsilon_{h_n}(\lambda) = \sum\limits_{k=1}^{n} \eta_a(H)^{\frac{3^{n-k}-1}{2}} \delta_{h_k}(\lambda)^{3^{n-k}}$.

5. The application to second order elliptic eigenvalue problem

In this section, for example, the multi-level correction method presented in this paper is applied to the second order elliptic eigenvalue problem. We also discuss two possible ways to implement the multi-level correction Algorithm 4.1. The first way is the "two-grid method" of Xu and Zhou introduced and studied in [25]. The second one proposed and studied by Andreev and Racheva in [1, 19] uses the same mesh but higher order finite elements.

In (2.13), the second order elliptic eigenvalue problem can be defined by

$$a(u, v) = \int_\Omega \nabla u \cdot \mathcal{A}\nabla v d\Omega, \qquad b(u, v) = \int_\Omega \rho u v d\Omega,$$

where $\Omega \subset \mathcal{R}^d$ $(d = 2, 3)$ is a bounded domain, $\mathcal{A} \in \left(W^{1,\infty}(\Omega)\right)^{d\times d}$ a uniformly positive definite matrix on Ω and $\rho \in W^{0,\infty}(\Omega)$ is a uniformly positive function on Ω. We pose Dirichlet boundary condition to the problem and it means here $V = H^1_0(\Omega)$ and $W = L^2(\Omega)$. In order to use the finite element discretization method, we employ the meshes defined in Section 3.

Here, we introduce two ways to implement the multi-level correction Algorithm 4.1. The first way uses finer meshes to construct the series of nested finite element spaces. The advantage of this approach is that it uses the same finite element method and does not require higher regularity of the exact eigenfunctions (see [19]). The second way is based on the same finite element mesh but using higher order finite elements. In order to improve the convergence order, the higher regularity of the exact eigenfunctions is required.

Let us discuss the methods to construct the series of finite element spaces $V_{h_k}(k = 2, 3, \cdots, n)$ for implementing the multi-level correction method.

Way 1. ("Multi-grid method"): In this case, V_{h_k} $(k = 2, 3, \cdots, n)$ is the same type of finite element as V_H and is defined on the finer mesh $\mathcal{T}_{h_k}$ with a smaller mesh size h_k. Here $\mathcal{T}_{h_k}$ is a finer mesh of Ω that can be generated by the (e.g., regular or bisectional) refinement just as in the multi-grid method(see, e.g., [25]) from $\mathcal{T}_{h_{k-1}}$ such that $h_k = \eta_a(H)h_{k-1}$. Assume the computing domain Ω is a convex domain. Then $\eta_a(H) = \mathcal{O}(H)$ and $\delta_{h_k} = \mathcal{O}(h_k) = \mathcal{O}(H^k)$ $(k = 1, 2, \cdots, n)$, and we can obtain the following error estimate for (λ_{h_n}, u_{h_n}):

$$|\lambda - \lambda_{h_n}| \lesssim \eta_a(H)^{2n-2}\delta_H^2(\lambda) = \mathcal{O}(H^{2n}) = \mathcal{O}(h_n^2), \tag{5.1}$$

$$\|u - u_{h_n}\|_a \lesssim \eta_a(H)^{n-1}\delta_H(\lambda) = \mathcal{O}(H^n) = \mathcal{O}(h_n). \tag{5.2}$$

From the error estimates above, we can find that the multi-level correction scheme can obtain the accuracy the same as solving the eigenvalue problem directly on the finest mesh $\mathcal{T}_{h_n}$. This improvement costs solving the source problems on the finer finite element spaces V_{h_k} $(k = 2, 3, \cdots, n)$ and the eigenvalue problems in coarse spaces V_{H,h_k} $(k = 2, 3, \cdots, n-1)$. This is better than solving the eigenvalue problem on the finest finite element space directly, because solving the source problem needs much less computation than solving the corresponding eigenvalue problem.

Remark 5.1. If we use the multi-level correction described in Remark 4.2, the mesh size of $\mathcal{T}_{h_k}$ can be chosen as

$$h_k = H^{\frac{3^k-1}{2}}, \qquad k = 2, \cdots, n. \tag{5.3}$$

It means $H = h_n^{\frac{2}{3^n-1}}$ which is a weaker requirement than $H = h_n^{\frac{1}{n}}$.

Way 2. ("Multi-space method"): In this case, V_{h_k} is defined on the same mesh $\mathcal{T}_H$ but uses a higher order finite element than $V_{h_{k-1}}$. In order to describe the scheme simply, we suppose the exact eigenfunction has sufficient regularity. We use the linear finite element space to solve the original eigenvalue problem (2.13) on V_H, and solve the source problem (3.3) in higher order finite element space with the way that the order of V_{h_k} is one order higher than $V_{h_{k-1}}$. Then we have the following error estimates for the final eigenpair approximation (λ_{h_n}, u_{h_n}):

$$|\lambda - \lambda_{h_n}| \lesssim \eta_a(H)^{2n-2}\delta_H^2(\lambda) = \mathcal{O}(H^{2n}), \tag{5.4}$$

$$\|u - u_{h_n}\|_a \lesssim \eta_a(H)^{n-1}\delta_H(\lambda) = \mathcal{O}(H^n). \tag{5.5}$$

The improved error estimates above just cost solving the source problems on the same mesh but in higher order finite element spaces and eigenvalue problems in the lowest order finite element space.

Remark 5.2. If we use the multi-level correction described in Remark 4.2, the order of V_{h_k} can be chosen as $\frac{3^k-1}{2}$. For example, we can choose fourth order finite element for V_{h_2} and thirteenth order finite element for V_{h_3}.

6. Numerical results

In this section, we give two numerical examples to illustrate the efficiency of the multi-level correction algorithm proposed in this paper.

6.1. **Model eigenvalue problem.** We solve the model eigenvalue problem (1.1) on the unit square $\Omega = (0, 1) \times (0, 1)$.

Multi-space way. Here we give the numerical results of the multi-level correction scheme in which the finer finite element spaces are constructed by improving the finite element orders on the same mesh. We first solve the eigenvalue problem (1.1) in the linear finite element space on the mesh $\mathcal{T}_H$. Then do the first correction step with the quadratic element and the cubic element for the second correction step.

Here, we adopt the meshes which are produced by the regular refinement from the initial mesh generated by Delaunay method to investigate the convergence behaviors. Figure 1 shows the initial mesh. Figure 2 gives the corresponding numerical results for the first eigenvalue $\lambda_1 = 2\pi^2$ and the corresponding eigenfunction. Figure 3 gives the numerical results for the first 6 eigenvalues: $2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2$ and $10\pi^2$.

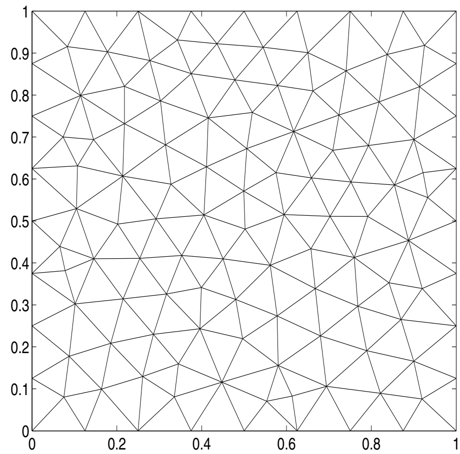

FIGURE 1. Initial mesh for multi-space way

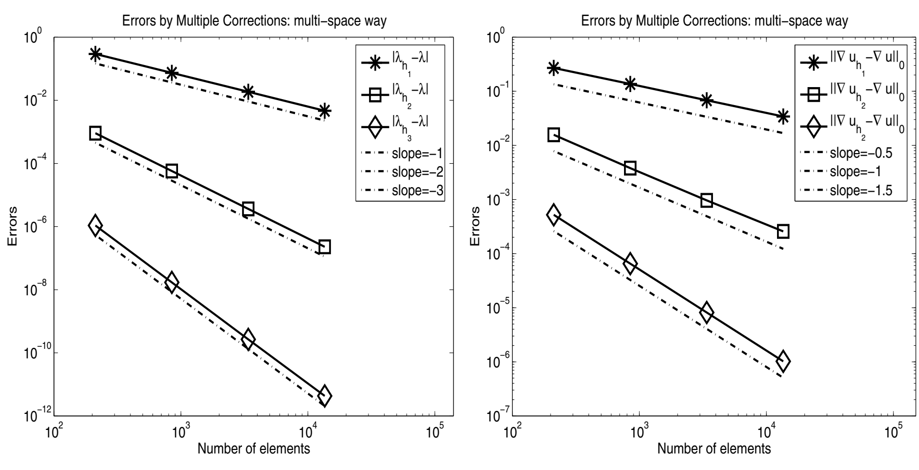

FIGURE 2. The errors for the eigenpair approximations by multi-level correction algorithm for the first eigenvalue $2\pi^2$ and the corresponding eigenfunction with multi-space way

From Figures 2 and 3, we can find that each correction step can improve the convergence order by two for the eigenvalue approximations and one for the eigenfunction approximations with the multi-space way when the exact eigenfunction is smooth enough.

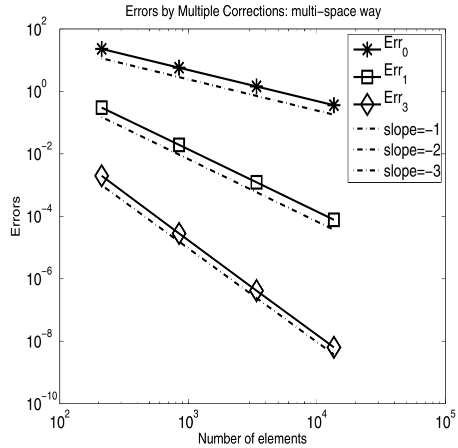

FIGURE 3. The errors for the eigenvalue approximations by multi-level correction algorithm for the first 6 eigenvalues with multi-space way, where the error is defined by $\mathrm{Err}_0 = \sum_{j=1}^{6} |\lambda_{j,h_0} - \lambda_j|$, $\mathrm{Err}_1 = \sum_{j=1}^{6} |\lambda_{j,h_1} - \lambda_j|$ and $\mathrm{Err}_2 = \sum_{j=1}^{6} |\lambda_{j,h_2} - \lambda_j|$

TABLE 1. The comparison of the first eigenvalue approximations obtained by the multi-level correction method and the extended two-grid method

h	Errors of initial approximations	Errors of the extended two-grid method	Errors of multi-level correction method
1/2	1.2261E+01	1.2819E-01	2.1362E-07
1/4	3.1266E+00	6.5478E-04	7.7200E-11

We also compare the results of the multi-level correction method with those of the extended two-grid method. The linear finite element is used to obtain the initial eigenvalue approximations and the fourth order element is applied to do the extended two-grid correction method (see [10]). For the aim of comparison, we use the multi-level correction method stated in Remark 4.2 to do two corrections. The linear finite element is also adopted to obtain the initial eigenvalue approximations and the fourth order element is applied to do the first correction. In order to eliminate the effect of the machine error, we only use the eighth order element to do the second correction even the thirteenth order element can be used in the second correction as predicted in Remark 5.2.

The results of the comparison are presented in Table 1. From Table 1, we can find that the multi-level correction method can obtain better results than the extended two-grid method.

Multi-grid way. Here we give the numerical results of the multi-level correction scheme where the finer finite element spaces are constructed by refining the existed mesh. We first solve the eigenvalue problem (1.1) in the linear finite element space on the mesh $\mathcal{T}_H$. Then, as an example,we chose the regular way to refine the mesh such that the size of the resultant mesh $h_k = O(H^k)$ to obtain the mesh $\mathcal{T}_{h_k}$ $(k = 2, \cdots, n)$ and solve the auxiliary source problem (3.3) in the linear finite element space V_{h_k} defined on $\mathcal{T}_{h_k}$ and the corresponding eigenvalue problem (3.4) in V_{H,h_k}.

Figure 4 gives the corresponding numerical results for the first eigenvalue $\lambda = 2\pi^2$ and the corresponding eigenfunction on the uniform meshes. Figure 5 gives the numerical results for the first 6 eigenvalues: $2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2$ and $10\pi^2$.

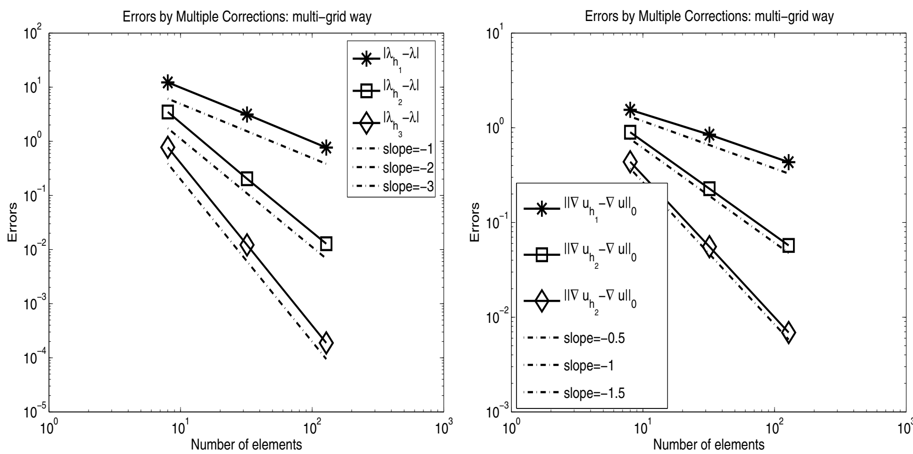

FIGURE 4. The errors for the eigenpair approximations by multi-level correction algorithm for the first eigenvalue $2\pi^2$ with multi-grid way

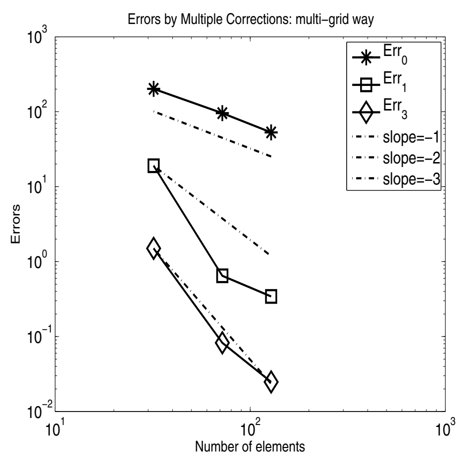

FIGURE 5. The errors for the eigenvalue approximations by multi-level correction algorithm for the first 6 eigenvalues with multi-grid way, where the error is defined by $\mathrm{Err}_0 = \sum_{j=1}^{6} |\lambda_{j,h_0} - \lambda_j|$, $\mathrm{Err}_1 = \sum_{j=1}^{6} |\lambda_{j,h_1} - \lambda_j|$ and $\mathrm{Err}_2 = \sum_{j=1}^{6} |\lambda_{j,h_2} - \lambda_j|$

From Figures 4 and 5, we can also find that each correction step can improve the convergence order by two for the eigenvalue approximations and one for the eigenfunction approximations with the multi-grid way.

6.2. **Eigenvalue problem on L-shape domain.** In the second example, we consider the model eigenvalue problem on the L-shape domain $\Omega = (-1,1) \times (-1,1)\backslash[0,1) \times (-1,0]$. Since Ω has a re-entrant corner, the singularity of eigenfunctions is expected. The convergence order for the eigenvalue approximation is less than 2 by the linear finite element method which is the order predicted by the theory for regular eigenfunctions.

We investigate the numerical results for the first eigenvalue. Since the exact eigenvalue is not known, we choose an adequately accurate approximation $\lambda = 9.6397238440219$ as the exact first eigenvalue for our numerical tests. We give the numerical results of the multi-level correction scheme in which the sequence of meshes $\mathcal{T}_H$, $\mathcal{T}_{h_2}, \cdots, \mathcal{T}_{h_n}$ is produced by the adaptive refinement (cf. [21, 26]). Figure 6 shows the initial mesh and the one after 12 adaptive iterations. Figure 7 gives the corresponding numerical results for the adaptive iterations. In order to show the accuracy of the multi-level correction method more clearly, we compare the results with those obtained by the direct adaptive finite element method. From Figure 7, we can find the multi-level correction method can also work on the adaptive family of meshes and obtain the optimal accuracy. Furthermore, the initial mesh has nothing to do with the finest one which is different from the two-gird [25] and the extended two-grid method [10]. The multi-level correction method can be coupled with the adaptive refinement naturally.

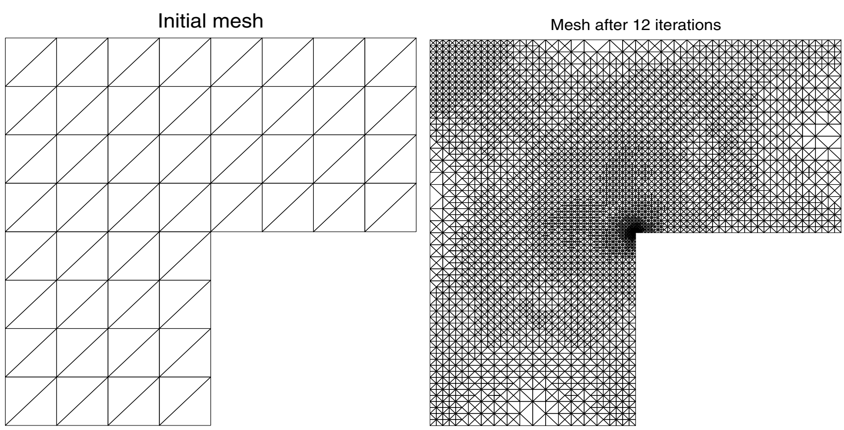

FIGURE 6. The initial mesh and the one after 12 adaptive iterations for Example 2

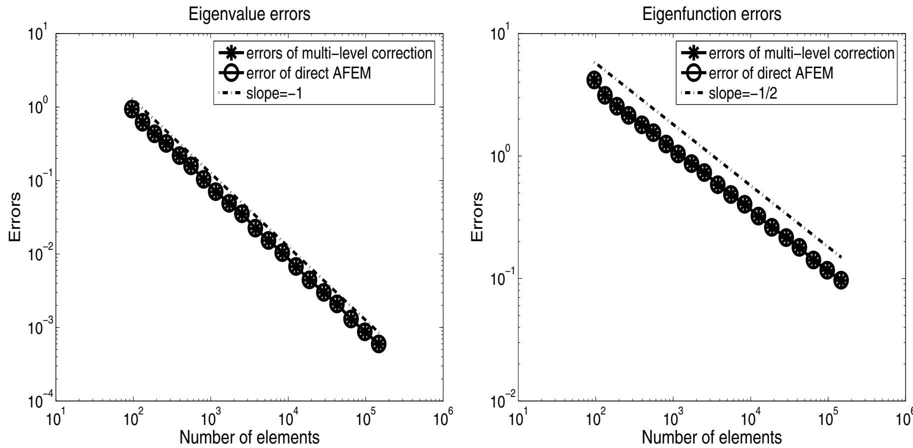

FIGURE 7. The errors of the smallest eigenvalue approximations and the a posteriori errors of the associated eigenfunction approximations by multi-level correction method and direct adaptive finite element method for Example 2

7. Concluding remarks

In this paper, we give a type of multi-level correction scheme to improve the accuracy of the eigenpair approximations. Comparing with the superconvergence method (for example PPR in [18, 21]), we would like to say the superconvergence method has better efficiency when the mesh is structural and the exact eigenfunction has higher regularity. As the multi-grid method relative to the superconvergence technique in solving boundary value problems, the multi-level correction method also has its value in solving eigenvalue problems.

We can use the better eigenvalue and eigenfunction approximation (λ_{h_n}, u_{h_n}) to construct an a posteriori error estimator of the eigenpair approximations for the eigenvalue problems (see, e.g., [7, 16]). Furthermore, our multi-level correction scheme can be coupled with the multi-grid method to construct a type of multi-grid and parallel method for eigenvalue problems (see, e.g, [26]). It can also be combined with the adaptive refinement technique for the singular eigenfunction cases. These will be our future work.

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