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A multi-valued neutrosophic qualitative flexible approach based on likelihood for multi-criteria decision-making problems

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ABSTRACT
In this paper, multi-criteria decision-making (MCDM) problems based on the qualitative flexible multiple criteria method (QUALIFLEX), in which the criteria values are expressed by multi-valued neutrosophic information, are investigated. First, multi-valued neutrosophic sets (MVNSs), which allow the truth-membership function, indeterminacy-membership function and falsity-membership function to have a set of crisp values between zero and one, are introduced. Then the likelihood of multi-valued neutrosophic number (MVNN) preference relations is defined and the corresponding properties are also discussed. Finally, an extended QUALIFLEX approach based on likelihood is explored to solve MCDM problems where the assessments of alternatives are in the form of MVNNs; furthermore an example is provided to illustrate the application of the proposed method, together with a comparison analysis.

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1. Introduction

Smarandache (1998, 1999, 2005) initially introduced the concept of neutrosophic sets (NSs), which are an extension of the standard interval [0, 1] of Atanassov's intuitionistic fuzzy sets (IFSs) (Atanassov, 1986). NSs are characterised by a truth-membership function, indeterminacy-membership function and falsity membership function that are represented by a set of crisp numbers between $]0^-, 1^+[$, the non-standard unit interval. However, without a specific description, NSs are difficult to apply in real-life situations. Hence, single-valued neutrosophic sets (SNSs) and interval neutrosophic sets (INSs) were defined, which can be denoted by three real numbers and intervals, respectively, in the real unit interval [0, 1] (Majumdar & Samant, 2014; Wang, Smarandache, Zhang, & Sunderraman, 2010; Ye, 2013, 2014a). Seemingly, SNSs and INSs are particular cases of NSs. In recent years, NSs and their particular cases have been applied to solve multi-criteria decision-making (MCDM) problems (Liu & Shi, 2015; Peng, Wang, Zhang, & Chen, 2014; Tian, Zhang, Wang, Wang, & Chen, 2015; Wu, Wang, Peng, & Chen, 2016; Ye, 2014b, 2014c), and also applied to medical diagnosis (Ma, Wang, Wang, & Wu, 2016; Ye, 2015a), clustering analysis (Ye, 2014d), image processing (Guo & Sengur, 2014), green product development (Tian, Wang, Wang, & Zhang, 2016a), engineering machine (Tian, Wang, Wang, & Zhang, 2016b) and graphs (Broumi, Talea, Bakali, & Smarandache, 2016a, 2016b). For example, Ye (2014a) proposed an MCDM method using the aggregation operators of SNSs. Moreover, Zhang, Ji, Wang, and Chen (2015) developed an MCDM method based on integrated weight under an interval neutrosophic environment. Based on the operations in Ye (2014a), Peng, Wang, Wang, Zhang, and Chen (2016) developed some aggregation operators of SNSs, and applied them to multi-criteria group decision-making (MCGDM) problems. Liu and Wang (2014) investigated the single-valued neutrosophic normalised weighted Bonferroni mean and applied it to MCDM problems. Liu, Chu, Li, and Chen (2014) developed some Hamacher aggregation operators with NSs. Ye (2014b, 2014c) proposed the similarity measures between SNSs and INSs to solve MCDM problems. Finally, Peng et al. (2014) defined the outranking relations with SNSs and Zhang, Ji, Wang, and Chen (2016) developed a neutrosophic normal cloud and both were applied to solve MCDM problems.

However, because of the ambiguity and complexity of decision-making in the real world, it is difficult for decision-makers to express precisely their preferences by using NSs and their particular cases, including SNSs and INSs. Under these circumstances, Wang and Li (2015) and Ye (2015b) developed the definition of multi-valued neutrosophic sets (MVNSs) and single-valued neutrosophic hesitant fuzzy sets (SVNHFSs), respectively, which are both extensions of SNSs and the hesitant fuzzy sets (HFSs) introduced by Torra and Narukawa (2009) and Torra (2010). Moreover, both MVNSs and SVNHFSs are represented by truth-membership, indeterminacy-membership and falsity-membership functions that have a set of crisp values between zero and one. Actually, there is no difference between MVNSs and SVNHFSs. Based on the definition of MVNSs, Peng, Wang, Wu, Wang, and Chen (2015) and Peng, Wang, and Wu (2016a) further defined multi-valued neutrosophic power aggregation operators and outranking relations, and applied them to resolve MCGDM or MCDM problems. Ji, Zhang, and Wang (2016) developed a projection-based TODIM method with multi-valued neutrosophic information.

The qualitative flexible multiple criteria (QUALIFLEX) method, developed by Paelinck (1976, 1977, 1978), is based on the pair-wise comparisons of alternatives with respect to each criterion under all possible alternative permutations, and

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identifies the optimal permutation that maximises the value of the concordance/discordance index (Martel & Matarazzo, 2005). Recently, based on likelihood-based preference relations, several extended QUALIFLEX methods were developed and applied to manage different types of fuzzy information, including interval-valued intuitionistic fuzzy (Chen, 2014a), interval type-2 fuzzy (Wang, Tsao, & Chen, 2015), hesitant fuzzy linguistic (Lee & Chen, 2015) and interval type-2 trapezoidal fuzzy information (Chen, 2014b, 2015). However, their extensions were only used to resolve MCDM problems with fuzzy information and few attempts have been made to extend QUALIFLEX to the multi-valued neutrosophic decision-making environment. Moreover, some decision-making problems, where the number of criteria significantly exceeds the number of alternatives, cannot be managed by most of existing neutrosophic decision-making methods, or they fail to obtain the distinct ranking of alternatives. Therefore, the purpose of this paper is to propose an extended QUALIFLEX method based on multi-valued neutrosophic numbers (MVNNs) to obtain more imprecise or uncertain decision-making information.

Therefore, in this paper, the preference relations of MVNNs are developed based on likelihood. Then, based on these likelihoods, an extended QUALIFLEX approach is established to deal with MCDM problems where the data are expressed by MVNNs. Consequently, an illustrative example is also provided to demonstrate the applicability of the proposed method. This paper introduces the concept of multi-valued likelihoods of the possibility of MVNNs preference relations, namely truth-membership likelihood, indeterminacy-membership likelihood and falsity-membership likelihood, and then determines the likelihoods between MVNNs numbers. Therefore, the proposed approach is different to the previous method referred to above, as it captures more imprecise or uncertain decision information and effectively addresses MCDM problems within a multi-valued neutrosophic environment.

The rest of paper is organised as follows. In Section 2, some basic concepts of NSs and SNSs are briefly reviewed. Then the definition of MVNSs is introduced, and the likelihood preference relations of MVNNs are defined in Section 3. Section 4 contains the extended QUALIFLEX method to solve MCDM problems with MVNNs. In Section 5, an illustrative example and a comparison analysis are presented to verify the proposed approach. Finally, conclusions are drawn in Section 6.

2. Preliminaries

In this section, the definitions of NSs, SNSs and HFSs are introduced, which will be utilised in the latter analysis.

2.1. Neutrosophic sets and simplified neutrosophic sets

Definition 1 (Smarandache, 1999)**:** Let X be a space of points, and x be an element in X. A NS A in X is characterised as follows:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \,|x \in X\}, \tag{1}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, and satisfying $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

When considering the applicability of NSs, Ye (2014a) reduced NSs of nonstandard intervals into SNSs of standard intervals, which can preserve the operations of NSs appropriately.

Definition 2 (Ye, 2014a)**:** Let X be a space of points (objects), and x be an element in X. An SNS A in X is characterised by:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \,|x \in X\}, \tag{2}$$

here $T_A(x)$, $I_A(x)$ and $F_A(x)$ are subsets of the real standard [0, 1], which represent the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively. In particular, an SNS is a special case of NSs. If X has only one element, then A is called a simplified neutrosophic number (SNN), which can be denoted by $A = \langle T_A(x), I_A(x), F_A(x)\rangle$.

Example 1: Assume that $X = \{x_1, x_2, x_3\}$, where x_1 is quality, x_2 is trustworthiness, and x_3 is the price of an equipment, is the universal set. The values of x_1, x_2 and x_3 are in [0, 1]. They can be collected from the questionnaire of some domain experts, whose option could be a degree of 'good', a degree of indeterminacy and a degree of 'poor'. A is an SNS of X defined as follows:

$$A = \{\langle\{0.4\}, \{0.2\}, \{0.5\}\rangle/x_1, \langle\{0.4\}, \{0.1\}, \{0.3\}\rangle/x_2, \langle\{0.5\}, \{0.2\}, \{0.3\}\rangle/x_3\}.$$

Definition 3 (Ye, 2014b)**:** The complement of an SNS A is denoted by A^C and is defined as

$$A^C = \left\{\left\langle x, T_A^C(x), I_A^C(x), F_A^C(x)\right\rangle |x \in X\right\}. \tag{3}$$

Here, $T_A^C(x) = F_A(x)$, $I_A^C(x) = \{1\} - I_A(x)$ and $F_A^C(x) = T_A(x)$ for every x in X.

2.2. Hesitant fuzzy sets

Definition 4 (Torra, 2010; Torra & Narukawa, 2009)**:** Let X be a reference set, and x be an element in X. An HFS E on X can be expressed as a mathematical symbol:

$$E = \{\langle x, h_E(x)\rangle |\, x \in X\}, \tag{4}$$

where $h_E(x)$ is a set of values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set E. In particular, if X has only one element, E is called a hesitant fuzzy number (HFN) (Xia & Xu, 2011), which can be denoted by $E = \{h_E(x)\}$. The set of all HFNs is represented by HFNS.

Example 2: Let $X = \{x_1, x_2, x_3\}$ be a reference set, then an HFS E can be obtained as follows:

$$E = \{\langle x_1, \{0.3, 0.1\}\rangle, \langle x_2, \{0.4, 0.2\}\rangle, \langle x_3, \{0.5, 0.6\}\rangle\}.$$

Definition 5 (Torra, 2010)**:** The complement of an HFS E is denoted by E^C and is defined as:

$$E^C = \left\{ \left\langle x, h_E^C(x) \right\rangle \middle| \, x \in X \right\}. \quad (5)$$

Here $h^C(x) = \bigcup_{\gamma \in h^C(x)} \{1-\gamma\}$ for any x in X.

Definition 6 (Peng, Wang, & Wu, 2016b)**:** Let h_A and h_B be two HFNs, all elements in HFNs be arranged in ascending order, and $\gamma_{h_i}^{\sigma(j)}$ be referred to as the j th value in h_i $(i = A, B)$. Then the following comparison methods can be provided:

(1) $h_A \le h_B$ if $\gamma_A^{\sigma(j)} \le \gamma_B^{\sigma(j)}$ and $\gamma_A^{\sigma(l_{h_A})} \le \gamma_B^{\sigma(l_{h_B})}$,

where $\gamma_A^{\sigma(j)} \in h_A, \gamma_B^{\sigma(j)} \in h_B, j = 1, 2, \cdots, l_h$, and $l_h = \min(l_{h_A}, l_{h_B})$ (l_{h_i} is the number of elements in h_i);

(2) $h_A = h_B$ if $h_A \le h_B$ and $h_B \le h_A$.

3. The likelihood of MVNNs preference relations

In this section, MVNSs are introduced, and the preference relations of MVNNs based on likelihood are defined.

3.1. MVNSs and their operations

Definition 7 (Wang & Li, 2015; Ye, 2015b)**:** Let X be a space of points, and x be an element in X. A MVNS A in X is characterised by:

$$A = \{\langle x, \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\rangle | x \in X\}, \quad (6)$$

where $\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ are three sets of precise values in [0,1] and in the form of HFSs, denoting the truth-membership function, indeterminacy-membership function and falsity-membership function, respectively, and satisfying $0 \le \gamma, \ \eta, \ \xi \le 1, \ 0 \le \gamma^+ + \eta^+ + \xi^+ \le 3$, where $\gamma \in \tilde{T}_A(x), \ \eta \in \tilde{I}_A(x), \ \xi \in \tilde{F}_A(x), \gamma^+ = \sup \tilde{T}_A(x), \eta^+ = \sup \tilde{I}_A(x)$ and $\xi^+ = \sup \tilde{F}_A(x)$.

If X has only one element, then A is called a MVNN, denoted by $A = \langle \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x) \rangle$. For convenience, a MVNN can be denoted by $A = \langle \tilde{T}_A, \tilde{I}_A, \tilde{F}_A \rangle$. The set of all MVNNs is represented by MVNNS.

Obviously, MVNSs are particular cases of NSs. If each of $\tilde{T}_A(x)$, $\tilde{I}_A(x)$ and $\tilde{F}_A(x)$ for any x has only one value, i.e. γ, η and ξ, and $0 \le \gamma + \eta + \xi \le 3$, then MVNSs are reduced to SNSs; if $\tilde{I}_A(x) = \emptyset$ for any x, then MVNSs are reduced to DHFSs; and if $\tilde{I}_A(x) = \tilde{F}_A(x) = \emptyset$ for any x, then MVNSs are reduced to HFSs. In other words, MVNSs are extensions of SNSs, DHFSs and HFSs.

Example 3: Assume that $X = \{x_1, x_2, x_3\}$, where x_1 is the technology, x_2 is the market potential, and x_3 is the risk, is the universal set. The values of x_1, x_2 and x_3 are in [0,1]. They can be collected from the questionnaire of some domain experts, whose option could be a degree of 'excellent', a degree of indeterminacy and a degree of 'weakness'. A is a MVNS of X defined as follows:

$$A = \{\langle\{0.2, 0.3\}, \{0.1\}, \{0.5\}\rangle/x_1, \langle\{0.4\}, \{0.2\}, \{0.3\}\rangle/x_2, \langle\{0.3\}, \{0.2, 0.4\}, \{0.6\}\rangle/x_3\}.$$

Definition 8 (Peng et al., 2015)**:** Let $A \in MVNNS$, then the complement of a MVNN can be denoted by A^C, which can be defined as follows:

$$A^C = \left\{ \left\langle \cup_{\xi \in \tilde{F}_A} \{\xi\}, \cup_{\eta \in \tilde{I}_A} \{1-\eta\}, \cup_{\gamma \in \tilde{T}_A} \{\gamma\} \right\rangle \right\}. \quad (7)$$

Definition 9: Let $A = \langle \tilde{T}_A, \tilde{I}_A, \tilde{F}_A \rangle$ and $B = \langle \tilde{T}_B, \tilde{I}_B, \tilde{F}_B \rangle$ be two MVNNs, all elements in $\tilde{T}_i$, $\tilde{I}_i$ and $\tilde{F}_i (i = A, B)$ be arranged in ascending order, and $\gamma_i^{\sigma(\cdot)}$, $\eta_i^{\sigma(\cdot)}$ and $\xi_i^{\sigma(\cdot)}$ be referred to as the $(\cdot)$ -th value in $\tilde{T}_i$, $\tilde{I}_i$ and $\tilde{F}_i (i = A, B)$, respectively. Then the following comparison methods can be provided.

(1) $A \le B$ if $\gamma_A^{\sigma(j)} \le \gamma_B^{\sigma(j)}$ and $\gamma_A^{\sigma(l_{\tilde{T}_A})} \le \gamma_B^{\sigma(l_{\tilde{T}_B})}$ $(j = 1, 2, \ldots, l_{\tilde{T}}, l_{\tilde{T}} = \min(l_{\tilde{T}_A}, l_{\tilde{T}_B}))$,

$\eta_A^{\sigma(k)} \ge \eta_B^{\sigma(k)}$ and $\eta_A^{\sigma(l_{\tilde{I}_A})} \ge \eta_B^{\sigma(l_{\tilde{I}_B})}$ $(k = 1, 2, \ldots, l_{\tilde{I}}, l_{\tilde{I}} = \min(l_{\tilde{I}_A}, l_{\tilde{I}_B}))$,

$\xi_A^{\sigma(m)} \ge \xi_B^{\sigma(m)}$ and $\xi_A^{\sigma(l_{\tilde{F}_A})} \ge \xi_B^{\sigma(l_{\tilde{F}_B})}$ $(m = 1, 2, \ldots, l_{\tilde{F}}, l_{\tilde{F}} = \min(l_{\tilde{F}_A}, l_{\tilde{F}_B}))$.

Here $l_{\tilde{T}}$, $l_{\tilde{I}}$ and $l_{\tilde{F}}$ are the number of elements in $\tilde{T}_i$, $\tilde{I}_i$ and $\tilde{F}_i (i = A, B)$, respectively.

(2) $A = B$ if $A \le B$ and $B \le A$.

Example 4: Let $A = \langle\{0.2, 0.5\}, \{0.1, 0.2\}, \{0.3\}\rangle$ and $B = \langle\{0.2, 0.5\}, \{0.1, 0.2\}, \{0.3\}\rangle$ be two MVNNs. According to Definition 9, $A = B$ can be easily obtained, which is consistent with our intuition.

3.2. The likelihood of MVNNs preference relations

Since the comparison methods presented in Definition 9 reflect the partial order of MVNNs, the likelihood of MVNNs reference relations is now defined in the following.

Definition 10: Let $A = \langle \tilde{T}_A, \tilde{I}_A, \tilde{F}_A \rangle$ and $B = \langle \tilde{T}_B, \tilde{I}_B, \tilde{F}_B \rangle$ be two MVNNs, then the truth-membership likelihood, indeterminacy-membership likelihood and falsity-membership likelihood$P(\tilde{T}_A, \tilde{T}_B)$, $P(\tilde{I}_A, \tilde{I}_B)$ and $P(\tilde{F}_A, \tilde{F}_B)$, respectively, of a MVNN preference relation $P(A, B)$ are defined as follows:

(1) $P(\tilde{T}_A, \tilde{T}_B)$

$$= \begin{cases} \frac{1}{l_{\tilde{T}_A} \cdot l_{\tilde{T}_B}} \sum\limits_{\gamma_A \in \tilde{T}_A} \sum\limits_{\gamma_B \in \tilde{T}_B} \frac{\gamma_A}{\gamma_A + \gamma_B}, & \tilde{T}_A \ne \{0\} \text{ or } \tilde{T}_B \ne \{0\} \\ 0.5. & \tilde{T}_A = \{0\} \text{ and } \tilde{T}_B = \{0\} \end{cases}, \quad (8)$$

(2) $P(\tilde{I}_A, \tilde{I}_B)$

$$= \begin{cases} 1 - \frac{1}{l_{\tilde{I}_A} \cdot l_{\tilde{I}_B}} \sum\limits_{\eta_A \in \tilde{I}_A} \sum\limits_{\eta_B \in \tilde{I}_B} \frac{\eta_A}{\eta_A + \eta_B}, & \tilde{I}_A \ne \{0\} \text{ or } \tilde{I}_B \ne \{0\} \\ 0.5. & \tilde{I}_A = \{0\} \text{ and } \tilde{I}_B = \{0\} \end{cases}; \quad (9)$$

(3) $P(\tilde{F}_A, \tilde{F}_B)$

$$= \begin{cases} 1 - \frac{1}{l_{\tilde{F}_A} \cdot l_{\tilde{F}_B}} \sum\limits_{\xi_A \in \tilde{F}_A} \sum\limits_{\xi_B \in \tilde{F}_B} \frac{\xi_A}{\xi_A + \xi_B}, & \tilde{F}_A \ne \{0\} \text{ or } \tilde{F}_B \ne \{0\} \\ 0.5. & \tilde{F}_A = \{0\} \text{ and } \tilde{F}_B = \{0\} \end{cases}. \quad (10)$$

The likelihood of an MVNN preference relation $P(A,B)$ is defined by

$$P(A,B)=\frac{1}{3}(P(\tilde{T}_A,\tilde{T}_B)+P(\tilde{I}_A,\tilde{I}_B)+P(\tilde{F}_A,\tilde{F}_B)). \quad (11)$$

Here $l_{(\cdot)}$ represents the number of elements in $(\cdot)$.

Example 5: Let $A=\langle\{0.5,0.6\},\{0.4\},\{0.2\}\rangle$ and $B=\langle\{0.5\},\{0.5\},\{0.1,0.3\}\rangle$ be two MVNNs. Then the following can be true:

$$P(\tilde{T}_A,\tilde{T}_B)=0.5225,\ P(\tilde{I}_A,\tilde{I}_B)=0.5556,\ P(\tilde{F}_A,\tilde{F}_B)=0.4667.$$

Thus, $P(A,B)=\frac{1}{3}(P(\tilde{T}_A,\tilde{T}_B)+P(\tilde{I}_A,\tilde{I}_B)+P(\tilde{F}_A,\tilde{F}_B))=0.5149$.

Apparently, the likelihood of MVNNs reference relations can deal with any two MVNNs.

Property 1: Let $A=\langle\tilde{T}_A,\tilde{I}_A,\tilde{F}_A\rangle$ and $B=\langle\tilde{T}_B,\tilde{I}_B,\tilde{F}_B\rangle$ be two MVNNs, and $P(\tilde{T}_A,\tilde{T}_B)$, $P(\tilde{I}_A,\tilde{I}_B)$ and $P(\tilde{F}_A,\tilde{F}_B)$ be the truth-membership, indeterminacy-membership and falsity-membership likelihoods of the relation $A\geq B$, respectively. The following properties can therefore be true:

(1) $0\leq P(\tilde{T}_A,\tilde{T}_B)\leq 1$;
(2) $0\leq P(\tilde{I}_A,\tilde{I}_B)\leq 1$;
(3) $0\leq P(\tilde{F}_A,\tilde{F}_B)\leq 1$;
(4) $P(\tilde{T}_A,\tilde{T}_B)+P(\tilde{T}_B,\tilde{T}_A)=1$;
(5) $P(\tilde{I}_A,\tilde{I}_B)+P(\tilde{I}_B,\tilde{I}_A)=1$;
(6) $P(\tilde{F}_A,\tilde{F}_B)+P(\tilde{F}_B,\tilde{F}_A)=1$;
(7) if $\tilde{T}_A=\tilde{T}_B$, then $P(\tilde{T}_A,\tilde{T}_B)=0.5$;
(8) if $\tilde{I}_A=\tilde{I}_B$, then $P(\tilde{I}_A,\tilde{I}_B)=0.5$;
(9) if $\tilde{F}_A=\tilde{F}_B$, then $P(\tilde{F}_A,\tilde{F}_B)=0.5$.

Proof: (1), (2) and (3) can easily be obtained. ■

(4) If $\tilde{T}_A=\tilde{T}_B=\{0\}$, then according to Equation (8) in Definition 10, $P(\tilde{T}_A,\tilde{T}_B)=P(\tilde{T}_B,\tilde{T}_A)=\frac{1}{2}$. Thus, $P(\tilde{T}_A,\tilde{T}_B)+P(\tilde{T}_B,\tilde{T}_A)=1$. If $\tilde{T}_A\neq\{0\}$ or $\tilde{T}_B\neq\{0\}$, then all elements in $\tilde{T}_i$ $(i=A,B)$ are arranged in ascending order, and $\gamma_i^{\sigma(\cdot)}$ $(i=A,B)$ is referred to as the $(\cdot)$-th value in $\tilde{T}_i$ $(i=A,B)$. Thus,

$$\begin{aligned}
&P(\tilde{T}_A,\tilde{T}_B)+P(\tilde{T}_B,\tilde{T}_A)\\
&=\frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{T}_B}}\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(1)}}+\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(2)}}+\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(3)}}\right.\\
&\quad+\cdots+\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}+\\
&\quad\cdots\cdots\cdots\\
&\quad\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(1)}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(2)}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(3)}}\\
&\quad\left.+\cdots+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}\right)+\frac{1}{l_{\tilde{T}_B}\cdot l_{\tilde{T}_A}}\left(\frac{\gamma_B^{\sigma(1)}}{\gamma_B^{\sigma(1)}+\gamma_A^{\sigma(1)}}\right.\\
&\quad+\frac{\gamma_B^{\sigma(1)}}{\gamma_B^{\sigma(1)}+\gamma_A^{\sigma(2)}}+\frac{\gamma_B^{\sigma(1)}}{\gamma_B^{\sigma(1)}+\gamma_A^{\sigma(3)}}+\cdots+\frac{\gamma_B^{\sigma(1)}}{\gamma_B^{\sigma(1)}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}\\
&\quad\cdots\cdots\cdots\\
&\quad+\frac{\gamma_B^{\sigma(l_{\tilde{T}_B})}}{\gamma_B^{\sigma(l_{\tilde{T}_B})}+\gamma_A^{\sigma(1)}}+\frac{\gamma_B^{\sigma(l_{\tilde{T}_B})}}{\gamma_B^{\sigma(l_{\tilde{T}_B})}+\gamma_A^{\sigma(2)}}+\frac{\gamma_B^{\sigma(l_{\tilde{T}_B})}}{\gamma_B^{\sigma(l_{\tilde{T}_B})}+\gamma_A^{\sigma(3)}}\\
&\quad\left.+\cdots+\frac{\gamma_B^{\sigma(l_{\tilde{T}_B})}}{\gamma_B^{\sigma(l_{\tilde{T}_B})}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}\right)\\
&=\frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{T}_B}}\left(\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(1)}}+\frac{\gamma_B^{\sigma(1)}}{\gamma_B^{\sigma(1)}+\gamma_A^{\sigma(1)}}\right)\right.\\
&\quad+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(2)}}+\frac{\gamma_B^{\sigma(2)}}{\gamma_B^{\sigma(2)}+\gamma_A^{\sigma(1)}}\right)\\
&\quad+\cdots+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}+\frac{\gamma_B^{\sigma(l_{\tilde{T}_B})}}{\gamma_B^{\sigma(l_{\tilde{T}_B})}+\gamma_A^{\sigma(1)}}\right)\\
&\quad\left.+\cdots+\left(\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}+\frac{\gamma_B^{\sigma(l_{\tilde{T}_B})}}{\gamma_B^{\sigma(l_{\tilde{T}_B})}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}\right)\right)\\
&=\frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{T}_B}}\left(l_{\tilde{T}_B}\cdot l_{\tilde{T}_A}\right)=1.
\end{aligned}$$

Similarly, $P(\tilde{I}_A,\tilde{I}_B)+P(\tilde{I}_B,\tilde{I}_A)=1$ and $P(\tilde{F}_A,\tilde{F}_B)+P(\tilde{F}_B,\tilde{F}_A)=1$ can be achieved.

(7) If $\tilde{T}_A=\tilde{T}_B=\{0\}$, then $P(\tilde{T}_A,\tilde{T}_B)=0.5$ is certainly valid. If $\tilde{T}_A\neq\{0\}$ or $\tilde{T}_B\neq\{0\}$, then all elements in $\tilde{T}_i$ $(i=A,B)$ are arranged in ascending order, and $\gamma_i^{\sigma(\cdot)}$ $(i=A,B)$ is referred to as the $(\cdot)$th value in $\tilde{T}_i$ $(i=A,B)$. Based on Definition 9, since $\tilde{T}_A=\tilde{T}_B$, $\gamma_A^{\sigma(j)}=\gamma_B^{\sigma(j)}$, $\gamma_A^{\sigma(l_{\tilde{T}_A})}=\gamma_B^{\sigma(l_{\tilde{T}_B})}$ and $l_{\tilde{T}_A}=l_{\tilde{T}_B}$ can be obtained. Thus,

$$\begin{aligned}
P(\tilde{T}_A,\tilde{T}_B)&=\frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{T}_B}}\sum_{\gamma_A\in\tilde{T}_A}\sum_{\gamma_B\in\tilde{T}_B}\frac{\gamma_A}{\gamma_A+\gamma_B}\\
&=\frac{1}{(l_{\tilde{T}_A})^2}\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(1)}}+\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(2)}}\right.\\
&\quad+\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(3)}}+\cdots+\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}\\
&\quad+\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_A^{\sigma(1)}}+\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_A^{\sigma(2)}}+\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_A^{\sigma(3)}}\\
&\quad+\cdots+\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}+\\
&\quad\cdots\cdots\cdots\\
&\quad\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_A^{\sigma(1)}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_A^{\sigma(2)}}\\
&\quad\left.+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_A^{\sigma(3)}}+\cdots+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}\right)\\
&=\frac{1}{\left(l_{\tilde{T}_A}\right)^2}\left(\frac{1}{2}\,l_{\tilde{T}_A}+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(2)}}+\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_A^{\sigma(1)}}\right)\right.\\
&\quad+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(3)}}+\frac{\gamma_A^{\sigma(3)}}{\gamma_A^{\sigma(3)}+\gamma_A^{\sigma(1)}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\cdots+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_A^{\sigma(1)}}\right)\\
&+\left(\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_A^{\sigma(3)}}+\frac{\gamma_A^{\sigma(3)}}{\gamma_A^{\sigma(3)}+\gamma_A^{\sigma(2)}}\right)\\
&+\cdots+\left(\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_A^{\sigma(2)}}\right)\\
&+\cdots+\left(\frac{\gamma_A^{\sigma(l_{\tilde{T}_A}-1)}}{\gamma_A^{\sigma(l_{\tilde{T}_A}-1)}+\gamma_A^{\sigma(l_{\tilde{T}_A})}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_A^{\sigma(l_{\tilde{T}_A}-1)}}\right)\Bigg)\Bigg)\\
=&\ \frac{1}{\left(l_{\tilde{T}_A}\right)^2}\left(\frac{1}{2}l_{\tilde{T}_A}+l_{\tilde{T}_A}\left(l_{\tilde{T}_A}-1\right)-\left(1+2+\cdots+\left(l_{\tilde{T}_A}-1\right)\right)\right)\\
=&\ \frac{1}{\left(l_{\tilde{T}_A}\right)^2}\cdot\left(\frac{1}{2}l_{\tilde{T}_A}^2\right)=0.5.
\end{aligned}
$$

Similarly, (8) and (9) can be obtained.

Property 2: Let $A=\langle\tilde{T}_A,\tilde{I}_A,\tilde{F}_A\rangle$ and $B=\langle\tilde{T}_B,\tilde{I}_B,\tilde{F}_B\rangle$ be two MVNNs, and $P(A,B)$ be the likelihood of the relation $A\geq B$. The following properties can therefore be true:

(1) $0\leq P(A,B)\leq 1$;
(2) $P(A,A)=0.5$;
(3) $P(A,B)+P(B,A)=1$;
(4) if $P(A,B)=P(B,A)$, then $P(A,B)=P(B,A)=0.5$;
(5) if $P(A,B)\geq 0.5$ and $P(B,C)\geq 0.5$, then $P(A,C)\geq 0.5$;
(6) if $A=B$, then $P(A,B)=0.5$;
(7) If $A\geq B$, then $P(A,B)\geq 0.5$.

Proof: Only (6) and (7) will be proved and the other process of proof will be omitted here. ■

(6) If $A=B$ and $\tilde{T}_A=\tilde{T}_B=\{0\}$, $\tilde{I}_A=\tilde{I}_B=\{0\}$ and $\tilde{F}_A=\tilde{F}_B=\{0\}$, then $P(A\geq B)=0.5$ can be easily obtained. If $A=B$, and $\tilde{T}_A\neq\{0\}$ or $\tilde{T}_B\neq\{0\}$, $\tilde{I}_A\neq\{0\}$ or $\tilde{I}_B\neq\{0\}$ and $\tilde{F}_A=\{0\}$ and $\tilde{F}_B=\{0\}$, then $\tilde{T}_A=\tilde{T}_B$, $\tilde{I}_A=\tilde{I}_B$ and $\tilde{F}_A=\tilde{F}_B$ can be achieved based on Definition 9. According to Property 1, $P(\tilde{T}_A,\tilde{T}_B)=P(\tilde{I}_A,\tilde{I}_B)=P(\tilde{F}_A,\tilde{F}_B)=0.5$can be obtained. Thus, $P(A,B)=\frac{1}{3}(P(\tilde{T}_A\geq\tilde{T}_B)+P(\tilde{I}_A\geq\tilde{I}_B)+\ P(\tilde{F}_A,\tilde{F}_B))=0.5$ is certainly valid.

(7) If $A\geq B$, then according to Definition 9, $\gamma_A^{\sigma(j)}\leq\gamma_B^{\sigma(j)}$ and $\gamma_A^{\sigma(l_{\tilde{T}_A})}\leq\gamma_B^{\sigma(l_{\tilde{T}_B})}$ $(j=1,2,\ldots,l_{\tilde{T}},l_{\tilde{T}}=\min(l_{\tilde{T}_A},l_{\tilde{T}_B}))$ can be obtained. Thus,

$$
\begin{aligned}
P(\tilde{T}_A,\tilde{T}_B)=&\ \frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{T}_B}}\Bigg(\Bigg(\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(1)}}+\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_B^{\sigma(2)}}\right.\\
&\left.+\cdots+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}\right)\\
&+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(2)}}+\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_B^{\sigma(1)}}\right)\\
&+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_A^{\sigma(3)}}+\frac{\gamma_A^{\sigma(3)}}{\gamma_A^{\sigma(3)}+\gamma_B^{\sigma(1)}}\right)\\
&+\cdots+\left(\frac{\gamma_A^{\sigma(1)}}{\gamma_A^{\sigma(1)}+\gamma_B^{\sigma(l_{\tilde{T}_A})}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(1)}}\right)\\
&+\left(\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_B^{\sigma(3)}}+\frac{\gamma_A^{\sigma(3)}}{\gamma_A^{\sigma(3)}+\gamma_B^{\sigma(2)}}\right)\\
&+\cdots+\left(\frac{\gamma_A^{\sigma(2)}}{\gamma_A^{\sigma(2)}+\gamma_B^{\sigma(l_{\tilde{T}_A})}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(2)}}\right)\\
&+\cdots+\left(\frac{\gamma_A^{\sigma(l_{\tilde{T}_A}-1)}}{\gamma_A^{\sigma(l_{\tilde{T}_A}-1)}+\gamma_B^{\sigma(l_{\tilde{T}_A})}}+\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(l_{\tilde{T}_A}-1)}}\right)\Bigg)\Bigg)\\
\geq&\ \frac{1}{\left(l_{\tilde{T}_A}\right)^2}\left(\frac{1}{2}l_{\tilde{T}_A}+l_{\tilde{T}_A}(l_{\tilde{T}_A}-1)-\left(1+2+\cdots+\left(l_{\tilde{T}_A}-1\right)\right)\right)\\
=&\ \frac{1}{\left(l_{\tilde{T}_A}\right)^2}\cdot\left(\frac{1}{2}l_{\tilde{T}_A}^2\right)=0.5.
\end{aligned}
$$

Definition 11: Let $A=\langle\tilde{T}_A,\tilde{I}_A,\tilde{F}_A\rangle$ and $B=\langle\tilde{T}_B,\tilde{I}_B,\tilde{F}_B\rangle$ be two MVNNs. The following preference relations can then be obtained:

(1) If $P(A,B)>0.5$, then A is superior to B, denoted by $A\succ B$;
(2) If $P(A,B)=0.5$, then A is indifferent to B, denoted by $A\sim B$;
(3) If $P(A,B)<0.5$, then A is inferior to B, denoted by $A\prec B$.

Example 6. Based on Example 5, $P(A,B)=0.5149>0.5$, and A is superior to B, i.e. $A\succ B$.

Property 3: Let $A=\langle\tilde{T}_A,\tilde{I}_A,\tilde{F}_A\rangle$ and $B=\langle\tilde{T}_B,\tilde{I}_B,\tilde{F}_B\rangle$ be two MVNNs. If $A<B$, then $A\prec B$.

Proof: Let all elements in $\tilde{T}_i$ $(i=A,B)$ be arranged in ascending order, and $\gamma_i^{\sigma(\cdot)}$ $(i=A,B)$ be referred to as the $(\cdot)$-th value in $\tilde{T}_i$ $(i=A,B)$. If $A<B$, then according to Definition 9, $\gamma_A^{\sigma(j)}<\gamma_B^{\sigma(j)}$ and $\gamma_A^{\sigma(l_{\tilde{T}_A})}<\gamma_B^{\sigma(l_{\tilde{T}_B})}$ $(j=1,2,\ldots,l_{\tilde{T}},l_{\tilde{T}}=\min(l_{\tilde{T}_A},l_{\tilde{T}_B}))$ can be obtained. Thus, $\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}<\frac{1}{2}$. So $\sum_{\gamma_A\in\tilde{T}_A}\sum_{\gamma_B\in\tilde{T}_B}\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}<\frac{1}{2}\cdot l_{\tilde{T}_A}\cdot l_{\tilde{T}_B}$, i.e. $P(\tilde{T}_A,\tilde{T}_B)=\frac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{T}_B}}\sum_{\gamma_A\in\tilde{T}_A}\sum_{\gamma_B\in\tilde{T}_B}\frac{\gamma_A^{\sigma(l_{\tilde{T}_A})}}{\gamma_A^{\sigma(l_{\tilde{T}_A})}+\gamma_B^{\sigma(l_{\tilde{T}_B})}}<\frac{1}{2}$. Similarly, $P(\tilde{I}_A,\tilde{I}_B)=1-\frac{1}{l_{\tilde{I}_A}\cdot l_{\tilde{I}_B}}\sum_{\eta_A\in\tilde{I}_A}\sum_{\eta_B\in\tilde{I}_B}\frac{\eta_A^{\sigma(l_{\tilde{I}_A})}}{\eta_A^{\sigma(l_{\tilde{I}_A})}+\eta_B^{\sigma(l_{\tilde{I}_B})}}<\frac{1}{2}$ and $P(\tilde{F}_A,\tilde{F}_B)=1-\frac{1}{l_{\tilde{F}_A}\cdot l_{\tilde{F}_B}}\sum_{\xi_A\in\tilde{F}_A}\sum_{\xi_B\in\tilde{F}_B}\frac{\xi_A^{\sigma(l_{\tilde{F}_A})}}{\xi_A^{\sigma(l_{\tilde{F}_A})}+\xi_B^{\sigma(l_{\tilde{F}_B})}}<\frac{1}{2}$ can be true. Therefore, $P(A,B)=\frac{1}{3}(P(\tilde{T}_A,\tilde{T}_B)+P(\tilde{I}_A,\tilde{I}_B)+P(\tilde{F}_A,\tilde{F}_B))<\frac{1}{3}(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})=\frac{1}{2}$. Based on Definition 11, $A\prec B$ can be obtained. ■

4. The likelihood-based QUALIFLEX approach with MVNNs

Assume there are n alternatives denoted by $A=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ and m criteria denoted by $C=\{c_1,c_2,\ldots,c_m\}$, and the weight vector of criteria is $W=(w_1,w_2,\ldots,w_m)$, where $w_j\geq 0$ $(j=1,2,\ldots,m)$ and $\sum_{j=1}^{m}w_j=1$. Let

$R = (\alpha_{ij})_{n\times m}$ be the multi-valued neutrosophic decision matrix, and $\alpha_{ij} = \langle \tilde{T}_{\alpha_{ij}}, \tilde{I}_{\alpha_{ij}}, \tilde{F}_{\alpha_{ij}} \rangle$ be the evaluation value of α_i for criterion c_j being in the form of MVNNs. Where $\tilde{T}_{\alpha_{ij}}$ indicates the truth-membership function, $\tilde{I}_{\alpha_{ij}}$ indicates the indeterminacy-membership function and $\tilde{F}_{\alpha_{ij}}$ indicates the falsity-membership function.

Based on the likelihood of preference relations on MVNNs, this study compares the MVNNs ratings and proceeds to establish an extended QUALIFLEX method, which can be used to solve MCDM problems within a multi-valued neutrosophic environment. The proposed approach can be summarised in the following series of steps.

Step 1: Normalise the decision-making matrix.

In general, there are maximising criteria and minimising criteria in MCDM problems. According to the method proposed by Xu and Hu (2010), the minimising criteria can be transformed into maximising criteria as follows:

$$\beta_{ij} = \begin{cases} \alpha_{ij}, & \text{for maximizing criteria } c_j \\ (\alpha_{ij})^c, & \text{for minimizing criteria } c_j \end{cases}, \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \tag{12}$$

Here, $(\alpha_{ij})^c$ is the complement of α_{ij} as defined in Definition 8.

Step 2: Determine all of the possible permutations of the alternatives.

Take the set of alternative α and assume that there exist $n!$ permutations of the ranking of the alternatives. Let P^τ denote the τth permutation as

$$P^\tau = (\ldots, \alpha_\delta, \ldots, \alpha_\theta, \ldots), \ \tau = 1, 2, \ldots, n!. \tag{13}$$

Where $\alpha_\delta, \alpha_\theta \in \alpha$ and the alternative α_δ is ranked higher than or equal to α_θ.

Step 3: Calculate the likelihood.

Based on Definition 10, the likelihood of the relation $P(\alpha_{\delta j}, \alpha_{\theta j})$ $(j = 1, 2, \ldots, m)$ for each pair of alternatives $(\alpha_\delta, \alpha_\theta)$ $(\alpha_\delta, \alpha_\theta \in \alpha)$ can be obtained as follows:

$$P(\alpha_{\delta j}, \alpha_{\theta j}) = \frac{1}{3}(P(\tilde{T}_{\alpha_{\delta j}}, \tilde{T}_{\theta j}) + P(\tilde{I}_{\alpha_{\delta j}}, \tilde{I}_{\theta j}) + P(\tilde{F}_{\alpha_{\delta j}}, \tilde{F}_{\theta j})). \tag{14}$$

As indicated in Definition 11, each pair of alternatives $(\alpha_\delta, \alpha_\theta)$ at the level of ranking with respect to the criterion $c_j \in C$ is considered. Then the ranking corresponding to the permutation P^τ is determined. Next, based on the likelihood of the relation $P(\alpha_{\delta j}, \alpha_{\theta j})$, the following can be true: (1) if $P(\alpha_{\delta j}, \alpha_{\theta j}) > 0.5$, then concordance exists; if $P(\alpha_{\delta j}, \alpha_{\theta j}) = 0.5$, then indifference exists; and if $P(\alpha_{\delta j}, \alpha_{\theta j}) < 0.5$, then discordance exists. The larger the $P(\alpha_{\delta j}, \alpha_{\theta j})$, the higher the concordance.

Step 4: Calculate the weighted concordance index.

According to the weights $w_j (j = 1, 2, \ldots, m)$ of the criteria, the weighted concordance index $\varphi^\tau(\alpha_\delta, \alpha_\theta)$ for each pair of alternatives $(\alpha_\delta, \alpha_\theta)$ $(\alpha_\delta, \alpha_\theta \in \alpha)$ can be defined as follows:

$$\varphi^\tau(\alpha_\delta, \alpha_\theta) = \sum_{j=1}^{m} \left(P\left(\alpha_{\delta j}, \alpha_{\theta j}\right) \cdot w_j\right). \tag{15}$$

Step 5: Calculate the comprehensive concordance/ discordance index.

Based on Step 4, the comprehensive concordance index φ_τ of the permutation P^τ $(\tau = 1, 2, \ldots, n!)$ can be obtained as follows:

$$\varphi^\tau = \sum_{\alpha_\delta, \alpha_\theta \in \alpha} \varphi^\tau(\alpha_\delta, \alpha_\theta). \tag{16}$$

The comprehensive concordance index φ^τ can serve as the evaluation criterion of the chosen hypothesis for ranking the alternatives.

Step 6: Determine the ranking order of all alternatives.

The bigger the comprehensive concordance index of the permutation value, the better the final ranking result of the alternatives. Therefore, the optimal ranking order of alternatives can be determined by comparing the values φ^τ of each permutation P^τ, which is the permutation with the maximal value φ^τ, namely:

$$P^* = \max_{\tau=1}^{n!} \{\varphi^\tau\}. \tag{17}$$

5. An illustrative example

In this section, an example of an MCDM problem (adapted from Shen, Olfat, Govindan, Khodaverdi, & Diabat, 2013) is used to demonstrate the application and effectiveness of the proposed decision-making approach.

In order to lower environmental impacts and increase ecological efficiency, ABC Automobile manufacturing company wants to implement green practices at all stages of the manufacturing process to achieve profit and market share objectives. Thus, how to choose a suitable green supplier from several potential suppliers is a MCDM problem. Assume that there are three possible green suppliers $a_i (i = 1, 2, 3)$ to be selected. Each supplier is evaluated based on nine criteria, which are denoted by $c_j (j = 1, 2, \ldots, 9)$: c_1 is the pollution produced; c_2 is the resource consumption; c_3 is the eco-design; c_4 is the green image; c_5 is the environmental management system; c_6 is the commitment to green supply chain management from managers; c_7 is the use of environmentally friendly technology; c_8 is the use of environmentally friendly materials; and c_9 is the staff environmental training. Moreover, c_1 is a minimising type and other criteria are of

Table 1. Criteria for selecting and evaluating green suppliers.

Criteria	Name	Definition
c_1	Pollution production	Average volume of air emission pollutant, waster, solid wastes and harmful materials releases per day during measurement period
c_2	Resource consumption	Resource consumption in terms of raw material, energy and water during the measurement
c_3	Eco-design	Design of products for reduced consumption of material/energy, design of products for reuse, recycle, recovery of material, design of products
c_4	Green image	The ratio of green customers to total customers
c_5	Environmental management system	Environmental certifications such as ISO 1400, environmental policies, planning of environmental objectives, checking and control of environmental activities.
c_6	Commitment of GSCM from managers	Senior and mid-level managers commitment and support to improve green supply chain management practices and environmental performance
c_7	Use of environmentally friendly technology	The application of the environmental science to conserve the natural environment and resources, and to curb the negative impacts of human involvement
c_8	Use of environmentally friendly materials	The level of green recyclable material used in packaging and manufacturing of goods
c_9	Staff environmental training	Staff training on environmental targers

the maximising type. More details can be found in Table 1. The weights of criteria are provided by the company as follows: $w = (0.1226,\ 0.0900,\ 0.1311,\ 0.1415,\ 0.1303,\ 0.1017,\ 0.0846,\ 0.0974,\ 0.1008)$. A decision team including an operations manager, a purchasing manager and an environmental manager is invited to assess the performance of these three potential suppliers using each criterion. The decision-makers evaluate three suppliers using the nine criteria, and the results are in the form of MVNNs. This method is only suitable if the amount of decision-makers is small and they could evaluate these criteria in the form of MVNNs. One decision-maker could give several evaluation values for three membership functions. However, when more than one decision-maker evaluates the same value, it is only counted once. The evaluations of suppliers using the criteria can be found in Table 2.

5.1. An illustration of the proposed method

The procedures for obtaining the optimal alternative, by using the developed method, are as follows.

Step 1: Normalise the decision-making matrix.

Since c_1 is a minimising type and other criteria are of the maximising type, according to Equation (11), the normalised MVNN decision matrix $\tilde{R}^k = (\beta_{ij}^k)_{4\times 3}$ can be obtained and is shown in Table 3.

Step 2: Determine all of the possible permutations of the alternatives.

Since $n = 3$, there are $6(3! = 6)$ permutations of the rankings for all alternatives that must be tested and which are expressed in the following:

$$P^1 = (\alpha_1, \alpha_2, \alpha_3)\,,\ P^2 = (\alpha_1, \alpha_3, \alpha_2)\,,\ P^3 = (\alpha_2, \alpha_1, \alpha_3)\,,$$
$$P^4 = (\alpha_2, \alpha_3, \alpha_1)\,,\ P^5 = (\alpha_3, \alpha_1, \alpha_2)\,,\ P^6 = (\alpha_3, \alpha_2, \alpha_1)\,.$$

Step 3: Calculate the likelihood.

Based on the truth-membership likelihood, indeterminacy-membership likelihood and falsity-membership likelihood, $P(\tilde{T}_{\alpha_{\delta j}}, \tilde{T}_{\alpha_{\theta j}})$, $P(\tilde{I}_{\alpha_{\delta j}}, \tilde{I}_{\alpha_{\theta j}})$ and $P(\tilde{F}_{\alpha_{\delta j}}, \tilde{F}_{\alpha_{\theta j}})$, respectively, the likelihood $P(\alpha_{\delta j}, \alpha_{\alpha_{\theta j}})$ can be achieved and is shown in Table 4.

Step 4: Calculate the weighted concordance index.

Based on Equation (14) and the likelihood $P(\alpha_{\delta j}, \alpha_{\theta j})$in Step 3, the weighted concordance index $\varphi^{\tau}(\alpha_\delta, \alpha_\theta)$ for each pair of alternatives $(\alpha_\delta, \alpha_\theta)$ $(\alpha_\delta, \alpha_\theta \in \alpha)$ can be obtained and is shown in Table 5.

Step 5: Calculate the comprehensive concordance/discordance index.

Table 2. The evaluations of the green suppliers by decision-makers under criteria.

Criteria	Suppliers α_1	α_2	α_3
c_1	⟨{0.3,0.7},{0.4},{0.2}⟩	⟨{0.3,0.5},{0.5},{0.2}⟩	⟨{0.4,0.7},{0.3},{0.1,0.2}⟩
c_2	⟨{0.5,0.6},{0.3},{0.4}⟩	⟨{0.4},{0.5},{0.2}⟩	⟨{0.6},{0.3},{0.1,0.2}⟩
c_3	⟨{0.4},{0.6},{0.4}⟩	⟨{0.5, 0.7},{0.3},{0.3}⟩	⟨{0.5},{0.4},{0.3}⟩
c_4	⟨{0.7},{0.3},{0.2}⟩	⟨{0.3,0.4},{0.2},{0.4}⟩	⟨{0.4,0.6},{0.1},{0.3}⟩
c_5	⟨{0.5},{0.2},{0.1}⟩	⟨{0.5},{0.4},{0.1}⟩	⟨{0.4,0.7},{0.2},{0.5}⟩
c_6	⟨{0.6,0.7},{0.3},{0.2}⟩	⟨{0.6},{0.5,0.6},{0.3}⟩	⟨{0.5,0.6},{0.2},{0.6}⟩
c_7	⟨{0.6,0.9},{0.6},{0.4}⟩	⟨{0.6,0.8},{0.4},{0.6}⟩	⟨{0.5, 0.8},{0.2},{0.4}⟩
c_8	⟨{0.6,0.8},{0.3},{0.5}⟩	⟨{0.7,0.9},{0.2},{0.4}⟩	⟨{0.6},{0.1,0.5},{0.3}⟩
c_9	⟨{0.7, 0.9},{0.3},{0.6}⟩	⟨{0.6,0.8},{0.2},{0.4}⟩	⟨{0.6,0.8},{0.3},{0.4}⟩

Table 3. Normalised decision matrix.

Criteria	Suppliers		
	α_1	α_2	α_3
c_1	⟨{0.2},{0.6},{0.3,0.7}⟩	⟨{0.2},{0.5},{0.3,0.5}⟩	⟨{0.1,0.2},{0.7},{0.4,0.7}⟩
c_2	⟨{0.5,0.6},{0.3},{0.4}⟩	⟨{0.4},{0.5},{0.2}⟩	⟨{0.6},{0.3},{0.1,0.2}⟩
c_3	⟨{0.4},{0.6},{0.4}⟩	⟨{0.5, 0.7},{0.3},{0.3}⟩	⟨{0.5},{0.4},{0.3}⟩
c_4	⟨{0.7},{0.3},{0.2}⟩	⟨{0.3,0.4},{0.2},{0.4}⟩	⟨{0.4,0.6},{0.1},{0.3}⟩
c_5	⟨{0.5},{0.2},{0.1}⟩	⟨{0.5},{0.4},{0.1}⟩	⟨{0.4,0.7},{0.2},{0.5}⟩
c_6	⟨{0.6,0.7},{0.3},{0.2}⟩	⟨{0.6},{0.5,0.6},{0.3}⟩	⟨{0.5,0.6},{0.2},{0.6}⟩
c_7	⟨{0.6,0.9},{0.6},{0.4}⟩	⟨{0.6,0.8},{0.4},{0.6}⟩	⟨{0.5, 0.8},{0.2},{0.4}⟩
c_8	⟨{0.6,0.8},{0.3},{0.5}⟩	⟨{0.7,0.9},{0.2},{0.4}⟩	⟨{0.6},{0.1,0.5},{0.3}⟩
c_9	⟨{0.7, 0.9},{0.3},{0.6}⟩	⟨{0.6,0.8},{0.2},{0.4}⟩	⟨{0.6,0.8},{0.3},{0.4}⟩

Based on Equation (15), the comprehensive concordance/discordance index φ^τ can be calculated as follows:

$$\varphi^1 = 1.4856;\ \varphi^2 = 1.5002;\ \varphi^3 = 1.4802;\ \varphi^4 = 1.4997;$$
$$\varphi^5 = 1.5198;\ \varphi^6 = 1.5144.$$

Step 6: Determine the optimal ranking of all alternatives.

According to the results in Step 5 and Equation (16), $\varphi^5 > \varphi^6 > \varphi^2 > \varphi^4 > \varphi^1 > \varphi^3$ and $P^* = \max_{\tau=1}^{n!}\{\varphi^\tau\} = P^5$ can be obtained. Thus, the final ranking of the three potential suppliers is: $\alpha_3 \succ \alpha_1 \succ \alpha_2$ and the best alternative is α_3, while the worst is α_2.

5.2. A comparison analysis

In order to validate the feasibility of the proposed decision-making method, a comparative study was conducted with other methods based on the same illustrative example. The comparison analysis includes two cases. One consists of the two methods that were outlined in Ye (2014a) and Peng et al. (2014), which are compared to the proposed method with simplified neutrosophic information. In the other, the method that was introduced in Ye (2015b) and Peng et al. (2015), which are

Table 4. The results of the likelihoods of $P(\alpha_{\delta j}, \alpha_{\theta j})$.

	$P(\tilde{T}_{\alpha_{1j}},\tilde{T}_{\alpha_{2j}})$	$P(\tilde{I}_{\alpha_{1j}},\tilde{I}_{\alpha_{2j}})$	$P(\tilde{F}_{\alpha_{1j}},\tilde{F}_{\alpha_{2j}})$	$P(\alpha_{1j},\alpha_{2j})$	$P(\tilde{T}_{\alpha_{1j}},\tilde{T}_{\alpha_{3j}})$	$P(\tilde{I}_{\alpha_{1j}},\tilde{I}_{\alpha_{3j}})$	$P(\tilde{F}_{\alpha_{1j}},\tilde{F}_{\alpha_{3j}})$	$P(\alpha_{1j},\alpha_{3j})$
c_1	0.5000	0.5455	0.4604	0.5020	0.5833	0.5385	0.5338	0.5519
c_2	0.5778	0.6250	0.3333	0.5120	0.4773	0.5000	0.2667	0.4146
c_3	0.4040	0.3333	0.4286	0.3886	0.4444	0.4000	0.4286	0.4243
c_4	0.6682	0.4000	0.6667	0.5783	0.5874	0.2500	0.6000	0.4791
c_5	0.5000	0.6667	0.5000	0.5556	0.4861	0.5000	0.8333	0.6065
c_6	0.5192	0.6458	0.6000	0.5884	0.5418	0.4000	0.7500	0.5639
c_7	0.5145	0.4000	0.6000	0.5048	0.5366	0.2500	0.5000	0.4289
c_8	0.4664	0.4000	0.4444	0.4369	0.5357	0.4375	0.3750	0.4494
c_9	0.5336	0.4000	0.4000	0.4445	0.5336	0.5000	0.4000	0.4779
	$P(\tilde{T}_{\alpha_{2j}},\tilde{T}_{\alpha_{1j}})$	$P(\tilde{I}_{\alpha_{2j}},\tilde{I}_{\alpha_{1j}})$	$P(\tilde{F}_{\alpha_{2j}},\tilde{F}_{\alpha_{1j}})$	$P(\alpha_{2j},\alpha_{1j})$	$P(\tilde{T}_{\alpha_{2j}},\tilde{T}_{\alpha_{3j}})$	$P(\tilde{I}_{\alpha_{2j}},\tilde{I}_{\alpha_{3j}})$	$P(\tilde{F}_{\alpha_{2j}},\tilde{F}_{\alpha_{3j}})$	$P(\alpha_{2j},\alpha_{3j})$
c_1	0.5000	0.4545	0.5396	0.498	0.5833	0.5833	0.5748	0.5805
c_2	0.4222	0.375	0.6667	0.4880	0.4000	0.3750	0.4167	0.3972
c_3	0.5960	0.6667	0.5714	0.6114	0.5417	0.5714	0.5000	0.5377
c_4	0.3318	0.6000	0.3333	0.4217	0.4155	0.3333	0.4286	0.3925
c_5	0.5000	0.3333	0.5000	0.4444	0.4861	0.3333	0.8333	0.5509
c_6	0.4808	0.3542	0.4000	0.4116	0.5227	0.2679	0.6667	0.4858
c_7	0.4855	0.6000	0.4000	0.4952	0.5224	0.3333	0.4000	0.4186
c_8	0.5336	0.6000	0.5556	0.5631	0.5692	0.5238	0.4286	0.5072
c_9	0.4664	0.6000	0.6000	0.5555	0.5000	0.6000	0.5000	0.5333
	$P(\tilde{T}_{\alpha_{3j}},\tilde{T}_{\alpha_{1j}})$	$P(\tilde{I}_{\alpha_{3j}},\tilde{I}_{\alpha_{1j}})$	$P(\tilde{F}_{\alpha_{3j}},\tilde{F}_{\alpha_{1j}})$	$P(\alpha_{3j},\alpha_{1j})$	$P(\tilde{T}_{\alpha_{3j}},\tilde{T}_{\alpha_{2j}})$	$P(\tilde{I}_{\alpha_{3j}},\tilde{I}_{\alpha_{2j}})$	$P(\tilde{F}_{\alpha_{3j}},\tilde{F}_{\alpha_{2j}})$	$P(\alpha_{3j},\alpha_{2j})$
c_1	0.4167	0.4615	0.4662	0.4481	0.4167	0.4167	0.4252	0.4195
c_2	0.5227	0.5000	0.7333	0.5854	0.6000	0.625	0.5833	0.6028
c_3	0.5556	0.6000	0.5714	0.5757	0.4583	0.4286	0.5000	0.4623
c_4	0.4126	0.7500	0.4000	0.5209	0.5845	0.6667	0.5714	0.6075
c_5	0.5139	0.5000	0.1667	0.3935	0.5139	0.6667	0.1667	0.4491
c_6	0.4582	0.6000	0.2500	0.4361	0.4773	0.7321	0.3333	0.5142
c_7	0.4634	0.7500	0.5000	0.5711	0.4776	0.6667	0.6000	0.5814
c_8	0.4643	0.5625	0.6250	0.5506	0.4308	0.4762	0.5714	0.4928
c_9	0.4664	0.5000	0.6000	0.5221	0.5000	0.4000	0.5000	0.4667

Table 5. The results of the weighted concordance/discordance index.

P^1	$\varphi^1(\alpha_1,\alpha_2)$	$\varphi^1(\alpha_1,\alpha_3)$	$\varphi^1(\alpha_2,\alpha_3)$	P^2	$\varphi^2(\alpha_1,\alpha_3)$	$\varphi^2(\alpha_1,\alpha_2)$	$\varphi^2(\alpha_3,\alpha_2)$
	0.5027	0.4902	0.4927		0.4902	0.5027	0.5073
P^3	$\varphi^3(\alpha_2,\alpha_1)$	$\varphi^3(\alpha_2,\alpha_3)$	$\varphi^3(\alpha_1,\alpha_3)$	P^4	$\varphi^4(\alpha_2,\alpha_3)$	$\varphi^4(\alpha_2,\alpha_1)$	$\varphi^4(\alpha_3,\alpha_1)$
	0.4973	0.4927	0.4902		0.4927	0.4973	0.5098
P^5	$\varphi^5(\alpha_3,\alpha_1)$	$\varphi^5(\alpha_3,\alpha_2)$	$\varphi^5(\alpha_1,\alpha_2)$	P^6	$\varphi^6(\alpha_3,\alpha_2)$	$\varphi^6(\alpha_3,\alpha_1)$	$\varphi^6(\alpha_2,\alpha_1)$
	0.5098	0.5073	0.5027		0.5073	0.5098	0.4973

Table 6. A comparison of different methods with simplified neutrosophic information.

Methods	Ranking of alternatives
Ye (2014a)	$\alpha_2 \succ \alpha_3 \succ \alpha_1$
Peng et al. (2014)	$\alpha_3 \succ \alpha_2 \succ \alpha_1$
Proposed method	$\alpha_3 \succ \alpha_2 \succ \alpha_1$

Table 7. A comparison of different methods with multi-valued neutrosophic information.

Methods	Ranking of alternatives
Ye (2015b)	$\alpha_1 \succ \alpha_3 \succ \alpha_2$ or $\alpha_3 \succ \alpha_1 \succ \alpha_2$
Peng et al. (2015)	$\alpha_1 \succ \alpha_3 \succ \alpha_2$ or $\alpha_3 \succ \alpha_1 \succ \alpha_2$
Proposed method	$\alpha_3 \succ \alpha_1 \succ \alpha_2$

extended to a multi-valued neutrosophic environment, are compared with the proposed approach with multi-valued neutrosophic information.

Case 1. The proposed approach is compared with two methods of Ye (2014a) and Peng et al. (2014) using simplified neutrosophic information.

Ye (2014a) defined the weighted arithmetic averaging operator and weighted geometric averaging operator to resolve MCDM problems with SNNs. Furthermore, Peng et al. (2014) developed an extended ELECTRE method for dealing with simplified neutrosophic MCDM problems, which is closest to the proposed method. However, their methods fail to resolve MCDM problems where the assessments of alternatives regarding the criteria are in the form of MVNNs. Thus, the decision-making information should be modified to facilitate the comparative analysis. With regard to the same illustrative example, all multi-valued neutrosophic evaluation values are translated into single-valued neutrosophic values by using the mean values of truth-membership, indeterminacy-membership and falsity-membership.

For the method of Ye (2014a), the weighted arithmetic averaging operator and the weighted geometric averaging operator are used to aggregate the nine criteria of each alternative, then the similarity measures of each alternative can be obtained to rank the alternatives. However, the concordance and discordance matrixes are calculated first in the method of Peng et al. (2014), and the outranking matrices are obtained to determine the final order of alternatives. Therefore, when the methods of Ye (2014a), Peng et al. (2014) and the proposed approach are used to solve the modified MCDM problem, the results can be found in Table 6.

From the results in Table 6, it can be seen that if the weighted arithmetic averaging and the weighted geometric averaging operators are utilised, respectively, then the final ranking is $\alpha_2 \succ \alpha_3 \succ \alpha_1$, and the best alternative is α_2. However, if the method of Peng et al. (2014) is utilised to manage the modified decision-making information, then the final ranking is $\alpha_3 \succ \alpha_2 \succ \alpha_1$, which is consistent with that of the proposed approach, and the best alternative is α_3.

Case 2. The proposed approach is compared with the method of Ye (2015b) and Peng et al. (2015) using multi-valued neutrosophic information.

The methods in Ye (2015b) and Peng et al. (2015) can deal with multi-valued neutrosophic information directly. Then the results can be found in Table 7.

From the results presented in Table 7, it can be seen that if the single-valued neutrosophic hesitant fuzzy weighted arithmetic averaging and single-valued neutrosophic hesitant fuzzy weighted geometric averaging operators in Ye (2015b) are utilised, respectively, then the final ranking is $\alpha_1 \succ \alpha_3 \succ \alpha_2$ or $\alpha_3 \succ \alpha_1 \succ \alpha_2$. Whereas if the power weighted arithmetic averaging and power weighted geometric averaging operators in Peng et al. (2015) are used, respectively, then the final ranking is either $\alpha_1 \succ \alpha_3 \succ \alpha_2$ or $\alpha_3 \succ \alpha_1 \succ \alpha_2$. Therefore, the final rankings obtained by utilising the methods of Ye (2015b) and Peng et al. (2015) are different from that using the proposed approach. Moreover, when the aggregation operators in the methods of Ye (2015b) and Peng et al. (2015) are used, then there are a large number of elements in the truth-membership, indeterminacy-membership and falsity-membership degrees for the three aggregated values. For example, 32 elements can be obtained for the truth-membership degree of alternative α_1.

Thus, from the comparison analysis presented above, two issues can be discussed.

First, the result of the proposed approach is different to that using the methods of Ye (2014a, 2015b) and Peng et al. (2015) with simplified neutrosophic and multi-valued neutrosophic information. Although different aggregation operators can be used to deal with the different relationships of the aggregated arguments, the number of operations and the sizes of the results will exponentially increase if more MVNNs are involved in the operations. Furthermore, different aggregation operators can lead to different results. The deterioration caused by these complexities may confine the application of aggregation operators.

Second, the proposed approach and the method of Peng et al. (2014) produce the same results for a MCDM problem with a suitable number of alternatives and criteria. Moreover, in the illustrative example, where there are few alternatives and a large number of criteria, if the method of Peng et al. (2014) is used, then the final ranking results of the alternatives cannot be obtained directly.

However, the proposed approach with MVNNs differs from the existing methods, which always involve operations whose impact on the final solution may be considerable as stated earlier, as the developed method can overcome these shortcomings. The loss and distortion of the preference information provided can be avoided, which makes the final results better correspond with real decision-making problems. Moreover, the proposed method is preferred to be used to resolving problems in which the number of criteria observably exceeds the number of alternatives. Therefore, the proposed approach can effectively deal with the preference information expressed by MVNNs, which is a prerequisite of guaranteeing the accuracy of the final rankings.

6. Conclusions

MVNSs can be applied to problems with uncertain, imprecise, incomplete and inconsistent information, which widely exist in scientific and engineering cases. In this paper, the preference relations of MVNNs based on likelihood were developed. Subsequently, based on the likelihood of MVNNs, an extended QUALIFLEX approach was proposed to deal with MCDM problems where the data are expressed by MVNNs. Finally, an illustrative example demonstrated the application of the proposed decision-making approach, and showed that the results are feasible and credible. The primary advantages of the approach developed in this paper over the other methods are not only its ability to deal effectively with preference information in the form of MVNNs, but also its ability to resolve MCDM problems with a few alternatives and a large set of criteria, which makes the final results better correspond with real decision-making problems. In future research, the related measures of MVNNs will be further investigated.

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