A MULTIBODY DYNAMICS BENCHMARK ON THE EQUATIONS OF MOTION OF AN UNCONTROLLED BICYCLE

A. L. Schwab
Lab. for Engineering Mechanics
Delft University of Technology
Mekelweg 2, NL-2628 CD Delft
The Netherlands
a.l.schwab@wbmt.tudelft.nl

J. P. Meijaard
School of MMME
The University of Nottingham
University Park
Nottingham NG7 2RD
United Kingdom
Jaap.Meijaard@nottingham.ac.uk

J. M. Papadopoulos
Paper Converting Machine Compan
Green Bay, Wisconsin, USA
jimpapadopoulos@pcmc.com

Abstract
In this paper we present the linearized equations of motion for a bicycle as a benchmark. The results obtained by pencil-and-paper and two programs are compared. The bicycle model we consider here consists of four rigid bodies, viz. a rear frame, a front frame being the front fork and handlebar assembly, a rear wheel and a front wheel, which are connected by revolute joints. The contact between the knife-edge wheels and the flat level surface is modelled by holonomic constraints in the normal direction and by non-holonomic constraints in the longitudinal and lateral directions. The rider is rigidly attached to the rear frame with hands free from the handlebar. This system has three degrees of freedom: the roll, the steer, and the forward speed. For the benchmark we consider the linearized equations for small perturbations of the upright steady forward motion. The entries of the matrices of these equations form the basis for comparison. Three different kinds of methods to obtain the results are compared: pencil-and-paper, the numeric multibody dynamics program SPACAR, and the symbolic software system AutoSim. Because the results of the three methods agree within the machine round-off error, we assume that the results are correct and can be used as a bicycle dynamics benchmark.

Key words
Bicycle Dynamics, Benchmark, Linearization, Stability, Nonholonomic Systems.

1 Introduction

A variety of vehicles can be statically unstable yet dynamically stable, for example a skateboard with a rigidly attached rider, a tricycle with raked steering axis, or a bicycle/motorcycle. Of these the bicycle is the most interesting, yet the hardest to analyse correctly. As a result the literature contains a great many flawed equations, and widespread qualitative explanations of uncontrolled self-stability which are inconsistent with careful analyses.

It is the purpose of this paper to present a high-precision benchmark for the linearized equations of motion for a clearly defined bicycle travelling at a range of speeds. Alternative formulations, or even non-linear simulation of a small perturbation, can therefore be checked with confidence. A second aim is to present exhaustively confirmed linearized equations of motion suitable for research and application.

The study of bicycle and motorcycle dynamics has attracted attention from mechanical engineers such as [Rankine, 1869; Klein and Sommerfeld, 1910; Timoshenko and Young, 1948; Den Hartog, 1948; Neĭmark and Fufaev, 1972; Kane, 1975] and many others, also outside the engineering discipline. Investigations have ranged from purely ad hoc analyses to full non-linear computer simulations.

The first publication of the full non-linear and also the linearized equations of motion for an upright uncontrolled bicycle was by [Whipple, 1899]. His linearization was found to be correct except for some typographical errors. Subsequently equations were derived by scores of people, some of these agree, others do not. The Master thesis by [Hand, 1988] gives a detailed review.

The organization of the paper is as follows. After this introduction the bicycle model is described. In Section 3 the linearized equations of motion are presented in an algorithmic manner. In Section 4 the results for the benchmark bicycle are presented and discussed. In Sections 5 and 6 the derivation of the equations of motion by means of the multibody dynamics software programs SPACAR and AutoSim are discussed and the results are compared. Next, in Section 7, some bicycle dynamics folklore is disproved by means of an example. The paper ends with some conclusions. An appendix reviews the literature on bicycle dynamics up

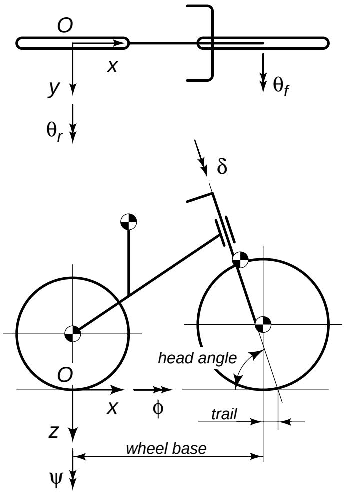

Figure 1. Bicycle model together with the coordinate system, the degrees of freedom, and the parameters.

Parameter	Symbol	Value
Wheel base	w	1.02 m
Trail	t	0.08 m
Head angle	α	$\arctan(3)$
Gravity	g	9.81 N/kg
Forward speed	v	*variable* m/s
Rear wheel		
Radius	R_{rw}	0.3 m
Mass	m_{rw}	2 kg
Mass moments of inertia	(A_{xx}, A_{yy}, A_{zz})	$(0.06, 0.12, 0.06)$ kgm^2
Rear frame		
Position centre of mass	(x_{rf}, y_{rf}, z_{rf})	$(0.3, 0, -0.9)$ m
Mass	m_{rf}	85 kg
Mass moments of inertia	$\begin{bmatrix} B_{xx} & 0 & B_{xz} \\ & B_{yy} & 0 \\ sym. & & B_{zz} \end{bmatrix}$	$\begin{bmatrix} 9.2 & 0 & 2.4 \\ & 11 & 0 \\ & & 2.8 \end{bmatrix}$ kgm^2
Front frame		
Position centre of mass	(x_{ff}, y_{ff}, z_{ff})	$(0.9, 0, -0.7)$ m
Mass	m_{ff}	4 kg
Mass moments of inertia	$\begin{bmatrix} C_{xx} & 0 & C_{xz} \\ & C_{yy} & 0 \\ sym. & & C_{zz} \end{bmatrix}$	$\begin{bmatrix} 0.0546 & 0 & -0.0162 \\ & 0.06 & 0 \\ & & 0.0114 \end{bmatrix}$ kgm^2
Front wheel		
Radius	R_{fw}	0.35 m
Mass	m_{fw}	3 kg
Mass moments of inertia	(D_{xx}, D_{yy}, D_{zz})	$(0.14, 0.28, 0.14)$ kgm^2

Table 1. Parameters for the benchmark bicycle from Figure 1.

until 1988 and shows the listings of the bicycle model input for the multibody dynamics software programs.

2 Bicycle Model

The mechanical model of the bicycle consists of four rigid bodies, viz. the rear frame with the rider rigidly attached to it, the front frame being the front fork and handle bar assembly and the two knife-edge wheels. These bodies are interconnected by revolute hinges at the steering head between the rear frame and the front frame and at the two wheel hubs. In the reference configuration, all bodies are symmetric relative to the bicycle midplane. The contact between the stiff non-slipping wheels and the flat level surface is modelled by holonomic constraints in the normal direction and by non-holonomic constraints in the longitudinal and lateral direction. There is no friction, apart from the idealized friction between the non-slipping wheels and the surface, nor propulsion and no rider control, the so-called hands free coasting operation. These assumptions make the model energy-conserving. In the reference position, the global Cartesian coordinate system is located at the rear-wheel contact point O, where the x-axis points in the longitudinal direction of the bicycle and the z-axis is directed downwards. Figure 1 shows the directions of the axes, where the terminology used mainly follows the SAE recommended practice as described in the report SAE-J607e [SAE, 2001], last revised in 1976.

The mechanical model of the bicycle has three degrees of freedom: the roll angle ϕ of the rear frame, the steering angle δ, and the rotation θ_r of the rear wheel with respect to the rear frame. The angles are defined as follows. The orientation of the rear frame with respect to the global reference frame O–xyz is given by a sequence of three angular rotations: a yaw rotation, ψ, about the z–axis, a roll rotation, ϕ, about the rotated x–axis, and a pitch rotation, θ, about the rotated y–axis. These rotations are materialized and depicted in Figure 5 by three hinges in series, ①, ②, and ③, mounted at the rear hub. The steering angle δ is the rotation of the front frame with respect to the rear frame about the steering axis. Due to the non-holonomic constraints there are four extra kinematic coordinates which describe, together with the degrees of freedom, the configuration of the system [Schwab and Meijaard, 2003]. The four kinematic coordinates are taken here as the Cartesian coordinates x and y of the rear-wheel contact point, the yaw angle ψ of the rear frame, and the rotation θ_f of the front wheel with respect to the front frame.

The dimensions and mechanical properties of the benchmark model are presented in Table 1. The system is symmetric about the vertical longitudinal plane and the wheels are rotationally symmetric about their axles. The mass moments of inertia are given at the centre of mass of the individual bodies and along the global xyz-axes.

3 Linearized Equations of Motion

This section gives an algorithmic presentation of the linearized equations of motion for the bicycle model under study as derived by [Papadopoulos, 1987]. The equations of motion are obtained by pencil-and-paper using D'Alembert's principle and linear and angular momentum balances. They are expressed in terms of small changes in the degrees of freedom ϕ, the rear frame roll angle, and δ, the steering angle, from the upright straight ahead configuration $\phi_0 = 0$, $\delta_0 = 0$, at a forward speed of $v = -\dot{\theta}_r R_{rw}$.

Let us consider the bicycle from Figure 1 and Table 1. The subscripts used are: rw for the rear wheel, rf for the rear frame, ff for the front frame, fw for the front wheel, t for the total system, f for the front assembly which is the front frame plus the front wheel, x, y, and z are the directions along the global xyz-axes, and $\boldsymbol{\lambda}$

is the direction of the steering axis pointing downward. Then the algorithm is as follows. For the system as a whole, calculate the total mass and the corresponding centre of mass with respect to the origin O as

$$m_t = m_{rw} + m_{rf} + m_{ff} + m_{fw}, \tag{1}$$

$$x_t = (x_{rf} m_{rf} + x_{ff} m_{ff} + w m_{fw})/m_t, \tag{2}$$

$$z_t = (-R_{rw} m_{rw} + z_{rf} m_{rf} + z_{ff} m_{ff} - R_{fw} m_{fw})/m_t. \tag{3}$$

For the system as a whole, calculate the relevant mass moments and products of inertia at the origin O along the global axes as

$$T_{xx} = A_{xx} + B_{xx} + C_{xx} + D_{xx} + m_{rw} R_{rw}^2 + m_{rf} z_{rf}^2 + m_{ff} z_{ff}^2 + m_{fw} R_{fw}^2, \tag{4}$$

$$T_{xz} = B_{xz} + C_{xz} - m_{rf} x_{rf} z_{rf} - m_{ff} x_{ff} z_{ff} + m_{fw} w R_{fw}, \tag{5}$$

$$T_{zz} = A_{zz} + B_{zz} + C_{zz} + D_{zz} + m_{rf} x_{rf}^2 + m_{ff} x_{ff}^2 + m_{fw} w^2. \tag{6}$$

Now determine the same properties for the front assembly, being the front frame and the front wheel, as

$$m_f = m_{ff} + m_{fw}, \tag{7}$$

$$x_f = (x_{ff} m_{ff} + w m_{fw})/m_f, \tag{8}$$

$$z_f = (z_{ff} m_{ff} - R_{fw} m_{fw})/m_f, \tag{9}$$

and calculate the relevant mass moments and products of inertia for the front assembly at the centre of mass of the front assembly along the global axes as

$$F_{xx} = C_{xx} + D_{xx} + m_{ff}(z_{ff} - z_f)^2 + m_{fw}(R_{fw} + z_f)^2, \tag{10}$$

$$F_{xz} = C_{xz} - m_{ff}(x_{ff} - x_f)(z_{ff} - z_f) + m_{fw}(w - x_f)(R_{fw} + z_f), \tag{11}$$

$$F_{zz} = C_{zz} + D_{zz} + m_{ff}(x_{ff} - x_f)^2 + m_{fw}(w - x_f)^2. \tag{12}$$

Let λ be the angle of the steering axis $\boldsymbol{\lambda} = (\sin(\lambda), 0, \cos(\lambda))^T$ with the global z-axis in the vertical plane,

$$\lambda = \pi/2 - \alpha. \tag{13}$$

Calculate the perpendicular distance that the centre of mass of the front assembly is ahead of the steering axis,

$$u = (x_f - w - t)\cos(\lambda) - z_f \sin(\lambda). \tag{14}$$

Calculate for the front assembly the relevant mass moments and products of inertia along the steering axis and the global axes at points where they intersect as

$$F_{\lambda\lambda} = m_f u^2 + F_{xx}\sin(\lambda)^2 + 2F_{xz}\sin(\lambda)\cos(\lambda) + F_{zz}\cos(\lambda)^2, \tag{15}$$

$$F_{\lambda x} = -m_f u z_f + F_{xx}\sin(\lambda) + F_{xz}\cos(\lambda), \tag{16}$$

$$F_{\lambda z} = m_f u x_f + F_{xz}\sin(\lambda) + F_{zz}\cos(\lambda). \tag{17}$$

Define the ratio of the mechanical trail (i.e. the perpendicular distance that the front wheel contact point is behind the steering axis) to the wheelbase as

$$f = t\cos(\lambda)/w. \tag{18}$$

Calculate for the rear and the front wheel the angular momentum along the y-axis divided by the forward speed, together with their sum as

$$S_r = A_{yy}/R_{rw}, \tag{19}$$

$$S_f = D_{yy}/R_{fw}, \tag{20}$$

$$S_t = S_r + S_f. \tag{21}$$

Define a frequently appearing static moment term as

$$S_u = m_f u + f m_t x_t. \tag{22}$$

Now the linearized equations of motion for the bicycle expressed in the degrees of freedom $\mathbf{q}^d = (\phi, \delta)^T$ have the form

$$\mathbf{M}\ddot{\mathbf{q}}^d + [\mathbf{C1}\cdot v]\dot{\mathbf{q}}^d + [\mathbf{K0} + \mathbf{K2}\cdot v^2]\mathbf{q}^d = \mathbf{f}^d, \tag{23}$$

with a constant mass matrix, $\mathbf{M}$, a "damping" matrix $\mathbf{C1}\cdot v$ which is linear in the forward speed, and a stiffness matrix which is the sum of a constant part, $\mathbf{K0}$, and a part, $\mathbf{K2}\cdot v^2$, which is quadratic in the forward speed. The linearized equation of motion for the third degree of freedom, the rotation θ_r of the rear wheel, is decoupled from the first two (23) and takes on the very simple form of: $\ddot{\theta}_r = 0$. This means that the forward speed remains constant for small changes in the upright configuration. The forces on the right-hand side, $\mathbf{f}^d$, are the applied forces which are energetically dual to the degrees of freedom $\mathbf{q}^d$. For the bicycle model the first is M_ϕ, the action-reaction roll moment between the fixed space and the rear frame. In practice such a torque could be applied by side wind, or by a parent teaching a child to ride. The second is M_δ, the action-reaction steering moment between the rear frame and the front frame. This is the torque that would be applied by a rider's hands, a steering spring-damper, or a controller. In the case of an ordinary uncontrolled bicycle, both of these moments are taken to be zero. The elements of the mass matrix are

$$
\begin{aligned}
M(1,1) &= T_{xx}, \\
M(1,2) &= F_{\lambda x} + f T_{xz}, \\
M(2,1) &= M(1,2), \\
M(2,2) &= F_{\lambda\lambda} + 2 f F_{\lambda z} + f^2 T_{zz}.
\end{aligned} \tag{24}
$$

The velocity-independent elements of the stiffness matrix are

$$
\begin{aligned}
K0(1,1) &= g m_t z_t, \\
K0(1,2) &= -g S_u, \\
K0(2,1) &= K0(1,2), \\
K0(2,2) &= -g S_u \sin(\lambda),
\end{aligned} \tag{25}
$$

and the elements of the stiffness matrix to be multiplied by the square of the forward speed are

$$
\begin{aligned}
K2(1,1) &= 0, \\
K2(1,2) &= (S_t - m_t z_t)\cos(\lambda)/w, \\
K2(2,1) &= 0, \\
K2(2,2) &= (S_u + S_f \sin(\lambda))\cos(\lambda)/w.
\end{aligned} \tag{26}
$$

Finally, the "damping" matrix which is to be multiplied by the forward speed is given by

$$
\begin{aligned}
C1(1,1) &= 0, \\
C1(1,2) &= f S_t + S_f \cos(\lambda) + T_{xz}\cos(\lambda)/w - f m_t z_t, \\
C1(2,1) &= -(f S_t + S_f \cos(\lambda)), \\
C1(2,2) &= F_{\lambda z}\cos(\lambda)/w + f(S_u + T_{zz}\cos(\lambda)/w).
\end{aligned} \tag{27}
$$

4 Results

Substitution of the parameter values from Table 1 results in the following values for the entries in the mass matrix from (24),

$$
\mathbf{M} = \begin{bmatrix} 80.812\,100\,000\,000\,02, & 2.323\,431\,426\,235\,49 \\ 2.323\,431\,426\,235\,49, & 0.301\,265\,709\,342\,56 \end{bmatrix}, \tag{28}
$$

the constant stiffness matrix from (25),

$$
\mathbf{K0} = \begin{bmatrix} -794.119\,500\,000\,000, & -25.739\,089\,291\,258 \\ -25.739\,089\,291\,258, & -8.139\,414\,705\,882 \end{bmatrix}, \tag{29}
$$

the stiffness matrix from (26) which is proportional to the square of the forward speed

$$
\mathbf{K2} = \begin{bmatrix} 0, & 76.406\,208\,759\,656\,57 \\ 0, & 2.675\,605\,536\,332\,18 \end{bmatrix}, \tag{30}
$$

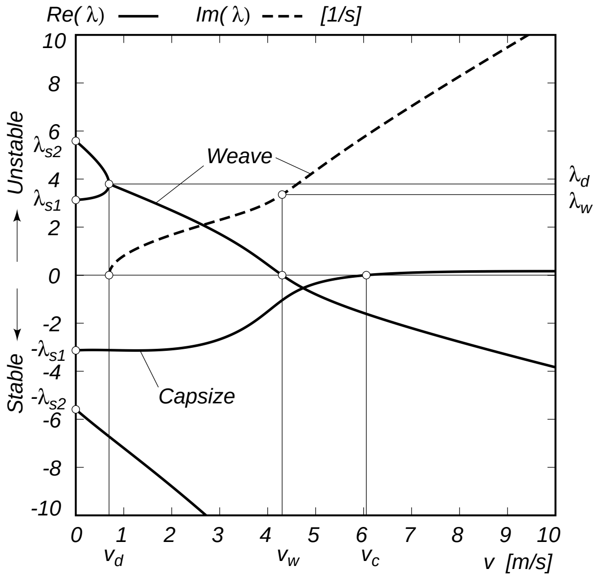

Figure 2. Eigenvalues λ from the linearized stability analysis for the benchmark bicycle from Figure 1 and Table 1 where the solid lines correspond to the real part of the eigenvalues and the dashed line corresponds to the imaginary part of the eigenvalues, in the forward speed range of $0 \le v \le 10$ m/s. The zero crossings of the real part of the eigenvalues are for the weave motion at the weave speed $v_w = 4.302$ m/s and for the capsize motion at capsize speed $v_c = 6.057$ m/s, and there is a double real root at $v_d = 0.694$ m/s, for more accurate values see Table 2. The asymptotically stable speed range for the bicycle is $v_w < v < v_c$.

and finally the the "damping" matrix from (27) which depends linearly on the forward speed

$$
\mathbf{C1} = \begin{bmatrix} 0, & 33.773\,869\,475\,930\,10 \\ -0.848\,234\,478\,256\,93, & 1.706\,965\,397\,923\,87 \end{bmatrix}. \tag{31}
$$

4.1 Linearized Stability, Eigenvalues

The stability of the bicycle in the upright steady motion at constant forward speed can be investigated by the homogeneous linearized equations of motion from (23). We start with the usual assumption of an exponential motion with respect to time for the small variations in the degrees of freedom $\mathbf{q}^d = (\phi, \delta)^T$ which then takes the form $\mathbf{q}^d = \mathbf{q}_0^d \exp(\lambda t)$. Substitution into the linearized equations of motion leads to an eigenvalue problem. For the bicycle model under study the characteristic equation of this eigenvalue problem is a polynomial in the eigenvalues λ of order four. The coefficients in this polynomial are themselves polynomials in the forward speed v, since some coefficients of the linearized equations of motion have a linear or quadratic dependency on v. The solutions of the characteristic polynomial for a range of forward speeds are the root loci of the eigenvalues λ, which are shown in Figure 2. Eigenvalues with a positive real part correspond to unstable motions whereas eigenvalues with a negative real part result in asymptotically stable mo-

v [m/s]	λ [1/s]
$v = 0$	$\lambda_{s1} = \pm 3.131\,435\,844\,365\,21$
$v = 0$	$\lambda_{s2} = \pm 5.587\,754\,114\,792\,34$
$v_d = 0.693\,712\,762\,387\,39$	$\lambda_d = 3.795\,471\,790\,345\,88$
$v_w = 4.301\,611\,037\,733\,12$	$\lambda_w = 0 \pm 3.353\,979\,714\,187\,50\,i$
$v_c = 6.057\,011\,283\,544\,49$	0

Table 2. Some characteristic values for the forward speed v and the eigenvalues λ from the linearized stability analysis for the benchmark bicycle from Figure 1 and Table 1 in the forward speed range of $0 \leq v \leq 10$ m/s. With two static eigenvalues at zero speed, a double root at the speed v_d, the weave speed v_w, and the capsize speed v_c.

tions. Complex conjugated eigenvalues give rise to oscillatory motions. For the bicycle model there are two significant eigenmodes, called capsize mode and weave mode. The capsize motion is a non-oscillatory motion in which, when unstable, the bicycle just falls over like a capsizing ship. The weave motion is an oscillatory motion in which the bicycle sways about the headed direction. At very low speed, $0 < v < 0.5$ m/s, there are two positive and two negative eigenvalues which correspond to an inverted pendulum-like motion of the bicycle. Then at $v_d = 0.694$ m/s two real eigenvalues become identical and start forming a conjugated pair; this is where the oscillatory weave motion emerges. At first this motion is unstable but at $v_w = 4.302$ m/s these eigenvalues cross the real axis in a Hopf bifurcation and the weave motion becomes stable until infinity. After this bifurcation the frequency of the weave motion is almost proportional to the forward speed. Meanwhile the capsize motion, which was stable for low speed, crosses the real axis in a pitchfork bifurcation at $v_c = 6.057$ m/s and the motion becomes mildly unstable. We call a motion mildly unstable when the eigenvalues have a absolute value which is smaller than 2 s^{-1}, in which case it is fairly easy to stabilize the motion manually. With further increase in speed, the capsize eigenvalue approaches zero. Some characteristic values for the forward speed v and the eigenvalues λ as introduced above are presented with fifteen significant digits in Table 2.

We conclude that the speed range for which the bicycle shows asymptotically stable behaviour is $v_w < v < v_c$, although from a practical point of view one could say that the bicycle is easy to balance for all speeds above 2 m/s.

4.2 Eigenvalues for Comparison

In order to test the equations of motion against any other set of equations, usually with a different choice of state variables, we present the eigenvalues with fifteen significant digits in a tabulated form. Eigenvalues are objective and coordinate free. Table 3 presents the real and imaginary part of the weave speed for the forward speed range of $0 \leq v \leq 10$ m/s, note that there is no weave motion at low speed. The eigenvalues for the capsize mode and stable steering mode, a mode with a steering angle much larger than the lean angle, are presented in Table 4.

v [m/s]	$\mathrm{Re}(\lambda_{weave})$ [1/s]	$\mathrm{Im}(\lambda_{weave})$ [1/s]
0	–	–
1	$3.544\,205\,145\,548\,87$	$0.803\,758\,373\,000\,36$
2	$2.693\,674\,773\,305\,74$	$1.678\,828\,917\,907\,97$
3	$1.720\,957\,788\,279\,10$	$2.296\,625\,407\,427\,06$
4	$0.436\,362\,119\,499\,78$	$3.008\,741\,465\,795\,03$
5	$-0.796\,974\,698\,035\,21$	$4.346\,861\,189\,884\,42$
6	$-1.574\,537\,004\,541\,48$	$5.738\,444\,449\,263\,20$
7	$-2.205\,683\,819\,126\,67$	$7.034\,232\,043\,107\,23$
8	$-2.777\,227\,223\,861\,88$	$8.275\,247\,335\,273\,91$
9	$-3.316\,436\,963\,837\,01$	$9.483\,978\,499\,142\,20$
10	$-3.835\,293\,220\,572\,69$	$10.672\,131\,916\,701\,23$

Table 3. Eigenvalues λ from the linearized stability analysis for the oscillatory weave motion for the benchmark bicycle from Figure 1 and Table 1 in the forward speed range of $0 \leq v \leq 10$ m/s.

v [m/s]	$\lambda_{capsize}$ [1/s]	λ_{ss} [1/s]
0	$-3.131\,435\,844\,365\,21$	$-5.587\,754\,114\,792\,34$
1	$-3.132\,456\,200\,083\,79$	$-7.198\,742\,879\,169\,33$
2	$-3.079\,168\,373\,984\,22$	$-8.793\,758\,748\,938\,05$
3	$-2.672\,380\,269\,446\,02$	$-10.497\,901\,671\,578\,35$
4	$-1.515\,016\,792\,101\,13$	$-12.328\,862\,599\,519\,56$
5	$-0.349\,966\,855\,680\,58$	$-14.270\,027\,689\,026\,00$
6	$-0.009\,940\,447\,809\,29$	$-16.297\,718\,272\,040\,15$
7	$0.102\,808\,114\,149\,01$	$-18.390\,961\,992\,983\,64$
8	$0.145\,690\,334\,393\,54$	$-20.533\,546\,191\,913\,53$
9	$0.161\,289\,013\,155\,47$	$-22.713\,514\,178\,876\,04$
10	$0.164\,852\,473\,666\,66$	$-24.922\,153\,914\,075\,30$

Table 4. Eigenvalues λ from the linearized stability analysis for the capsize motion and the stable steering (ss) motion for the benchmark bicycle from Figure 1 and Table 1 in the forward speed range of $0 \leq v \leq 10$ m/s.

5 Equations of Motion Derived with the Numeric Program SPACAR

SPACAR is a program system written in Fortran-77 for dynamic analysis of multibody systems, based on a finite element technique. Starting from the principles as laid out by [Besseling, 1964], this software was initiated in the seventies by [Van Der Werff, 1977], and has been further developed among others by [Jonker, 1988; Jonker and Meijaard, 1990; Meijaard, 1991; Schwab, 2002]. The SPACAR program can handle mechanical systems of rigid and flexible bodies that are interconnected by complex joints in both open and closed kinematic loops and may have rolling contacts. The dynamical equations are given for a set of minimal coordinates rather than with the aid of Lagrangian multipliers. Besides doing forward dynamic analysis, the system is also capable of deriving the numeric coefficients for the linearized equations of motion in any given configuration and state of motion of the system. With the help of a rather limited number of finite element types it is possible to model a wide class of systems. Typical types of elements are beam, truss and hinge elements, while more specialized elements can be used to model complex joint connections, transmissions of motion [Schwab and Meijaard, 1999a], and rolling contact as in road vehicles and track-guided vehicles [Schwab and Meijaard, 1999b; Schwab and Meijaard, 2003].

The SPACAR model for the benchmark bicycle is sketched in Figure 5, whereas the input file for the SPACAR program describing this model is presented in Appendix B. The model consists of two knife-edge rigid wheel elements, two rigid bodies for the front and the rear frame, and six hinge elements for describing relative rotations. The elements describing the three degrees of freedom are the relative rotations in: hinge ② for the roll angle ϕ, hinge ⑨ for the steering angle δ, and hinge ④ for the rotation θ_r of the rear wheel with respect to the rear frame. The four kinematic coordinates can be found in the model as the x and y components of node $\vec{9}$ which is the rear-wheel contact point, the relative rotation in hinge ① for the yaw angle ψ, and the relative rotation in hinge ⑫ for the rotation θ_f of the front wheel with respect to the front frame.

The entries in the matrices of the linearized equations of motion (23) as determined by the numeric program SPACAR agree with the values presented in Section 4, where at most the fifteenth digit may differ a unit.

5.1 Non-linear Dynamic Response

When an uncontrolled bicycle is in its stable speed range, roll and steer perturbations die away in a seemingly damped fashion. However, the system conserves energy and the perturbation energy has been transformed into energy of forward travel. As the forward speed is affected only to second order, linearized equations do not capture this. Therefore a non-linear dynamic analysis with SPACAR is performed on the benchmark bicycle model to demonstrate this phenomenon. The initial conditions at $t = 0$ are (an upright position) $(\phi, \delta, \theta_r) = (0, 0, 0)$ at a forward speed of $v = 4.5$ m/s, which is within the stable speed range of the linearized analysis, and a small perturbation of the angular roll velocity of $\dot{\phi} = 0.5$ rad/s. Then, in Figure 3, the dynamic response clearly shows a small increase of the forward velocity v while the perturbed lateral motions die out. The same figure shows that the period for the roll and the steer motion is approximately $T_0 = 1.73$ s, which compares well with the 1.734 s from the linearized stability analysis. Note also the small phase lag of the steering motion $\dot{\delta}$ relative to the roll motion $\dot{\phi}$.

6 Linearized Equations of Motion Derived with the Symbolic Program Autosim

With the multibody dynamics program AutoSim [Sayers, 1991a; Sayers, 1991b], the equations of motion for a mechanical system can be derived in a symbolic form. The program is mainly designed for analysing systems of rigid bodies that are interconnected by prismatic

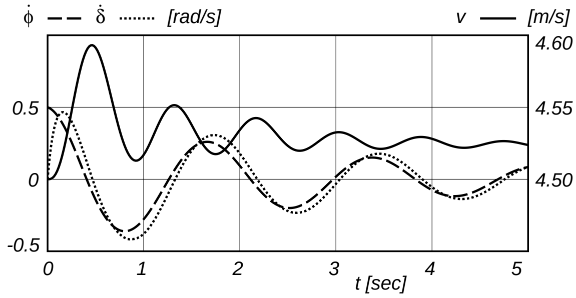

Figure 3. Non-linear dynamic response of the benchmark bicycle from Figure 1 and Table 1, with the angular roll velocity $\dot{\phi}$, the angular steering velocity $\dot{\delta}$, and the forward speed $v = -\dot{\theta}_r R_{rw}$ for the initial conditions: $(\phi, \delta, \theta_r)_0 = (0, 0, 0)$ and $(\dot{\phi}, \dot{\delta}, v)_0 = (0.5\ \mathrm{rad/s}, 0, 4.5\ \mathrm{m/s})$ for a time period of 5 seconds.

and revolute joints and are arranged in a tree topology. Additional constraints can be imposed on the system for taking into account kinematic closed loops, special joints or non-holonomic constraints. Additional holonomic constraints, however, cannot be solved in general in a symbolic form for the dependent coordinates: an iterative numerical solution for these coordinates is needed, which destroys the purely symbolic nature of the equations. Non-holonomic constraints are generally linear in the velocities and can be solved for the dependent velocities.

The methods used for deriving the equations of motion are mainly based on Kane's approach [Kane, 1968], with some minor modifications. The program is written in Lisp [Steele, 1990] and consists of a set of definitions of functions, macros and data structures. The definitions give procedures for handling algebraic expressions, for modelling of components of multibody systems such as bodies, points, joints and forces, for formulating the equations of motion and for generating output. The input file for an analysis is a Lisp program and the full language is available to the user. The modeller has a fairly good control over the formulation of the equations of motion, while user-defined forces are easy to add.

The equations of motion are obtained in the form

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{S}(\mathbf{q}, t)\mathbf{u},\\ \dot{\mathbf{u}} &= [\mathbf{M}(\mathbf{q}, t)]^{-1}\mathbf{Q}(\mathbf{q}, \mathbf{u}, t).\end{aligned} \tag{32}$$

Here, $\mathbf{q}$ are the generalized coordinates, $\mathbf{u}$ are the generalized velocities, $\mathbf{S}$ is the kinematic matrix that relates the rates of the generalized coordinates to the generalized speeds, $\mathbf{M}$ is the system matrix, and $\mathbf{Q}$ contains all force terms and velocity dependent inertia terms.

A standard option for linearization is available, which, however, is not applicable for systems with closed kinematic loops (e.g. the front-wheel ground contact of a bicycle). Fortunately, for the highly symmetric bicycle model, the dependent coordinate, the pitch angle, remains zero to first order, for which special case the linearization option gives the right results. The output consists of a MatLab script file that calculates the matrices of the linearized equations.

The input file used for the bicycle model is listed in Appendix C. The generalized coordinates and velocities are the same as those in the SPACAR model. Two massless intermediate reference frames have been introduced: the yawing frame describes the in-plane translation and yawing of the rear frame, and the rolling frame describes the rolling of the rear frame with respect to the yawing frame. These additional frames automatically satisfy the holonomic constraint at the rear wheel, and also give a better control over the choice of the generalized coordinates. The holonomic constraint at the front wheel and the four non-holonomic constraints are explicitly defined in the input file.

The entries in the matrices of the linearized equations of motion (23) as determined by the program AutoSim agree with the values presented in Section 4, where at most the fifteenth digit may differ a unit.

7 Bicycle Dynamics Folklore

The world of bicycle dynamics is filled with folklore. For instance, some publications persist in the necessity of positive trail or gyroscopic effect of the wheels for the existence of a forward speed range with uncontrolled stable operation. Here we will show, by means of a counter example, that this is not necessarily the case.

Consider the bicycle model from Section 2 but with the following dimensions and mechanical properties. The wheel base is 1.2 m at zero trail, and the head angle is 85 degrees. Both wheels are massless and have a diameter of 0.35 m. The mass distribution of the rear frame is modelled by two point masses, one of 40 kg upfront at $(x, z) = (1.5, -0.6)$ m and one of 40 kg at the rear contact point. The latter has to insure contact at the rear wheel, but gives no contribution to the linearized equations of motion. The front fork has a mass of 1 kg located at front hub, $(x, z) = (1.2, 0.35)$ m and zero mass moment of inertia. Gravity is 9.81 N/kg.

The eigenvalue analysis on the linearized equations of motion results in a weave speed of $v_w = 2.815$ m/s for this model and no capsize speed, see Figure 4. Inspection of the eigenvalues for a wide forward speed range shows that the capsize motion is always stable and that *all* eigenvalues above the weave speed have a negative real part. In other words, this bicycle with zero trail and zero gyroscopic effect shows asymptotically stable uncontrolled motion for the broad forward speed range of $2.815 \leq v \leq \infty$ m/s.

8 Conclusions

If we compare the results obtained by the three methods, it appears that the coefficients for the linearized equations agree with each other and the difference are only caused by the finite precision of the numeric calculations: the relative errors are less than 1 part in 10^{14}.

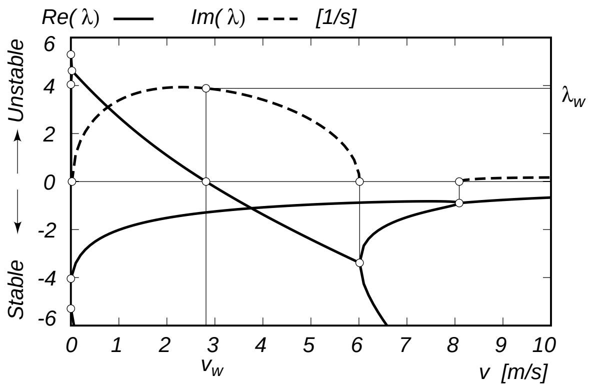

Figure 4. Eigenvalues λ from the linearized stability analysis for a bicycle with zero trail and no gyroscopic effects from Section 7. The solid lines correspond to the real part of the eigenvalues and the dashed line corresponds to the imaginary part of the eigenvalues, in the forward speed range of $0 \leq v \leq 10$ m/s. The zero crossing of the real part of the eigenvalues is for the weave motion at the weave speed $v_w = 4.302$ m/s and there are three double real roots at $v = 0.022$, 6.014, and 8.089 m/s. The asymptotically stable speed range for this bicycle is $v_w < v < \infty$.

This gives us confidence that the presented results are correct and the problem can be used as a benchmark test for multibody dynamics simulations.

Starting from the given basic model for the bicycle, more elaborate models can be developed. These may include the finite width of the tires, control torque at the handle bar, relative motion between the rider and the rear frame and tire models that include wheel slips and compliance.

Acknowledgement

Special thanks are extended to Andy Ruina for stimulating, supporting, hosting, and guiding this work on bicycle dynamics, and to Andrew Dressel for his work on the graphical user interface to the JBike6 program: the Matlab implementation of the linearized equations of motion presented here. The second author, J. P. Meijaard, was supported by the Engineering and Physical Sciences research Council (EPSRC) of the U.K.

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A Literature review

This review is a condensed account of the review presented in the MSc thesis by [Hand, 1988]. Some prior publications where missed. In general the literature had the major defects that few or no authors compared their result with those of others. Of course comparison is difficult because of different choice of variables and combination of equations, different parametrization of the bicycle and lack of model equivalence. Moreover, there are two additional problems. One is that people are making messy complicated equations which were never reduced and use bad notation. The other is that some models are special cases like vertical steering axis or only point masses, zero mass moments of inertia.

Of the twenty sets of equations discussed in this overview only three are fully general and perfectly correct [Döhring, 1955; Singh and Goel, 1971; Weir, 1972]. Five others were little less general as for example principal axis aligned with the steering axis, or had minor and easily corrected errors [Whipple, 1899; Carvallo, 1901; Klein and Sommerfeld, 1910; Timoshenko and Young, 1948; Sharp, 1971]. Three [Collins, 1963; Singh, 1964; Roland, 1973] were too difficult to evaluate and we have reservations about the first two. The remaining eight had missing terms or disagreed in other ways. Only one author [Weir, 1972] explicitly stated that he compared equations.

[Whipple, 1899] He is the first and fully nonlinear (which we did not validate). His linearized equations agrees very well except for minor typographical errors.

[Carvallo, 1901]) Not general, massless fork and handle bar, equations are right.

[Klein and Sommerfeld, 1910] Not general, massless fork and handle bar, equations are right.

[Bower, 1915] Not general, point masses in special places and vertical steering axis, equations are incorrect.

[Pearsall, 1922] Special case, inertia along vertical axis, and equations are incorrect.

[Loĭcjanskiĭ and Lur̆e, 1934] No review, used by Neimark and Fufaev.

[Timoshenko and Young, 1948] Very simplified, using point masses and a controlled steering angle, only the lean equations, vertical steering axis, and when we linearize this equation it is correct.

[Döhring, 1955] He took away the restrictions on the model by Sommerfeld and Klein, and it's all correct.

[Collins, 1963] General model with driving and drag forces, his equations imply a correct lean equation but we were unable to determine whether they also imply the correct steering equation.

[Singh, 1964] Rederived Collins equations, added a suspect tire model, and it disagreed with respect to Collins, incorrect.

[Neĭmark and Fufaev, 1972] General model, ignores the pitch of the rear frame, incorrect.

[Singh and Goel, 1971] Use Döhring's equations and add steering damping, seemingly correct.

[Sharp, 1971] General model but with parallel inertia to the steering axis and tire models. The linear analysis is probably correct, in elimination of the tire model he introduces an algebra error, otherwise it is correct.

[Roland and Massing, 1971] General model with tire model, nonlinear. Unable to linearize, supply missing definitions and appropriate linearize and combine equations to make a comparison.

[Roland, 1973]) Uses the same equations as Roland & Massing 1971, corrects typos and includes a missing figure.

[Weir, 1972]) General model with tire models. He is the only author to state explicitly that he compared his equations to any compared work, Sharp 1971, with which his equations are in agreement. Correct.

[Eaton, 1973] No review.

[Sing and Goel, 1975] Used Sharp's equations and extended, we did not compare.

[Sharp and Jones, 1977] Sharp 1971 but with a different tire model, when we removed the tire model it agreed.

[Van Zytveld, 1975] No review.

[Weir and Zellner, 1978] In the mistaken belief that the Weir 1972 derivation was incorrect they deleted a necessary term and introduced some typos, incorrect.

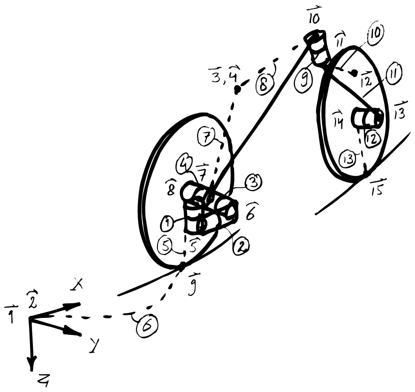

Figure 5. Sketch of the bicycle model for SPACAR input together with node numbers, with straight arrows for positions, curved arrows for orientations, and element numbers encircled.

[Gobas, 1978]) General model, added forward acceleration but the steering equation does not agree, incorrect.

[Adiele, 1979] General model with tires, nonlinear, we linearized, they are incorrect.

[Psiaki, 1979] No review.

[Lowel and McKell, 1982] Very specialist model, point masses and no front mass, vertical axis and its incorrect.

[Maers, 1988] No review.

B SPACAR Input

The sketch of this model is shown in Figure 5.

```
* benchmark1, linearized equations of motion
* elements
hinge    1   2  5      0  0  1
hinge    2   5  6      1  0  0
hinge    3   6  4      0 -1  0
hinge    4   4  8      0 -1  0
wheel    5   7  8  9   0 -1  0
pinbody  6   1  2  9
pinbody  7   3  4  7
pinbody  8   3  4 10
hinge    9   4 11      1  0  3
pinbody 10  10 11 12
pinbody 11  10 11 13
hinge   12  11 14      0 -1  0
wheel   13  13 14 15   0 -1  0
* nodes
x  1  0     0  0
x  3  0.3   0 -0.9
x  7  0     0 -0.3
x  9  0     0  0
x 10  0.8   0 -0.9
x 12  0.9   0 -0.7
x 13  1.02 0 -0.35
x 15  1.02 0  0
* boundary conditions
fix    1  1 2 3
fix    2  1 2 3 4
rlse   1  1
line   2  1
rlse   3  1
inpute 4  1
enhc   5  4   6  1
enhc   5  5   6  2
rlse   6  1 2 3
line   9  1
rlse  12  1
enhc  13  4   1  1
enhc  13  5  12  1
* mass & stiffness
mass  3  85
mass  4  9.2    0  2.4     11   0  2.8
mass  7  2
mass  8  0.06   0  0       0.12 0  0.06
mass 11  0.0546 0 -0.0162  0.06 0  0.0114
mass 12  4
mass 13  3
mass 14  0.14 0 0  0.28 0  0.14
* applied force, take g=9.81
force  3  0 0 833.85
force  7  0 0  19.62
force 12  0 0  39.24
force 13  0 0  29.43
* initial conditions and settings
ed  4 1  1.0
epskin  1e-6
epsint  1e-5
epsind  1e-5
timestep  1 1e-5
hmax  0.01
end
eof
```

C AutoSim Input

```
;;;; This is the file fiets.lsp, with the benchmark1 model.
;; Set up preliminaries:
(reset)
(si)
(add-gravity :direction [nz] :gees g)
(set-names g "Acceleration of gravity" )
(set-defaults g 9.81) ; this value is used in the benchmark,
                      ; though g is a little smaller.
;; The name of the model is set to the string "fiets"
(setsym *multibody-system-name* "fiets")
;; Introduce a massless moving reference frame. This frame
;; has x and y translational degrees of freedoms and a yaw
;; rotational degree of freedom.
( add-body yawframe :name "moving yawing reference frame"
  :parent n :translate (x y) :body-rotation-axes z
  :parent-rotation-axis z :reference-axis x :mass 0
  :inertia-matrix 0 )
;; Introduce another massless moving reference frame. This
;; frame has a rolling (rotational about a longitudinal
;; axis) degree of freedom.
( add-body rollframe :name "moving rolling reference frame"
  :parent yawframe :body-rotation-axes (x)
  :parent-rotation-axis x :reference-axis y :mass 0
  :inertia-matrix 0 )
;; Add the rear frame of the bicycle. The rear frame has a
;; pitching (rotation about the local lateral y-axis of the
;; frame) degree of freedom.
( add-body rear :name "rear frame" :parent rollframe
  :joint-coordinates (0 0 "-Rrw") :body-rotation-axes y
  :parent-rotation-axis y :reference-axis z
  :cm-coordinates (bb 0 "Rrw-hh") :mass Mr
  :inertia-matrix ((Irxx 0 Irxz) (0 Iryy 0) (Irxz 0 Irzz)) )
( set-names
  Rrw "Rear wheel radius"
  bb "Longitudinal distance to the c.o.m. of the rear frame"
  hh "Height of the centre of gravity of the rear frame"
  Mr "Mass of the rear frame"
  Irxx "Longitudinal moment of inertia of the rear frame"
  Irxz "Minus product of inertia of the rear frame"
  Iryy "Transversal moment of inertia of the rear frame"
  Irzz "Vertical moment of inertia of the rear frame" )
( set-defaults Rrw 0.30 bb 0.3 hh 0.9
  Mr 85.0 Irxx 9.2 Irxz 2.4 Iryy 11.0 Irzz 2.8 )
;; Add the rear wheel of the vehicle. This body rotates
;; about the y axis of its physical parent, the rear frame.
( add-body rw :name "rear wheel" :parent rear
  :body-rotation-axes y :parent-rotation-axis y
  :reference-axis z :joint-coordinates (0 0 0) :mass Mrw
  :inertia-matrix (irwx "2.0*irwx" irwx) )
( set-names
  Mrw "mass of the rear wheel"
  irwx "rear wheel in-plane moment of inertia" )
(set-defaults Mrw 2.0 irwx 0.06)
;; Now we proceed with the front frame.
;; Define the steering and reference axes of the front frame:
;; Add in the front frame: define some points
( add-point head :name "steering head point B" :body n
  :coordinates (xcohead 0 zcohead) )
( add-point frontcmpoint :name "c.o.m. of the front frame"
  :body n :coordinates (xfcm 0 zfcm) )
( set-names
```

```
  epsilon "steering head angle"
  xcohead "x coordinate of the steering head point B"
  zcohead "z coordinate of the steering head point B"
  xfcm "x coordinate of the c.o.m. of the front frame"
  zfcm "z coordinate of the c.o.m. of the front frame" )
( set-defaults epsilon 0.321750554396642163
  xcohead 0.80 zcohead -0.90 xfcm 0.90 zfcm -0.70 )
( add-body front :name "front frame" :parent rear
  :body-rotation-axes z :parent-rotation-axis
  "sin(epsilon)*[rearx]+cos(epsilon)*[rearz]"
  :reference-axis "cos(epsilon)*[rearx]-sin(epsilon)*[rearz]"
  :joint-coordinates head :cm-coordinates frontcmpoint
  :mass Mf
  :inertia-matrix ((Ifxx 0 Ifxz) (0 Ifyy 0) (Ifxz 0 Ifzz))
  :inertia-matrix-coordinate-system n )
( set-names
  Mf "Mass of the front frame assembly"
  Ifxx "Longitudinal moment of inertia of the front frame"
  Ifxz "Minus product of inertia of the front frame"
  Ifyy "Transversal moment of inertia of the front frame"
  Ifzz "Vertical moment of inertia of the front frame" )
( set-defaults Mf 4.0
  Ifxx 0.0546 Ifxz -0.0162 Ifyy 0.06 Ifzz 0.0114 )
;; Add in the front wheel:
( add-point fw_centre :name "Front wheel centre point"
  :body n :coordinates (ll 0 "-Rfw") )
( add-body fw :name "front wheel" :parent front
  :body-rotation-axes y :parent-rotation-axis y
  :reference-axis "[nz]"  :joint-coordinates fw_centre
  :mass Mfw :inertia-matrix (ifwx "2.0*ifwx" ifwx) )
( set-names
  ll "Wheel base"
  Rfw "Radius of the front wheel"
  Mfw "Mass of the front wheel"
  ifwx "In-plane moment of inertia of the front wheel" )
(set-defaults ll 1.02 Rfw 0.35 Mfw 3.0 ifwx 0.14)
;; The system is now complete,
;; except for the contact constraints at the wheels.
;; The holonomic constraint at the rear wheel is
;; automatically satisfied. The rear wheel slip is zero.
( add-speed-constraint
  "dot(vel(yawframe0),[yawframex])+Rrw*(ru(rear)+ru(rw))"
  :u "tu(yawframe,1)" )
( add-speed-constraint "dot(vel(yawframe0),[yawframey])"
  :u "tu(yawframe,2)" )
;; For the front wheel we have a holonomic constraint for
;; the contact and two non-holonomic slip constraints.
;; The slip velocities are defined now.
(setsym singammafw "dot([fwy],[nz])")
(setsym cosgammafw "sqrt(1-@singammafw**2)")
(setsym fw_rad "([nz] - [fwy]*@singammafw)/@cosgammafw")
( setsym slipfw_long
  "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[nx])" )
;; No longitudinal slip on front wheel;
;; eliminate rotational velocity about the axis
(add-speed-constraint "@slipfw_long" :u "ru(fw)")
;; normal constraint; eliminate the pitch angle
( setsym slipfw_n
  "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[nz])" )
(add-speed-constraint "@slipfw_n" :u "ru(rear)")
( add-position-constraint
  "dot(pos(fw0),[nz])+Rfw*@cosgammafw" :q "rq(rear)" )
;; No lateral slip on front wheel;
;; eliminate yaw rate of the yawing frame
( setsym slipfw_lat
  "dot(vel(fw0)+Rfw*cross(rot(fw),@fw_rad),[ny])" )
(add-speed-constraint "@slipfw_lat" :u "ru(yawframe)")
(dynamics)
(linear)
```