A MULTIDIMENSIONAL HISTOGRAM EQUALIZATION BY FITTING AN ISOTROPIC GAUSSIAN MIXTURE TO A UNIFORM DISTRIBUTION

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ABSTRACT

In this paper, a novel method to extend the grayscale histogram equalization (GHE) for color images in a multi-dimension is proposed. Unlike most current techniques, the proposed method can generate a uniform histogram, thus minimizing the disparity between the histogram and uniform distribution. A histogram of any dimension is regarded as a mixture of isotropic Gaussians. This method is a natural extension of the GHE to a multi-dimension. An efficient algorithm for the histogram equalization is provided. The results show that this approach is valid, and a psycho-visual study on a target distribution will improve the practical use of the proposed method.

*Keywords***—** *Color, Image enhancement, Image processing*

1. INTRODUCTION

In many image-processing applications, the grayscale histogram equalization (GHE) is one of the simplest and most effective primitives for contrast enhancement. For a given grayscale image, its histogram $\{p_i\}_{i=1}^n$ is defined as the relative frequency of an intensity appearance, x_i, in the domain, $D=[0,1]$. The GHE can be performed easily by using a cumulative histogram [1][2]. The intensity of x_i for every pixel is transformed to x_i^*, as follows:

$$x_i^* = \sum_{x_k \le x_i} p_k \ . \quad (1)$$

The histogram specification, a generalization of this technique, allows any user-supplied histogram to be used.

Extending the GHE to a multidimensional case is not straightforward, and various methods have been proposed to address the difficulty. The simplest extension is to apply the GHE independently to the different bands of the color image. Another approach is to spread the histogram along its principal component axes [3] or along the brightness component of the image [4][5]. Using color difference color space, [7] equalizes the conditional histogram of saturation given luminance and hue. All of these techniques use marginal or conditional color histograms only and, therefore, do not consider the correlation between the different bands. With a different observation on GHE, namely, it matches the cumulative distribution function (CDF) of histogram to that of target distribution, [6] tries to match the CDFs of the histogram and target distribution in multi-dimension. The recent method in [8] provides a nearly uniform color histogram. This method involves deforming the mesh in the RGB color space to fit a given histogram, and then mapping it approximately to a uniform histogram. This technique, however, requires excessive computational time.

In this study, a new method for the multidimensional extension of the GHE is proposed. Our preliminary work for color images was presented in [9]. Accordingly, a given histogram with any dimensions, as a probability density function (PDF), is regarded as an isotropic Gaussian mixture. To fit the PDF to the target distribution, their squared error is derived and minimized with respect to centers of the Gaussian mixture. The proposed method is formulated as a nonlinear optimization problem with bound constraints; therefore, the multidimensional extension is quite natural. This paper is organized as follows: section 2 formulates the multidimensional histogram equalization; section 3 shows several examples, including the relationship between the proposed and the conventional GHE; finally, section 4 concludes the paper.

2. MATHEMATICAL FORMULATION

To acquire smoothness in the formulation, each probability mass of a histogram with any dimensions is approximated by an isotropic Gaussian density having the same mass and center. For example, the line, plane, and volume densities for 1-D, 2-D, and 3-D histograms can be considered. Then, their centers are rearranged to fit a mixture of Gaussian densities into a uniform density within the domain. As a result, the histogram equalization is formulated as a nonlinear optimization problem with bound constraints.

2.1. Grayscale Histogram Equalization

Given a histogram represented by a positive probability $\{p_i\}_{i=1}^n$ at distinct points $\{x_i\}_{i=1}^n$ in the domain $D=[0,1]$, its

PDF can be regarded as a mixture of Gaussian distributions with the scaling factor σ, i.e.

$$f(x)=\sum_{i=1}^{n} p_i N(x \mid x_i, \sigma^2 p_i^2), \tag{2}$$

where $N(x \mid x_i, \sigma^2 p_i^2)$ is the normal distribution function of x with the mean x_i and variance $\sigma^2 p_i^2$. The histogram equalization can be defined as adjusting $\{x_i\}_{i=1}^{n}$ to fit $f(x)$ into $g(x)$, the target PDF. The disparity measure between the two PDFs is defined as

$$\Phi = \int_D \{g(x)-f(x)\}^2 dD. \tag{3}$$

While the target distribution may be chosen from the features of a vision system or an image displaying device, they are set to be uniform for the remainder of this paper: $g(x)=1$. This gives

$$\begin{aligned}\Phi &= \frac{1}{2}\int_D \left\{1-\sum_{i=1}^{n} p_i N(x \mid x_i, \sigma^2 p_i^2)\right\}^2 dD \\ &= \frac{1}{2} - \int_D \left\{\sum_{i=1}^{n} p_i N(x \mid x_i, \sigma^2 p_i^2) \right. \\ &\quad \left. -\frac{1}{2}\sum_{i,j} p_i p_j N(x \mid x_i, \sigma^2 p_i^2) N(x \mid x_j, \sigma^2 p_j^2)\right\} dD.\end{aligned}$$

The closed form of this integral can be obtained precisely in D; however, doing so is complex and there is no further benefit in terms of accuracy. To approximate the objective function, $D=(-\infty,\infty)$ is assumed temporarily. At this point,

$$\Phi \approx -\frac{1}{2}+\frac{1}{2}\sum_{i,j}\frac{p_i p_j}{\sigma\sqrt{2\pi(p_i^2+p_j^2)}}\exp\left\{-\frac{(x_i-x_j)^2}{2\sigma^2(p_i^2+p_j^2)}\right\}.$$

By eliminating the constant coefficients, the objective function is redefined as

$$\Phi := \frac{1}{2}\sum_{i,j}\frac{p_i p_j}{\sqrt{p_i^2+p_j^2}}\exp\left\{-\frac{(x_i-x_j)^2}{2\sigma^2(p_i^2+p_j^2)}\right\}. \tag{4}$$

Since Φ is smooth in D, the gradient of Φ, with respect to x_i, is calculated by

$$\frac{\partial \Phi}{\partial x_i} = -\sum_{j=1}^{n}\frac{(x_i-x_j)}{\sigma^2(p_i^2+p_j^2)}\Phi_{ij}, \tag{5}$$

where Φ_{ij} denotes the inner term of the summation of (4). Again, taking the derivative of (5) with respect to x_i and x_j, for $i \neq j$, the Hessian is obtained:

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \frac{\Phi_{ij}}{\sigma^2(p_i^2+p_j^2)}\left\{1-\frac{(x_i-x_j)^2}{\sigma^2(p_i^2+p_j^2)}\right\},$$

$$\frac{\partial^2 \Phi}{\partial x_i^2} = -\sum_{j\neq i}\frac{\partial^2 \Phi}{\partial x_i \partial x_j}.$$

Consequently, the GHE is formulated as a minimization problem of Φ with bound constraints. Furthermore, the given histogram will be a good initial guess to find its nearest local optimizer in order to alleviate the color change from the original image.

2.2. Multidimensional Histogram Equalization

Let us denote $N(\mathbf{x} \mid \mathbf{x}_i, r^2\mathbf{I})$ as a multivariate, say d-dimensional, normal distribution function of $\mathbf{x}$ with the mean $\mathbf{x}_i$ and isotropic covariance $r^2\mathbf{I}$, where $\mathbf{I}$ denotes an identity matrix. Equating the probability p to the volume of hyper-sphere with radius r in d-dimension, we found that $r^d \propto p$. Given the positive probability $\{p_i\}_{i=1}^{n}$ at the distinct points $\{\mathbf{x}_i\}_{i=1}^{n}$ in the domain $D=[0,1]^d$, the PDF is constructed by an isotropic Gaussian mixture with the scaling factor σ:

$$f(\mathbf{x}) = \sum_{i=1}^{n} p_i N(\mathbf{x} \mid \mathbf{x}_i, \sigma^2 p_i^{2/d}\mathbf{I}). \tag{6}$$

Similar to the previous section, Φ is defined for a uniform target distribution:

$$\Phi := \frac{1}{2}\sum_{i,j}\frac{p_i p_j}{(p_i^{2/d}+p_j^{2/d})^{d/2}}\exp\left\{-\frac{\|\mathbf{x}_i-\mathbf{x}_j\|^2}{2\sigma^2(p_i^{2/d}+p_j^{2/d})}\right\}. \tag{7}$$

The gradient and Hessian of Φ are

$$\frac{\partial \Phi}{\partial \mathbf{x}_i} = -\sum_{j=1}^{n}\frac{\mathbf{x}_i-\mathbf{x}_j}{\sigma^2(p_i^{2/d}+p_j^{2/d})}\Phi_{ij}, \text{ and}$$

$$\frac{\partial^2 \Phi}{\partial \mathbf{x}_i \partial \mathbf{x}_j} = \frac{\Phi_{ij}}{\sigma^2(p_i^{2/d}+p_j^{2/d})}\left\{\mathbf{I}-\frac{(\mathbf{x}_i-\mathbf{x}_j)(\mathbf{x}_i-\mathbf{x}_j)^T}{\sigma^2(p_i^{2/d}+p_j^{2/d})}\right\} \text{ and}$$

$$\frac{\partial^2 \Phi}{\partial \mathbf{x}_i^2} = -\sum_{j\neq i}\frac{\partial^2 \Phi}{\partial \mathbf{x}_i \partial \mathbf{x}_j},$$

where Φ_{ij} denotes the inner term of the summation of (7).

3. EXPERIMENTAL RESULTS

To test the validity of the proposed method, grayscale and color images are equalized and the results are displayed. To compute the output, a nonlinear optimization with bound constraints is implemented with the commercial KNITRO 5.0 and MATLAB 7.0 software on an Intel Pentium® D CPU 3GHz. Through extensive experimentation, we found that the scaling factor $\sigma = 0.25$ works well with all the images we processed.

3.1. Relation with Grayscale Histogram Equalization

Even though the formalism in section 2 may appear far from the simple GHE of Equation (1), it is a rather natural extension. Figure 1(b) shows the result of the proposed GHE of Figure 1(a). Figure 1(c) shows that the proposed method provides an almost identical transformation of the cumulative histogram of Equation (1). KNITRO 5.0 is able

(a) Original image (b) Grayscale-equalized image (c) Cumulative histogram (dotted) and proposed transform (solid)

Figure 1. Grayscale histogram equalization for the Lena image.

to provide a solution within one second via the interior point conjugate gradient method [10]. The computational time was 0.23 seconds for the Lena image, whose histogram is nonzero at 219 gray values out of 256.

3.2. Color Image Examples

Figure 2(a) is the original red-green image of a moth's head, an image example from [8]. It has two bands, R and G, with 16 levels and 335 x 228 pixels. Its histogram is nonzero at 160 colors out of 256. The contrast of Figure 2(a) is relatively low because its histogram, shown in Figure 2(b), is denser along the diagonal. Through our algorithm, the histogram is expanded more toward the off-diagonal so that it is mapped to fit the uniform distribution shown in Figure 2(d). The equalized image, Figure 2(c), shows a higher contrast than the original image. Using MATLAB, a large-scale nonlinear optimization was implemented. This was completed in 8.9 seconds.

Figure 3 shows a color image with 768 x 512 pixels and 16 levels for each band. As a result of the color histogram equalization, Figure 3(b) shows more diversity in color than 3(a). Due to lack of space, the three-dimensional histogram is not depicted here. In fact, it is equalized uniformly. It took 412.6 seconds to equalize the histogram with nonzero values at 1501 colors out of 4096.

4. CONCLUSION

In this paper, a new method for extending the grayscale histogram equalization to a multi-dimension is proposed. The histogram of a given image is regarded as an isotropic Gaussian mixture, and its squared error from the target distribution is minimized. The proposed GHE results in a solution nearly identical to the conventional one, which demonstrates its validity. The error analysis for the canonical configuration which consists of only three consecutive probabilities in the histogram would be further investigated. The results of the histogram equalization for two color images are displayed.

Even though a uniform target distribution was assumed for a simple development, a practical target distribution should be investigated through a psycho-visual study on the human eye and display device. While this approach can equalize a histogram efficiently, there is room for improvement. Through a numerical implementation, it was found that the point to be optimized is affected most by its near-neighbors, and the effect of the remaining points is negligible. To hasten the computing time, maintaining a proximity list of all points and the numerical calculations would be needed only for the near points [11]. We hope to apply such a subdivision scheme to build and update the proximity list in a future study.

5. ACKNOWLEDGEMENT

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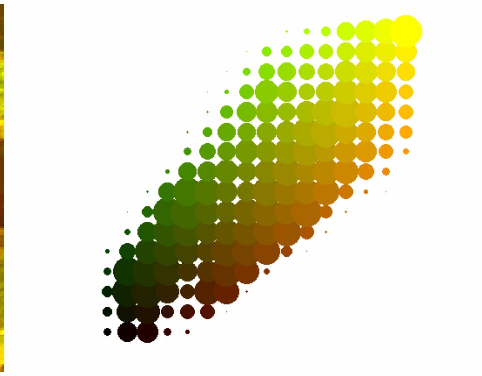
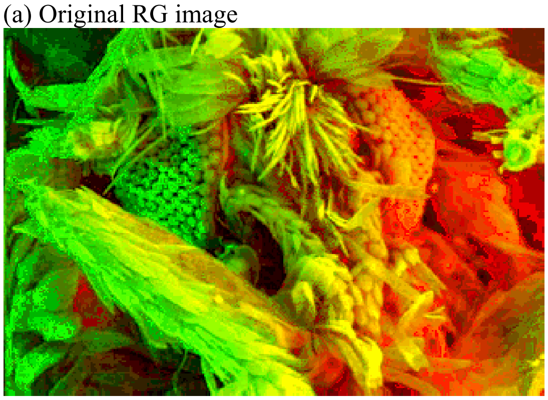
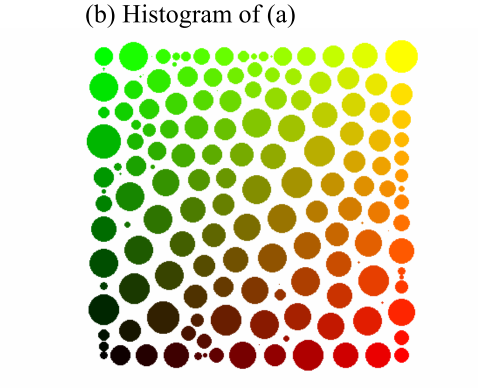
(a) Original RG image



(b) Histogram of (a)

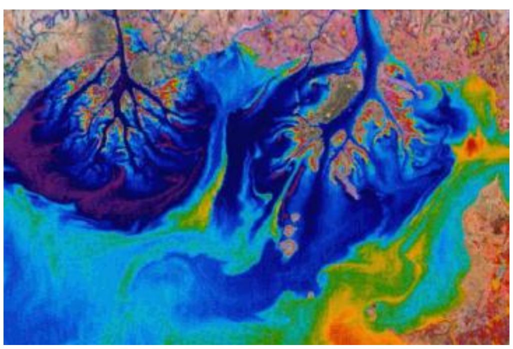


(c) Color-equalized image

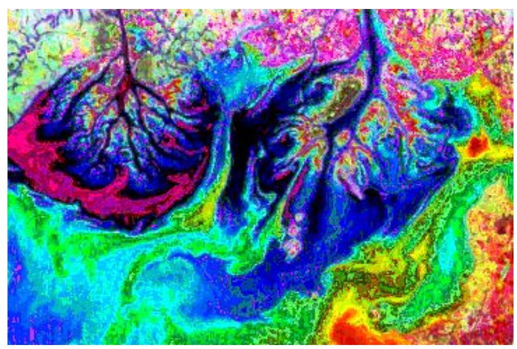


(d) Equalization of (b)

Figure 2. Histogram equalization of a two-band color image.



(a) Original image



(b) Color-equalized image

Figure 3. Histogram equalization of a color image.

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