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Aplikace matematiky, Vol. 31 (1986), No. 5, 365–378

Persistent URL: http://dml.cz/dmlcz/104214

31 (1986) APLIKACE MATEMATIKY No. 5, 365—378

A MULTI-LEVEL METHOD WITH CORRECTION BY AGGREGATION FOR SOLVING DISCRETE ELLIPTIC PROBLEMS

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(Received May 24, 1985)

Summary. The author studies the behaviour of a multi-level method that combines the Jacobi iterations and the correction by aggregation of unknowns. Our considerations are restricted to a simple one-dimensional example, which allows us to employ the technique of the Fourier analysis. Despite of this restriction we are able to demonstrate differences between the behaviour of the algorithm considered and of multigrid methods employing interpolation instead of aggregation.

Key words: multilevel method, correction by aggregation, multigrid method.

1. INTRODUCTION

The aim of this paper is to give some insight into the multi-level method with correction by aggregation of unknowns. This method belongs to the broad class of multi-level or multigrid methods for solving linear systems arising from the discretization of elliptic boundary value problems [1]. The consideration of the above mentioned aggregation correction multi-level method was motivated by the following reasons.

First, initial approximation by aggregation was observed to be very efficient in the solution of the three-dimensional elasticity problems arising in geomechanical modelling [5].

Second, in the finite element analysis of complicated and, especially, three-dimensional engineering problems we are usually not able to produce a sequence of nested grids, which is needed for the standard multigrid algorithms. Hence, the multi-level algorithms with generating coarser levels from the finest one, as in the algebraic multigrid approach [4], are of interest. Generating coarser levels by means of aggregation of unknowns is probably the simplest way of creating them, and, consequently, the aggregation correction multi-level method is attractive from this point of view.

The paper is organized as follows. In Section 2 we describe a simple one-dimensional model problem, which allows us to demonstrate some essential features of the method considered. The technique of our analysis is clarified in Section 3, cf.[2]. The spectral radius, energy and spectral norms of the two-level iteration operator are studied in the following sections.

2. MODEL PROBLEM, ALGORITHMS

Let us consider a one-dimensional model problem

$$-u'' = f \quad \text{in} \quad \Omega = (0, 1)\,; \quad u(0) = u(1) = 0\,.$$

In order to apply the finite difference or finite element method, a computational grid Ω_h is defined as follows:

$$\Omega_h = \{x_k = kh : k = 0, \ldots, n\}\ ; h = 1/n.$$

We shall consider the discretization resulting in the system of linear equations

$$(2.1) \qquad L_h u^h = f^h \quad \text{with} \quad L_h = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix},$$

where L_h is an $(n-1)\,x(n-1)$ matrix, u^h, f^h are $(n-1)$-dimensional algebraic vectors corresponding to the grid functions on Ω_h.

Let us introduce the notation

$$(2.2) \qquad L_h = L_m\,, \quad u^h = u^m\,, \quad f^h = f^m\,,$$

where m is the number of the highest (finest) level. Next, the lower (coarser) level systems of equations

$$(2.3) \qquad L_l u^l = f^l\,, \quad l = m-1, \ldots, 0$$

will be formed by means of a restriction I_l^{l-1} and a prolongation I_{l-1}^l:

$$(2.4) \qquad L_{l-1} = I_l^{l-1} L_l I_{l-1}^l\,, \quad f^{l-1} = I_l^{l-1} f^l\,.$$

In this paper I_l^{l-1} will be an aggregation opetator, I_{l-1}^l will be a disaggregation operator. It means that I_l^{l-1}, I_{l-1}^l are up to a multiplicative constant represented by rectangular matrices with zero elements everywhere except at most one 1 per column of I_l^{l-1} and one 1 per row of I_{l-1}^l.

The Fourier analysis described in the next section forces us to introduce the special choice of $n = n_m = 3^m n_0$ and the following definition of I_l^{l-1} and I_{l-1}^l:

(2.5) $$I_l^{l-1} = \frac{1}{3}\begin{bmatrix} 0 & 1 & 1 & 1 & & & & & & & \\ & & & 1 & 1 & 1 & & & & & \\ & & & & & & \cdots & & & & \\ & & & & & & & \cdots & & & \\ & & & & & & & 1 & 1 & 1 & 0 \end{bmatrix}$$

is an $(n_{l-1} - 1)\, x(n_l - 1)$ matrix, $n_l = 3^l n_0$ and I_{l-1}^l is the transpose of I_l^{l-1}.

From (2.1) and (2.4) we obtain

(2.6) $$L_l = \frac{1}{h_l^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix},$$

where $h_l = 1/n_l = 3^{m-l}h_m, \quad l = m, \ldots, 0.$

For $l = m, \ldots, 1$ the problem on level l (2.2), (2.3) can be solved by the iterative algorithm whose one cycle is defined as follows (cf. [2], [3]).

Two-Level Algorithm: starting with initial u^l

(2.7a) perform ν_1 smoothing steps $u^l \leftarrow R_l(u^l, f^l)$,

(2.7b) solve the problem $L_{l-1}u^{l-1} = I_l^{l-1}(f^l - L_l u^l)$,

(2.7c) perform the correction $u^l \leftarrow u^l + I_{l-1}^l u^{l-1}$,

(2.7d) perform ν_2 smoothing steps $u^l \leftarrow R_l(u^l, f^l)$.

In the above described cycle, ν_1, ν_2 are constants, R_l denotes the operator of the smoothing procedure. With respect to the Fourier analysis of the convergence we shall consider for smoothing only the Jacobi relaxation method, so that

(2.8) $$R_l(u^l, f^l) = S_l u^l + T_l f^l = u^l + \tfrac{1}{2}h_l^2\omega(f^l - L_l u^l),$$

where ω is a relaxation parameter, $\omega \in (0, 1\rangle$.

In order to avoid the (exact) solution of (2.7b) in the two-level algorithm we can solve the problem on the lower level approximately by the same method. In this way we obtain the following

Multi-Level Algorithm defined recursively:

For $l = m, \ldots, 0$ the problem on level l

$$L_l u^l = f^l$$

is solved iteratively. One iteration is defined as follows:

If $l = 0$, then $u^0 \leftarrow L_0^{-1}f^0$.

If $l > 0$, then starting with initial u^l

(2.9a) perform ν_1 smoothing steps $u^l \leftarrow R_l(u^l, f^l)$, (2.9b) perform γ cycles on level $l-1$ for the problem

$$L_{l-1}\, u^{l-1} = I_l^{l-1}(f^l - L_l u^l)$$

starting from the initial approximation $u^{l-1} = 0$,

(2.9c) perform the correction $u^l \leftarrow u^l + I_{l-1}^l u^{l-1}$,

(2.9d) perform ν_2 smoothing steps $u^l \leftarrow R_l(u^l, f^l)$.

Let us note that γ in (2.9b) is a further parameter.

Both algorithms presented are stationary linear iterative methods. Let M_l, $\overline{M}_l$ be the iteration operators corresponding to the two-level and multi-level algorithms, respectively. Then for $l = m, \ldots, 1$

$$M_l = S_l^{\nu_2} Q_l S_l^{\nu_1}, \tag{2.10a}$$

$$Q_l = I_l - P_l, \quad I_l \text{ is the } (n_l - 1) \times (n_l - 1) \text{ identity}, \tag{2.10b}$$

$$P_l = I_{l-1}^l L_{l-1}^{-1} I_l^{l-1} L_l \tag{2.10c}$$

and

$$\overline{M}_1 = M_1,\ \overline{M}_l = M_l + S_l^{\nu_2} I_{l-1}^l \overline{M}_{l-1}^{\gamma} L_{l-1}^{-1} I_l^{l-1} L_l S_l^{\nu_1}. \tag{2.11}$$

For the proof of the relations (2.10) and (2.11) see [2], [3].

3. FOURIER ANALYSIS

Due to the use of uniform grids the eigenvectors of L_l are algebraic vectors Φ_k^l with the entries

$$(\Phi_k^l)_i = \sin \frac{k\pi}{n_l} i : \quad k, i = 1, \ldots, n_l - 1, \tag{3.1a}$$

and the corresponding eigenvalues of L_l are

$$\lambda_k^l = 4n_l^2 \sin^2 \frac{k\pi}{2n_l}. \tag{3.1b}$$

The Fourier analysis of the iteration operator M_l consists in its representation with respect to the basis $\{\Phi_k^l\}$. The crucial point for this is the representation of the transfer operators I_l^{l-1}, I_{l-1}^l with respect to the bases $\{\Phi_k^l\}$, $\{\Phi_k^{l-1}\}$. The following identities make this representation possible:

$$\begin{aligned} I_l^{l-1}\Phi_k^l &= \tfrac{1}{3} c_k^l \Phi_k^{l-1}, \\ I_l^{l-1}\Phi_{2N-k}^l &= \tfrac{1}{3} c_{2N-k}^l \Phi_k^{l-1}, \\ I_l^{l-1}\Phi_{2N+k}^l &= \tfrac{1}{3} c_{2N+k}^l \Phi_k^{l-1}, \end{aligned} \tag{3.2a}$$

$$I_l^{l-1}\Phi_N^l = I_l^{l-1}\Phi_{2N}^l = 0, \tag{3.2b}$$

$$I_{l-1}^l \Phi_k^{l-1} = \tfrac{1}{9}(c_k^l \Phi_k^l + c_{2N-k}^l \Phi_{2N-k}^l + c_{2N+k}^l \Phi_{2N+k}^l), \tag{3.2c}$$

where $N = n_{l-1}$, $k = 1, \ldots, N-1$ and

$$
\tag{3.3}
\begin{aligned}
c_k^l &= 1 + 2\cos\frac{k\pi}{3N}\,,\\
-c_{2N-k}^l &= 1 + 2\cos\frac{(2N-k)\pi}{3N}\,,\\
c_{2N+k}^l &= 1 + 2\cos\frac{(2N+k)\pi}{3N}\,.
\end{aligned}
$$

A verification of these identities may be found in the appendix.

As the consequence of (3.2), the spaces

$$
\tag{3.4}
\begin{aligned}
E_{l,k} &= \operatorname{span}\{\Phi_k^l, \Phi_{2N-k}^l, \Phi_{2N+k}^l\}\,, \quad k = 1, \ldots, N-1\,,\\
E_{l,N} &= \operatorname{span}\{\Phi_N^l\}\,, \quad E_{l,2N} = \operatorname{span}\{\Phi_{2N}^l\}
\end{aligned}
$$

reduce the operators S_l, P_l, Q_l, M_l defined in Section 2. In what follows, let $S_{l,k}$, $P_{l,k}$, $Q_{l,k}$, $M_{l,k}$ denote respectively the restrictions of the operators S_l, P_l, Q_l, M_l to the invariant subspace $E_{l,k}$.

Now, we are interested in the matrix representation of $P_{l,k}$, $Q_{l,k}$, $M_{l,k}$ with respect to the bases of $E_{l,k}$ formed by the eigenvectors of L_l. This matrix representation will be denoted by a hat. Let us start with $k = 1, \ldots, N-1$. Then

$$
\hat{P}_{l,k} = \frac{1}{27\lambda_k^{l-1}} \begin{bmatrix} c_k^l \\ c_{2N-k}^l \\ c_{2N+k}^l \end{bmatrix} \left[\lambda_k^l c_k^l,\ \lambda_{2N-k}^l c_{2N-k}^l,\ \lambda_{2N+k}^l c_{2N+k}^l\right].
$$

From the definitions of λ_k^{l-1}, λ_k^l, c_k^l and from the expression of $\sin 3\alpha$ in terms of the trigonometric functions of the argument α we immediately obtain the following relations:

$$
\frac{\lambda_k^l c_k^l}{9\lambda_k^{l-1}} = \frac{1}{c_k^l}\,, \quad \frac{\lambda_{2N-k}^l c_{2N-k}^l}{9\lambda_k^{l-1}} = \frac{1}{c_{2N-k}^l}\,, \quad \frac{\lambda_{2N+k}^l c_{2N+k}^l}{9\lambda_k^{l-1}} = \frac{1}{c_{2N+k}^l}\,.
$$

Hence, we have

$$
\tag{3.5}
\hat{P}_{l,k} = \tfrac{1}{3} \begin{bmatrix} c_k^l & & \\ & c_{2N-k}^l & \\ & & c_{2N+k}^l \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_k^l & & \\ & c_{2N-k}^l & \\ & & c_{2N+k}^l \end{bmatrix}^{-1},
$$

$$
\tag{3.6}
\hat{Q}_{l,k} = \tfrac{1}{3} \begin{bmatrix} c_k^l & & \\ & c_{2N-k}^l & \\ & & c_{2N+k}^l \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} c_k^l & & \\ & c_{2N-k}^l & \\ & & c_{2N+k}^l \end{bmatrix}^{-1}
$$

and

$$
\tag{3.7}
\hat{M}_{l,k} = \begin{bmatrix} s_k^l & & \\ & s_{2N-k}^l & \\ & & s_{2N+k}^l \end{bmatrix}^{\nu_2} \hat{Q}_{l,k} \begin{bmatrix} s_k^l & & \\ & s_{2N-k}^l & \\ & & s_{2N+k}^l \end{bmatrix}^{\nu_1},
$$

where

$$s_z^l = 1 - \omega \frac{h_l^2}{2} \lambda_z^l = 1 - 2\omega \sin^2 \frac{z\pi}{6N}, \tag{3.8}$$

$$\text{for} \quad z = 1, \ldots, 3N - 1, \quad \omega \in (0, 1\rangle .$$

Finally, let us note that

$$\hat{M}_{l,N} = [s_N^l]^{\nu_2 + \nu_1} = [1 - \tfrac{1}{2}\omega]^{\nu_2 + \nu_1}, \tag{3.9a}$$

$$\hat{M}_{l,2N} = [s_{2N}^l]^{\nu_2 + \nu_1} = [1 - \tfrac{3}{2}\omega]^{\nu_2 + \nu_1}. \tag{3.9b}$$

4. SPECTRAL RADIUS OF THE TWO-LEVEL OPERATOR

To give some insight into the asymptotic convergence behaviour of the two-level method we shall consider the quantities

$$\varrho_l = \varrho_l(\omega, \nu) = \varrho(M_l) \; - \text{the spectral radius of } M_l, \tag{4.1a}$$

$$\varrho^* = \varrho^*(\omega, \nu) = \sup \{\varrho_l : l = 1, 2, \ldots\}. \tag{4.1b}$$

Note that both ϱ_l and ϱ^* depend on the parameter ω of the smoothing procedure and on the numbers of smoothing iterations ν_1, ν_2, more precisely on their sum $\nu = \nu_1 + \nu_2$.

Taking into account the reduction of the operator M_l described in Section 2, we obtain

$$\varrho_l = \max \{\varrho_{l,1}, \ldots, \varrho_{l,N-1}, \varrho_{l,N}, \varrho_{l,2N}\}, \tag{4.2}$$

where $\varrho_{l,k}$ denotes the spectral radius of $M_{l,k}$, $N = n_{l-1}$.

Let $k = 1, \ldots, N - 1$, then according to (3.7) the matrix $\hat{M}_{l,k}$ is equivalent to $\hat{G}_{l,k}$, where

$$\hat{G}_{l,k} = \frac{1}{3} \begin{bmatrix} 2s_k^\nu & -s_k^{\nu_2} s_{2N-k}^{\nu_1} & -s_k^{\nu_2} s_{2N+k}^{\nu_1} \\ -s_{2N-k}^{\nu_2} s_k^{\nu_1} & 2s_{2N-k}^\nu & -s_{2N-k}^{\nu_2} s_{2N+k}^{\nu_1} \\ -s_{2N+k}^{\nu_2} s_k^{\nu_1} & -s_{2N+k}^{\nu_2} s_{2N-k}^{\nu_1} & 2s_{2N+k}^\nu \end{bmatrix}, \tag{4.3}$$

$\nu = \nu_1 + \nu_2$ and for the sake of brevity s_k^ν means $(s_k^l)^\nu$, etc.

Hence, the characteristic equation of $\hat{M}_{l,k}$ has the form

$$3\mu(3\mu^2 - 2\mu B_{l,k} + C_{l,k}) = 0, \tag{4.4a}$$

where

$$B_{l,k} = s_k^\nu + s_{2N-k}^\nu + s_{2N+k}^\nu; \tag{4.4b}$$

$$C_{l,k} = s_k^\nu s_{2N-k}^\nu + s_{2N-k}^\nu s_{2N+k}^\nu + s_{2N+k}^\nu s_k^\nu. \tag{4.4c}$$

The roots of (4.4) are

$$\mu_1 = 0, \quad \mu_{2,3} = \tfrac{1}{3}(B_{l,k} \pm D_{l,k}), \tag{4.5}$$

where

$$(4.6)\qquad D_{l,k}^2 = B_{l,k}^2 - 3C_{l,k} = $$
$$= \tfrac{1}{2}[(s_k^\nu - s_{2N-k}^\nu)^2 + (s_{2N-k}^\nu - s_{2N+k}^\nu)^2 + (s_{2N+k}^\nu - s_k^\nu)^2] \geqq 0\,.$$

From (4.2), (4.5) and (3.9) we conclude that

$$(4.7)\qquad \varrho_l = \max \begin{Bmatrix} \tfrac{1}{3}|B_{l,k} + D_{l,k}| : k = 1, \dots, N-1 \\ \tfrac{1}{3}|B_{l,k} - D_{l,k}| : k = 1, \dots, N-1 \\ |1 - \tfrac{1}{2}\omega|^\nu ; \quad |1 - \tfrac{3}{2}\omega|^\nu \end{Bmatrix}.$$

In what follows we focus our attention on the asymptotic convergence factor ϱ^*. We shall start with upper estimates of ϱ^* in the cases of $\nu = 1$ and $\nu > 1$ even.

Let us consider the case $\nu = 1$. From (3.8), (4.4)–(4.7) we obtain

$$(4.8a)\qquad B_{l,k} = 3(1 - \omega)\,,$$
$$(4.8b)\qquad D_{l,k} = \tfrac{3}{2}\omega\,.$$

The proof of these relations is postponed to the appendix. Consequently, for $\nu = 1$ we have

$$(4.9)\qquad \varrho^* = \varrho_l = \max\{|1 - \tfrac{1}{2}\omega|\,, \ |1 - \tfrac{3}{2}\omega|\} \geqq \tfrac{1}{2}\,.$$

Moreover, $\varrho^* = \varrho^*(\omega) = 1/2$ for $\omega = 1$.

Further, we shall consider the cases of $\nu = 2$ and generally of ν even. For $\nu = 2$ we have

$$(4.10a)\qquad B_{l,k} = 3[(1 - \omega)^2 + \tfrac{1}{2}\omega^2] > 0\,,$$
$$(4.10b)\qquad D_{l,k}^2 = B_{l,k}^2 - 3C_{l,k} < B_{l,k}^2\,.$$

The proof of the former relation will be given in the appendix; the latter relation is obvious.

From (4.10), it follows that

$$\tfrac{1}{3}|B_{l,k} \pm D_{l,k}| \leqq \tfrac{2}{3}B_{l,k} = 2[(1 - \omega)^2 + \tfrac{1}{2}\omega^2]\,,$$
$$(4.11)\qquad \varrho^* \leqq \max\{|1 - \tfrac{1}{2}\omega|^2; |1 - \tfrac{3}{2}\omega|^2; 2[(1 - \omega)^2 + \tfrac{1}{2}\omega^2]\}\,.$$

The minimum value of the above estimate is $2/3$ and it is attained for $\omega = 2/3$. The estimate (4.11) holds true for all $\nu > 1$ even, since

$$s_k^\nu \leqq s_k^2\,, \quad s_{2N-k}^\nu \leqq s_{2N-k}^2\,, \quad s_{2N+k}^\nu \leqq s_{2N+k}^2\,.$$

Furthermore, considering the behaviour of the spectral radius of $M_{l,1}$ for $l \to \infty$, we can establish lower bounds for $\varrho^* = \varrho^*(\omega, \nu)$.

From (3.8) we obtain

$$s_1^l \to 1 \quad \text{for} \quad l \to \infty\,,$$
$$s_{2N-1}^l\,, \quad s_{2N+1}^l \to \xi = 1 - \tfrac{3}{2}\omega \quad \text{for} \quad l \to \infty\,,$$

and $\xi \in \langle -0{\cdot}5, 1)$ for $\omega \in (0, 1\rangle$.

Thus

$$B_{l,1} \to 1 + 2\xi^\nu \quad \text{for} \quad l \to \infty,$$
$$D_{l,1} \to 1 - \xi^\nu \quad \text{for} \quad l \to \infty$$
and according to (4.1), (4.2), (4.7)
$$\varrho^*(\omega, \nu) \geqq \max\{\tfrac{1}{3}|2 + \xi^\nu| ; |\xi^\nu|\}.$$

Consequently, for ν odd we have
$$\inf_\omega \varrho^*(\omega, \nu) \geqq \inf_{-2^{-\nu} \leqq \xi^\nu < 1} \max\{\tfrac{1}{3}|2 + \xi^\nu|; |\xi^\nu|\} \geqq \frac{2}{3}\left(1 - \frac{1}{2^{\nu+1}}\right)$$
and for ν even
$$\inf_\omega \varrho^*(\omega, \nu) = \inf_{0 \leqq \xi^\nu < 1} \max\{\tfrac{1}{3}|2 + \xi^\nu|; |\xi^\nu|\} \geqq \tfrac{2}{3}.$$

Now, let us summarize the results obtained in the following

Theorem 1. *Let $\varrho^* = \varrho^*(\omega, \nu)$ be the supremum of the spectral radii of the two-level operators introduced in* (4.1).

Then for $\omega \in (0, 1\rangle$,

(4.12a) $$\varrho^*(\omega, \nu) \geqq \tfrac{2}{3}(1 - 2^{-\nu-1}) \quad \textit{if } \nu \textit{ is odd},$$

(4.12b) $$\varrho^*(\omega, \nu) \geqq \tfrac{2}{3} \quad \textit{if } \nu \textit{ is even}.$$

Moreover, the lower bounds are attained in the cases $\omega = 1$, $\nu = 1$ and $\omega = 2/3$, ν even.

Remark. Let us point out that $\varrho^*(\omega, \nu)$ does not tend to zero if ν goes to infinity as is typical for the multi-level methods based on interpolation transfer operators, cf. [2], [3].

5. ENERGY NORM OF THE TWO-LEVEL OPERATOR

In this section we are interested in the quantities

(5.1) $$\tau_l = \tau_l(\omega, \nu_1, \nu_2) = \|M_l\|_E = \|\tilde{M}_l\|_S,$$

(5.2) $$\tau^* = \tau^*(\omega, \nu_1, \nu_2) = \sup\{\tau_l : l = 1, 2, \ldots\},$$

where

(5.3) $$\tilde{M}_l = \tilde{M}_l(\omega, \nu_1, \nu_2) = L_l^{1/2} M_l L_l^{-1/2} = = S_l^{\nu_2}[I - L_l^{1/2} I_{l-1}^l L_{l-1}^{-1} I_l^{l-1} L_l^{1/2}] S_l^{\nu_1},$$

and

(5.4) $$\|M\|_S = \sqrt{(\varrho(M^\mathrm{T} M))}$$

is the spectral norm of the operator defined by the matrix M, M^T denotes the transpose of M.

Theorem 2. *Let* $0 < \omega \leqq 1$, let ν_1, ν_2 *be nonnegative integers*, $\nu = \nu_1 + \nu_2$. *Then*

$$\tau^*(\omega, \nu_1, \nu_2) = \tau^*(\omega, \nu_2, \nu_1)\,, \tag{5.5a}$$

$$\tau^*(\omega, \nu/2, \nu/2) = \varrho^*(\omega, \nu) \quad \textit{for} \quad \nu \textit{ even}\,, \quad \nu \geqq 2\,, \tag{5.5b}$$

$$\tau^*(\omega, 0, \nu) = \tau^*(\omega, \nu, 0) = \sqrt{(\varrho^*(\omega, 2\nu))}\,, \tag{5.5c}$$

$$\tau^*(\omega, \nu_1, \nu_2) \leqq \tau^*(\omega, 1, 1) = \varrho^*(\omega, 2) \quad \textit{for} \quad \nu_1, \nu_2 \geqq 1\,. \tag{5.5d}$$

Corollary. *For $\omega = 2/3$ we obtain the following optimal results*:

$$\tau^*(2/3, 0, \nu) = \tau^*(2/3, \nu, 0) = \sqrt{(2/3)}\,, \tag{5.6a}$$

$$\tau^*(2/3, \nu/2, \nu/2) = 2/3 \quad \textit{for} \quad \nu \textit{ even}, \nu \geqq 2\,, \tag{5.6b}$$

$$\tau^*(2/3, \nu_1, \nu_2) \leqq 2/3 \quad \textit{for} \quad \nu_1, \nu_2 \geqq 1\,. \tag{5.6c}$$

Proof. The relation (5.5a) is a consequence of the identity

$$\tilde{M}_l(\omega, \nu_1, \nu_2) = [\tilde{M}_l(\omega, \nu_2, \nu_1)]^{\mathrm{T}}\,.$$

The next relation (5.5b) follows by the symmetry of $\tilde{M}_l(\omega, \nu/2, \nu/2)$. More precisely, we have

$$\|\tilde{M}_l(\omega, \nu/2, \nu/2)\|_S = \varrho(\tilde{M}_l(\omega, \nu/2, \nu/2)) = \varrho(M_l(\omega, \nu/2, \nu/2))\,.$$

According to (5.5a) we have $\tau^*(\omega, 0, \nu) = \tau^*(\omega, \nu, 0)$. Further, by virtue of the identity

$$[\tilde{M}_l(\omega, \nu, 0)]^{\mathrm{T}}\, \tilde{M}_l(\omega, \nu, 0) = \tilde{M}_l(\omega, \nu, \nu)$$

we obtain

$$\|M_l(\omega, \nu, 0)\|_E = \sqrt{(\varrho(\tilde{M}_l(\omega, \nu, \nu)))} = \sqrt{(\varrho(M_l(\omega, \nu, \nu)))}\,,$$

so that (5.5c) holds true.
Finally, for $\nu_1, \nu_2 \geqq 1$ we have

$$\|\tilde{M}_l(\omega, \nu_1, \nu_2)\|_S \leqq \|S_1\|_S^{\nu_2 - 1}\, \|\tilde{M}_l(\omega, 1, 1)\|_S\, \|S_l\|_S^{\nu_1 - 1}\,,$$

where $\|S_l\|_S = \varrho(S_l) \leqq 1$. This yields the last relation (5.5d).

6. SPECTRAL NORM OF THE TWO-LEVEL OPERATOR

Now, we shall study the quantities

$$\sigma_l = \sigma_l(\omega, \nu_1, \nu_2) = \|M_l\|_S\,, \tag{6.1}$$

$$\sigma^* = \sigma^*(\omega, \nu_1, \nu_2) = \sup\,\{\sigma_l\colon l = 1, 2, \ldots\}\,. \tag{6.2}$$

Let us notice three facts. First, the spectral norm can be expressed in the form

$$\|M_l\|_S = \sup_{u \neq 0} \frac{\|M_l u\|}{\|u\|}\,,$$

where $\|u\|$is the l_2-norm of the $(n - 1)$-dimensional algebraic vector u. Second, the operator M_l is reduced by l_2-orthogonal subspaces $E_{l,k}$, $k = 1, \ldots, N, 2N$. Finally, all eigenvectors Φ_k^l can be l_2-normalized by the common factor.

From these facts we can conclude that

(6.3) $$\sigma_l = \max\{\sigma_{l,1}, \ldots, \sigma_{l,N}, \sigma_{l,2N}\},$$

where

(6.4) $$\sigma_{l,k} = \|M_{l,k}\|_S = \|\hat{M}_{l,k}\|_S .$$

Theorem 3. *Let* $0 < \omega \leqq 1$ *and let* ν_1, ν_2 *be nonnegative integers. Then*

(6.5a) $$\sigma^*(\omega, \nu_1, \nu_2) = \infty \quad \text{for} \quad \omega \neq 2/3, \quad \nu_1, \nu_2 \ \text{arbitrary},$$

(6.5b) $$\sigma^*(\omega, \nu_1, \nu_2) \leqq \sqrt{\tfrac{2}{3}} \quad \text{for} \quad \omega = 2/3, \quad \nu_1 \geqq 1 .$$

Proof. The matrix $\hat{M}_{l,k}$ has the entry

$$[\hat{M}_{l,k}]_{1,2} = -\frac{c_k}{c_{2N-k}} s_k^{\nu_2} s_{2N-k}^{\nu_1}$$

whose absolute value goes to infinity in the case $\omega \neq 2/3$, k, ν_1, ν_2 fixed and $l \to \infty$. Consequently, the statement (6.5a) must hold true.

Furthermore, for $\omega = 2/3$ we have

$$c_k^l = 3s_k^l, \quad k = 1, \ldots, n_l - 1$$

and $\hat{M}_{l,k}$ assumes the form

$$\frac{1}{3}\begin{bmatrix} 2s_k^\nu & -s_k^{\nu_2+1}s_{2N-k}^{\nu_1-1} & -s_k^{\nu_2+1}s_{2N+k}^{\nu_1-1} \\ -s_{2N-k}^{\nu_2+1}s_k^{\nu_1-1} & 2s_{2N-k}^\nu & -s_{2N-k}^{\nu_2+1}s_{2N+k}^{\nu_1-1} \\ -s_{2N+k}^{\nu_2+1}s_k^{\nu_1-1} & -s_{2N+k}^{\nu_2+1}s_{2N-k}^{\nu_1-1} & 2s_{2N+k}^\nu \end{bmatrix}.$$

Thus, as the square of the spectral norm of a matrix M can be estimated from above by the sum of squares of its entries, we obtain

$$\sigma_{l,k} \leqq \tfrac{6}{9} s_k^2 + s_{2N-k}^2 + s_{2N+k}^2)$$

for $k = 1, \ldots, N-1$ and $\nu_1 \geqq 1$. With respect to (4.10) it follows that

$$\sigma_{l,k}^2 \leqq \tfrac{2}{3} .$$

Finally, from (3.9) we have $\sigma_{l,N} = 2/3$ and $\sigma_{l,2N} = 0$, so that (6.5b) holds.

Remark. For better understanding of (6.5a) we can consider the following example. Let us start with the error $e^{l,0}$

$$e_i^{l,0} = \varepsilon \quad \text{for} \quad i = 2, \quad n_l - 2, \quad \varepsilon \neq 0,$$
$$e_i^{l,0} = 0 \quad \text{for} \quad i = 1, 3, \ldots, n_l - 3, \quad n_l - 1 .$$

Then performing the correction steps (2.7b), (2.7c) we obtain the new error $e^{l,1}$

$$e_i^{l,1} = 0, \quad \text{for} \quad i = 1, 2, n_l - 2, n_l - 1,$$
$$e_i^{l,1} = -\varepsilon \quad \text{for} \quad i = 3, \ldots, n_l - 3 .$$

The smoothing of $e^{l,1}$ by ν Jacobi iterations (2.8) produces the next error $e^{l,1+\nu}$,

$$e_i^{l,1+\nu} = -\varepsilon \quad \text{for} \quad i = 3 + \nu, \ldots, n_l - \nu - 3 .$$

Therefore

$$\sigma_l(\omega, 0, \nu) \geqq \frac{\|e^{l,1+\nu}\|}{\|e^{l,0}\|} \geqq \frac{\sqrt{(n_l - 2\nu - 5)}}{\sqrt{2}}$$

and

$$\sigma^*(\omega, 0, \nu) = \infty .$$

Let us note that a little less unfavourable example was proposed in [4] to demonstrate the features of the so called one-sided interpolation transfers.

Remark. The use of Gauss-Seidel or conjugate gradients for smoothing after the aggregation correction steps does not change the above behaviour.

7. CONCLUDING REMARKS

The results concerning the spectral radius and the energy norm of the aggregation correction (AC) two-level iteration operator indicate a substantial acceleration of the convergence rate as compared with the simple Jacobi relaxation method. On the other hand, one AC two-level cycle is computationally more expensive than one Jacobi iteration, so that an estimate of the number of arithmetic operations to obtain given accuracy is needed for the appreciation of the AC method.

To this end, let us consider the multi-level AC process (2.9). In [6] we proved that under the assumptions $\omega = 2/3$, $\nu_1 = \nu_2 > 0$ the number of arithmetic operations required to reduce the energy norm of the initial error by a given factor is proportional to n^α, where n is the number of unknowns and $\alpha = \ln 4/\ln 3 \doteq 1{\cdot}262$.

From the above result we can conclude that the AC multi-level method is more efficient than the simple relaxation methods but less efficient than multigrid methods with better interpolation transfers. Thus, the multi-level AC method can be useful for problems where a better interpolation transfer is excluded for some reasons.

Acknowledgement. I thank Jan Mandel for valuable comments to this paper.

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APPENDIX

Let us start with three identities verifiable by elementary means. In the sequel, k, N are arbitrary, $x = 1$ or $x = 2$:

$$\text{(i)} \qquad \cos\frac{xk\pi}{3N} + \cos\frac{x(2N-k)\pi}{3N} + \cos\frac{x(2N+k)\pi}{3N} = 0\,,$$

$$\text{(ii)} \qquad \sin\frac{xk\pi}{3N} - \sin\frac{x(2N-k)\pi}{3N} + \sin\frac{x(2N+k)\pi}{3N} = 0\,,$$

$$\text{(iii)} \qquad \cos^2\frac{k\pi}{3N} + \cos^2\frac{(2N-k)\pi}{3N} + \cos^2\frac{(2N+k)\pi}{3N} = \frac{3}{2}\,.$$

Now, let us proceed to the formulae to be proved. For the proof of the first two, (3.2a), (3.2b), it will suffice to compare the entries of the vector identities. For example,

$$(I_l^{l-1}\Phi_k^l)_i = \tfrac{1}{3}\left[\sin\frac{k\pi}{3N}(3i-1) + \sin\frac{k\pi}{3N}3i + \sin\frac{k\pi}{3N}(3i+1)\right] =$$
$$= \tfrac{1}{3}\left(1 + 2\cos\frac{k\pi}{3N}\right)\sin\frac{k\pi}{3N}3i = \tfrac{1}{3}c_k^l(\Phi_k^{l-1})_i$$

for $i = 1, \ldots, N-1$, so that $I_l^{l-1}\Phi_k^l = \frac{1}{3}c_k^l\Phi_k^{l-1}$. The proof of the other identities is similar.

Next, we shall prove (3.2c). Let $N = n_{l-1}$, $j = 1, \ldots, N-1$. For the proof we shall compare the i-th entries of the vector identity. It is necessary to distinguish the cases $i = 3j$, $i = 3j-1$, $i = 3j+1$. In the first case we have

$$(c_k^l\Phi_k^l + c_{2N-k}^l\Phi_{2N-k}^l + c_{2N+k}^l\Phi_{2N+k}^l)_{3j} = \left[1 + 2\cos\frac{k\pi}{3N}\right]\sin\frac{k\pi}{3N}3j -$$
$$-\left[1 + 2\cos\frac{(2N-k)\pi}{3N}\right]\sin\frac{(2N-k)\pi}{3N}3j +$$
$$+\left[1 + 2\cos\frac{(2N+k)\pi}{3N}\right]\sin\frac{(2N+k)\pi}{3N}3j =$$
$$= 3\sin\frac{k\pi}{N}j + 2\sin\frac{k\pi}{N}j\left[\cos\frac{k\pi}{3N} + \cos\frac{(2N-k)\pi}{3N} + \cos\frac{(2N+k)\pi}{3N}\right] =$$
$$= 3\sin\frac{k\pi}{N}j = 9(I_{l-1}^l\Phi_k^{l-1})_{3j}$$

since (i) holds and

$$\sin\frac{k\pi}{3N}3j = -\sin\frac{(2N-k)\pi}{3N}3j = \sin\frac{(2N+k)\pi}{3N}3j = \sin\frac{k\pi}{N}j\,.$$

In the second case

$$(c_k^l\Phi_k^l + c_{2N-k}^l\Phi_{2N-k}^l + c_{2N+k}^l\Phi_{2N+k}^l)_{3j-1} =$$

$$= \left[1 + 2\cos\frac{k\pi}{3N}\right]\left[\sin\frac{k\pi}{N}j\cos\frac{k\pi}{3N} - \cos\frac{k\pi}{N}j\sin\frac{k\pi}{3N}\right] +$$

$$-\left[1 + 2\cos\frac{(2N-k)\pi}{3N}\right]\left[-\sin\frac{k\pi}{N}j\cos\frac{(2N-k)\pi}{3N} - \cos\frac{k\pi}{N}j\sin\frac{(2N-k)\pi}{3N}\right] +$$

$$+\left[1 + 2\cos\frac{(2N+k)\pi}{3N}\right]\left[\sin\frac{k\pi}{N}j\cos\frac{(2N+k)\pi}{3N} - \cos\frac{k\pi}{N}j\sin\frac{(2N+k)\pi}{3N}\right] =$$

$$= 3\sin\frac{k\pi}{N}j = 9(I_{l-1}^l\Phi_k^{l-1})_{3j-1}\,,$$

since (i), (ii) for $x = 1$ and $x = 2$, and (iii) hold true. The proof in the case $i = 3j + 1$ is similar.

Finally, let us focus our attention on the relations (4.8a), (4.8b) and (4.10a). From the definition (3.8) we obtain

$$s_k^l + s_{2N-k}^l + s_{2N+k}^l =$$

$$= 3 - 2\omega\left[\sin^2\frac{k\pi}{6N} + \sin^2\frac{(2N-k)\pi}{6N} + \sin^2\frac{(2N+k)\pi}{6N}\right] =$$

$$= 3 - 2\omega\left[\tfrac{3}{2} - \tfrac{1}{2}\cos\frac{k\pi}{3N} - \tfrac{1}{2}\cos\frac{(2N-k)\pi}{3N} - \tfrac{1}{2}\cos\frac{(2N+k)\pi}{3N}\right] = 3(1-\omega)\,.$$

Further,

$$(s_k^l)^2 + (s_{2N-k}^l)^2 + (s_{2N+k}^l)^2 =$$

$$= 3 - 4\omega\left[\sin^2\frac{k\pi}{6N} + \sin^2\frac{(2N-k)\pi}{6N} + \sin^2\frac{(2N+k)\pi}{6N}\right] +$$

$$+ 4\omega^2\left[\sin^4\frac{k\pi}{6N} + \sin^4\frac{(2N-k)\pi}{6N} + \sin^4\frac{(2N+k)\pi}{6N}\right] = 3 - 6\omega +$$

$$+ \omega^2\left[\left(1 - \cos\frac{k\pi}{3N}\right)^2 + \left(1 - \cos\frac{(2N-k)\pi}{3N}\right)^2 + \left(1 - \cos\frac{(2N+k)\pi}{3N}\right)^2\right] =$$

$$= 3[(1-\omega)^2 + \tfrac{1}{2}\omega^2]\,.$$

Consequently,

$$s_k^l s_{2N-k}^l + s_{2N-k}^l s_{2N+k}^l + s_{2N+k}^l s_k^l =$$
$$= \tfrac{1}{2}\left[\left(s_k^l + s_{2N-k}^l + s_{2N+k}^l\right)^2 - \left(s_k^l\right)^2 - \left(s_{2N-k}^l\right)^2 - \left(s_{2N+k}^l\right)^2\right] = 3\left[\left(1 - \omega^2\right) - \tfrac{1}{4}\omega^2\right].$$

Apparently, the above identities yield the relations to be proved.

Souhrn

VÍCEÚROVŇOVÁ METODA S OPRAVOU POMOCÍ AGREGACE PRO ŘEŠENÍ DISKRÉTNÍCH ELIPTICKÝCH ÚLOH

RADIM BLAHETA

V práci se studuje chování víceúrovňové metody, která kombinuje Jacobiho iterace a korekci pomocí agregace proměnných. Studium je omezeno na jednoduchý jednorozměrný modelový příklad, který umožňuje plně využít techniku Fourierovy analýzy. Přes uvedené omezení je v práci ukázán rozdíl v chování studované metody proti chování vícesíťových metod využívajících interpolaci místo agregace.

Резюме

МНОГОСЕТОЧНЫЙ МЕТОД РЕШЕНИЯ ДИСКРЕТНЫХ ЕЛЛИПТИЧЕСКИХ ЗАДАЧ С ПОПРАВКОЙ ПОСРЕДСТВОМ АГРЕГАЦИИ НЕИЗВЕСТНЫХ

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В статье изучается поведение многосеточного алгорифма использующего простые итерации и поправки посредством аргерации неизвестных. Поведение алгорифма продемонстрировано на примере простой одномерной краевой задачи, для которой вполне применим аппарат анализа Фурье. Этот пример также показывает различие между поведением рассматриваемого алгорифма и поведением многосеточных алгорифмов, использующих интерполяцию вместо агрегации.

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