A Type of Multigrid Method for Eigenvalue Problem*

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Abstract

In this paper, a new type of iteration method is proposed to solve eigenvalue problem by finite element method. With this new scheme, solving eigenvalue problem is transformed to solving a series of source problems on multilevel meshes by multigrid method. Besides, all other efficient iteration methods for solving source problems can serve as source problem solver. The computational work of this new scheme can reach optimal order the same as solving the corresponding source problem. Therefore, this type of iteration scheme improves the efficiency of eigenvalue problem solving. Some numerical experiments are presented to validate the efficiency of the new method.

Keywords. Eigenvalue problem, multigrid, multi-level correction, finite element method.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

Solving large scale eigenvalue problems becomes a fundamental problem in modern science and engineering society. However, it is always a very difficult task to solve high-dimensional eigenvalue problems which come from physical and chemistry sciences. Although many efficient algorithms, such as multigrid method and many other precondition techniques ([7, 17]), for solving source problems have been developed, there is no such efficient numerical method to solve eigenvalue problems with optimal computation work property.

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The multigrid method and other efficient preconditioners provide an optimal order algorithm for solving boundary value problems. The error bounds of the approximate solution obtained from these efficient numerical algorithms are comparable to the theoretical bounds determined by the finite element discretization. But the amount of computational work involved is only proportional to the number of unknowns in the discretized equations. For more details of the multigrid and multilevel methods, please read the papers: Bank and Dupont [3], Bramble and Pasciak [5], Bramble and Zhang [6], Xu [17], Scott and Zhang [15] and books: Brenner and Scott [7], Hackbusch [11], McCormick [12] and Bramble [4] and the references cited therein.

For the solution of eigenvalue problem, [18] gives a type of two-grid discretization method to improve the efficiency of solving eigenvalue problems. By the two-grid method, the solution of eigenvalue problem on a fine mesh is reduced to a solution of eigenvalue problem on a coarse mesh and a solution of a source problem on the fine mesh. Recently, we propose a new type of multilevel correction method to solve eigenvalue problem which can be implemented on multilevel grids ([13]). In the multilevel correction scheme, the solution of eigenvalue problem on a fine mesh can be reduced to a series of solutions of eigenvalue problem on a very coarse mesh and a series of solutions of source problems on the multilevel meshes. The multilevel correction method gives a way to construct a type of multigrid method for eigenvalue problem.

The aim of this paper is to present this new type of multigrid scheme for solving eigenvalue problems based on multilevel correction method ([13]). With this method solving eigenvalue problem will not be much more difficult than the solution of the corresponding source problem. The multigrid method for eigenvalue problem is based on a series of finite element spaces with different approximation properties which can be built with the same way as the multilevel method ([17]). It is worth to noting that besides the multigrid method acting as the linear algebraic solver for source problems here, other types of numerical algorithms such as BPX multilevel preconditioners, algebraic multigrid method and domain decomposition preconditioners ([7]) can also been adopted as the linear algebraic solvers.

The standard Galerkin finite element method for eigenvalue problem has been extensively investigated, e.g. Babuška and Osborn [1, 2], Chatelin [8] and references cited therein. Here we adopt some basic results in these papers for our analysis. The corresponding error and computational work estimates of the type of multigrid scheme for eigenvalue problem will be analyzed. Based on the analysis, the new method can obtain optimal errors with an optimal computational work. The eigenvalue multigrid procedure can be described as follows: (1) solve the eigenvalue problem in the coarsest finite element space; (2) solve an additional source problem with multigrid method on the refined mesh using the previous obtained eigenvalue multiplying the corresponding eigenfunction as the load vector; (3) solve eigenvalue problem again on the finite element space which is constructed by combining the coarsest finite element space with the obtained eigenfunction approximation in step (2). Then go to step (2) for next loop until stop.

In order to describe our method clearly, we give the following simple Laplace eigenvalue problem to illustrate the main idea in this paper (see section 4).

Find (λ, u) such that

$$\left\{\begin{array}{rcll} -\Delta u &=& \lambda u, & \text{in } \Omega,\\ u &=& 0, & \text{on } \partial\Omega,\\ \int_\Omega u^2 d\Omega &=& 1, & \end{array}\right. \tag{1.1}$$

where $\Omega \subset \mathcal{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and Δ denote the Laplace operator.

Let V_H denote the coarsest linear finite element space defined on the coarsest mesh $\mathcal{T}_H$. Additionally, we also need to construct a series of finite element spaces $V_{h_2}, V_{h_3}, \cdots, V_{h_n}$ which are defined on the corresponding series of multilevel meshes $\mathcal{T}_{h_k}$ $(k = 2, 3, \cdots n)$ such that $V_H := V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_n}$ with $h_k = h_{k-1}/\beta$ ([7, 9]). Our multigrid algorithm to obtain the approximation of the eigenpair can be defined as follows (see section 3 and section 4):

1. Solve an eigenvalue problem in the coarsest space V_H:
   Find $(\lambda_H, u_H) \in \mathcal{R} \times V_H$ such that $\|u_H\|_0 = 1$ and
   $$\int_\Omega \nabla u_H \nabla v_H d\Omega \;=\; \lambda_H \int_\Omega u_H v_H d\Omega, \quad \forall v_H \in V_H.$$

2. Set $h_1 = H$ and Do $k = 1, \cdots, n-2$
   - Solve the following auxiliary source problem with multigrid method:
     Find $\widehat{u}_{h_{k+1}} \in V_{h_{k+1}}$ such that
     $$\int_\Omega \nabla \widehat{u}_{h_{k+1}} \nabla v_{h_{k+1}} d\Omega \;=\; \lambda_{h_k} \int_\Omega u_{h_k} v_{h_{k+1}} d\Omega, \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}.$$
     Then let $\widetilde{u}_{h_{k+1}} := MG(V_{h_{k+1}}, u_{h_k}, \lambda_{h_k} u_{h_k}, m_{k+1})$ and the multigrid solution has the error estimate $\|\widehat{u}_{h_k} - \widetilde{u}_{h_k}\| \leq C h_k$, where $\widetilde{u}_{h_{k+1}}$ is the solution by multigrid method, u_{h_k} denotes the initial solution, $\lambda_{h_k} u_{h_k}$ the right hand side term, m_{k+1} the multigrid iteration time.
   - Define a new finite element space $V_{H,h_{k+1}} = V_H + \text{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the following eigenvalue problem:
     Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{H,h_{k+1}}$ such that $\|u_{h_{k+1}}\|_0 = 1$ and
     $$\int_\Omega \nabla u_{h_{k+1}} \nabla v_{H,h_{k+1}} d\Omega = \lambda_{h_{k+1}} \int_\Omega u_{h_{k+1}} v_{H,h_{k+1}} d\Omega, \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}.$$

end Do

3. Solve the following auxiliary source problem with multigrid method:

   Find $\widehat{u}_{h_n} \in V_{h_n}$ such that

$$\int_\Omega \nabla \widehat{u}_{h_n} \nabla v_{h_n} d\Omega \;=\; \lambda_{h_{n-1}} \int_\Omega u_{h_{n-1}} v_{h_n} d\Omega, \quad \forall v_{h_n} \in V_{h_n}.$$

   Define $u_{h_n} := MG(V_{h_n}, u_{h_{n-1}}, \lambda_{h_{n-1}} u_{h_{n-1}}, m_n)$ and the multigrid solution has the error estimate $\|\widehat{u}_{h_n} - u_{h_n}\| \le Ch_n$. Then compute the Rayleigh quotient

$$\lambda_{h_n} = \frac{\|\nabla u_{h_n}\|_0^2}{\|u_{h_n}\|_0^2}.$$

If, for example, λ_H is the approximation of the first eigenvalue of the problem at the first step and Ω is a convex domain, then we can establish the following results by taking $\beta\eta_a(H) < 1$ (see section 3 and section 4 for details)

$$\|\nabla(u - u_{h_n})\|_0 = \mathcal{O}(h_n), \quad \text{and} \quad |\lambda_{h_n} - \lambda| = \mathcal{O}(h_n^2).$$

These two estimates means that we obtain asymptotic optimal errors.

In this method, we replace solving eigenvalue problem on the finest finite element space by solving a series of boundary value problems with multigrid scheme in the corresponding series of finite element spaces and a series of eigenvalue problems in the coarsest finite element space. So this multigrid method can improve the efficiency of solving eigenvalue problems as it does for solution problems (with computational work $\mathcal{O}(N_n)$).

An outline of the paper goes as follows. In Section 2, we introduce finite element method for eigenvalue problem and the corresponding basic error estimates. A type of one correction step is given in section 3. In section 4, we propose a type of multigrid algorithm for solving eigenvalue problem by finite element method. Two numerical examples are presented to validate our theoretical analysis in section 5. Some concluding remarks are given in the last section.

2 Discretization by finite element method

In this section, we introduce some notation and error estimates of the finite element approximation for eigenvalue problems. The letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \le C_1 y_1$, $x_2 \ge c_2 y_2$ and $c_3 x_3 \le y_3 \le C_3 x_3$ for some constants C_1, c_2, c_3 and C_3 that are independent of mesh sizes.

Let $(V, \|\cdot\|)$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, respectively. Let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ be two symmetric bilinear forms on $X \times X$ satisfying

$$a(w,v) \lesssim \|w\|\|v\|, \quad \forall w \in V \text{ and } \forall v \in V, \tag{2.1}$$
$$\|w\|^2 \lesssim a(w,w), \quad \forall w \in V \text{ and } 0 < b(w,w), \quad \forall w \in V \text{ and } w \neq 0. \tag{2.2}$$

From (2.1) and (2.2), we know that $\|\cdot\|_a := a(\cdot,\cdot)^{1/2}$ and $\|\cdot\|$ are two equivalent norms on V. We assume that the norm $\|\cdot\|$ is relatively compact with respect to the norm $\|\cdot\|_b := b(\cdot,\cdot)^{1/2}$. We shall use $a(\cdot,\cdot)$ and $\|\cdot\|_a$ as the inner product and norm on V in the rest of this paper.

Set

$$W := \text{the completion of } V \text{ with respect to } \|\cdot\|_b.$$

Thus W is a Hilbert space with the inner product $b(\cdot,\cdot)$ and compactly imbedded in V. Construct a "negative space" by V' = the dual of V with a norm $\|\cdot\|_{-a}$ given by

$$\|w\|_{-a} = \sup_{v \in V, \|v\|_a = 1} b(w,v). \tag{2.3}$$

Then $W \subset V'$ compactly, and for $v \in V$, $b(w,v)$ has a continuous extension to $w \in V'$ such that $b(w,v)$ is continuous on V' by Hahn-Banach theorem ([10]). We assume that $V_h \subset V$ is a family of finite-dimensional spaces that satisfy the following assumption:

For any $w \in V$

$$\lim_{h \to 0} \inf_{v \in V_h} \|w - v\|_a = 0. \tag{2.4}$$

Let P_h be the finite element projection operator of V onto V_h defined by

$$a(w - P_h w, v) = 0, \quad \forall w \in V \text{ and } \forall v \in V_h. \tag{2.5}$$

Obviously

$$\|P_h w\|_a \leq \|w\|_a, \quad \forall w \in V. \tag{2.6}$$

For any $w \in V$, by (2.4) we have

$$\|w - P_h w\|_a = o(1), \quad \text{as } h \to 0. \tag{2.7}$$

Define $\eta_a(h)$ as

$$\eta_a(h) = \sup_{f \in V, \|f\|_a = 1} \inf_{v \in V_h} \|Tf - v\|_a, \tag{2.8}$$

where the operator $T : V' \mapsto V$ is defined as

$$a(Tf, v) = b(f, v), \quad \forall f \in V' \text{ and } \forall v \in V. \tag{2.9}$$

In order to derive the error estimate of eigenpair approximation in negative norm $\|\cdot\|_{-a}$, we need the following negative norm error estimate of the finite element projection operator P_h.

Lemma 2.1. *([2, Lemma 3.3 and Lemma 3.4])*

$$\eta_a(h) \;=\; o(1), \quad \text{as } h \to 0, \tag{2.10}$$

and

$$\|w - P_h w\|_{-a} \;\lesssim\; \eta_a(h)\|w - P_h w\|_a, \quad \forall w \in V. \tag{2.11}$$

In our methodology description, we are concerned with the following general eigenvalue problem:

Find $(\lambda, u) \in \mathcal{R} \times V$ such that $b(u, u) = 1$ and

$$a(u, v) \;=\; \lambda b(u, v), \quad \forall v \in V. \tag{2.12}$$

For the eigenvalue λ, there exists the following Rayleigh quotient expression ([1, 2, 18])

$$\lambda = \frac{a(u, u)}{b(u, u)}. \tag{2.13}$$

From [2, 8], we know the eigenvalue problem (2.12) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity.

Now, let us define the finite element approximations of the problem (2.12). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we can construct a finite element space denoted by $V_h \subset V$. In order to apply multigrid scheme, we start the process on the original mesh $\mathcal{T}_H$ with the mesh size H and the original coarsest finite element space V_H defined on the mesh $\mathcal{T}_H$.

Then we can define the approximation of eigenpair (λ, u) of (2.12) by the finite element method as:

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ such that $b(u_h, u_h) = 1$ and

$$a(u_h, v_h) \;=\; \lambda_h b(u_h, v_h), \qquad \forall v_h \in V_h. \tag{2.14}$$

From (2.14), we know the following Rayleigh quotient expression for λ_h holds ([1, 2, 18])

$$\lambda_h = \frac{a(u_h, u_h)}{b(u_h, u_h)}. \tag{2.15}$$

Similarly, we know from [2, 8] the eigenvalue problem (2.12) has eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{k,h} \leq \cdots \leq \lambda_{N_h,h},$$

and the corresponding eigenfunctions

$$u_{1,h}, u_{2,h}, \cdots, u_{k,h}, \cdots, u_{N_h,h},$$

where $b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \leq i, j \leq N_h$ (N_h is the dimension of the finite element space V_h).

From the minimum-maximum principle ([1, 2]), the following upper bound result holds

$$\lambda_i \leq \lambda_{i,h}, \quad i = 1, 2, \cdots, N_h.$$

Let $M(\lambda_i)$ denote the eigenspace corresponding to the eigenvalue λ_i which is defined by

$$M(\lambda_i) = \big\{w \in V : w \text{ is an eigenvalue of (2.12) corresponding to } \lambda_i \text{ and } \|w\|_b = 1\big\}. \tag{2.16}$$

Then we define

$$\delta_h(\lambda_i) = \sup_{w \in M(\lambda_i)} \inf_{v \in V_h} \|w - v\|_a. \tag{2.17}$$

For the eigenpair approximations by finite element method, there exist the following error estimates.

Proposition 2.1. *([1, Lemma 3.7, (3.29b)], [2, P. 699] and [8])*
(i) For any eigenfunction approximation $u_{i,h}$ of (2.14) ($i = 1, 2, \cdots, N_h$), there is an eigenfunction u_i of (2.12) corresponding to λ_i such that $\|u_i\|_b = 1$ and

$$\|u_i - u_{i,h}\|_a \leq C_i \delta_h(\lambda_i). \tag{2.18}$$

Furthermore,

$$\|u_i - u_{i,h}\|_{-a} \leq C_i \eta_a(h) \|u_i - u_{i,h}\|_a. \tag{2.19}$$

(ii) For each eigenvalue, we have

$$\lambda_i \leq \lambda_{i,h} \leq \lambda_i + C_i \delta_h^2(\lambda_i) \tag{2.20}$$

Here and hereafter C_i is some constant depending on i but independent of the mesh size h.

3 One correction step with multigrid method

In this section, we present a type of correction step to improve the accuracy of the current eigenvalue and eigenfunction approximations. This correction method contains solving an auxiliary source problem with multigrid method in the finer finite element space and an eigenvalue problem on the coarsest finite element space. For simplicity of notation, we set $(\lambda, u) = (\lambda_i, u_i)$ $(i = 1, 2, \cdots, k, \cdots)$ and $(\lambda_h, u_h) = (\lambda_{i,h}, u_{i,h})$ $(i = 1, 2, \cdots, N_h)$ to denote an eigenpair and the corresponding approximation of problem (2.12) and (2.14), respectively.

To analyze our method, we introduce the error expansion of eigenvalue by the Rayleigh quotient formula which comes from [1, 2, 14, 18].

Theorem 3.1. *Assume (λ, u) is the true solution of the eigenvalue problem (2.12), $0 \neq \psi \in V$. Let us define*

$$\widehat{\lambda} = \frac{a(\psi, \psi)}{b(\psi, \psi)}. \tag{3.1}$$

Then we have

$$\widehat{\lambda} - \lambda = \frac{a(u-\psi, u-\psi)}{b(\psi, \psi)} - \lambda \frac{b(u-\psi, u-\psi)}{b(\psi, \psi)}. \tag{3.2}$$

Proof. First from (2.13), (3.1) and direct computation, we have

$$\begin{aligned}
\widehat{\lambda} - \lambda &= \frac{a(\psi, \psi) - \lambda b(\psi, \psi)}{b(\psi, \psi)} \\
&= \frac{a(\psi - u, \psi - u) + 2a(\psi, u) - a(u, u) - \lambda b(\psi, \psi)}{b(\psi, \psi)} \\
&= \frac{a(\psi - u, \psi - u) + 2\lambda b(\psi, u) - \lambda b(u, u) - \lambda b(\psi, \psi)}{b(\psi, \psi)} \\
&= \frac{a(\psi - u, \psi - u) - \lambda b(\psi - u, \psi - u)}{b(\psi, \psi)}. \qquad (3.3)
\end{aligned}$$

Then we obtain the desired result (3.2). □

Assume we have obtained an eigenpair approximation $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$. Now we introduce a type of correction step to improve the accuracy of the current eigenpair approximation (λ_{h_1}, u_{h_1}). Let $V_{h_2} \subset V$ be a finer finite element space such that $V_{h_1} \subset V_{h_2}$. Based on this finer finite element space, we define the following correction step.

Algorithm 3.1. *One Correction Step*

1. *Define the following auxiliary source problem:*

   Find $\widehat{u}_{h_2} \in V_{h_2}$ *such that*

$$a(\widehat{u}_{h_2}, v_{h_2}) = \lambda_{h_1} b(u_{h_1}, v_{h_2}), \quad \forall v_{h_2} \in V_{h_2}. \tag{3.4}$$

   Solve this equation with multigrid method to obtain a new eigenfunction approximation $\widetilde{u}_{h_2} \in V_{h_2}$ *with error estimate* $\|\widehat{u}_{h_2} - \widetilde{u}_{h_2}\|_a \leq C\delta_{h_2}(\lambda)$ *and define* $\widetilde{u}_{h_2} := MG(V_{h_2}, u_{h_1}, \lambda_{h_1} u_{h_1}, m_2)$*, where* V_{h_2} *denotes the computing space,* u_{h_1} *is the initial solution,* $\lambda_{h_1} u_{h_1}$ *the right hand side and* m_2 *the iteration time of the multigrid scheme.*

2. *Define a new finite element space* $V_{H,h_2} = V_H + \mathrm{span}\{\widetilde{u}_{h,2}\}$ *and solve the following eigenvalue problem:*

   Find $(\lambda_{h_2}, u_{h_2}) \in \mathcal{R} \times V_{H,h_2}$ *such that* $b(u_{h_2}, u_{h_2}) = 1$ *and*

$$a(u_{h_2}, v_{H,h_2}) = \lambda_{h_2} b(u_{h_2}, v_{H,h_2}), \quad \forall v_{H,h_2} \in V_{H,h_2}. \tag{3.5}$$

Define $(\lambda_{h_2}, u_{h_2}) = Correction(V_H, \lambda_{h_1}, u_{h_1}, V_{h_2})$.

Theorem 3.2. *Assume the current eigenpair approximation* $(\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1}$ *has the following error estimates*

$$\|u - u_{h_1}\|_a \lesssim \varepsilon_{h_1}(\lambda), \tag{3.6}$$
$$\|u - u_{h_1}\|_{-a} \lesssim \eta_a(H)\|u - u_{h_1}\|_a, \tag{3.7}$$
$$|\lambda - \lambda_{h_1}| \lesssim \varepsilon_{h_1}^2(\lambda). \tag{3.8}$$

Then after one correction step, the resultant approximation $(\lambda_{h_2}, u_{h_2}) \in \mathcal{R} \times V_{h_2}$ *has the following error estimates*

$$\|u - u_{h_2}\|_a \lesssim \varepsilon_{h_2}(\lambda), \tag{3.9}$$
$$\|u - u_{h_2}\|_{-a} \lesssim \eta_a(H)\|u - u_{h_2}\|_a, \tag{3.10}$$
$$|\lambda - \lambda_{h_2}| \lesssim \varepsilon_{h_2}^2(\lambda), \tag{3.11}$$

where $\varepsilon_{h_2}(\lambda) := \eta_a(H)\varepsilon_{h_1}(\lambda) + \varepsilon_{h_1}^2(\lambda) + \delta_{h_2}(\lambda)$.

Proof. From problems (2.5), (2.12) and (3.4), and (3.6), (3.7) and (3.8), the following estimate holds

$$\begin{aligned}
\|\widehat{u}_{h_2} - P_{h_2}u\|_a^2 &\lesssim a(\widehat{u}_{h_2} - P_{h_2}u, \widehat{u}_{h_2} - P_{h_2}u) = b(\lambda_{h_1}u_{h_1} - \lambda u, \widehat{u}_{h_2} - P_{h_2}u) \\
&\lesssim \|\lambda_{h_1}u_{h_1} - \lambda u\|_{-a}\|\widehat{u}_{h_2} - P_{h_2}u\|_a \\
&\lesssim (|\lambda_{h_1} - \lambda|\|u_{h_1}\|_{-a} + \lambda\|u_{h_1} - u\|_{-a})\|\widehat{u}_{h_2} - P_{h_2}u\|_a \\
&\lesssim \big(\varepsilon_{h_1}^2(\lambda) + \eta_a(H)\varepsilon_{h_1}(\lambda)\big)\|\widehat{u}_{h_2} - P_{h_2}u\|_a.
\end{aligned}$$

Then we have

$$\|\widehat{u}_{h_2} - P_{h_2}u\|_a \lesssim \varepsilon_{h_1}^2(\lambda) + \eta_a(H)\varepsilon_{h_1}(\lambda). \quad (3.12)$$

Combining (3.12) and the error estimate of finite element projection

$$\|u - P_{h_2}u\|_a \lesssim \delta_{h_2}(\lambda),$$

we have

$$\|\widehat{u}_{h_2} - u\|_a \lesssim \varepsilon_{h_1}^2(\lambda) + \eta_a(H)\varepsilon_{h_1}(\lambda) + \delta_{h_2}(\lambda). \quad (3.13)$$

From (3.13) and $\|\widehat{u}_{h_2} - \widetilde{u}_{h_2}\|_a \lesssim \delta_{h_2}(\lambda)$, the following estimate holds

$$\|\widetilde{u}_{h_2} - u\|_a \lesssim \varepsilon_{h_1}^2(\lambda) + \eta_a(H)\varepsilon_{h_1}(\lambda) + \delta_{h_2}(\lambda). \quad (3.14)$$

Now we come to estimate the eigenpair solution (λ_{h_2}, u_{h_2}) of problem (3.5). Based on the error estimate theory of eigenvalue problem by finite element method ([1, 2]), the following estimates hold

$$\|u - u_{h_2}\|_a \lesssim \sup_{w\in M(\lambda)} \inf_{v\in V_{H,h_2}} \|w - v\|_a \lesssim \|u - \widetilde{u}_{h_2}\|_a, \quad (3.15)$$

and

$$\|u - u_{h_2}\|_{-a} \lesssim \widetilde{\eta}_a(H)\|u - u_{h_2}\|_a, \quad (3.16)$$

where

$$\widetilde{\eta}_a(H) = \sup_{f\in V, \|f\|_a=1} \inf_{v\in V_{H,h_2}} \|Tf - v\|_a \leq \eta_a(H). \quad (3.17)$$

From (3.14), (3.15), (3.16) and (3.17), we can obtain (3.9) and (3.10). The estimate (3.11) can be derived by Theorem 3.1 and (3.9). □

4 Multigrid scheme for the eigenvalue problem

In this section, we introduce a type of multigrid scheme based on the *One Correction Step* defined in Algorithm 3.1. This type of multigrid method can obtain the optimal error estimate as same as solving the eigenvalue problem directly in the finest finite element space.

In order to do multigrid scheme, we define a sequence of triangulations $\mathcal{T}_{h_k}$ of Ω determined as follows. Suppose $\mathcal{T}_{h_1}$ is given and let $\mathcal{T}_{h_k}$ be obtained from $\mathcal{T}_{h_{k-1}}$ via regular refinement (produce β^d congruent elements) such that

$$h_k = \frac{1}{\beta}h_{k-1}.$$

Based on this sequence of meshes, we construct the corresponding linear finite element spaces such that

$$V_H := V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_n},$$

and the following relation of approximation errors holds

$$\delta_{h_k}(\lambda) = \frac{1}{\beta}\delta_{h_{k-1}}(\lambda), \quad k = 2, \cdots, n. \tag{4.1}$$

Algorithm 4.1. *Eigenvalue Multigrid Scheme*

1. *Construct a coarse finite element space V_H and solve the following eigenvalue problem:*

   Find $(\lambda_H, u_H) \in \mathcal{R} \times V_H$ such that $b(u_H, u_H) = 1$ and

$$a(u_H, v_H) \;=\; \lambda_H b(u_H, v_H), \qquad \forall v_H \in V_H. \tag{4.2}$$

2. *Set $h_1 = H$ and construct a series of finer finite element spaces $V_{h_2}, \cdots, V_{h_n}$ such that $\eta_a(H) \gtrsim \delta_{h_1}(\lambda) \geq \delta_{h_2}(\lambda) \geq \cdots \geq \delta_{h_n}(\lambda)$ as (4.1).*

3. *Do $k = 0, 1, \cdots, n-2$*

   Obtain a new eigenpair approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$ by a correction step

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = Correction(V_H, \lambda_{h_k}, u_{h_k}, V_{h_{k+1}}). \tag{4.3}$$

   end Do

4. *Solve the following source problem with multigrid method:*

   Find $\widehat{u}_{h_n} \in V_{h_n}$ such that

$$a(\widehat{u}_{h_n}, v_{h_n}) \;=\; \lambda_{h_{n-1}} b(u_{h_{n-1}}, v_{h_n}), \quad \forall v_{h_n} \in V_{h_n}. \tag{4.4}$$

   And let $u_{h_n} := MG(V_{h_n}, u_{h_{n-1}}, \lambda_{h_{n-1}} u_{h_{n-1}}, m_n)$ and has error estimate $\|u_{h_n} - \widehat{u}_{h_n}\|_a \lesssim \delta_{h_n}(\lambda)$. Then compute the Rayleigh quotient of u_{h_n}

$$\lambda_{h_n} \;=\; \frac{a(u_{h_n}, u_{h_n})}{b(u_{h_n}, u_{h_n})}. \tag{4.5}$$

Finally, we obtain an eigenpair approximation $(\lambda_{h_n}, u_{h_n}) \in \mathcal{R} \times V_{h_n}$.

Theorem 4.1. *After implementing Algorithm 4.1, the resultant eigenpair approximation (λ_{h_n}, u_{h_n}) has the following error estimate*

$$\|u_{h_n} - u\|_a \;\lesssim\; \delta_{h_n}(\lambda), \tag{4.6}$$

$$|\lambda_{h_n} - \lambda| \;\lesssim\; \delta_{h_n}^2(\lambda), \tag{4.7}$$

with the condition $\beta\eta_a(H) < 1$.

Proof. From $\eta_a(H) \gtrsim \delta_{h_1}(\lambda) \geq \delta_{h_2}(\lambda) \geq \cdots \geq \delta_{h_n}(\lambda)$ and Theorem 3.2, we have

$$\varepsilon_{h_{k+1}}(\lambda) \lesssim \eta_a(H)\varepsilon_{h_k}(\lambda) + \delta_{h_{k+1}}(\lambda), \quad \text{for } 1 \leq k \leq n-2. \tag{4.8}$$

Then by recursive relation, we can obtain

$$\begin{aligned}
\varepsilon_{h_{n-1}}(\lambda) &\lesssim \eta_a(H)\varepsilon_{h_{n-2}}(\lambda) + \delta_{h_{n-1}}(\lambda) \\
&\lesssim \eta_a(H)^2\varepsilon_{h_{n-3}}(\lambda) + \eta_a(H)\delta_{h_{n-2}}(\lambda) + \delta_{h_{n-1}}(\lambda) \\
&\lesssim \sum_{k=1}^{n-1} \eta_a(H)^{n-1-k}\delta_{h_k}(\lambda).
\end{aligned} \tag{4.9}$$

Based on the proof in Theorem 3.2, (4.1) and (4.9), the final eigenfunction approximation u_{h_n} has the error estimate

$$\begin{aligned}
\|u_{h_n} - u\|_a &\lesssim \varepsilon_{h_{n-1}}^2(\lambda) + \eta_a(H)\varepsilon_{h_{n-1}}(\lambda) + \delta_{h_n}(\lambda) \\
&\lesssim \sum_{k=1}^{n} \eta_a(H)^{n-k}\delta_{h_k}(\lambda) \\
&= \sum_{k=1}^{n} (\beta\eta_a(H))^{n-k}\delta_{h_n}(\lambda) \\
&\lesssim \frac{\beta\eta_a(H)}{1-\beta\eta_a(H)}\delta_{h_n} \\
&\lesssim \delta_{h_n}(\lambda).
\end{aligned} \tag{4.10}$$

This is the estimate (4.6). From Theorem 3.1 and (4.10), we can obtain the estimate (4.7). □

5 Work estimates of eigenvalue multigrid scheme

In this section, we turn our attention to the estimate of computation work for *Eigenvalue Multigrid Scheme 4.1.* We will show that Algorithm 4.1 makes solving eigenvalue problem need almost the same work as solving source problem.

First, we define the dimension of each level finite element space as $N_k := \dim V_{h_k}$. Then we have

$$N_k = \left(\frac{1}{\beta}\right)^{d(n-k)} N_n, \quad k = 1, 2, \cdots, n. \tag{5.1}$$

Theorem 5.1. *Assume the eigenvalue problem solving in the coarsest space $V_H = V_{h_1}$ need work $\mathcal{O}(N_1)$ and the work of the multigrid solver $MG(V_{h_k}, u_{h_k}, \lambda_{h_k}u_{h_k}, m_k)$ in each level space V_{h_k} be $\mathcal{O}(N_k)$ for $k = 2, 3, \cdots, n$. Then the work involved in the Eigenvalue Multigrid Scheme 4.1 is $\mathcal{O}(N_n)$.*

Proof. Let W_k denote the work in the correction step in the k-th finite element space V_{h_k}. Then with the correction definition, we have

$$W_k = \mathcal{O}(N_k + N_1). \tag{5.2}$$

Iterating (5.2) and using the fact (5.1), we obtain

$$\begin{aligned} W_n &= \sum_{k=1}^{n} W_k = \mathcal{O}\Big(\sum_{k=1}^{n}\big(N_k + N_1\big)\Big) \\ &= \mathcal{O}\Big(\sum_{k=1}^{n} N_k\Big) = \mathcal{O}\Big(\sum_{k=1}^{n}\big(\frac{1}{\beta}\big)^{d(n-k)} N_n\Big) \\ &= \mathcal{O}(N_n). \end{aligned}$$

This is the desired result and we complete the proof. □

6 Numerical results

In this section, two numerical examples are presented to illustrate the efficiency of multigrid scheme proposed in this paper. We solve the model eigenvalue problem (1.1) on the unit square $\Omega = (0,1)\times(0,1)$ and unit brick $\Omega = (0,1)\times(0,1)\times(0,1)$.

6.1 Model eigenvalue problem in two dimensional domain

Here we give the numerical results of the multigrid scheme on the two dimensional domain $\Omega = (0,1)\times(0,1)$. The sequence of finite element spaces is constructed by using linear element on the series of mesh which are produce by regular refinement with $\beta = 2$ (connecting the midpoints of each edge). In this example, we use two coarse meshes which are generated by Delaunay method as the initial mesh to investigate the convergence behaviors. Figure 1 shows the corresponding initial meshes: one is coarse and the other is fine.

Eigenvalue Multigrid Scheme 4.1 is applied to solve the eigenvalue problem. For comparison, we also solve the eigenvalue problem by the direct method. Figure 2 gives the corresponding numerical results for the first eigenvalue $\lambda_1 = 2\pi^2$ and the corresponding eigenfunction on the two initial meshes illustrated in Figure 1. From Figure 2, we find the multigrid scheme can obtain the optimal error estimates as same as the direct eigenvalue solving method for the eigenvalue and the corresponding eigenfunction approximations.

We also check the convergence behavior for multi eigenvalue approximations with *Eigenvalue Multigrid Scheme 4.1*. Here the first six eigenvalues $\lambda = 2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2, 10\pi^2$ are investigated. We adopt the right one in Figure 1 as the initial mesh and the corresponding numerical results are shown in Figure 3. Figure 3 also exhibits the optimal convergence of the multigrid scheme. 4.1.

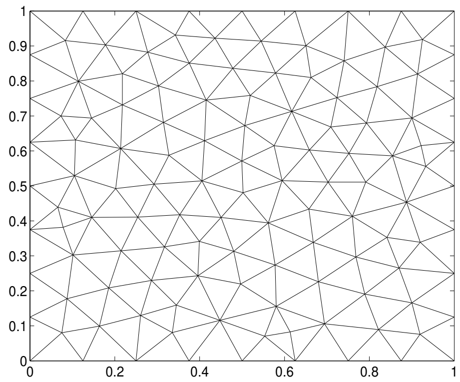

Figure 1: The coarse and fine initial meshes for the unit square

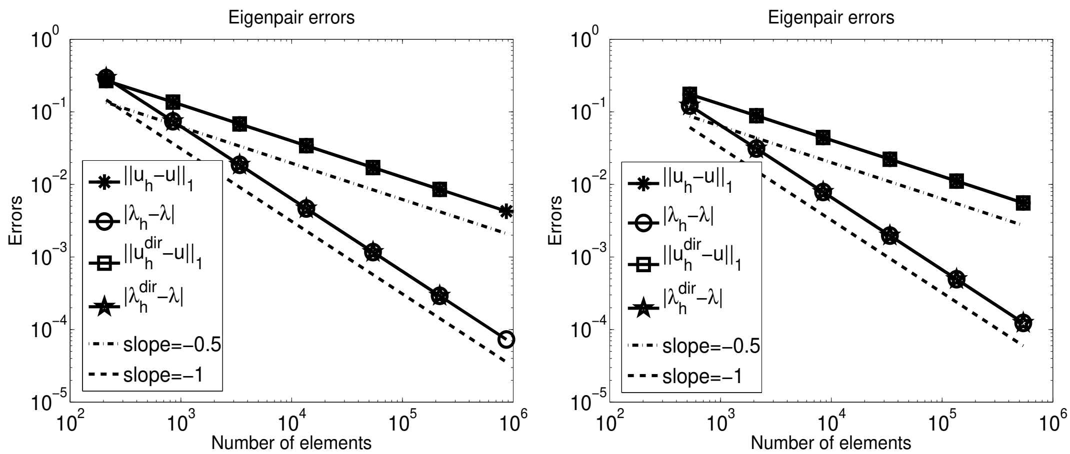

Figure 2: The errors of the multigrid algorithm for the first eigenvalue $2\pi^2$ and the corresponding eigenfunction, where $u_h^{\rm dir}$ and $\lambda_h^{\rm dir}$ denote the eigenfunction and eigenvalue approximation by direct eigenvalue solving

6.2 Model eigenvalue problem in three dimensional domain

Here we give the numerical results of the multigrid scheme for solving the model eigenvalue problem on the three dimensional unit brick domain. We first solve the eigenvalue problem (1.1) by linear finite element space on the coarse mesh $\mathcal{T}_H$. Then refine the mesh by the regular way to produce a series of meshes $\mathcal{T}_{h_k}$ $(k = 2, \cdots, n)$ with $\beta = 2$ (connecting the midpoints of each edge) and solve the auxiliary source problem (3.4) in the finer linear finite element space V_{h_k} defined on $\mathcal{T}_{h_k}$ and the corresponding eigenvalue problem in V_{H,h_k}.

In this example, we use two coarse meshes which are shown in Figure 4 as the initial meshes to investigate the convergence behaviors. Figure 5 gives the corresponding numerical results for the first eigenvalue $\lambda = 3\pi^2$ and the corresponding eigenfunction. Here we also compare the numerical results with the direct algorithm.

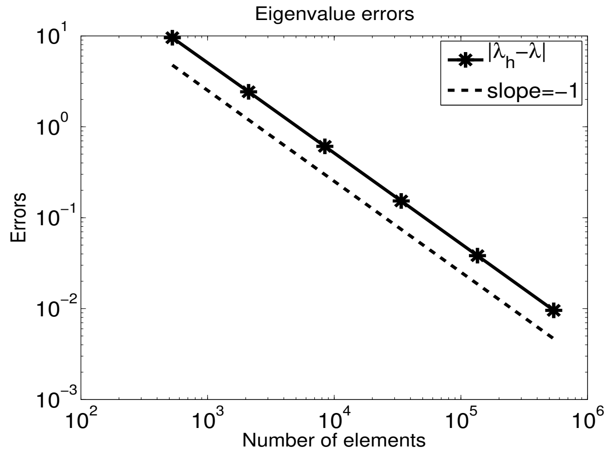

Figure 3: The errors of the multigrid algorithm for the first six eigenvalues on the unit square

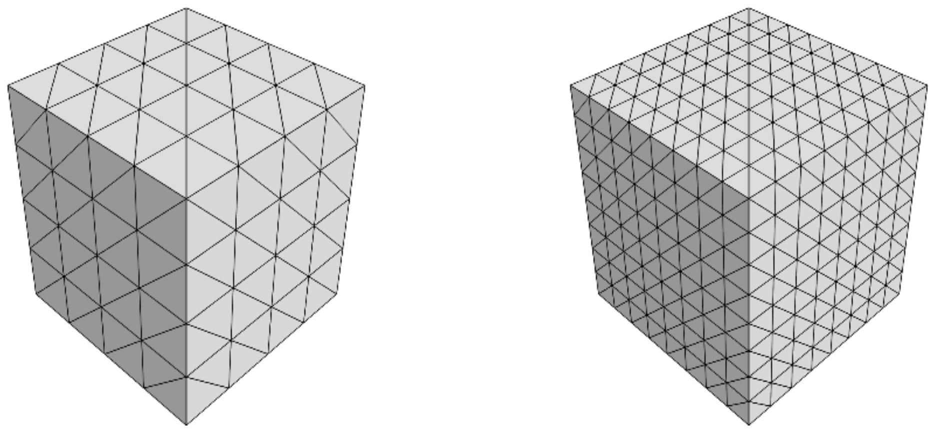

Figure 4: The coarse and fine initial meshes for the unit brick

From Figure 5, we find the multigrid scheme can also obtain the optimal error estimates for the eigenvalue and the corresponding eigenfunction approximations.

As in the two dimensional case, we also check the convergence behavior for multi eigenvalue approximations with the multigrid scheme 4.1. Here the first four eigenvalues $\lambda = 3\pi^2, 6\pi^2, 6\pi^2, 6\pi^2$ are investigated. We adopt the right one in Figure 4 as the initial mesh and the corresponding numerical results are shown in Figure 6. Figure 6 also exhibits the optimal convergence of *Eigenvalue Multigrid Scheme 4.1.*

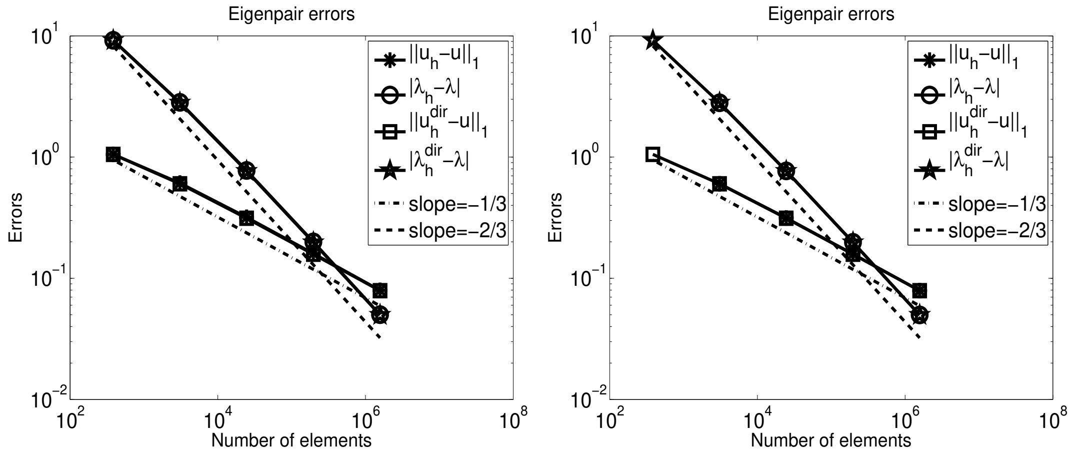

Figure 5: The errors of the multigrid algorithm for the first eigenvalue $3\pi^2$ and the corresponding eigenfunction, where $u_h^{\rm dir}$ and $\lambda_h^{\rm dir}$ denote the eigenfunction and eigenvalue approximation by direct eigenvalue solving

7 Concluding remarks

In this paper, we give a new type of multigrid scheme to solve the eigenvalue problems. The idea here is to use the multilevel correction method to transform solving eigenvalue problem to a series of solving source problems with multigrid method. We can replace the multigrid method by other types of efficient iteration methods such as algebraic multigrid method and other types preconditioned schemes based on the subspace decomposition and subspace corrections ([7, 17]) and the domain decomposition method ([16]).

Furthermore, our framework here can also be coupled with parallel method and the adaptive refinement technique. The ideas here can be extended to other types of linear and nonlinear eigenvalue problems. These will be investigated in our future work.

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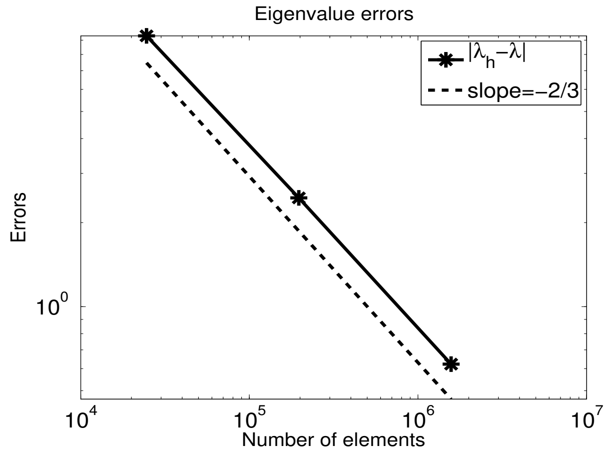

Figure 6: The errors of the multigrid algorithm for the first four eigenvalues on the unit brick

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