A Multiline Method of Network Analyzer Calibration

Roger B. Marks, *Member, IEEE*

***Abstract*—This paper presents a new method for the calibration of network analyzers. The essential feature is the use of multiple, redundant transmission line standards. The additional information provided by the redundant standards is used to minimize the effects of random errors, such as those caused by imperfect connector repeatability. The resulting method exhibits improvements in both accuracy and bandwidth over conventional methods.**

The basis of the statistical treatment is a linearized error analysis of the TRL (thru-reflect-line) calibration method. This analysis, presented here, is useful in the assessment of calibration accuracy. It also yields new results relevant to the choice of standards.

I. Introduction

PERHAPS the most precise means of network analyzer calibration is the TRL (thru-reflect-line) method [1]. This technique uses as standards two transmission lines, one of which is designated the "thru," as well as an arbitrary one-port "reflect" termination. Certain inevitable errors, especially connector nonrepeatability, limit the accuracy of any calibration method. The susceptibility of the TRL method to these errors has not been previously studied in detail.

The TRL error analysis developed below is valuable because it suggests tactics for error minimization. One novel result requires a modification of conventional design rules for the choice of line lengths when the lines are lossy. Loss is shown to increase the calibration accuracy, so that a lossy line may provide a usable calibration over a broader band than conventionally assumed.

In addition to error minimization, the results of the error analysis suggest a strategy for error *reduction*. The proposed method makes use of multiple, redundant line standards, by which we mean more than one line *in addition to* the thru.

The conventional TRL method, while employing multiple lines, does not use them simultaneously. The method is typically applied over a bandwidth no larger than 8:1. For wider bandwidths, a second line is normally employed, and the band is split. This analysis will show that the limitation of each line measurement to only a portion of the band neglects a large amount of data that *could* be used to reduce the overall error. Another drawback of the split-band method, namely the calibration discontinuity at the frequency break point, can also be eliminated.

Redundant standards have been used in previous calibration methods. Least-squares solutions have been applied to both four-port [2] and six-port reflectometers [3], [4]. An automatic network analyzer (ANA) calibration method [5] allows incompletely characterized standards of the TRL type but requires the iterative solution of an overdetermined, nonlinear system of at least 12 simultaneous equations with nine complex unknowns. Unfortunately, without accurate estimates of the relevant covariances, none of these least-squares methods ensures optimal (in some cases even unbiased) estimates of the calibration constants.

In contrast, the present method makes use of the known, linear solution to the simple TRL problem and linearizes the errors. Iterative solutions are avoided because all computations are linear. The computations are compact because the calibration constants are determined individually instead of *en masse*. The order of the linear systems is simply the number of lines, excluding the thru, and some of the matrices are analytically invertible. Perhaps the most significant distinction, however, is that, to linear order, the current method provides optimal, minimum-variance estimates of the calibration constants themselves.

One other use of multiple transmission lines is our own earlier version [6], [7], based on a crude estimate of the covariance matrices instead of the expressions derived here. The current method has previously been presented in conference [8].

The current algorithm hinges on the determination of the linearized covariance matrix. That matrix can be explicitly evaluated only with certain assumptions on the nature of the errors and their correlations. The assumptions made here are appropriate to random repeatability errors in connectors. In order to model other errors, such as systematic transmission line imperfections, certain modifications may need to be made. The errors need not be normally distributed.

II. Error Analysis

The problem is most conveniently analyzed in terms of cascade parameters. We choose a definition such that the cascade matrix of a two-port with cascade matrix A

Manuscript received November 5, 1990; revised March 15, 1991.
The author is with the National Institute of Standards and Technology, Mail Code 813.01, 325 Broadway, Boulder, CO 80303.
IEEE Log Number 9100134.

connected in series *to the left* of another two-port with cascade matrix B is simply AB.

In order to take advantage of symmetries in the problem at hand, we require the capability of reversing the direction of the cascade. The reverse cascade matrix, which we denote by the overbar, is related to the matrix inverse by

$$\bar{A} = \widetilde{A^{-1}} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

The *measured* cascade parameters of standard i are simply

$$M^i = XT^i\bar{Y} \tag{2}$$

where X and Y are the unknown cascade matrices to be determined. The overbar in (2) reflects the fact that the matrix Y is defined to cascade "right to left." This representation takes advantage of the symmetry; as a consequence, all of the results shown here for determination of X lead directly to results for Y by the simple exchange of the port numbers "1" and "2."

In (2), T^i is the *actual* cascade matrix of standard i. If the standard is an ideal transmission line and the connectors are perfect, then T^i is simply given by

$$L^i \equiv \begin{bmatrix} e^{-\gamma l_i} & 0 \\ 0 & e^{+\gamma l_i} \end{bmatrix} \equiv \begin{bmatrix} E_1^i & 0 \\ 0 & E_2^i \end{bmatrix} \tag{3}$$

where γ is the propagation constant and l_i is the length of line i. If, however, the standard is a nonideal line, we can represent its cascade matrix as

$$T^i = (I + \delta^{1i}) L^i \overline{(I + \delta^{2i})} \tag{4}$$

where the (presumably) small matrices δ^{1i} and δ^{2i} represent imperfections. The introduction of *two* perturbing matrices allows the association of an error term with each port: δ^{1i} with port 1 and δ^{2i} with port 2. This is a convenient representation of nonideal connectors. Later on, we will make some assumption about the statistical distributions of these errors. For the moment, we assume only that they are small.

To complete the symmetry of the description, the port 2 perturbation $(I + \delta^{2i})$, like Y, is defined to cascade "right to left."

Given any pair of line measurements, two equations of the form (2) lead to

$$M^{ij}X = XT^{ij} \tag{5}$$

where

$$M^{ij} \equiv M^j (M^i)^{-1} \tag{6}$$

and

$$T^{ij} \equiv T^j (T^i)^{-1}. \tag{7}$$

Consider first the case in which all of the δ's vanish and T^{ij} reduces to

$$L^{ij} \equiv L^j (L^i)^{-1} = \begin{bmatrix} E_1^j / E_1^i & 0 \\ 0 & E_2^j / E_2^i \end{bmatrix}$$
$$\equiv \begin{bmatrix} E_1^{ij} & 0 \\ 0 & E_2^{ij} \end{bmatrix} = \begin{bmatrix} e^{-\gamma(l_j - l_i)} & 0 \\ 0 & e^{+\gamma(l_j - l_i)} \end{bmatrix}. \tag{8}$$

In this case, as pointed out in [9], (5) has the form of an eigenvalue problem; the diagonal elements (E_1^{ij} and E_2^{ij}) of T^{ij} are the eigenvalues and the columns of X the eigenvectors, respectively, of M^{ij}. Since each of the eigenvectors is of arbitrary magnitude, (5) determines two of the parameters of X.

If, instead, the lines or connectors are imperfect, then T^{ij} is not diagonal and the problem is more complicated since the eigenvectors of M^{ij} are no longer the quantities of interest. We therefore assume each δ to be a small parameter and consider the resulting perturbation problem for the eigenvectors and eigenvalues. In practice, we always determine the eigenvalues and eigenvectors of M^{ij}. These, however, are easily related to the eigenvalues and eigenvectors of T^{ij}. If V^{ij} and Λ^{ij}, respectively, are the eigenvector and eigenvalue matrices of T^{ij}, then

$$T^{ij}V^{ij} = V^{ij}\Lambda^{ij} \tag{9}$$

from which follows

$$M^{ij}U^{ij} = U^{ij}\Lambda^{ij} \tag{10}$$

where

$$U^{ij} = XV^{ij}. \tag{11}$$

In other words, M^{ij} and T^{ij} have the same eigenvalues, and their eigenvectors are related by (11). Thus we can analyze the effects of the perturbations on the matrix M^{ij} via their effects on T^{ij}.

First expand the port 2 perturbation in terms of δ^{2i}, the components of which we assume to have magnitude much less than 1. To first order,

$$\overline{I + \delta^{2i}} = \left(\widetilde{I + \delta^{2i}}\right)^{-1} \approx \left(\widetilde{I - \delta^{2i}}\right) I - \widetilde{\delta^{2i}}. \tag{12}$$

Using (12) in (4) and assuming that the components of δ^{1i} are also small, we obtain the first-order result

$$(T^i)^{-1} = \left(I - \widetilde{\delta^{2i}}\right)^{-1} (L^i)^{-1} (I + \delta^{1i})^{-1}$$
$$\approx \left(I + \widetilde{\delta^{2i}}\right) (L^i)^{-1} (I - \delta^{1i}). \tag{13}$$

Insert (13) into (7) and collect the first-order terms:

$$T^{ij} \approx L^{ij} + \delta^{1j}L^{ij} - L^{ij}\delta^{1i} + L^j\left[\widetilde{\delta^{2i}} - \widetilde{\delta^{2j}}\right](L^i)^{-1} \tag{14}$$

which can be expressed as

$$T^{ij} \approx L^{ij} + \epsilon^{ij} \tag{15}$$

where the linear difference term is defined as

$$\epsilon^{ij} \equiv \begin{bmatrix} (\delta_{11}^{1j}-\delta_{11}^{1i}+\delta_{22}^{2i}-\delta_{22}^{2j})E_1^{ij} & \delta_{12}^{1j}E_2^{ij}-\delta_{12}^{1i}E_1^{ij}+(\delta_{21}^{2i}-\delta_{21}^{2j})E_1^{i}E_1^{j} \\ \delta_{21}^{1j}E_1^{ij}-\delta_{21}^{1i}E_2^{ij}+(\delta_{12}^{2i}-\delta_{12}^{2j})E_2^{i}E_2^{j} & (\delta_{22}^{1j}-\delta_{22}^{1i}+\delta_{11}^{2i}-\delta_{11}^{2j})E_2^{ij} \end{bmatrix}. \tag{16}$$

The errors are conveniently expressed in terms of the scattering parameters of the imperfections. To first order in the elements of δ, the scattering matrices S^{1i} and S^{2i} of the port 1 and port 2 imperfections, respectively, are[1]

$$S^{1i} \approx \begin{bmatrix} \delta_{12}^{1i} & 1+\delta_{11}^{1i} \\ 1-\delta_{22}^{1i} & -\delta_{21}^{1i} \end{bmatrix} \tag{17}$$

and

$$S^{2i} \approx \begin{bmatrix} -\delta_{21}^{2i} & 1-\delta_{22}^{2i} \\ 1+\delta_{11}^{2i} & \delta_{12}^{2i} \end{bmatrix} \tag{18}$$

in terms of which (16) can be written, to first order, as

$$\epsilon^{ij} \equiv \begin{bmatrix} \left[\dfrac{S_{12}^{1j}S_{12}^{2j}}{S_{12}^{1i}S_{12}^{2i}}-1\right]E_1^{ij} & S_{11}^{1j}E_2^{ij}-S_{11}^{1i}E_1^{ij}+(S_{11}^{2j}-S_{11}^{2i})E_1^{i}E_1^{j} \\ S_{22}^{1i}E_1^{ij}-S_{22}^{1j}E_2^{ij}+(S_{22}^{2i}-S_{22}^{2j})E_2^{i}E_2^{j} & \left[\dfrac{S_{21}^{1i}S_{21}^{2i}}{S_{21}^{1j}S_{21}^{2j}}-1\right]E_2^{ij} \end{bmatrix}. \tag{19}$$

To first order, the off-diagonal elements of ϵ (and δ) represent errors in the measurement of the *reflection* coefficients of the line, whereas their diagonal elements represent errors in the measurement of the *transmission* coefficients of the line.

A. Eigenvalue Perturbation and the Propagation Constant

The eigenvalues λ_1^{ij} and λ_2^{ij} of T^{ij} (and of M^{ij}) are

$$\lambda_1^{ij},\lambda_2^{ij} = \frac{1}{2}\left[(T_{11}^{ij}+T_{22}^{ij}) \pm \sqrt{(T_{11}^{ij}-T_{22}^{ij})^2+4T_{12}^{ij}T_{21}^{ij}}\right] \tag{20}$$

but we know that the zero-order eigenvalues are E_1^{ij} and E_2^{ij}. We assume that the eigenvalue λ_1^{ij} is associated with E_1^{ij}, not with E_2^{ij}. In practice, this can be ensured by comparing an estimate of E_1^{ij}, based on an estimate of the line lengths and the propagation constant, with the two computed eigenvalues. We seek an expansion of λ_1^{ij} and λ_2^{ij} about their zero-order values, of the form

$$\lambda_p^{ij} \approx E_p^{ij} + \sum_{m,n} \epsilon_{mn}^{ij} \left.\frac{\partial \lambda_p^{ij}}{\partial T_{mn}^{ij}}\right|_{\delta=0} \tag{21}$$

where the notation $\delta = 0$ indicates the zero-order limit $\delta^{1i}=\delta^{2i}=\delta^{1j}=\delta^{2j}=0$. Carrying out the differentiation leads to

$$\left.\frac{\partial \lambda_1^{ij}}{\partial T_{11}^{ij}}\right|_{\delta=0} = 1 \tag{22}$$

and

$$\left.\frac{\partial \lambda_2^{ij}}{\partial T_{22}^{ij}}\right|_{\delta=0} = 1 \tag{23}$$

with all other derivatives vanishing to zero order. Thus (21) reduces to

$$\lambda_1^{ij} \approx E_1^{ij} + \epsilon_{11}^{ij} = E_1^{ij}(1+\delta_{11}^{1j}-\delta_{11}^{1i}+\delta_{22}^{2i}-\delta_{22}^{2j}) \tag{24}$$

and

$$\lambda_2^{ij} \approx E_2^{ij} + \epsilon_{22}^{ij} = E_2^{ij}(1+\delta_{22}^{1j}-\delta_{22}^{1i}+\delta_{11}^{2i}-\delta_{11}^{2j}). \tag{25}$$

Equations (24) and (25) express the sensitivity of the eigenvalues to the perturbations. Notice that they are independent of the off-diagonal elements of the matrix ϵ^{ij}. Linear errors in the eigenvaues are induced only by errors in the *transmission* coefficients S_{12} and S_{21} of the line measurements. Small *reflections* do not affect the result.

Recall that λ_1^{ij} and λ_2^{ij} are the eigenvalues of M^{ij}, determined by the measured data. Also, $E_1^{ij}=1/E_2^{ij}=\exp(-\gamma[l_j-l_i])$. Provided the line lengths l_i and l_j are known, both (24) and (25) provide an estimate of E_1^{ij} and thence the propagation constant γ. At first glance, these may appear to be independent measurements. However, if the two imperfections are *reciprocal*, their S parameters must satisfy the condition $S_{12}=S_{21}$. In view of (17) and (18), the estimates of E_1^{ij} from (24) and (25) are then identical to first order and can be used interchangeably. On the other hand, the two estimates may differ to second order, so there may be slight advantage in using the average of the two expressions. We therefore define

$$\lambda^{ij} \equiv \frac{1}{2}(\lambda_1^{ij}+1/\lambda_2^{ij}) \approx E_1^{ij}\left[1+\frac{1}{2}(\delta_{11}^{1j}-\delta_{11}^{1i}+\delta_{22}^{1i}-\delta_{22}^{1j} + \delta_{11}^{2j}-\delta_{11}^{2i}+\delta_{22}^{2i}-\delta_{22}^{2j})\right]. \tag{26}$$

To first order, the geometric mean is identical to the algebraic mean. The advantage of the geometric mean is that it can be computed knowing only the ratios of the elements of M^{ij}, making it practical for use with dual-reflectometer network analyzers which may not provide

[1]The ports of the "imperfections" are numbered in accordance with those of the ANA, so that port 1 (port 2) is nearer port 1 (port 2) of the ANA.

an explicit measurement of M^{ij}. Its drawback is that it requires a root choice.

From (26), we solve for an estimate of the propagation constant:

$$\gamma^{ij} \equiv \frac{\ln(\lambda^{ij})}{l_i - l_j} \approx \gamma + \Delta\gamma^{ij} \tag{27}$$

where the explicit linear error term is

$$\Delta\gamma^{ij} = \frac{1}{2(l_i - l_j)}\left[\delta_{11}^{1j} - \delta_{11}^{1i} + \delta_{22}^{1i} - \delta_{22}^{1j} + \delta_{11}^{2j} - \delta_{11}^{2i} + \delta_{22}^{2i} - \delta_{22}^{2j}\right]. \tag{28}$$

This simple result clearly demonstrates that the error in γ is minimized by maximizing the difference in line lengths. As discussed in the following subsection, the error in the *calibration constants* is of entirely different form.

B. Eigenvector Perturbation and the Calibration Constants

Equation (9) defines V^{ij} as the eigenvector matrix of T^{ij}; that is, its columns are eigenvectors of T^{ij}. In order to normalize these eigenvectors, choose one element of each column to be unity, thereby defining V^{ij} by

$$V^{ij} \equiv \begin{bmatrix} 1 & \mu^{ij} \\ v^{ij} & 1 \end{bmatrix}. \tag{29}$$

One can determine μ^{ij} and v^{ij}:

$$\mu^{ij} = \frac{1}{2T_{21}^{ij}}\left[T_{11}^{ij} - T_{22}^{ij} + \left(\lambda_2^{ij} - \lambda_1^{ij}\right)\right] \tag{30}$$

and

$$v^{ij} = \frac{1}{-2T_{12}^{ij}}\left[T_{11}^{ij} - T_{22}^{ij} + \left(\lambda_2^{ij} - \lambda_1^{ij}\right)\right] \tag{31}$$

where the eigenvalues λ^{ij} are given by (20). If the errors δ vanish, then T^{ij} reduces to the diagonal matrix L^{ij} and, as a result, μ^{ij} and v^{ij} also vanish.

The eigenvector matrix U^{ij} of M^{ij} is equal to XV^{ij}. Representing X by

$$X \equiv r\begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \tag{32}$$

we find that

$$XV^{ij} = r\begin{bmatrix} a + v^{ij}b & b + \mu^{ij}a \\ c + v^{ij} & 1 + \mu^{ij}c \end{bmatrix}. \tag{33}$$

We may renormalize these eigenvectors by dividing the first column by its first element and the second column by its second element. Thus,

$$U^{ij} \equiv \begin{bmatrix} 1 & \alpha^{ij} \\ \beta^{ij} & 1 \end{bmatrix} \tag{34}$$

where

$$\alpha^{ij} = \frac{b + \mu^{ij}a}{1 + \mu^{ij}c} \tag{35}$$

and

$$\beta^{ij} = \frac{c + v^{ij}}{a + v^{ij}b}. \tag{36}$$

Computation of the eigenvectors of the measured matrix M^{ij}, which requires only the ratios of the elements of M^{ij}, produces α^{ij} and β^{ij}. In the errorless case $\delta = 0$, both μ^{ij} and v^{ij} vanish, and α^{ij} and β^{ij} reduce to b and c/a, two parameters of the unknown matrix X which were to be determined. The determination of b and c/a is part of the standard TRL process.

More generally, we wish to observe the first-order expansions of α^{ij} and β^{ij} with respect to small perturbations. These take the form

$$\alpha^{ij} \approx b + \sum_{m,n} \epsilon_{mn}^{ij} \left.\frac{\partial\alpha^{ij}}{\partial T_{mn}^{ij}}\right|_{\delta=0} \tag{37}$$

and

$$\beta^{ij} \approx \frac{c}{a} + \sum_{m,n} \epsilon_{mn}^{ij} \left.\frac{\partial\beta^{ij}}{\partial T_{mn}^{ij}}\right|_{\delta=0}. \tag{38}$$

The partial derivatives are expanded:

$$\frac{\partial\alpha^{ij}}{\partial T_{mn}^{ij}} = \frac{\partial\alpha^{ij}}{\partial\mu^{ij}}\frac{\partial\mu^{ij}}{\partial T_{mn}^{ij}} + \frac{\partial\alpha^{ij}}{\partial v^{ij}}\frac{\partial v^{ij}}{\partial T_{mn}^{ij}} \tag{39}$$

but the last term vanishes. A similar expression holds for β^{ij}.

The differentiations with respect to μ^{ij} and v^{ij} can be carried out using (35) and (36); in the zero-order limit, the results are

$$\left.\frac{\partial\alpha^{ij}}{\partial\mu^{ij}}\right|_{\delta=0} = a - bc \tag{40}$$

and

$$\left.\frac{\partial\beta^{ij}}{\partial v^{ij}}\right|_{\delta=0} = \frac{a - bc}{a^2}. \tag{41}$$

The other two derivatives vanish.

In order to evaluate the derivatives of μ^{ij} and v^{ij} with respect to the elements of T^{ij}, return to (30) and (31). The zero-order limits are

$$\left.\frac{\partial\mu^{ij}}{\partial T_{12}^{ij}}\right|_{\delta=0} = \frac{1}{E_2^{ij} - E_1^{ij}} \tag{42}$$

and

$$\left.\frac{\partial v^{ij}}{\partial T_{21}^{ij}}\right|_{\delta=0} = \frac{1}{E_1^{ij} - E_2^{ij}}. \tag{43}$$

All of the other derivatives vanish in this limit.

Combining all of these results with (37) and (38) yields

$$\alpha^{ij} \approx b + \Delta\alpha^{ij} \tag{44}$$

and

$$\beta^{ij} \approx \frac{c}{a} + \Delta\beta^{ij} \tag{45}$$

where the explicit linear error terms are

$$\Delta\alpha^{ij} \equiv \epsilon_{12}^{ij}\frac{a-bc}{E_2^{ij}-E_1^{ij}} \tag{46}$$

and

$$\Delta\beta^{ij} \equiv \epsilon_{21}^{ij}\frac{a-bc}{a^2\left(E_2^{ij}-E_1^{ij}\right)}. \tag{47}$$

Only the off-diagonal terms in ϵ contribute first-order errors in α^{ij} and β^{ij}. Thus linear errors are induced only by errors in the *reflection* coefficients S_{11} and S_{22} of the line measurements, not by transmission errors. A perfect match or attenuator therefore provides the same calibration as a perfect transmission line standard. From the previous section, however, we know that only the transmission line allows the determination of the propagation constant, required for movement of the reference plane.

Consider now the magnitude of the error term (46):

$$|\Delta\alpha^{ij}| \leqslant \left[|\delta_{12}^{1j}||E_2^{ij}|+|\delta_{12}^{1i}||E_1^{ij}| + |\delta_{21}^{2i}-\delta_{21}^{2j}||E_1^iE_1^j|\right]\frac{|a-bc|}{|E_2^{ij}-E_1^{ij}|} \tag{48}$$

with a similar expression arising from (47). Since, for small line loss, the terms E_1^{ij} and E_2^{ij} have magnitude near unity, the dominant factor in determining the typical magnitude of the error terms is the denominator. We can write

$$E_2^{ij}-E_1^{ij} = 2j\sin\left[\phi - j\rho\right] \tag{49}$$

where $j=\sqrt{-1}$ unless used as an index. We have defined $\rho \equiv (l_j - l_i)\mathrm{Re}(\gamma)$ and $\phi \equiv (l_j - l_i)\mathrm{Im}(\gamma)$ as the loss and phase shift, respectively, associated with the difference in line lengths. Equation (49) leads to

$$|E_2^{ij}-E_1^{ij}|^2 = 4\sin^2\phi\cosh^2\rho + 4\cos^2\phi\sinh^2\rho. \tag{50}$$

For a *lossless* line, $\rho = 0$ and

$$|\Delta\alpha^{ij}|, |\Delta\beta^{ij}| \propto \frac{1}{2|\sin\phi|}. \tag{51}$$

This result, that the error in the lossless case is inversely proportional to the sine of the phase difference, was given by Hoer [10]. In particular, the error becomes infinite when the difference in phase delays is an integral multiple of 180°, illustrative of the well-known fact that the TRL procedure is ill conditioned near these points.

In the lossy case, (51) is invalid near the maximum error points, because (50) never vanishes. For the case of low loss,

$$\frac{1}{|E_2^{ij}-E_1^{ij}|} \lesssim \frac{1}{2|\sinh\rho|} \approx \frac{1}{2|\rho|}. \tag{52}$$

This result has interesting implications. For example, consider a pair of lines with a 180° phase difference at some frequency. Another pair of lines with twice the difference in length will have a 360° phase difference and will also produce an ill-conditioned calibration problem. However, the calibration error of the latter line pair is expected to be only about *half* that of the first pair since ρ is doubled.

This analysis of lossy lines yields some new design rules, even in the case of conventional TRL calibration. For example, it is common practice to limit a single pair of lines to between 20° and 160°. For a lossless line, the expected error at the band edges is approximately 2.92 times that at the optimal (90°) point. On the other hand, for a lossy line with $\rho = 1/2.92 \approx 0.35$, (52) predicts a worst-case error, at 0° and 180°, that is no worse than the lossless line at 20° and 160°. The accuracy increases linearly with both the attenuation factor and the length difference. Of course, for large losses, we need to consider the numerator as well as the denominator of (46). A more exact calculation of the probable error is discussed in Section III. In general, if the loss is great enough, the phase difference criterion is irrelevant and the calibration bandwidth may be greatly extended. This is an advantage of the LRM calibration method [11], which uses a "match," equivalent to an infinitely lossy attenuator, in place of the line. One drawback of LRM is that it fails to provide the propagation constant required to shift the reference plane.

For the lossy case, it is appropriate to define the *effective* phase difference, ϕ_{eff}:

$$\phi_{\mathrm{eff}} = \arcsin\left|\frac{1}{2}\left(E_2^{ij}-E_1^{ij}\right)\right| \tag{53}$$

since it is $1/|\sin(\phi_{\mathrm{eff}})|$ rather than $1/|\sin\phi|$ which actually predicts the error. We stipulate that $\phi_{\mathrm{eff}} = 90°$ if the argument of the arcsin is greater than 1. Since in the TRL case an estimate of E_1^{ij} is available from the procedure of subsection II-A, ϕ_{eff} is easily computed as part of the calibration routine.

All of the results of this section are applicable to the determination of the second cascade matrix Y by the interchange of Y and X and that of the port identifications 1 and 2.

C. Completion of the Calibration

In order to complete the calibration by determining the remaining constants a and r, additional measurements are required. The simplest requires the measurement of a single unknown reflect termination on both ports [1]. The analysis of the error involved in this procedure is not included here.

III. Redundant Measurements

The results (27) for the propagation constant and (44) and (45) for the calibration constants form the basis of a statistical treatment of redundant measurements. By way of introduction, consider how we might use a redundant set of "noisy" measurements of some parameter. If the measurements are equally noisy, we might expect simple data averaging to yield the most accurate estimate. On the other hand, we may find that some measurements are inherently more noisy than others. Rather than either ignoring these data or blindly including them in the average, the best estimate of the parameter comes from a *weighted* average in which the noisy data are given less significance. The complete theory includes not only the variations in accuracy of the various measurements but also the correlations among them. It is this theory that we require here and which will be briefly sketched in its simplest relevant form.

Suppose that we wish to determine some parameter x. To do so, we make N measurements b_n of the quantity $a_n x$, where a_n is presumed known. Each of these measurements differs from the "true" value $a_n x$ by some amount e_n:

$$b_n = a_n x + e_n. \tag{54}$$

Assume that e_n is a random variable whose expectation value (denoted by $\langle e_n \rangle$) vanishes; that is, no systematic errors are present. Let $\boldsymbol{b}$ and $\boldsymbol{a}$ represent column vectors whose elements are b_n and a_n. According to the Gauss–Markov theorem [12], the best unbiased linear estimate of x is

$$\underline{x} = (\sigma_x)^2 \boldsymbol{a}^\dagger V^{-1} \boldsymbol{b} \tag{55}$$

where "†" indicates Hermitian adjoint, the covariance matrix V is defined by

$$V_{mn} \equiv \langle e_m^* e_n \rangle \tag{56}$$

and σ_x, defined by

$$\sigma_x \equiv \frac{1}{\sqrt{\boldsymbol{a}^\dagger V^{-1} \boldsymbol{a}}} \tag{57}$$

is the standard deviation of $\underline{x}$. The estimate (55) is "best" because it minimizes the variance of $\underline{x}$. No assumption about the distribution of errors is required in establishment of this theorem.

A. Propagation Constant

The theory is applicable to the results of subsection II-A with the substitutions

$$x = \gamma \tag{58}$$

$$b_{ij} = \ln(\lambda^{ij}) \tag{59}$$

$$a_{ij} = l_i - l_j \tag{60}$$

and

$$e_{ij} = \kappa_j - \kappa_i \tag{61}$$

where the term

$$\kappa_i = \frac{1}{2}\left[\delta_{11}^{1i} - \delta_{22}^{1i} + \delta_{11}^{2i} - \delta_{22}^{2i}\right] \tag{62}$$

includes errors caused only by a single line. Note that a multi-index ij is used in place of the single index n used previously.

In attempting to gather additional information concerning γ, we might first seek to reiterate the analysis of subsection II-A but with the port identities interchanged. Unfortunately, (28) is invariant with respect to this change, so no additional information is to be had. The same is true of the interchange of the line identities i and j. Only additional measurements provide additional information.

Now consider the addition of a third line measurement M^k to the two we have discussed. This provides two further measurements, b_{ik} and b_{jk}, in addition to (59). Unfortunately, these three are not linearly independent, since

$$e_{ij} + e_{jk} = e_{ik}. \tag{63}$$

The Gauss–Markov theorem cannot be applied to a set of linearly dependent measurements, since the resulting covariance matrix is singular. Only two of the three measurements may be used. In general, we find that N linearly independent measurements of the propagation constant arise from the measurement of $N+1$ transmission lines (including the thru). We can consider these N measurements to arise from the pairing of a particular line $i = 0$ with each of the other lines.

In order to evaluate V, we must make some assumptions about the nature of the errors. However, in this case, these assumptions do not need to be very restrictive, since the covariances are simply

$$\begin{aligned}\langle e_{ij}^* e_{kl} \rangle &= \langle (\kappa_j - \kappa_i)^* (\kappa_l - \kappa_k) \rangle \\ &= \langle \kappa_j^* \kappa_l \rangle + \langle \kappa_i^* \kappa_k \rangle - \langle \kappa_i^* \kappa_l \rangle - \langle \kappa_j^* \kappa_k \rangle. \end{aligned} \tag{64}$$

In the absence of any *a priori* reason to assume a correlation between the errors in the measurements of two different lines, we will ignore terms on the right with nonmatching indices. Further, we assume for the purposes of this paper that each of the lines is equally prone to error. Thus,

$$\langle \kappa_i^* \kappa_j \rangle = \delta_{ij}^K (\sigma_\kappa)^2 \tag{65}$$

where δ_{ij}^K is the Kronecker delta and σ_κ is the standard deviation in any of the κ_i.

We choose a simple ordering scheme in which a single line measurement is common to all pairs. Specifically, in (59), we let $i = 0$ represent the common line and let j run from 1 through N. This provides the required N linearly independent measurements and allows for the ready computation of the covariance matrix:

$$V_{mn} = \left(1 + \delta_{mn}^K\right)(\sigma_\kappa)^2 \tag{66}$$

which is explicitly invertible as

$$(V^{-1})_{mn} = \left(\delta_{mn}^K - \frac{1}{N+1}\right)\frac{1}{(\sigma_\kappa)^2}. \tag{67}$$

Numerical matrix inversion is not required. Furthermore, knowledge of σ_κ is not required in the evaluation of $\underline{\gamma}$, since it appears in both the numerator and denominator of (55). Knowledge of σ_κ serves only to predict the *absolute* variance of $\underline{\gamma}$. On the other hand, we can always predict *relative* estimates of σ_γ and thereby compare various sets of calibration standards.

B. Calibration Constants

The application of the Gauss–Markov theorem to the results of subsection II-B requires the substitutions

$$x = \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \equiv \begin{Bmatrix} b \\ c/a \end{Bmatrix} \tag{68}$$

$$b_{ij} = \begin{Bmatrix} \alpha_{ij} \\ \beta_{ij} \end{Bmatrix} \tag{69}$$

$$a_{ij} = 1 \tag{70}$$

and

$$e_{ij} = \begin{Bmatrix} \Delta\alpha^{ij} \\ \Delta\beta^{ij} \end{Bmatrix}. \tag{71}$$

The three measurements $\Delta\alpha^{ij}$, $\Delta\alpha^{ik}$, and $\Delta\alpha^{jk}$ are jointly linearly dependent, and likewise $\Delta\beta^{ij}$, $\Delta\beta^{ik}$, and $\Delta\beta^{jk}$. Thus we use an ordering similar to that of the previous section, forming N linearly independent measurements by pairing some line $i = 0$ with each of the other N lines.

For both b and c/a, the computation of V requires the evaluation of 16 covariances among the elements of the matrices δ. These have the form

$$\langle \delta_{mn}^{ij*} \delta_{pq}^{kl} \rangle. \tag{72}$$

Some of these terms are calculable under fairly general assumptions. Under the postulate that the errors in different lines are uncorrelated, half of the terms, those with $j \neq l$, vanish. More problematic are terms of the form

$$\langle \delta_{12}^{1i*} \delta_{21}^{2i} \rangle, \langle \delta_{21}^{1i*} \delta_{12}^{2i} \rangle. \tag{73}$$

These terms may or may not vanish according to whether or not the errors in the measurement are due to imperfect lines or imperfect connectors. If the lines are imperfect, we may well expect the port 1 and port 2 errors to be correlated. On the other hand, errors caused only by imperfect repeatability of the two connectors ought not to be correlated. For this paper, we select the latter viewpoint and thereby ignore the terms (73).

The only terms remaining are

$$\langle \delta_{12}^{1i*} \delta_{12}^{1i} \rangle, \quad \langle \delta_{12}^{2i*} \delta_{12}^{2i} \rangle, \quad \langle \delta_{21}^{1i*} \delta_{21}^{1i} \rangle, \quad \langle \delta_{21}^{2i*} \delta_{21}^{2i} \rangle. \tag{74}$$

Making the reasonable assumption that ports 1 and 2 are equally "noisy," the first and second terms and the third and fourth terms of (74) are identical. In the absence of any detailed information as to the nature of the connector imperfection, we also make the plausible assumption that the first and third terms are equal:

$$\langle \delta_{12}^{1i*} \delta_{12}^{1i} \rangle = \langle \delta_{12}^{2i*} \delta_{12}^{2i} \rangle = \langle \delta_{21}^{1i*} \delta_{21}^{1i} \rangle = \langle \delta_{21}^{2i*} \delta_{21}^{2i} \rangle = (\sigma_\delta)^2. \tag{75}$$

This amounts to the assumption that S_{11} and S_{22} of the connector are equally "noisy," as would be true for a lossless imperfection.

Lest any of these assumptions leave us uncomfortable, we note that *any* estimate of the form (55), regardless of the validity of V, remains unbiased, if not the optimum estimate.

Using these conjectures, we compute the covariance matrix for α:

$$V_{ij,il}^{\alpha} = \frac{|a-bc|^2(\sigma_\delta)^2}{(E_2^{ij} - E_1^{ij})^*(E_2^{il} - E_1^{il})} \Big[E_1^{ij*} E_1^{il} + \delta_{jl}^K |E_2^{ij}|^2 + (1 + \delta_{jl}^K) |E_1^i|^2 E_1^{j*} E_1^l \Big] \tag{76}$$

and for β:

$$V_{ij,il}^{\beta} = \frac{|a-bc|^2(\sigma_\delta)^2}{|a|^4 (E_2^{ij} - E_1^{ij})^*(E_2^{il} - E_1^{il})} \Big[E_2^{ij*} E_2^{il} + \delta_{jl}^K |E_1^{ij}|^2 + (1 + \delta_{jl}^K) |E_2^i|^2 E_2^{j*} E_2^l \Big]. \tag{77}$$

Once again, the constant factors σ_δ, $|a - bc|$, and $|a|$ are required only to evaluate the *variance* in the estimates of α and β; they do not affect the estimates themselves. Furthermore, we can define the *normalized* standard deviations

$$\sigma_{\alpha 0} = \frac{\sigma_\alpha}{|a - bc| \sigma_\delta} \tag{78}$$

and

$$\sigma_{\beta 0} = \frac{\sigma_\beta}{|a|^2 |a - bc| \sigma_\delta} \tag{79}$$

which do not depend on any of these factors but only on the properties of the calibration standards. In fact (78) and (79) can be readily calculated using only the factors E_1^i, which can be computed using the propagation constant, estimated in the previous section, and the known line lengths. For the most part, these terms appear in the form E_1^{ij} and E_2^{ij}; these are directly estimable as λ_1^{ij} and λ_2^{ij} without reference to line lengths. The normalization is such that $\sigma_{\alpha 0} = \sigma_{\beta 0} = 1$ for a calibration using a single pair of lossless transmission lines with the optimum phase difference of 90°. In addition, $\sigma_{\alpha 0} = \sigma_{\beta 0}$ for any number of lossless lines.

The matrices (76) and (77) are straightforward to compute using measured data. Although they must be inverted numerically, the process is not extremely time consuming since the dimension is equal to the number of lines, excluding the thru. One numerical problem may arise if the effective phase difference, ϕ_{eff}, of any line pair is near 0°, for in that case V is nearly singular. A solution to the problem is to choose the line i, common to all line pairs, to have the greatest minimum ϕ_{eff}. For example, if we used lines of length 0, 2, 2, 6, and 6 cm, we would be required to use the pairs 0–2, 0–2, 0–6, and 0–6. Any other choice of i would result in a singularity. For best performance, i should in general be chosen anew at each frequency.

This choice of line pairs affects only numerical stability, not the actual variance. Assume, for example, that a 90° line is equivalent to 4 cm. Then, in the example above, we would be including the 45° and 135° pairs but excluding the 90° pairs. Although it may appear that the most effective information is being ignored, this is not in fact the case, for the Gauss–Markov estimate automatically

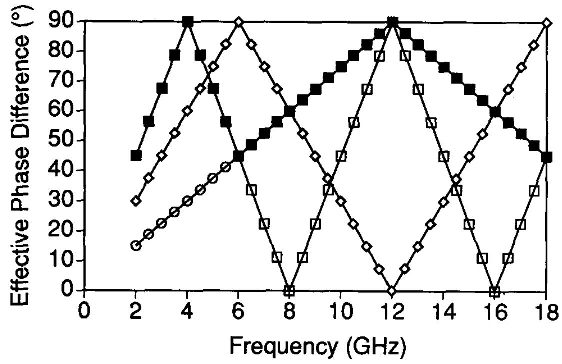


Fig. 1. Effective phase delays for three ideal TEM lines chosen according to convention for 2–18 GHz. The open squares represent the 1.875 cm line, the open circles the 0.625 cm line, and the open diamonds the difference between the two. The points represented by the solid squares are used by conventional TRL.

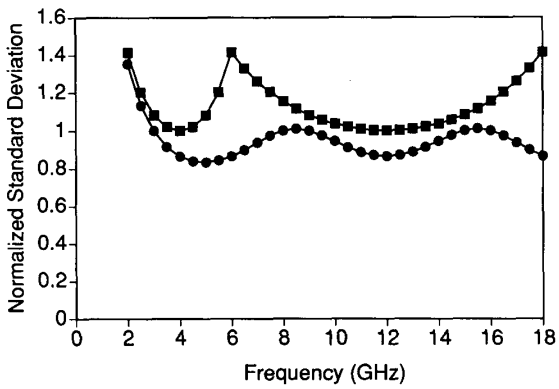


Fig. 2. Normalized standard deviation for the lines of Fig. 1. The squares represent the conventional TRL method, the circles the multiline TRL method.

takes advantage of the measurement information in the optimum fashion.

IV. Performance of the Multiline Calibration Method

In order to draw concrete comparisons between the conventional and multiline TRL methods, consider a lossless TEM calibration over the band 2–18 GHz. Convention holds that a pair of lines ought to be used over no more than an 8:1 bandwidth, so we would normally use three line standards (including the thru). A conventional design uses a zero-length thru along with lines of lengths 0.625 cm and 1.875 cm. This results in a minimum effective phase difference of 45° at 2.0, 6.0, and 18.0 GHz, as illustrated in Fig. 1. These three frequencies are local maxima in the normalized standard deviation $\sigma_{\alpha 0} = \sigma_{\beta 0}$, as illustrated by the squares in Fig. 2. The circles in the same figure represent the normalized standard deviation using the same lines but applying the multiline TRL described in this paper. The standard deviation of the

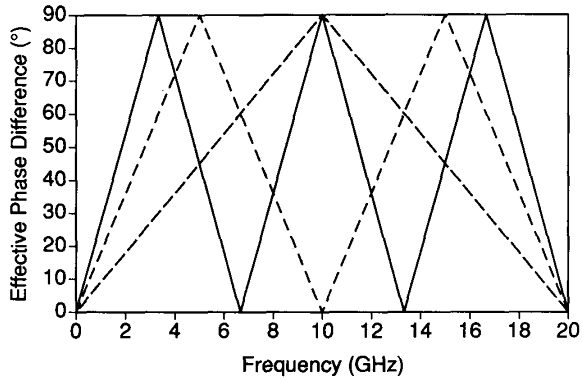


Fig. 3. Effective phase delays for three ideal TEM lines chosen for good performance over 2–18 GHz using the multiline TRL. The lengths are 0, 0.75, and 2.25 cm.

multiline calibration is not only smaller but also considerably smoother. In particular, the sharp peaks are eliminated. This improvement is attained without any increase in the number of standards.

In fact, the multiline standard deviation shown in Fig. 2 is not the optimum, since the line lengths were designed for conventional TRL. Although we have no proof, an optimal design seems to require one line to be a quarter wavelength long at the band center and the other line to be three times that length. For the 2–18 GHz band, these line lengths are 0.75 cm and 2.25 cm. The effective phase differences for these lines are shown in Fig. 3. The presence of the third curve, related to the difference between the two lines, produces the symmetry; note that lines of 1.5 and 2.25 cm would produce the same result. The standard deviation is symmetric about the band center and peaks at the edges. For the 2–18 GHz band, the maximum normalized standard deviation is 1.18, as opposed to 1.35 for the multiline method using the standards of Fig. 1 and 1.41 using the conventional TRL method.

Next we consider an actual set of coplanar waveguide lines patterned on gallium arsenide. Lines 1, 2, and 3 had lengths of 1.2850, 0.7415, and 0.2985 cm, so that the lengths associated with the pairs 1–2, 1–3, and 2–3 were 0.5435, 0.9865, and 0.4430 cm, respectively. These lengths were not designed as an optimum calibration set but instead were chosen to illustrate the effectiveness of the method.

The dark line of Fig. 4 shows the relative phase constant (the imaginary part of the propagation constant divided by its free-space value) of these lines as determined by the method of subsection III-A. Three other curves are plotted; these represent the relative phase constant as determined using each of the three line pairs alone. Pair 1–3, which has the greatest length difference, produces the smoothest of the three curves, as predicted by (28). The symmetry of the other two curves about that of pair 1–3 reflects the linear dependence of the errors (see (63)).

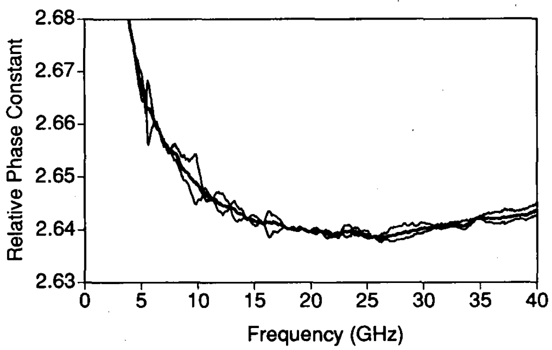


Fig. 4. Relative phase constant of a coplanar waveguide, determined using lines 1 and 2, lines 1 and 3, and lines 2 and 3, as well as from the multiline method using all three lines (dark curve). The curve representing lines 1 and 3 virtually duplicates the multiline curve.

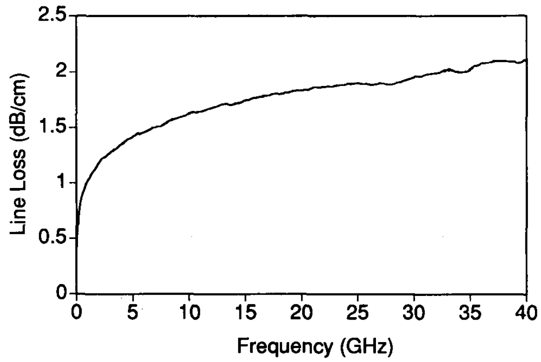


Fig. 5. Line loss of the coplanar waveguide, from the multiline method.

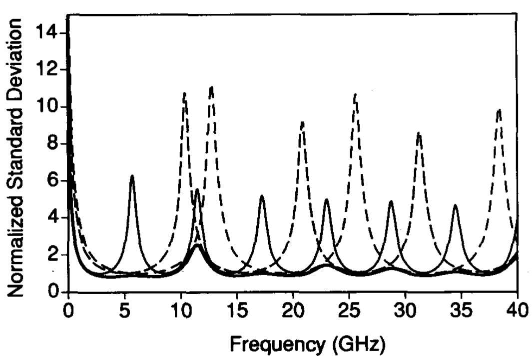


Fig. 6. Normalized standard deviation (average of $\sigma_{\alpha 0}$ and $\sigma_{\beta 0}$) determined using lines 1 and 2, lines 1 and 3, and lines 2 and 3, as well as from the multiline method using all three lines (dark curve).

Fig. 5 shows the line loss factor, in dB/cm. Only the minimum-variance estimate is shown. This graph is useful in the interpretation of later results.

Fig. 6 plots the average of the two normalized standard deviations $\sigma_{\alpha 0}$ and $\sigma_{\beta 0}$, as computed from the method of subsection III-B, for the three individual pairs alone as well as for the minimum-variance estimate (dark curve). The periodic peaks in the single-pair curves occur when the effective phase delay is near 0°. The highest peaks belong to the pairs of least length difference, as predicted by (52). The longest line pair provides, in the worst case, about twice the accuracy of the shorter pairs over this band. The decline in the height of consecutive peaks would appear to be caused by the increasing loss factor (see Fig. 5).

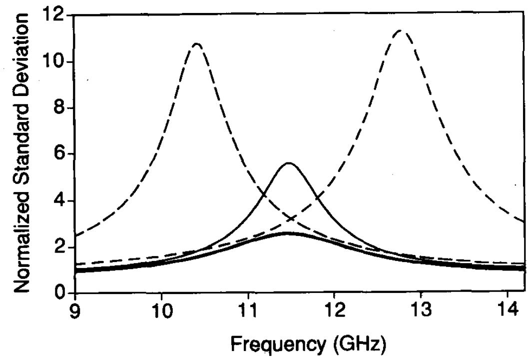


Fig. 7. Close-up view of Fig. 6.

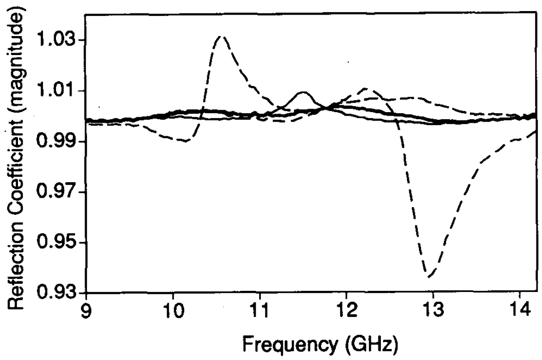


Fig. 8. Magnitude of reflection coefficient of a short circuit measured using the four calibrations illustrated in Fig. 7. Multiline calibration is represented by the dark curve.

A closeup view of Fig. 6 is presented in Fig. 7. In this case, the individual curves peak in a fairly narrow frequency band, so that one might expect to have difficulty, even with the multiline estimate. In fact, the predicted standard deviation does rise, but only to a moderate value of 2.55. The effects of these peaks are clearly illustrated in Fig. 8, which plots the magnitude of the measured reflection coefficient of a nominal short circuit as determined using each of these four calibrations. The declines in accuracy predicted by Fig. 7 are reflected in large deviations in the three measurements using single-pair calibrations. The locations and magnitudes of the peaks concur with the theory. The multiline estimate also agrees with the prediction by remaining quite flat.

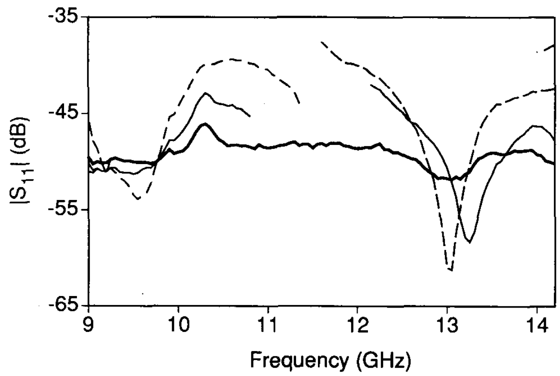


Fig. 9. Return loss of a line measured using the four calibrations illustrated in Fig. 7. Multiline calibration is the dark curve. Data from the three single-line calibrations are shown only where the effective phase delay is between 20° and 160°. This is the portion of the band over which single-line TRL is commonly used.

A final illustration (Fig. 9) uses the same calibrations as Fig. 8 in the measurement of a transmission line not belonging to the calibration set. The return loss using the multiline method is approximately 50 dB. For the three single-line calibrations, we include data *only* over the portion of the band in which the effective phase delay is between 20° and 160°. The point here is that the single-line calibration is inferior even over the limited bandwidth over which it is commonly applied.

V. Conclusion

This paper presents a detailed error analysis of the TRL method. This allows the development of the covariance matrix underlying a minimum-variance method using multiple transmission line standards. Furthermore, the analysis offers a means of assessing the errors in both the conventional and multiline methods.

The multiline method offers a number of advantages: better accuracy than any of the individual line calibrations, more uniform accuracy across the band, and the avoidance of band segmentation and associated frequency discontinuities in calibration constants. Moreover, the cost of implementation is small. In fact, for a wide calibration band over which more than one line is needed in any case, the proposed method simply provides a more efficient utilization of available information at essentially no cost. The actual expense of redundant standards depends on the transmission line medium of interest. Whereas coaxial transmission lines may be costly to produce and time-consuming to measure, the same is not true of planar standards such as microstrip and coplanar waveguide. Multiple planar standards are inexpensively produced on a single wafer, and wafer-probing techniques allow for rapid measurement.

Although we have focused on using standards of various lengths, there is nothing to prevent the use of multiple lines of the same length or even multiple measurements of the same line. As long as connector repeatability errors are the source of the calibration error, multiple measurements should improve the overall accuracy, the standard deviation varying roughly as the inverse square root of the number of measurements. One purpose for utilizing a large number of measurements is the assessment of connector repeatability. The tools for this assessment can be derived as a straightforward extension of the methods discussed in this paper.

An important set of assumptions was made in deriving the covariance matrix. These assumptions were founded on the model of connector repeatability errors. Other error models may lead to different covariance matrices. In particular, errors due to random imperfections in the lines themselves may need to account for correlations between errors at the opposite ends of each line. Nevertheless, the effect of these changes on the actual estimates is expected to be small.

Calibration methods similar to TRL are also amenable to the analysis presented here. In particular, methods using a match or attenuator instead of the line fit into the general scheme presented, although an accurate model of the errors may lead to a different estimate of the covariance matrix.

Acknowledgment

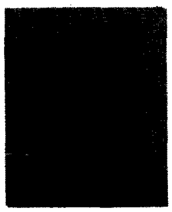
The author appreciates the contributions of D. F. Williams, G. F. Engen, J. R. Juroshek, and D. A. Hill, who offered critical comments concerning the manuscript, and K. R. Phillips, who performed the measurements.

References

[1] G. F. Engen and C. A. Hoer, "Thru-reflect-line: An improved technique for calibrating the dual six-port automatic network analyzer," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-27, pp. 987–993, Dec. 1979.
[2] R. F. Bauer and P. Penfield, "De-embedding and unterminating," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 282–288, Mar. 1974.
[3] G. F. Engen, "A least squares solution for use in the six-port measurement technique," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 1473–1477, Dec. 1980.
[4] S. P. Jachim and W. Daril Gutscher, "A statistical method for calibrating the six-port reflectometer using nonideal standards," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 1825–1828, Nov. 1989.
[5] D. Williams, "De-embedding and unterminating microwave fixtures with nonlinear least squares," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 787–791, June 1990.
[6] R. Marks and K. Phillips, "Wafer-level ANA calibrations at NIST," presented at 34th ARFTG Conference, 1989.
[7] G. Reeve, R. Marks, and D. Blackburn, "MMIC related metrology at the National Institute of Standards and Technology," *IEEE Trans. Instrum. Meas.*, vol. 39, pp. 958–961, Dec. 1990.
[8] R. Marks, "Multi-line calibration for MMIC measurement," presented at 36th ARFTG Conference, 1990.
[9] R. A. Soares, P. Gouzien, P. Legaud, and G. Follot, "A unified mathematical approach to two-port calibration techniques and some applications," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 1669–1674, Nov. 1989.
[10] C. A. Hoer, "On-line accuracy assessment for the dual six-port ANA: Treatment of systematic errors," *IEEE Trans. Instrum. Meas.*, vol. 36, pp. 514–519, June 1987.
[11] H.-J. Eul and B. Schiek, "Thru-match-reflect: One result of a rigorous theory for de-embedding and network analyzer

calibration," in *Proc. 18th European Microwave Conf.*, Sept. 1988, pp. 909–914.

[12] M. Zelen, "Linear estimation and related topics," in *A Survey of Numerical Analysis*, J. Todd, Ed. New York: McGraw-Hill, 1961.



Roger B. Marks (S'86–M'87) was born in Charleston, WV, on May 16, 1958. He received the A.B. degree in physics from Princeton University in 1980 and the Ph.D. degree in applied physics from Yale University in 1988.

He subsequently served as a postdoctoral research associate at the Laboratory for Electromagnetic Research at the Delft University of Technology, in The Netherlands. He has been with the Electromagnetic Fields Division of the National Institute of Standards and Technology (formerly the National Bureau of Standards) in Boulder, CO, since May 1989. He works with the NIST MMIC program, analyzing planar structures and developing methods for their measurement. His technical interests include mathematical and numerical methods for the analysis of scatterers, resonators, and waveguides.