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# A MULTIVARIATE APPROACH TO CONSTRUCTION CONTRACT BIDDING MARK-UP STRATEGIES

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**Corrigendum**

The paper contains the following errors:

1. Eqn (21) should read

$$\hat{\beta}_j = \frac{\sum_{i=1}^{r} \delta_{ij} (y_{ij} - \alpha_i) / \sigma_i^{\,2}}{\sum_i \delta_{ij} / \sigma_i^{\,2}}$$

In practice, however, convergence problems occur with this formulation and the original eqn (21) was used as an approximation.

2. Eqn (23) should read

$$\hat{\sigma}_i^{\,2} = \frac{1}{n_i} \sum_i \sum_j \delta_{ij} (y_{ij} - \alpha_i - \beta_j)^2$$

3. Fig 1 legend should read:

$c_e(x)$
$f_{55}(x)$
$X_e = X_{ev}$

4. Figs 2 and 3 $f_{ss}(x)$ should read $f_{55}(x)$.

# A Multivariate Approach to Construction Contract Bidding Mark-up Strategies

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A multivariate approach to contract bidding strategies in the construction industry is presented. This represents a radical departure from previous work in the field by using all available data on competing bidders. 'Optimal', 'no loss' and 'break even' mark up strategies are derived and methods of parameter estimation proposed. A case study shows how the three strategic mark up values are calculated against known competitors.

*Key Words:* Bidding Strategy; Multivariate Analysis; Construction Contracts

## INTRODUCTION

Many contracts for goods and services are let on the basis of sealed bid auctions. The usual conditions for the auction are that interested suppliers may enter, by a stipulated date and time, a bid for the amount they wish to be paid should they become a party to the contract. All bids are delivered separately and simultaneously in sealed envelopes to the procurer who then opens the envelopes. The procurer inspects the bids and generally enters into a written contract with the lowest bidder based on the amount stated in the bid.

Construction contracts are typical in this respect. Each bidding contractor estimates his likely costs of carrying out the work detailed in the project plans and schedules and adds a percentage mark-up to form the bid value. The value of the mark-up crucially influences the chances of a bidder winning the contract and the subsequent profit should the contract be secured and the work be completed. Clearly, a low mark-up value should increase the chance of winning but decrease the profit, whilst a high mark-up should increase the profit but decrease the chances of winning.

Strategic mark-up bidding assumes that the bidder applies a mark-up which happens to produce a satisfactory balance between the probability of the winning the contract and the profit generated as a result of winning the contract. A special case of strategic mark-up bidding is optimal bidding, defined as applying a mark-up which happens to maximise expected profit, ie., the product of the probability of winning the contract and the profit generated as a result of winning the contract[1].

The literature on strategic mark-up bidding is quite extensive and several reviews have been published (eg [2]). All the work to date has been based on two bivariate models. The Friedman[1] model compares the strategic bidder with individual competitors while Hanssmann and Rivett[3] compare the strategic bidder with lowest bidders. However, the Friedman model has been frequently criticised as demanding unrealistic amounts of data to estimate the model parameters (eg [4]) especially for construction contract auctions (eg. [5], and [6]). The Hanssmann and Rivett model partially solves this by reducing the number of parameters in the model and thus the data demands, but with loss of predictive power.

Multivariate methods offer a means of better utilisation of all available data, depending on the adequacy of certain assumptions concerning the statistical properties of bids. In this case an individual bidder is not restricted to data for auctions in which he has been a participant, as is the case with bivariate approaches. Instead, he is able to incorporate data for **all** auctions in which his competitors, and potential competitors, have been participants, irrespective of the individual bidder's participation. This increases, by several orders of magnitude, the amount of data available for estimating the model parameters.

Recent empirical studies[7] indicate that, with suitably transformed data, the assumptions implicit in multivariate approaches may not be unduly violated in construction contract auctions.

This paper considers, via a case study, the use of one such multivariate approach for deriving 'optimal' and other strategic mark up values against known competitors. Firstly the multivariate approach is introduced and the probability of a bidder underbidding his competitors formulated. This is then extended, by the inclusion of a mark-up decision variable, into a formulation for 'optimal', 'no loss' and 'breakeven' strategies. Maximum likelihood estimators are proposed for obtaining values of the basic parameters in the model from which it is shown how the other parameters may also be estimated. The paper describes how these parameters were estimated from the 'live' case study data and the three strategic mark-up values obtained against known competitors.

## THE MULTIVARIATE APPROACH

Profit depends on the value of the mark up multiplier, *v*. A low mark up increases the chance of acquiring a contract, but with little profit, while conversely a high mark up gives a larger profit, but with little chance of acquiring the contract. We propose a model for the probability of obtaining a contract as a function of bid, *x*, or equivalently, *v*. Since $v=x/c$ where *c* is the cost estimate, we can choose an additive formulation if we work on a log scale.

If for a particular contract, $y_i$ , $i$=1,2 ... ,$n$ are the (log transformed) bids, treated as continuous random variables with joint probability density function $f(y_1, \dots ,y_n)$ then

$$P(y_1 < y_i, \forall i \neq 1) = \int_{y_1=-\infty}^{\infty} \int_{y_2=y_1}^{\infty} \int_{y_3=y_1}^{\infty} f\left(y_1 y_2 y_3 \dots y_n\right) dy_n \dots dy_3 dy_2 dy_1 \quad \mathbf{(1)}$$

where $f(\dots)$ is the joint probability density function of the *n* bids (*n* is assumed to be known). Now assuming the variables are **independent**, it follows from (1) that

$$P(y_1 < y_i, \forall i \neq 1) = \int_{-\infty}^{\infty} f_1(y_1) \prod_{i=2}^{n} \left\{ \int_{y_i=y_1}^{\infty} f_i(y_i) dy_i \right\} dy_1 \quad \mathbf{(2)}$$

*Optimal bidding*

Skitmore[7] has proposed the model

$$y_i \sim N\left(\mu_i, \sigma_i^2\right), \text{ independently} \quad \mathbf{(3)}$$

where $y_i = \ln(x_i - mw)$, $x_i$ is the bid, *w* is a parameter estimated by the value of the lowest bid entered for the contract and *m* is a modifying constant. In an empirical analysis of two independent sets of construction contract auctions, Skitmore has also shown that any value of the modifying constant in the range 0≤$m$≤0.8 will provide a good normalising transformation[7].

Here we adopt the special case where $m$=0, and thus $y_i = \ln(x_i)$. If $\mu_1 = 0$, the probability of bidder 1 entering the lowest bid now becomes

$$\int_{-\infty}^{\infty} \frac{e^{-z_1^2/2}}{\sqrt{2\pi}} \prod_{i=2}^{n} K(z_i) dz_1 \tag{4}$$

where

$$K(x) = \int_{x}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz \tag{5}$$

and

$$z_i = \frac{\sigma_1 z_1 - \mu_i}{\sigma_i} \tag{6}$$

Applying a mark up multiplier of $v$ to $x_i$ affects the mean in model (3) by an amount of **ln**($v$), with the probability, $P(v)$, of entering the lowest bid given by (4) but with $z_i$ in (6) now replaced by

$$z_i = \frac{\sigma_1 z_1 + \ln v - \mu_i}{\sigma_i} \tag{7}$$

Replacing a bidder's bids with his/her cost estimates, $v$ then represents a mark up multiplier and the objective is to find the mark up $v_o$ to maximise expected profit. Since profit can be taken as zero for those contracts which we do not win, we need to compute

$$\text{Expected profit} = E[P] = E\left(\frac{X_e v - A}{A}\middle| win\right) P(v), \tag{8}$$

where $X_e$ = cost estimate (a random variable); A = unknown, but assumed fixed, actual cost.

Assuming cost estimates are unconditionally unbiased[8], the actual cost can be estimated by the expected value of $X_e$, $E[X_e] = \mu_e$, say. Substituting in (8) gives the estimated expected profit as

$$E[P] = E\left(\frac{X_e v - \mu_e}{\mu_e}\middle| win\right) P(v) \tag{9}$$

which is explicitly

$$E[P] = \int \frac{yv - \mu_e}{\mu_e} C_e(y) dy \tag{10}$$

where $C_e(x) = f_e(x \mid win)P(v)$ and $f_e(x \mid win)$ is the probability density function (pdf) of $X_e$ given that the contract is won. From Bayes' formula, we have the conditional distribution of

$$z_1 = \frac{\ln vx - \mu_1}{\sigma_1} \tag{11}$$

(a standardised, transformed bid), given that the contract is won, given by

$$p(z_1 | win) = \frac{p(win | z_1) p(z_1)}{p(win)} \tag{12}$$

and $p(win)$ is simply the normalisation constant, $P(v)$, given by (4), (5) and (7). In the case of model (3)

$$C_e(x) = \frac{e^{-z_1^2/2}}{\sqrt{2\pi}} \left(\prod_{j=2}^{n} K(z_j)\right) \frac{dz_1}{dx} \tag{13}$$

where

$$z_1 = \frac{\ln x - \ln v}{\sigma_1} \tag{14}$$

$$z_1 = \frac{\sigma_1 z_1 + \ln v - \mu_j}{\sigma_j}, \qquad j \geq 2 \tag{15}$$

and

$$\frac{dz_1}{dx} = \frac{1}{\sigma_1 x}. \tag{16}$$

*Loss Oriented Strategies - 'No loss' and 'Breakeven'*

As an alternative to maximising expected profit, a bidder may prefer to restrict the probability of making a loss in some way. One such strategy we term a 'no loss' strategy where the mark up, $v_n$, required is the one which results in a (conditional) probability of 0.05 of making a loss. This is given where

$$\int_{-\infty}^{\mu_e} \frac{C_e(x)dx}{P(v)} = p, \tag{17}$$

where $P(v)$ is given by (4) and $C_e(x)$ is given by (13) and we solve $v$ for $p$=0.05.

By adjusting $p$, the probability of making a loss can be set to any desired value (the 'breakeven' mark-up, $v_b$, is obtained by solving (17) for $p$=0.50).

*Estimation of $\mu_i$ and $\sigma_i$*

From data on $r$ bidders over $c$ contracts, Skitmore's[7] empirical analysis of construction contract bids found that bids may be adequately modelled by

$$\ln x_{ij} = y_{ij} \; iid \; N\left(\alpha_i + \beta_j, \sigma_i^2\right) \tag{18}$$

where $x_{ij}$ is the $i$th bidder's bid for the $j$th contract, $\alpha_i$ is a bidder location parameter, $\beta_j$ is a contract datum parameter (so $\alpha_i + \beta_j \approx \mu_{ij}$). We estimate $\alpha_i$, $\beta_j$, and $\sigma_i$ by maximising the likelihood of (18). The log-likelihood is

$$\ln L = -\sum_{i=1}^{r} \frac{n_i \ln \sigma_i^2}{2} - \frac{1}{2}\sum_{i=1}^{r} \frac{1}{\sigma_i^2} \sum_{j=1}^{c} \delta_{ij}\left(y_{ij} - \alpha_i - \beta_j\right)^2 \tag{19}$$

where $\delta_{ij}$=1 if bidder $i$ bids for contract $j$, and 0 if bidder $i$ does not bid for contract $j$,

$$n_i = \sum_{j=1}^{c} \delta_{ij} \tag{20}$$

The maximum likelihood estimates of the $\alpha_i$, $\beta_j$ and $\sigma_i^2$ are

$$\hat{\beta}_j = \sum_{i=1}^{l} \delta_{ij} \frac{y_{ij} - \alpha_i}{n_i} \tag{21}$$

$$\hat{\alpha}_i = \sum_{j=1}^{c} \delta_{ij} \frac{y_{ij} - \beta_j}{n_i} \tag{22}$$

$$\hat{\sigma}_i^2 = \sum_{j=1}^{c} \delta_{ij} \left(\frac{y_{ij} - \alpha_i - \beta_j}{n_i}\right)^2 \tag{23}$$

The numerical procedure for solving these equations involves initialising all $\alpha_i$ = 0 and iterating (21) and (22) to convergence. The estimates of $\sigma_i^2$ provided by (23) are adjusted for bias (approximately) by multiplying by the factor

$$\frac{n_i}{(n_i-1)\left(1-\frac{c-1}{N-r}\right)} \quad \textbf{(24)}$$

where

$$N=\sum_{j=1}^{c} n_i \quad \textbf{(25)}$$

and for once only bidders ($n_i$ = 1) a weighted average of the unbiased variance estimates with weights

$$\frac{n_i-r-1}{N-1} \quad \textbf{(26)}$$

For computational purposes it is unnecessary to introduce once only bidders until after convergence of the iteration procedure.

*Application to estimating v for a future contract*

Since the $\alpha_i$ are bidder effects, we can assume that they will have the same value for a future contract and so we can use the estimates obtained above. The $\beta$ value for a future contract is unknown to us but is tyhe same for all bidders in the contract and need not concern us. The only problem remaining in (10) is the unconditional mean cost estimate $\mu_e$.

*Estimation of $\mu_e$*

As $\mu_e$ is the expected value of the (unconditional) cost estimate, ie. setting ln($v$)=0, we can find an explicit value. The pdf is that of a log normal distribution, so $\mu_e$ is (recalling that $\mu_I$=0)

$$\mu_e = \exp\ \sigma^2/2 \quad \textbf{(27)}$$

## A CASE STUDY

To illustrate the practical application of this approach, a set of 'live' bidding data has been analysed. These were donated by a construction company operating in the London area and covered all the company’s building contract bidding activities during a 12 month period in the early 1980s for a total of 86 contracts. To preserve confidentiality, all the bidders were assigned a code at random, the bidder providing the data having code 304. Bidder 304's cost estimates were included in the data set instead of his bids. Some of the data were incomplete; that is the value of some bids or the identity of the bidders were not known by the company. In several cases it was possible to supplement these data from a bidding information agency in the London area. The 51 resulting contracts for which a full set of bids, together wit the identity of the bidder, were available for analysis are given in Table 1.
By first transforming the cost estimates to bids to

$$y_{ij} = \ln x_{ij} \quad \textbf{(28)}$$

and then applying the iterative eqns (21) and (22), the estimates $\alpha_i$ and $\sigma_i$ were obtained for all participants, $\alpha_{304} = 0$.

For simplicity, bidder 55 was chosen for analysis, bidder 55 having entered bids for 20 of the contracts in the data. As a result of the iterative procedure, the values of $\hat{\alpha}$ and $\hat{\sigma}$ for bidder 304's cost estimates and bidder 55's bids were found to be 0, 0.00137, 0.06922, and 0.00257 respectively.

The probability function for bidder 304's profit was obtained from (4) for a series of mark up multipliers, $v$, the area under which was computed by a suitable quadrature method for each percentage point. Expected profit was calculated by substituting for $\mu_e$ in (27), and then finding the $v$ which maximises $E[P]$ in (10). The estimated values of $\mu_e$, $P(v)$, and $E[P]$ for a range of mark up multipliers are given in Table 2.

Figs 1 to 3 show the probability curves for bidders 304 and 55 together with bidder 304's profit with 0, 5 and 10% mark-up values.

It can be seen that the profit probability curve becomes progressively flatter with the mean value, $E[P]/P(v)$, progressively diminishing relative to the mark up multiplier - a phenomenon often termed 'the winner's curse' (see Thaler[9] for a review of the literature).

Visual inspection of Table 2 indicates the maximum expected profit to be around 2.7% at a mark up multiplier of 8%. By using a method of successive approximation, the exact optimal mark up multiplier was found to be 7.80737% (2.68163% expected profit). The mark-up multipliers for the 'no loss' ($p$=0.05) and 'breakeven' ($p$=0.5) strategies were found to be 7.50510 and 0.57660 respectively.

The results for bidders 55 and 221 are given in Table 3. The $\hat{\alpha}$ and $\hat{\sigma}$ values for bidder 221 were 0.07971 and 0.00385 respectively and the 'no loss', 'break even' and 'optimal' mark up multipliers were found to be 8.39080, 1.05870 and 6.61799 (1.45403 expected profit) respectively.

## CONCLUSIONS

The application of multivariate methods to sealed bid mark-up strategies offers a potential improvement on previous bivariate methods in providing a means of better utilisation of all available data. Since it has been shown that, with suitably transformed data, the necessary assumptions concerning the statistical properties of bids may not be unduly violated in construction contract auctions[7], a multivariate model has been proposed for deriving 'optimal' and other strategic mark up multipliers. A means of estimating the model parameters was described and this has been applied in a case study to obtain the strategic mark-up multipliers required against known competitors.

The major untested assumptions in the method are that (1) the variables are independent (ie., bidders do not change their behaviour depending on who their competitors are), this could not be true if competitors used the method proposed here; (2) the variables are intertemporaly fixed (ie., bidders do not change their behaviour over time), (3) variability is stochastic (ie., the error term is truly random), (4) the estimated cost is unconditionally unbiased (ie., the conditions supporting 'winner's curse' apply), (5) the number and identity of competitors are known in advance of bidding. Assumptions (1)-(3) are also implicit in the Friedman model, and assumption (4) has been considered to be reasonable by Flanagan and Norman[8], who have extensive knowledge of the construction industry. Although not formally allowed in many sealed bid auctions, assumption (5) is generally accepted as being reasonable in construction contract auctions where such information may be purchased for the purpose. In the absence of certain knowledge of (5) however, probabilistic methods are available, and these have been examined to some extent by Friedman and others.

The multivariate approach described in this paper is still very much in its formative stages and it is clear that some work is yet needed before it can be applied with confidence in 'real-world' auctions. In addition to testing the assumptions in the model, the paucity of data in the field from which to estimate parameter values suggests that the resulting strategic mark-up multiplier estimates, though unbiased, may not be very accurate. The consequent opportunity loss may well be a significant factor. If this is the case, then it will be appropriate for future work in the field to consider the minimisation of such opportunity loss as a strategic option in itself.

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TABLE 1*: The 51 bids available for analysis*

| Project | Bdr | Bid | Bdr | Bid | Bdr | Bid | Bdr | Bid | Bdr | Bid | Bdr | Bid | Bdr | Bid | Bdr | Bid | Bdr | Bid |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 150 | 1454515 | 55 | 1514865 | 304 | 1475398 | 134 | 1468775 | 154 | 1447867 | 73 | 1457977 | 1 | 1386652 | | | | |
| 2 | 304 | 535608 | 291 | 502042 | 154 | 529744 | 157 | 516376 | 1 | 505291 | | | | | | | | |
| 3 | 75 | 1333142 | 217 | 1331156 | 304 | 1366863 | 281 | 1266892 | 115 | 1276787 | 93 | 1277652 | 360 | 1865545 | 1 | 1271146 | | |
| 4 | 304 | 696743 | 292 | 696972 | 237 | 701062 | 79 | 637815 | 361 | 697826 | 280 | 637815 | | | | | | |
| 5 | 55 | 404110 | 304 | 422297 | 97 | 413224 | 117 | 389196 | 362 | 417489 | 157 | 389848 | 1 | 389214 | | | | |
| 6 | 134 | 2116877 | 99 | 2169966 | 293 | 2187991 | 304 | 2161120 | 221 | 2198655 | 137 | 2296108 | 8 | 2165611 | 117 | 2153344 | 294 | 2133608 |
| | 1 | 2058210 | | | | | | | | | | | | | | | | |
| 7 | 304 | 3065742 | 150 | 3119689 | 170 | 3141641 | 134 | 3153800 | 191 | 3249927 | 55 | 3269768 | 187 | 3335993 | 1 | 2919754 | | |
| 8 | 221 | 7925257 | 304 | 7351929 | 247 | 7374650 | 20 | 6900000 | 1 | 7035339 | | | | | | | | |
| 9 | 118 | 871520 | 137 | 899935 | 304 | 902378 | 291 | 914393 | 83 | 950737 | 221 | 996483 | | | | | | |
| 10 | 304 | 1063337 | 251 | 1154023 | 173 | 1102272 | 201 | 1079657 | 1 | 1012702 | | | | | | | | |
| 11 | 154 | 1759614 | 281 | 1792123 | 157 | 1838532 | 170 | 1918066 | 304 | 1947733 | 308 | 1784215 | 1 | 1811845 | | | | |
| 12 | 304 | 1126816 | 201 | 1146398 | 154 | 1169795 | 24 | 1227296 | 280 | 1312527 | 221 | 1399472 | 1 | 1053099 | | | | |
| 13 | 304 | 698005 | 268 | 625501 | 308 | 630288 | 55 | 666545 | 1 | 652341 | | | | | | | | |
| 14 | 364 | 588810 | 365 | 584833 | 79 | 639229 | 145 | 646341 | 304 | 682802 | 154 | 691474 | | | | | | |
| 15 | 303 | 1429218 | 291 | 1493849 | 304 | 1511033 | 12 | 1521628 | 366 | 1526377 | 55 | 1717715 | | | | | | |
| 16 | 6 | 842319 | 304 | 870894 | 185 | 883617 | | | | | | | | | | | | |
| 17 | 367 | 284947 | 356 | 292692 | 368 | 294694 | 152 | 303700 | 85 | 307282 | 134 | 313203 | 369 | 315727 | 118 | 333597 | 370 | 334353 |
| | 304 | 348969 | | | | | | | | | | | | | | | | |
| 18 | 150 | 461444 | 304 | 483862 | 308 | 482241 | 55 | 447021 | 154 | 493417 | 311 | 455480 | | | | | | |
| 19 | 280 | 2858191 | 371 | 2947007 | 134 | 2950723 | 304 | 2999999 | 60 | 3093587 | 6 | 3099528 | 266 | 3278229 | 170 | 3325198 | 55 | 3333793 |
| | 1 | 2884614 | | | | | | | | | | | | | | | | |
| 20 | 276 | 7831865 | 304 | 7837276 | 256 | 7859122 | 55 | 7904172 | 152 | 8047230 | 293 | 8145323 | 117 | 8279564 | 134 | 8657685 | 1 | 7646123 |
| 21 | 55 | 3971051 | 304 | 3854074 | 372 | 4724785 | 154 | 3955009 | 373 | 3944772 | 79 | 3731543 | 237 | 4001188 | 1 | 3705840 | | |
| 22 | 292 | 573485 | 291 | 596737 | 134 | 597730 | 201 | 613528 | 304 | 615015 | 170 | 621223 | 1 | 580203 | | | | |
| 23 | 304 | 1610942 | 163 | 1623447 | 173 | 1646286 | 268 | 1663742 | 152 | 1700000 | 1 | 1558574 | | | | | | |
| 24 | 64 | 1196036 | 374 | 1199328 | 187 | 1206837 | 304 | 1226589 | 291 | 1262082 | 152 | 1271000 | 170 | 1295954 | 254 | 1302161 | 1 | 1179413 |
| 25 | 137 | 2636397 | 150 | 2654728 | 187 | 2673906 | 55 | 2685127 | 304 | 2762123 | 152 | 2845567 | | | | | | |
| 26 | 24 | 469663 | 268 | 476784 | 286 | 485870 | 55 | 486485 | 122 | 504026 | 263 | 529468 | 304 | 540814 | 1 | 515061 | | |
| 27 | 201 | 1526553 | 152 | 1533719 | 148 | 1698797 | 304 | 1876612 | 1 | 1770389 | | | | | | | | |
| 28 | 201 | 2106139 | 304 | 2175928 | 308 | 2210065 | 280 | 2223710 | 221 | 2255246 | 117 | 2296623 | 266 | 2331830 | 1 | 2062491 | | |

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 29 | 102 | 499888 | 55 | 559596 | 217 | 592026 | 170 | 602042 | 304 | 608957 | 134 | 619065 | | | | | | |
| 30 | 304 | 2639525 | 308 | 2842407 | 280 | 2874130 | 55 | 2861665 | 152 | 2736300 | 256 | 2770720 | 1 | 2538005 | | | | |
| 31 | 304 | 732572 | 365 | 599429 | 145 | 623906 | 79 | 691759 | 154 | 744332 | 364 | 607065 | | | | | | |
| 32 | 134 | 546641 | 268 | 539565 | 55 | 608242 | 24 | 538382 | 170 | 599934 | 304 | 559351 | 1 | 530190 | | | | |
| 33 | 221 | 792966 | 99 | 811788 | 308 | 819971 | 55 | 847621 | 137 | 847892 | 304 | 853793 | | | | | | |
| 34 | 152 | 2085151 | 107 | 2130217 | 280 | 2150583 | 115 | 2203956 | 137 | 2219653 | 154 | 2241687 | 304 | 2325900 | | | | |
| 35 | 268 | 821617 | 115 | 844579 | 303 | 848459 | 304 | 871927 | 106 | 872215 | 375 | 935765 | 1 | 830407 | | | | |
| 36 | 304 | 792474 | 24 | 747374 | 217 | 778559 | 252 | 743788 | 268 | 808345 | 170 | 835465 | 1 | 754737 | | | | |
| 37 | 304 | 7279854 | 60 | 7650271 | 308 | 7029448 | 150 | 6631664 | 193 | 7089879 | 170 | 7230120 | 247 | 6986341 | 191 | 7143710 | 266 | 6794553 |
| | 1 | 7067819 | | | | | | | | | | | | | | | | |
| 38 | 304 | 592096 | 150 | 573997 | 217 | 518613 | 121 | 508985 | 376 | 544480 | 1 | 550787 | | | | | | |
| 39 | 348 | 538600 | 377 | 567031 | 378 | 621365 | 268 | 699839 | 72 | 825451 | 190 | 991468 | 304 | 1001254 | | | | |
| 40 | 154 | 2087946 | 276 | 2104017 | 186 | 2183122 | 304 | 2205359 | 280 | 2212382 | 112 | 2267987 | 221 | 2332476 | 294 | 2400000 | | |
| 41 | 247 | 1503739 | 24 | 1536654 | 304 | 1576905 | 154 | 1583595 | 294 | 1616432 | 157 | 1704995 | 1 | 1530976 | | | | |
| 42 | 191 | 3624453 | 221 | 3694803 | 304 | 3732133 | 170 | 3751115 | 193 | 3773967 | 55 | 3866339 | 134 | 3922937 | 281 | 4122448 | 1 | 3641105 |
| 43 | 157 | 629164 | 173 | 695284 | 311 | 723315 | 266 | 729305 | 304 | 743578 | 379 | 768189 | | | | | | |
| 44 | 304 | 2252833 | 24 | 2264310 | 112 | 2274380 | 191 | 2323385 | 55 | 2384494 | 1 | 2187217 | | | | | | |
| 45 | 163 | 1202916 | 55 | 1268733 | 221 | 1291365 | 304 | 1294986 | | | | | | | | | | |
| 46 | 217 | 2968891 | 280 | 2772626 | 186 | 2822857 | 134 | 2972189 | 276 | 2821600 | 304 | 2857275 | 221 | 2793000 | 1 | 2787585 | | |
| 47 | 286 | 1398400 | 152 | 1401500 | 237 | 1427140 | 304 | 1436804 | 301 | 1453070 | 55 | 1511643 | 371 | 1591986 | 83 | 1665760 | 1 | 1381542 |
| 48 | 294 | 698161 | 24 | 709676 | 291 | 758565 | 304 | 789355 | 134 | 797926 | 55 | 842684 | 252 | 751677 | 1 | 751767 | | |
| 49 | 31 | 248733 | 291 | 251007 | 252 | 251415 | 380 | 261286 | 304 | 264933 | | | | | | | | |
| 50 | 293 | 358840 | 217 | 362370 | 304 | 386983 | 381 | 421797 | 154 | 456272 | 1 | 351803 | | | | | | |
| 51 | 317 | 527692 | 311 | 570874 | 75 | 588854 | 173 | 609221 | 308 | 636451 | 304 | 694297 | 1 | 645858 | | | | |

TABLE 2*: Results for a range of mark up multipliers (against bidder 55)*

| $v$(%) | $v$ | ln$(v)$ | $P(v)$ | $E[P]$ | $E[P]/P(v)$ |
|---|---|---|---|---|---|
| 0.0 | 1.00 | 0.00000 | 0.86476 | -0.48082 | -0.55602 |
| 1.0 | 1.01 | 0.00995 | 0.82730 | 0.25725 | 0.31095 |
| 2.0 | 1.02 | 0.01980 | 0.78427 | 0.91031 | 1.16071 |
| 3.0 | 1.03 | 0.02956 | 0.73610 | 1.46750 | 1.99362 |
| 4.0 | 1.04 | 0.03922 | 0.68351 | 1.92084 | 2.81025 |
| 5.0 | 1.05 | 0.04879 | 0.62748 | 2.26607 | 3.61136 |
| 6.0 | 1.06 | 0.05827 | 0.56918 | 2.50317 | 4.39784 |
| 7.0 | 1.07 | 0.06766 | 0.50989 | 2.63646 | 5.17060 |
| 8.0 | 1.08 | 0.07696 | 0.45094 | 2.67433 | 5.93061 |
| 9.0 | 1.09 | 0.08618 | 0.39357 | 2.62858 | 6.67881 |
| 10.0 | 1.10 | 0.09531 | 0.33892 | 2.51345 | 7.41610 |

TABLE 3*: Results for a range of mark up multipliers (against bidders 55 and 221)*

| $v$(%) | $v$ | ln$(v)$ | $P(v)$ | $E[P]$ | $E[P]/P(v)$ |
|---|---|---|---|---|---|
| 0.0 | 1.00 | 0.00000 | 0.99219 | -0.70731 | -0.92451 |
| 1.0 | 1.01 | 0.00995 | 0.99951 | -0.08586 | -0.12074 |
| 2.0 | 1.02 | 0.01980 | 1.00625 | 0.43161 | 0.66160 |
| 3.0 | 1.03 | 0.02956 | 1.01238 | 0.83968 | 1.42326 |
| 4.0 | 1.04 | 0.03922 | 1.01787 | 1.13790 | 2.16519 |
| 5.0 | 1.05 | 0.04879 | 1.02272 | 1.33097 | 2.88848 |
| 6.0 | 1.06 | 0.05827 | 1.02692 | 1.42837 | 3.59430 |
| 7.0 | 1.07 | 0.06766 | 1.03050 | 1.44345 | 4.28383 |
| 8.0 | 1.08 | 0.07696 | 1.03350 | 1.39210 | 4.95823 |
| 9.0 | 1.09 | 0.08618 | 1.03598 | 1.29136 | 5.61863 |
| 10.0 | 1.10 | 0.09531 | 1.03798 | 1.15792 | 6.26609 |

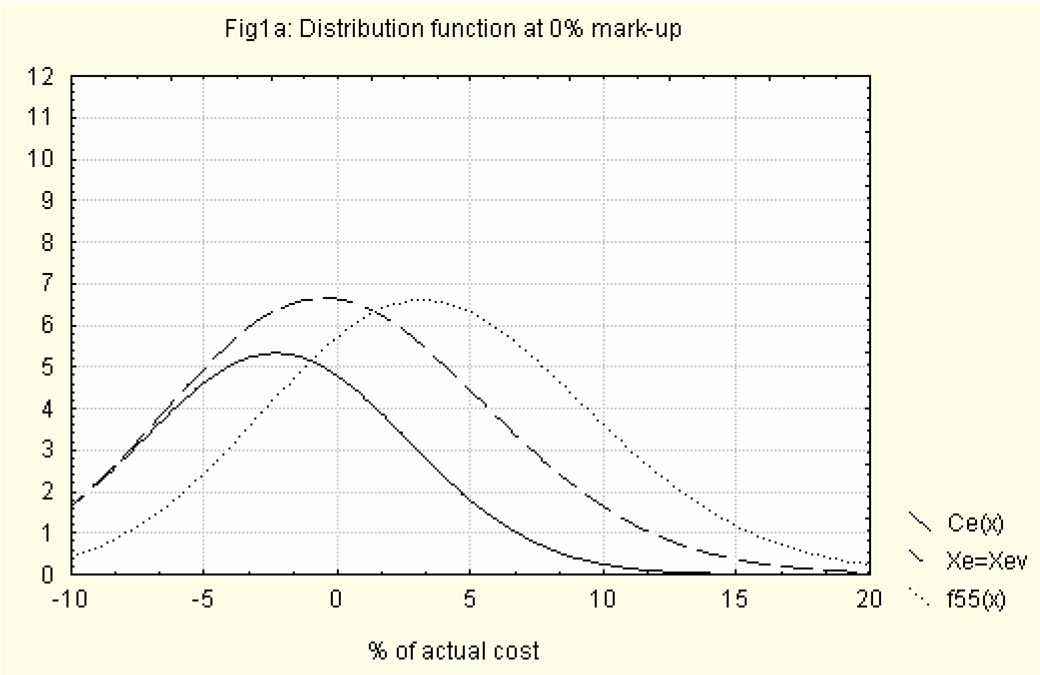

Fig1a: Distribution function at 0% mark-up
12
11
10
9
8
7
6
5
4
3
2
1
0
-10
-5
0
5
10
15
20
Ce(x)
Xe=Xev
f55(x)
% of actual cost


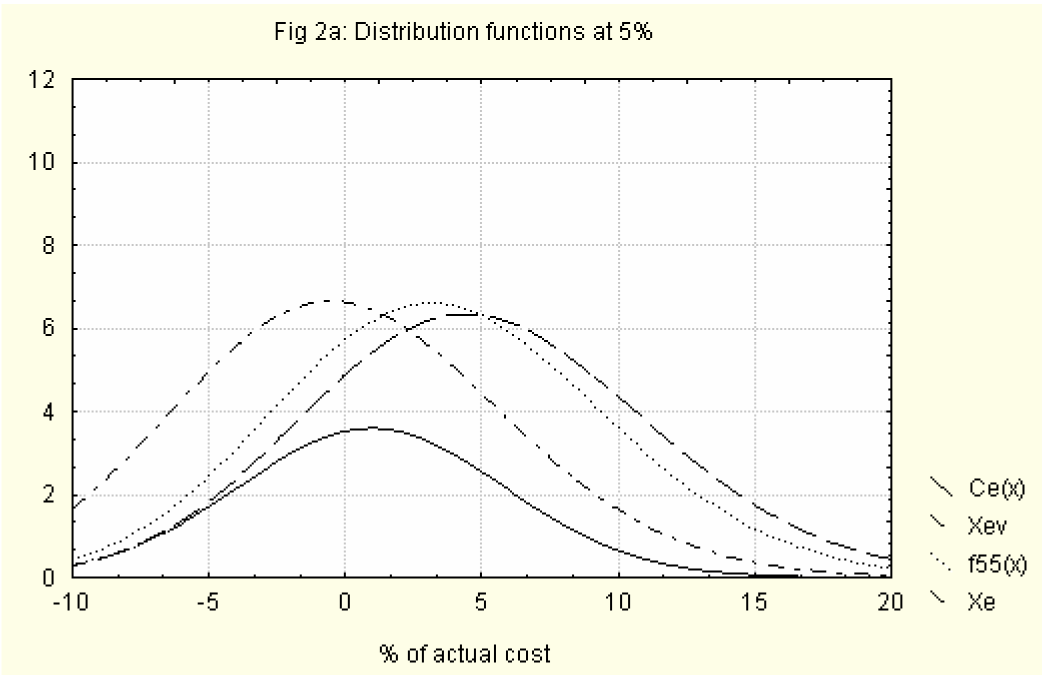

Fig 2a: Distribution functions at 5%
12
10
8
6
4
2
0
-10
-5
0
5
10
15
20
Ce(x)
Xev
f55(x)
Xe
% of actual cost

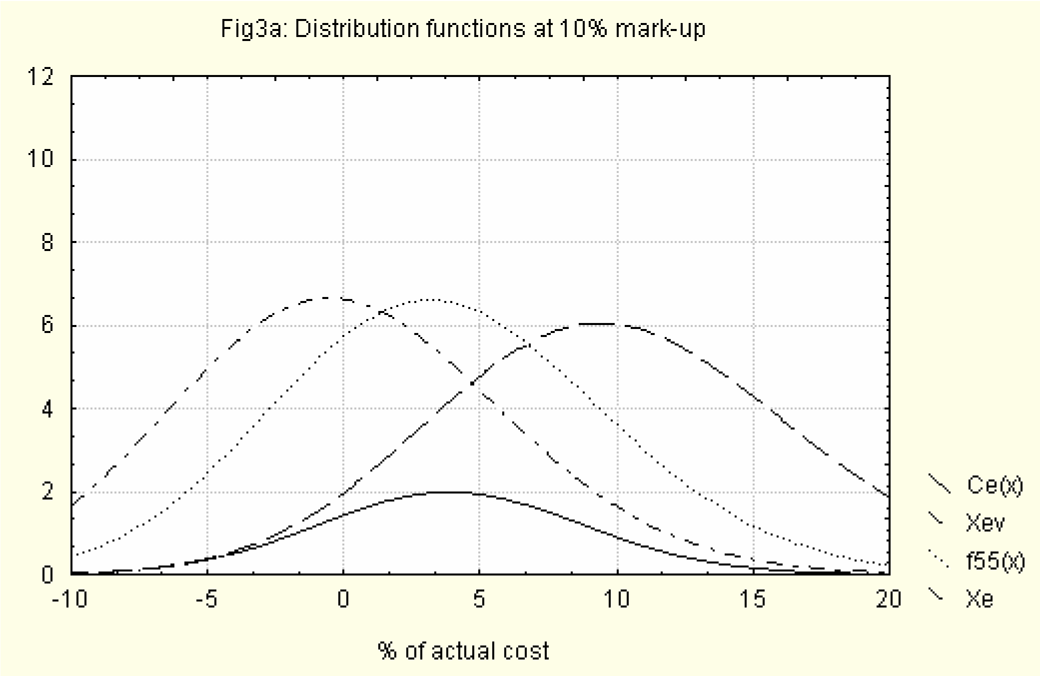
Fig3a: Distribution functions at 10% mark-up
12
10
8
6
4
2
0
-10
-5
0
5
10
15
20
% of actual cost
Ce(x)
Xev
f55(x)
Xe