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A MULTIVARIATE DEFINITION FOR INCREASING HAZARD RATE DISTRIBUTION FUNCTIONS

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1. Introduction. Increasing hazard rate (IHR) distribution functions of one variable have been discussed in the literature for many years and many of their properties have been obtained, for example, see [2]. In this paper, a definition which extends the notion of IHR to multivariate distributions is given and it is shown to satisfy certain desirable multivariate properties.

2. Multivariate IHR Distributions. Consider the random vector $(X_1, X_2, \cdots, X_n)$ with distribution function $F(x_1, x_2, \cdots, x_n) = P[X_1 \leqq x_1, X_2 \leqq x_2, \cdots, X_n \leqq x_n]$. We say that the set of random variables $X_1, X_2, \cdots X_n$ is *right corner set increasing*, written RCSI $(X_1, X_2, \cdots X_n)$ if $P[X_1 > x_1, \cdots X_n > x_n \mid X_1 > x_1', \cdots X_n > x_n']$ is nondecreasing in $x_1', \cdots, x_n'$ for every choice of $x_1, \cdots x_n$. This generalises some notions of dependence that were studied by Lehmann [4] and Esary and Proschan [3]. Setting $\bar{F}(x_1, x_2, \cdots x_n) = P[X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n]$ we have:

DEFINITION. A distribution function $F(x_1, x_2, \cdots, x_n)$ on the nonnegative orthant is *multivariate* IHR if it satisfies the conditions:

(i) $\bar{F}(x_1+t, \cdots, x_n+t)/\bar{F}(x_1, \cdots, x_n) \leqq \bar{F}(x_1'+t, \cdots x_n'+t)/\bar{F}(x_1', \cdots, x_n')$ for all $x_i \geqq x_i' \geqq 0$ and all $t \geqq 0$.

(ii) RCSI $(X_1, \cdots X_n)$.

REMARKS. a. In the context of life-testing, condition (i) is essentially a "wear-out" condition analogous to the univariate case.

b. Condition (ii) characterises a positive dependence between the random variables which has much intuitive appeal if one thinks of the X_i as lifetimes of devices in a common environment with corresponding high or low stress levels.

c. It is of interest to know what possible distribution functions can satisfy (i) when the inequality sign is replaced by an equality. The general form of such a distribution in the bivariate case was determined by Marshall and Olkin [5], and they showed further that if the marginal distributions are exponential, then the distribution is the bivariate exponential distribution: $\bar{F}(x_1, x_2) = \exp[-(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_{12} \max(x_1, x_2))]$.

For the n-dimensional case, the requirement that the $(n-1)$-dimensional marginals be multivariate exponential (MVE) yields the n-dimensional MVE. This result is similar to the univariate case in which the "boundary" of the class of IHR distributions is the class of exponential distributions.

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We now derive some properties of multivariate IHR distributions.

(P_1). *Any subset of multivariate* IHR *random variables is multivariate* IHR.

PROOF. Let us suppose that $(X_1, X_2, \cdots, X_n)$ are multivariate IHR; setting $x_1 = x_1' = 0$ in (i), we have:

$$\bar{F}(t, x_2+t, \cdots, x_n+t)/\bar{F}(0, x_2, \cdots, x_n) \leqq \bar{F}(t, x_2'+t, \cdots, x_n'+t)/\bar{F}(0, x_2', \cdots, x_n').$$

Also from RCSI $(X_1, X_2, \cdots X_n)$, we have:

$$\bar{F}(t, x_2+t, \cdots, x_n+t)/\bar{F}(0, x_2+t, \cdots, x_n+t) \geqq \bar{F}(t, x_2'+t, \cdots, x_n'+t)/\bar{F}(0, x_2'+t, \cdots, x_n'+t)$$

where $x_i \geqq x_i'$ for $i = 2, 3, \cdots, n$ and $t > 0$. Combining these two inequalities, we get:

$$\bar{F}(0, x_2+t, \cdots, x_n+t)/\bar{F}(0, x_2, \cdots, x_n) \leqq \bar{F}(0, x_2'+t, \cdots, x_n'+t)/\bar{F}(0, x_2', \cdots, x_n')$$

Moreover, it is clear that RCSI $(X_1, X_2, \cdots X_n)$ implies RCSI $(X_2, \cdots, X_n)$. Thus we have shown that $(X_2, \cdots X_n)$ are multivariate IHR and the proof is completed by induction.

(P_2). *The union of two mutually independent sets of multivariate* IHR *random variables is itself a set of multivariate* IHR *random variables.*

PROOF. The result is a trivial consequence of the factorization of the distribution function into the two corresponding parts.

(P_3). *A single multivariate* IHR *random variable is* IHR *in the usual sense.*

PROOF. This is an immediate consequence of condition (i). Condition (ii) is trivially true for a single random variable.

(P_4). *Sets of minimums of multivariate* IHR *random variables are multivariate* IHR.

PROOF. Suppose that $(Y_1, \cdots Y_m)$ are multivariate IHR. Let $X_i = \min_{j \in J_i} Y_j$, $i = 1, \cdots, n$, where $J_i \subset \{1, \cdots, m\}$. Let $I_j = \{i \mid j \in J_i\}$, $y_j = \max_{i \in I_j} x_i$, and $y_j' = \max_{i \in I_j} x_i'$, $j = 1, \cdots. m$. Then:

$$P[X_1 > x_1+t, \cdots, X_n > x_n+t \mid X_1 > x_1, \cdots, X_n > x_n]$$
$$= P[Y_1 > y_1+t, \cdots, Y_m > y_m+t \mid Y_1 > y_1, \cdots, Y_m > y_m]$$

is nonincreasing in $x_1, \cdots, x_n$ since $y_1, \cdots, y_m$ are nondecreasing in $x_1, \cdots, x_n$ and $Y_1, \cdots, Y_m$ are multivariate IHR. Thus condition (i) holds for $X_1, \cdots, X_n$. In addition:

$$P[X_1 > x_1, \cdots, X_n > x_n \mid X_1 > x_1', \cdots, X_n > x_n']$$
$$= P[Y_1 > y_1, \cdots, Y_m > y_m \mid Y_1 > y_1', \cdots, Y_m > y_m']$$

and this is nondecreasing in $x_1', \cdots, x_n'$ because $y_1', \cdots, y_m'$ are nondecreasing in $x_1', \cdots, x_n'$ and RCSI $(Y_1, \cdots, Y_m)$. Hence RCSI $(X_1, \cdots, X_n)$ and so $X_1, \cdots, X_n$ are multivariate IHR.

Property P_4 generalises an important structural property of the univariate IHR distribution. Together with properties P_2 and P_3, it shows that the multivariate distributions constructed by taking minimums over an independent basis of univariate IHR distributions are themselves multivariate IHR, and in particular that the multivariate exponential distribution defined by Marshall and Olkin [5] is in the class.

An alternative definition of the multivariate IHR concept is given by (i)*, but this leads to an undesirable negative dependence:

$$\text{(i)*} \qquad \bar{F}(x_1+t_1, x_2+t_2, \cdots, x_n+t_n)/\bar{F}(x_1, x_2, \cdots, x_n) \leqq \bar{F}(x_1'+t_1, x_2'+t_2, \cdots, x_n'+t_n)/\bar{F}(x_1', x_2', \cdots, x_n')$$

for all $x_i \geqq x_i' \geqq 0$ and all $t_i \geqq 0$. Setting $t_1 = x_2 = x_3 = \cdots = x_n = x_1' = x_2' = \cdots = x_n' = 0$, we get $\bar{F}(x_1, t_2, \cdots, t_n) \leqq \bar{F}(x_1, 0, \cdots, 0)\bar{F}(0, t_2, \cdots, t_n)$, and by repeating the process we get

$$\bar{F}(x_1, x_2, \cdots, x_n) \leqq \textstyle\prod_{i=1}^n \bar{F}_i(x_i), \qquad \text{for all } x_i,$$

where F_i is the marginal distribution of X_i.

This inequality represents some kind of negative dependence between the random variables which is undesirable in reliability applications. For example, the concept of *positive quadrant dependence* (Lehmann [4]), implies the reverse inequality and hence with this definition of multivariate IHR would force the random variables to be independent.

3. A subclass of bivariate IHR distributions. A restricted class of multivariate IHR distributions satisfying P_1 through P_4 could be generated from an independent basis with appropriate marginals, see [1]. For example, in the bivariate case, take $X = \min(U, W)$, $Y = \min(V, W+a)$, where U, V, W are arbitrary independent IHR random variables and $a \geqq 0$ is an arbitrary constant. Such a class would preclude more general types of bivariate IHR distributions which are possible under the definition in Section 2. Nevertheless, Theorem 3.1 and its corollary show that bivariate distributions of the above type are of interest.

THEOREM 3.1. *If $Y = \phi(X)$, where ϕ is a nondecreasing function which is not identically zero or infinity, and X has a marginal exponential distribution, then the pair (X, Y) has a bivariate* IHR *distribution if and only if*: $\phi(x) = x+a$, $a \geqq 0$.

PROOF. Let ψ be the inverse function of ϕ, which we can regard as implicitly defined by the probability identity:

$$P[X > x, \phi(X) > y] = P[X > x, X > \psi(y)]$$

for all $x \geqq 0$, $y \geqq 0$. Since X is an exponential random variable, ψ may as well be taken to be a nonnegative function with, necessarily, $\psi(0) = 0$. Then

$$\bar{F}(x, y) = P[X > x, Y > y] = \exp[-\lambda \max(x, \psi(y))],$$

where λ is a positive constant. Thus $\bar{F}(\cdot,\cdot)$ will satisfy condition (i) if and only if the condition

$$\text{(A)} \qquad \max\{x+t,\ \psi(y+t)\}-\max\{x,\ \psi(y)\} \geqq \max\{x'+t,\ \psi(y'+t)\}-\max\{x',\ \psi(y')\}$$

is satisfied for all $x \geqq x' \geqq 0$, $y \geqq y' \geqq 0$, and $t > 0$.

Given that condition (A) holds, then:

$$(1) \qquad \psi(y+t) \leqq \psi(y)+t \quad \text{for all} \quad y \geqq 0 \quad \text{and} \quad t > 0.$$

Proof of (1). Suppose there exists an $a \geqq 0$ and $t > 0$ such that $\psi(a+t) > \psi(a)+t$. Let $x' = \psi(a)$ and $x = \psi(a+t)-t$. Then $x > x' \geqq 0$. Let $y' = y = a$. Then condition (A) reduces to $t \geqq \psi(a+t)-\psi(a)$, and there is a contradiction.

$$(2) \quad \max\{x+t,\ \psi(y+t)\}-\max\{x,\ \psi(y)\} = t \quad \text{for all} \quad x \geqq 0, \quad y \geqq 0, \quad \text{and} \quad t > 0.$$

Proof of (2). From (1) $\psi(t) \leqq t$. Let $x' = y' = 0$. Then condition (A) reduces to $t \leqq \max\{x+t,\ \psi(y+t)\}-\max\{x,\ \psi(y)\}$. Also from (1), $\max\{x+t,\ \psi(y+t)\} \leqq \max\{x+t,\ \psi(y)+t\} = \max\{x,\ \psi(y)\}+t$. Then (2) follows.

It is then clear that condition (2), which can be written as $\max\{x+t,\ \psi(y+t)\} = \max\{x+t,\ \psi(y)+t\}$, is equivalent to condition (A). If condition (2) holds, then

$$(3) \qquad \text{If} \quad \psi(y) > 0, \quad \text{then} \quad \psi(y+t) = \psi(y)+t \quad \text{for all} \quad t > 0.$$

Proof of (3). Choose an x such that $0 \leqq x < \psi(y)$. Then condition (2) becomes $\max\{x+t,\ \psi(y+t)\} = \psi(y)+t$. Since $x+t < \psi(y)+t$, then $\psi(y+t) = \psi(y)+t$.

It is easy to see that condition (3) is equivalent to condition (2). Then the only nonnegative functions ψ with $\psi(0) = 0$ that satisfy condition (i) have the form:

$$\begin{aligned} \psi(y) &= 0 && \text{if} \quad 0 < y \leqq a, \\ &= y-a && \text{if} \quad a \leqq y; \end{aligned}$$

i.e. correspond to functions: $\phi(x) = x+a$, $a \geqq 0$. Condition (ii), RCSI (X, Y) follows from the monotonicity of ϕ.

Corollary. *If $X = \min(U, W)$ and $Y = \min(V, \phi(W))$, where U, V and W are independent exponential random variables and ϕ is a nondecreasing function which is not identically zero or infinity, then the pair (X, Y) has a bivariate* IHR *distribution if and only if*: $\phi(x) = x+a$, $a \geqq 0$.

Proof. If $\lambda_1, \lambda_2, \lambda_{12}$ are the parameters of U, V, W respectively, then:

$$\bar{F}(x, y) = P(X > x,\ Y > y) = \exp[-(\lambda_1 x+\lambda_2 y+\lambda_{12}\max(x,\ \psi(y)))].$$

Thus the requirement that $\bar{F}(\cdot,\cdot)$ satisfy condition (i) leads to the inequality (A) since the terms in λ_1 and λ_2 cancel. The result follows as in the theorem.

Thus for a bivariate IHR distribution with an exponential marginal distribution, the presence of a singular part due to positive probability of a functional relationship between X and Y is restricted by the form that the functional relationship may take. However, it does not seem that this result carries over for bivariate IHR distributions with non-exponential marginals.

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REFERENCES

[1] ARNOLD, B. C. (1967). A note on multivariate distributions with specified marginals. *J. Amer. Statist. Assoc.* **62** 1460–1461.

[2] BARLOW, R. E. and PROSCHAN, F. (1965). *Mathematical Theory of Reliability*. Wiley, New York.

[3] ESARY, J. D. and PROSCHAN, F. (1967). Relationships among some notions of bivariate dependence. Boeing Scientific Research Laboratories Document D1-82-0597.

[4] LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

[5] MARSHALL, A. W. and OLKIN, I. (1967). A multivariate expònential distribution. *J. Amer. Statist. Assoc.* **62** 30–44.