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A multivariate exponentially weighted moving average control chart for monitoring process variability

ARTHUR B. YEH[1], **DENNIS K. J. LIN**[2], **HONGHONG ZHOU**[3] & **CHANDRAMOULISWARAN VENKATARAMANI**[4], [1]*Bowling Green State University, Ohio, USA,* [2]*Penn State University, Pennsylvania, USA,* [3]*University of Michigan, Michigan, USA and* [4]*University of Pennsylvania, Pennsylvania, USA*

ABSTRACT *This paper introduces a new multivariate exponentially weighted moving average (EWMA) control chart. The proposed control chart, called an EWMA V-chart, is designed to detect small changes in the variability of correlated multivariate quality characteristics. Through examples and simulations, it is demonstrated that the EWMA V-chart is superior to the* |S|*-chart in detecting small changes in process variability. Furthermore, a counterpart of the EWMA V-chart for monitoring process mean, called the EWMA M-chart is proposed. In detecting small changes in process variability, the combination of EWMA M-chart and EWMA V-chart is a better alternative to the combination of MEWMA control chart (Lowry* et al.*, 1992) and* |S|*-chart. Furthermore, the EWMA M-chart and V-chart can be plotted in one single figure. As for monitoring both process mean and process variability, the combined MEWMA and EWMA V-charts provide the best control procedure.*

1 Introduction

In many industrial applications, the quality of a product typically depends on several correlated quality characteristics. For example, in a fabric-production process, the quality of fibres produced depends on correlated variables, such as the weight of textile fibres and a measure of breaking strength—called the single strand break

Correspondence: A. B. Yeh, Department of Applied Statistics and Operations Research, Bowling Green State University, Bowling Green, OH 43403, USA.

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factor—among other characteristics. Monitoring these two quality characteristics independently by two univariate control charts can be misleading. The process could be out-of-control when both variables are monitored simultaneously using a multivariate control chart even though neither univariate control chart shows any out-of-control signal. Since the work by Hotelling (1947), various multivariate control chart techniques have been proposed to deal with the issue of simultaneous monitoring of correlated variables. Extensive reviews of the literature on this topic can be found in Alt (1984), Jackson (1985), Wierda (1994) and Lowry & Montgomery (1995). However, it is only in the last two decades that we have seen more industry-wide applications of multivariate control charts, due mainly to the advancement in data-acquisition and computer technologies. The industrial usage of multivariate control charts will continue to grow as new technologies become more powerful and accessible.

Given a multivariate process of interest, the process mean is typically monitored by the Hotelling-T^2 chart, and the process variability is usually monitored by an $|S|$-chart based on the determinant of the sample variance–covariance matrix. Both the T^2-chart and $|S|$-chart, however, are sensitive only to moderate to large changes in population parameters, as in the univariate processes. Various types of multivariate control charts have been proposed to improve upon the T^2-chart, especially in detecting small changes in process mean. Examples include several multivariate cumulative sum (MCUSUM) control charts, which were studied in Woodall & Ncube (1985), Healy (1987), Crosier (1988), Pignatiello and Runger (1990) and Hawkins (1991), and the multivariate exponentially weighted moving average (MEWMA) control chart proposed by Lowry *et al.* (1992). The MEWMA control chart, in particular, is shown to provide flexibility in designing the control chart and performs as well as various MCUSUM control charts in detecting small changes in the process mean.

In this paper, we propose a new multivariate EWMA control chart specifically designed to detect small changes in process variability. This new control chart, called the EWMA V-chart, along with its counterpart for monitoring process mean, called the EWMA M-chart, will be discussed in detail in Section 2. As will be seen, if the process encounters a small change in process variability, the proposed EWMA V-chart is more sensitive, in the sense of having a smaller average run length (ARL), than the traditional $|S|$-chart in detecting such a change. Furthermore, under the same scenario, the MEWMA control chart tends to have a smaller ARL than the $|S|$-chart. Therefore, when a small change in process variability takes place, the combined MEWMA- and $|S|$-charts provide a mechanism that could potentially lead to misleading diagnostics, while the combined EWMA M- and V-charts provide better detection. In fact, the combined MEWMA and EWMA V-charts provide the best control charting mechanism for monitoring changes both in process mean and process variability.

The rest of the paper is organized as follows. Having discussed the EWMA V-chart in Section 2, we present two examples in Section 3 in which the proposed EWMA M- and V-charts are used to monitor the process. Some technical derivations are relegated to Appendix A. The combined MEWMA- and $|S|$-charts are also applied to the examples and the results are compared with the ones obtained using our proposed control charts. Section 4 is devoted to simulation studies and performance comparison between EWMA M-chart, EWMA V-chart, MEWMA chart and the $|S|$-chart. In Section 5, we conclude by addressing some relevant issues and discussing some open problems.

2 The multivariate EWMA V-chart

Let $X = (X_1, X_2, \ldots, X_p)'$ be a p-dimensional random variable that represents p correlated quality characteristics obtained from a process of interest. We assume that when the process is in control the distribution of X is $N_p(\mu_0, \Sigma_0)$, a p-dimensional normal distribution with mean vector μ_0 and variance–covariance matrix Σ_0, where μ_0 and Σ_0 are unknown. We also assume that μ_0 and Σ_0 can be estimated from a set of k training samples each with size n, and the process was in control when these k training samples were taken. From the training samples, we compute $\bar{\bar{X}} = \Sigma_{i=1}^{k} \bar{X}_i / k$ and $\bar{S} = \Sigma_{i=1}^{k} S_i / k$, where $\bar{X}_i = \Sigma_{h=1}^{n} X_{ih}/n$ and $S_i = \Sigma_{h=1}^{n}(X_{ih} - \bar{X}_i)(X_{ih} - \bar{X}_i)'/n$ are the sample mean vector and sample variance–covariance matrix of the ith training sample, $i = 1, 2, \ldots k$, respectively.

To monitor the quality of the process, we repeatedly take independent samples of size $n, X_{t1}, X_{t2}, \ldots, X_{tn}, t \geqslant 1$. Let $\bar{X}_t$ and $S_t, t \geqslant 1$, be the sample mean vector and sample variance–covariance matrix when the monitoring begins. Define for $t \geqslant 1$,

$$v_t = P\left(\prod_{i=1}^{p} F_{n-i, N-k+1-i} \leqslant \left(\prod_{i=1}^{p} \frac{N-k+1-i}{n-i} \right) \times \frac{|nS_t|}{|N\bar{S}|} \right), \tag{1}$$

where $|nS_t|$ (similarly $|N\bar{S}|$) denotes the determinant of the matrix nS_t and $N = n \times k$. Here, $F_{n-i,N-k+1-i}$ denotes an F distribution with $n-i$ and $N-k+1-i$ degrees of freedom. Note that v_t is the probability that the random variable $\Pi_{i=1}^{p} F_{n-i,N-k+1-i}$ is less than or equal to the observed statistic

$$\left\{ \prod_{i=1}^{p} \frac{N-k+1-i}{n-i} \right\} \times \left(\frac{|nS_t|}{|N\bar{S}|} \right).$$

It is shown in Appendix A that when the process is in control, v_t is distributed as $U(0,1)$, a uniform distribution supported on $(0,1)$. The exact distribution of the test statistic used in equation (1) is derived in Appendix A for the bivariate case $(p = 2)$. As for $p \geqslant 3$, an approximating distribution is also derived. This leads us to define an EWMA control chart based on the v_ts. Specifically for $t \geqslant 1$, let

$$S_v(t) = w \times (v_t - 0.5) + (1 - w) \times S_v(t-1), \tag{2}$$

where $0 < w < 1$ and $S_v(0) = 0$. Note that $(v_t - 0.5)$ is distributed as $U(-0.5, 0.5)$, therefore, $S_v(t)$ is just an EWMA of a series of independently and identically distributed (i.i.d.) $U(-0.5, 0.5)$s. It is easy to see that, for $t \geqslant 1$ and a given w,

$$\mathrm{E}(S_v(t)) = 0 \qquad \text{and} \qquad \mathrm{Var}(S_v(t)) = \frac{1}{12}\left(\frac{w}{2-w}\right)(1 - (1-w)^{2t}).$$

Also note that since $S_v(t)$ is symmetric at 0, the two control limits and the centre line can be chosen as the following:

$$UCL = L \times \sqrt{\frac{1}{12}\left(\frac{w}{2-w}\right)(1-(1-w)^{2t})} \tag{3}$$

$$CL = 0 \tag{4}$$

TABLE 1. Different L values for EWMA V-chart with in-control ARL = 200 and 400

w	ARL = 200	ARL = 400
0.2	2.49 (200.16)	2.69 (399.96)
0.4	2.37 (197.98)	2.49 (399.95)
0.6	2.16 (200.24)	2.23 (401.45)
0.8	1.93 (200.59)	1.97 (399.21)

$$LCL = -L \times \sqrt{\frac{1}{12}\left(\frac{w}{2-w}\right)(1-(1-w)^{2t})}, \tag{5}$$

where $t = 1, 2, \ldots$ and L is chosen to control the in-control ARL.

We shall call the $S_v(t)$-based control chart the EWMA V-chart. It is designed specifically to detect small changes in process variability. Note that when the subgroup size varies, we can easily modify the test statistic used in equation (1) to compute the corresponding v_t, and consequently $S_v(t)$ can be computed accordingly. Listed in Table 1 are the values of L for two different in-control ARL values 200 and 400, and four different w values 0.2, 0.4, 0.6 and 0.8. The L values were obtained based on Monte Carlo simulation, assuming that the in-control process has a bivariate standard normal distribution, i.e. $\mu_0 = 0$ and $\Sigma_0 = I$, where I is a 2×2 identity matrix. The numbers that appear in the parentheses represent the simulated in-control ARLs based on 20 000 simulations. The simulations were also carried out under different sample sizes $n = 4$, 6, 8 and 10. Sample size has little effect on L, at least under the different ns chosen for our simulations.

The idea of using $S_v(t)$ to monitor small changes in process variability can be extended to the case of monitoring small changes in process mean. Define for $t \geqslant 1$,

$$m_t = P\left(F_{p,N-k-p+1} \leqslant \frac{(N-k-p+1)}{p(k+1)}(\bar{X}_t - \bar{\bar{X}})'\bar{S}^{-1}(\bar{X}_t - \bar{\bar{X}})\right). \tag{6}$$

If the process is in control, then m_t is distributed as $U(0, 1)$ (see, for example, Anderson, 1984). Therefore, a counterpart of the EWMA V-chart can be defined as, for $t \geqslant 1$ and a given w,

$$S_m(t) = w \times (m_t - 0.5) + (1-w) \times S_m(t-1), \tag{7}$$

where $0 < w < 1$ and $S_m(0) = 0$. We shall call the $S_m(t)$-based control chart the EWMA M-chart, as it is specifically designed to detect small changes in process mean.

Note that the other existing EWMA-type procedure for monitoring small changes in a multivariate process mean is the MEWMA control chart proposed by Lowry *et al* (1992). For a given sample size n and a given w, the MEWMA calculates EWMAs of sample mean vectors as in, for $t \geqslant 1$,

$$Z_t = w(\bar{X}_t - \mu_0) + (1-w)Z_{t-1}$$

where $Z_0 = 0$. The MEWMA control chart gives an out-of-control signal as soon as

$$T_t = Z_t' \sum_t^{-1} Z_t > H$$

where $\Sigma_t = \{w[1-(1-w)^{2t}]/(2-w)\}(\Sigma_0/n)$, and H is chosen to achieve a specified in-control ARL. For a more detailed account of the MEWMA control chart, see Lowry *et al.* (1992) and Prabhu & Runger (1997).

It should be noted that if the process is in control, then $S_m(t)$ has the same distribution as $S_v(t)$. Therefore, the control limits (3) and (5) chosen for the EWMA V-chart can also be applied to the EWMA M-chart, and the resulting in-control ARLs for both control charts are the same. Furthermore, it is shown in Yeh & Lin (2002) that, given $\bar{\bar{X}}$ and $\bar{S}$, $S_m(t)$ and $S_v(t)$ are also independent. Thus, the combined EWMA M- and V-charts for a chosen L have an in-control ARL equal to half the in-control ARL for the individual control chart. In this regard, the combined EWMA M- and V-charts provide a unified approach to the monitoring of multivariate processes, especially when small changes in process parameters are of interest.

Remark 1

In our settings, we assume that μ_0 and Σ_0 are unknown and can be estimated based on k training samples each of size n, taken when the process was in control. For the EWMA M-chart, to ensure that $\bar{S} > 0$ (positively definite), it is assumed that $k(n-1) > p$. If μ_0 and Σ_0 are known, then χ^2_p instead of $F_{p,n\times k-k-p+1}$ is used to obtain m_t in equation (6). In an earlier study, Lowry & Montgomery (1995) recommended that for $2 \leqslant p \leqslant 5$ and $n \leqslant 10$, $n \times k$ should be in the range between 200 and 250, and for larger p values such as 10 and 20, and $n \leqslant 10$, $n \times k$ should be in the range between 500 and 600. Thus, in the bivariate case for instance, a collection of 50 in-control training samples, each of size 5, is recommended to start the EWMA M-chart. Note that the control limits of the EWMA M-chart will not be affected by different choices of n and k since the exact distribution is used in equation (6) to obtain the probability integral transformation m_t.

The EWMA V-chart requires $n > p$ to ensure that $S_i > 0$, $i = 1, 2, \ldots, k$, and $S_t > 0$, $t \geqslant 1$. For $p = 2$, if Σ_0 is known, then χ^2_{2n-4} instead of $F_{2n-4,2(n\times k-k-1)}$ will be used to obtain v_t in equation (1) (see Appendix A). This case is very similar to that of the EWMA M-chart. Therefore, we recommended that $n \times k$ be in the range between 200 and 250 for $n \leqslant 10$. For instance, in the bivariate case, 50 training samples and 20 to 25 training samples are recommended to start the EWMA V-chart for sample sizes equal to 5 and 10, respectively. Similarly, since the exact distribution is used in obtaining v_t in equation (1), the control limits of the EWMA V-chart will not be affected.

As for $p \geqslant 3$, we derive a normal approximation to the distribution used in equation (1) to obtain v_t (see Appendix A). The approximation is essentially based on the asymptotic distribution of a sum of p logarithmic chi-square distributions with various degrees of freedom. In general, the normal approximation is reasonably good for $n \geqslant 10$ and becomes better for larger values of $n \times k$ and p (see, for example, Bartlett & Kendall, 1946, and Gnanadesikan & Gupta, 1970). In the context of establishing the EWMA V-chart, we recommend that at least 50 training samples be used. Each sample should contain at least ten observations for $3 \leqslant p \leqslant 8$, and larger sample sizes are needed for larger ps. Note that the exact distribution in equation (1) when $p \geqslant 3$ tends to have heavier tails than the normal distribution, which makes $S_v(t)$ more likely to fall in the rejection regions.

Remark 2

Unlike the univariate case where the variance is a scalar, the p-dimensional variance–covariance matrix Σ_0 has $p(p+1)/2$ potentially different parameters, including p diagonal variances and $p(p-1)/2$ off-diagonal covariances. Therefore, it is sometimes desirable to summarize the variation expressed by Σ_0 using a single numerical measure. One such widely used measure is the generalized variance $|\Sigma_0|$, defined as the determinant of the variance–covariance matrix. In addition, the sample variance–covariance matrix $|S|$, used to estimate $|\Sigma_0|$, has very interesting geometrical interpretations. For any given sample $X_1, X_2, \ldots, X_n$ in R^p, consider the p-dimensional parallelotope (Anderson, 1984) whose p principal edges correspond to the p rows of $(X_1 - \bar{X}_n, X_2 - \bar{X}_n, \ldots, X_n - \bar{X}_n)$. The $|S|$ is proportional to the square of the volume of such a parallelotope. Furthermore, for any given constant c, $|S|$ is also proportional to the square of the volume of the ellipsoid generated by $\{X \in R^p : (X - \bar{X}_n)' S^{-1} (X - \bar{X}_n) \leqslant c^2\}$, which is the form of the confidence region for the mean vector under the normality assumption.

3 Examples

3.1 A fabric-production example

The first example, taken from Mitra (1993), is related to a fabric-production process. The two quality characteristics of interest are the single-strand and break factor and the weight of textile fibres. Originally, 20 samples each of size 4 were obtained from the process. The combined T^2- and $|S|$-charts are shown in Fig. 1, and no out-of-control signal is detected. From these 20 samples, we obtained

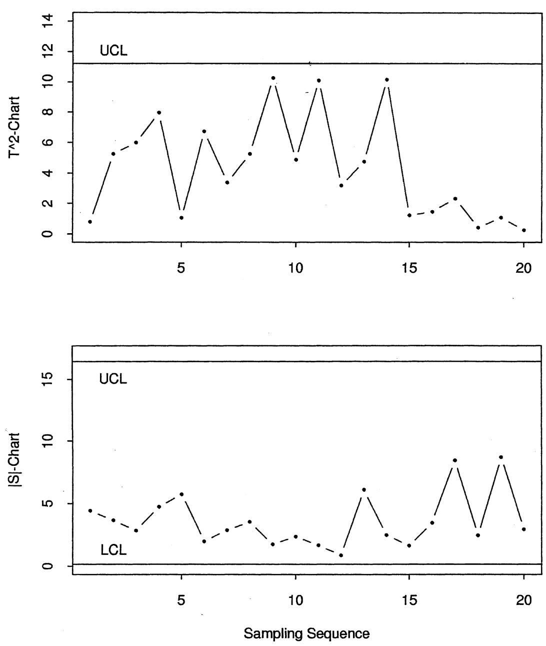

FIG. 1. The T^2- and $|S|$-charts for the fabric-production example.

$$\bar{\bar{X}} = \begin{pmatrix} 82.4625 \\ 20.1750 \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} 7.5215 & -0.3542 \\ -0.3542 & 3.2917 \end{pmatrix}$$

Assuming that the in-control process has a bivariate normal distribution with $\mu_0 = \bar{\bar{X}}$ and $\Sigma_0 = \bar{S}$, we generated 80 new samples each of size 4 from the distribution. The first ten samples were generated from the in-control process, and starting from sample 11, we applied a 50% increase to the standard deviation of the first variable, the single-strand break factor, while keeping the mean unchanged. The variance–covariance matrix of the new process starting from sample 11 is equal to

$$\Sigma_{\text{new}} = \begin{pmatrix} 16.9234 & -0.5315 \\ -0.5315 & 3.2917 \end{pmatrix}$$

and in this case $|\Sigma_{\text{new}}| = 2.25 \times |\Sigma_0|$.

The combined MEWMA- and $|S|$-charts are shown in Fig. 2. No out-of-control signal is detected on the $|S|$-chart, while an out-of-control signal is detected at sample 44 on the MEWMA chart. Therefore, one is more likely to conclude that the process mean, but not the process variability, is out of control. The combined EWMA M- and V-charts are shown in Fig. 3. An out-of-control signal is detected at sample 30 on the EWMA V-chart, while an out-of-control signal also shows up at sample 32 on the EWMA M-chart. Clearly, from Fig. 3, the process variability is out of control. However, care should be exercised in interpreting the out-of-control signal on the EWMA M-chart since the process variability is already out of control. Note that in Fig. 3, we actually plotted

$$S_v(t) \times \sqrt{\frac{12(2-w)}{w(1-(1-w)^{2t})}}$$

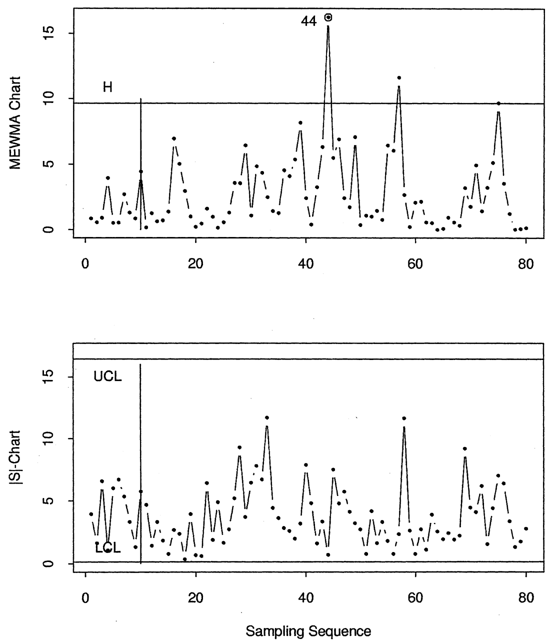

FIG. 2. The MEWMA- and $|S|$-charts with a change in variability starting at sample 11.

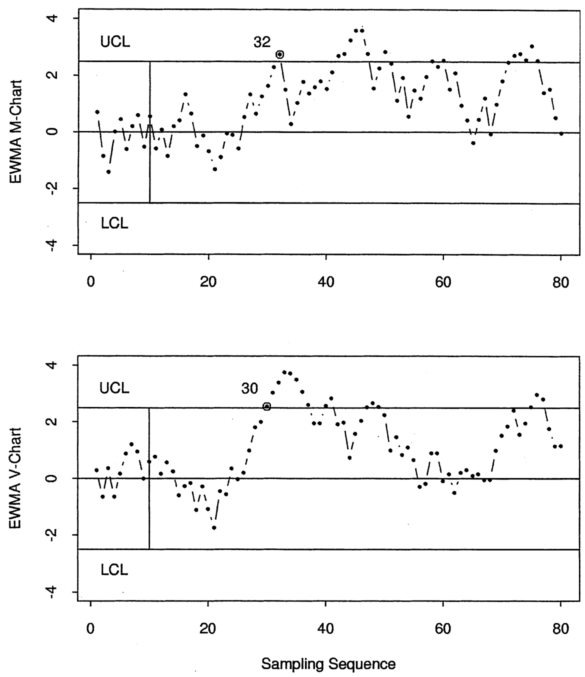

FIG. 3. The EWMA M- and V-charts with a change in variability starting at sample 11.

on the y-axis so that the UCL and LOL are 2.49 and -2.49 (see Table 1), respectively. Also note that $w = 0.2$ is used in both combined control charts.

Remark 3

Unlike in the univariate case, it is more difficult to interpret the EWMA V-chart or the $|S|$-chart when an out-of-control signal is detected. If a sample point is plotted outside of the control limits, it is primarily due to a change in the determinant of the sample variance–covariance matrix (increase or decrease in the determinant). However, this does not necessarily imply that there is an increase or a decrease in process variability. It is worth mentioning that Johnson & Wichern (1998) gave three sample variance–covariance matrices for bivariate data that all have the same determinant and yet have very different correlations.

Remark 4

In order better to understand what specific changes in process variability have taken place when an out-of-control signal shows up on the EWMA V-chart, one might consider performing (after an out-of-control signal is detected) a series of hierarchical likelihood ratio-based testing procedures proposed by Manly & Rayner (1987) (also see section 8.3 of Wierda, 1994). These testing procedures are designed to test, in a series of steps, whether (1) the out-of-control and in-control variance–covariance matrices differ in correlations; (2) the out-of-control and in-control variance–covarlance matrices differ only in variances; and (3) the out-of-control variance–covariance matrix is proportional to the in-control variance–covariance matrix.

The testing proceeds as follows. If the result of test (1) is significant, it is concluded that the correlations of the two variance–covariance matrices differ. If the result of test (1) is not significant, proceed to test (2). If the result is significant, it is concluded that the two variance–covariance matrices differ only in variances, whereas the correlations are equal. If the result of test (2) is not significant, perform test (3). If the result is significant, it is concluded that the two variance–covariance matrices are proportional. If the result is not significant, it is then concluded that the two matrices are equal.

3.2 A transmission assembly example

The second example, also taken from Mitra (1993), concerns the quality of a component used in the assembly of a transmission mechanism. The two quality characteristics of interest are tensile strength and diameter of the component. Twenty samples each of size 4, were taken from the original process, and the combined T^2- and $|S|$-charts are shown in Fig. 4. No out-of-control signal is detected either on the T^2-chart or on the $|S|$-chart. We obtained, from these 20 samples,

$$\bar{\bar{X}} = \begin{pmatrix} 71.2625 \\ 19.3000 \end{pmatrix}, \quad \bar{S} = \begin{pmatrix} 18.9042 & -1.4792 \\ -1.4792 & 2.6750 \end{pmatrix}$$

Assuming that the in-control process has a bivariate normal distribution with $\mu_0 = \bar{\bar{X}}$ and $\Sigma_0 = \bar{S}$ we generated another 80 samples each of size 4. We did not change the parameters when the first 10 samples were generated. Starting from

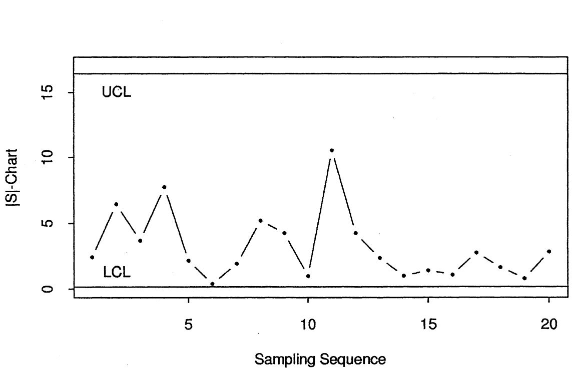

FIG. 4. The T^2- and $|S|$-charts for the transmission component example.

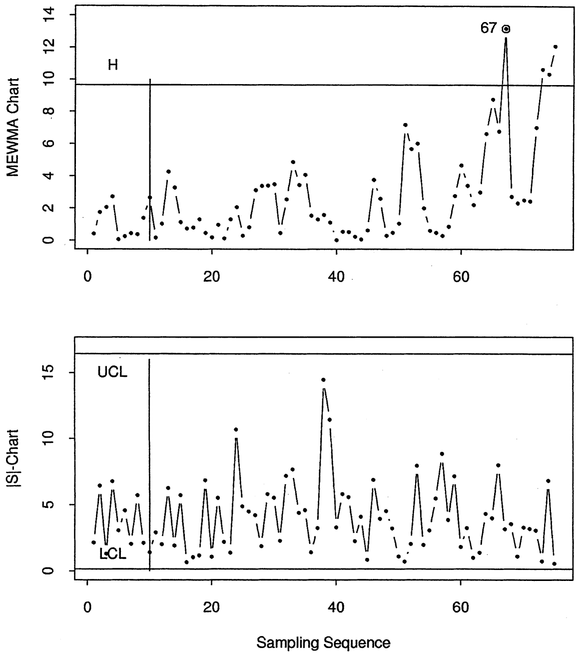

FIG. 5. The MEWMA- and $|S|$-charts with a change in variability starting at sample 11.

sample 11, we increased the variance of the first variable, the tensile strength, by approximately 25%, again while keeping the process mean unchanged. The resulting variance–covariance for sample 11 and onward is

$$\Sigma_{\text{new}} = \begin{pmatrix} 23.5378 & -1.6506 \\ -1.6506 & 2.6750 \end{pmatrix}$$

whose determinant is approximately $1.2451 \times |\Sigma_0|$. Shown in Fig. 5 and 6, respectively, are the combined MEWMA- and $|S|$-charts and the combined EWMA M- and V-charts for the 80 samples generated.

As shown in Fig. 5, an out-of-control signal is detected at sample 67 on the MEWMA chart, while no out-of-control signal appears on the $|S|$-chart. One is more likely to be misled by the signal on the combined MEWMA- and $|S|$-charts. On the other hand, as seen in Fig. 6, an out-of-control signal is detected at sample 42 on the EWMA V-chart, which should call for immediate action in determining the causes of the change in process variability.

4 Numerical studies

4.1 The performance of the proposed control charts

In this section, we present the results of simulated ARLs for bivariate processes. The in-control process is assumed to have a standard bivariate normal distribution, and the in-control ARL is assumed to be 200. All results were obtained based on 5000 simulations. When a sample was generated from a given out-of-control distribution, the sample was used to evaluate all the competing control charts

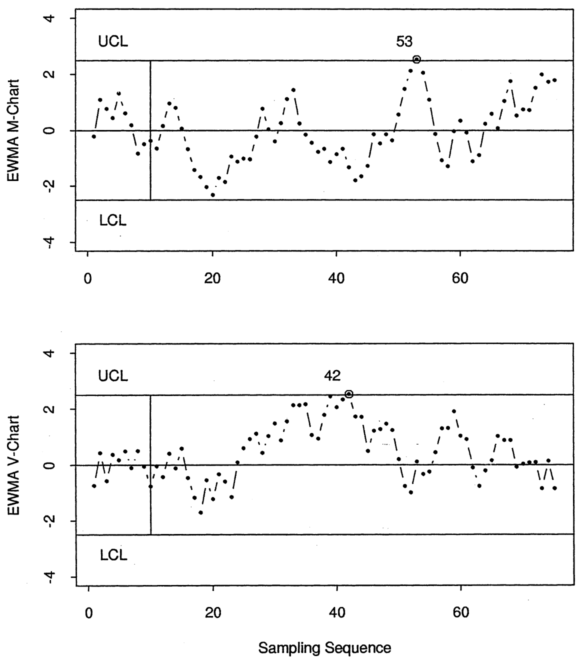

FIG. 6. The EWMA M- and V-charts with a change in variability starting at sample 11.

considered in our simulation studies. The standard errors of the simulations were all between 1% and 1.5% of the values of the simulated ARLs.

Assuming that $\Sigma_0 = I$, for given n and w, we simulated ARLs for different mean vectors $\mu = (\mu_1, \mu_2)$. These include 0.25σ to 1.25σ shifts in both variables, and similar shifts in just the first variable. The simulations involved two sample sizes $n = 4$ and 8, and three different smoothing constants $w = 0.2$, 0.4 and 0.6. The results are summarized in Table 2. Take, for example, the case when $n = 4$ and $w = 0.2$, the MEWMA control chart performs better than the EWMA M-chart for all the cases considered. In general, both the EWMA M-chart and the MEWMA control chart have better performance, as indicated by smaller ARLs, for larger sample sizes and smaller w values.

Next, assuming that $\mu_0 = (0, 0)$, for given n and w, we simulated ARLs for various combinations of standard deviations and correlation $(\sigma_1, \sigma_2, \rho)$. The changes considered here are a 10%, 25% and 50% increase in standard deviation of both variables or just the first variable, and four different correlations $\rho = 0$, -0.2, 0.5 and 0.8. We have chosen these out-of-control cases to represent a variety of parameter changes in the process variability of a bivariate process, including small to moderate increases in the standard deviation of one variable or both variables, and low to high correlations. These changes also represent a variety of different determinant values ranging from small to large determinant values. The simulations involved different n and w values similar to those considered in mean shift cases. The results are summarized in Table 3 for $n = 4$ and $w = 0.2$, and in Table 4 for $n = 8$ and $w = 0.2$. Additional tables for $(n, w) = (4, 0.4)$, $(4, 0.6)$, $(8, 0.4)$ and $(8, 0.6)$ are included in Tables B1–B4 in Appendix B.

Note that in the case when Σ_0 is known, the performance of the EWMA V-chart depends on $|\Sigma|/|\Sigma_0|$, the ratio of the determinant of the out-of-control variance–

TABLE 2. Comparisons of ARL of EWMA M-chart and MEWMA chart for mean shifts ($p = 2$)

(μ_1, μ_2)	EWMA M-chart		MEWMA chart	
($w = 0.2$)	($n = 4$)	($n = 8$)	($n = 4$)	($n = 8$)
(0.25,0.25)	80.43	33.13	16.81	9.05
(0.5,0.5)	12.82	5.72	5.04	2.84
(0.75,0.75)	5.10	3.33	2.59	1.59
(1.0,1.0)	3.49	3.03	1.69	1.15
(1.25,1.25)	3.09	3.01	1.31	1.02
(1.25,1.25)	107.55	79.54	26.99	17.38
(0.25,0)	33.70	12.59	9.08	4.87
(0.75,0)	10.79	5.11	4.39	2.57
(1.0,0)	5.72	3.50	2.83	1.72
(1.25,0)	4.01	3.09	2.03	1.30
($w = 0.4$)				
(0.25,0.25)	88.36	39.30	22.75	10.93
(0.5,0.5)	13.93	5.79	5.97	2.90
(0.75,0.75)	4.73	3.16	2.52	1.60
(1.0,1.0)	3.37	2.43	1.82	1.16
(1.25,1.25)	2.65	2.07	1.34	1.02
(0.25,0)	136.33	87.63	41.58	22.36
(0.5,0)	38.91	14.48	12.70	5.57
(0.75,0)	11.94	5.24	5.27	2.63
(1.0,0)	5.81	3.38	3.11	1.73
(1.25,0)	3.93	2.69	2.06	1.30
($w = 0.6$)				
(0.25,0.25)	93.66	44.23	29.50	14.93
(0.5,0.5)	16.08	5.95	6.82	3.21
(0.75,0.75)	5.16	2.72	2.96	1.62
(1.0,1.0)	2.97	2.11	1.79	1.16
(1.25,1.25)	2.28	2.04	1.32	1.02
(0.25,0)	149.39	95.68	52.48	30.70
(0.5,0)	44.76	16.08	14.53	6.83
(0.75,0)	13.44	5.05	5.94	2.85
(1.0,0)	5.96	2.98	3.29	1.78
(1.25,0)	3.64	2.29	2.18	1.31

covariance matrix Σ and the determinant of the in-control variance–covariance matrix Σ_0. These tables show that the EWMA V-chart significantly outperforms the $|S|$-chart in almost all cases, except when $|\Sigma|/|\Sigma_0|$ is large, in which case the EWMA V-chart and the $|S|$-chart have very similar ARLs. Plotted in Fig. 7 are the ARLs versus $|\Sigma|/|\Sigma_0|$ for both the EWMA V-chart and the $|S|$-chart under $n = 4$ and 8. We also plotted in Fig. 8 the ARLs of the EWMA V-chart under $w = 0.2, 0.4$ and 0.6 when $n = 4$. In general, for a given n, the EWMA V-chart performs better when smaller values of w are used, and for a given w, larger sample sizes lead to better performance of the EWMA V-chart. Furthermore, the EWMA V-chart with a smaller w value is more effective in detecting smaller changes in process variability.

The EWMA V-chart also outperforms the MEWMA chart in the majority of cases considered, except when (i) there is a small increase in standard deviation in just the first variable and (ii) the correlation is moderate and either both standard deviations increase by a small percentage or only the standard deviation of the first variable increases. However, in these exceptions, the possible diagnostics from the

TABLE 3. Comparisons of ARL for changes in variability ($n = 4$, $w = 0.2$, $p = 2$)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
(1.10,1.00,0)	128.42	114.06	112.61	123.60	1.210
(1.10,1.10,0)	93.40	64.44	88.93	94.05	1.464
(1.25,1.00,0)	72.96	51.73	68.18	80.96	1.563
(1.50,1.00,0)	31.80	20.36	32.09	35.09	2.250
(1.25,1.25,0)	31.81	17.38	39.22	29.24	2.441
(1.50,1.00,0)	12.60	7.03	15.04	8.32	5.063
(1.10,1.00,0.5)	125.67	141.25	83.94	146.53	0.908
(1.10,1.10,0.5)	107.29	133.87	65.24	133.60	1.098
(1.25,1.00,0.5)	87.96	126.77	53.88	129.65	1.172
(1.50,1.00,0.5)	39.33	41.73	28.92	68.85	1.688
(1.25,1.25,0.5)	40.16	33.01	33.39	57.74	1.831
(1.50,1.50, −0.2)	14.57	9.31	14.69	12.47	3.797
(1.10,1.00, −0.2)	129.36	126.33	107.15	128.95	1.162
(1.10,1.10, −0.2)	96.72	73.58	82.99	98.89	1.406
(1.25,1.00, −0.2)	77.57	58.06	66.22	88.19	1.500
(1.50,1.00, −0.2)	53.29	22.03	30.89	38.60	2.160
(1.25,1.25, −0.2)	33.18	19.23	37.30	32.17	2.344
(1.50,1.50, −0.2)	12.70	7.22	15.06	8.74	4.860
(1.10,1.00,0.8)	92.32	24.95	60.77	139.17	0.436
(1.10,1.10,0.8)	96.37	39.38	47.79	141.92	0.527
(1.25,1.00,0.8)	90.48	46.63	41.11	145.85	0.563
(1.50,1.00,0.8)	52.14	125.12	24.29	148.43	0.810
(1.25,1.25,0.8)	56.55	138.84	27.16	148.54	0.879
(1.50,1.50,0.8)	21.04	32.57	13.43	57.91	1.823

MEWMA control chart could be misleading. Furthermore, if we examine the combined MEWMA- and $|S|$-charts, the MEWMA control chart tends to have smaller ARLs than the $|S|$-chart, except when there are moderate to large increases in standard deviations of both variables. On the other hand, when there is a change in process variability, the combined EWMA M- and V-charts provide a monitoring mechanism for correctly identifying the actual changes taking place in the process, except when $|\Sigma|/|\Sigma_0|$ is close to 1. In fact, combining the MEWMA control chart and the EWMA V-chart gives the best control charting procedure for monitoring both process mean and process variability. From the results summarized in Tables 3 and 4, note also that both the MEWMA chart and the EWMA M-chart are sensitive to changes in process variability. Therefore, in practical applications, caution needs to be taken when interpreting the out-of-control signals shown on these two control charts. When these two control charts are combined with control charts for monitoring process variability (such as the EWMA V-chart), if out-of-control signals show up on both charts, efforts should first be directed toward identifying assignable causes for a possibly out-of-control process variability.

Remark 5

We have also carried out ARL simulations for detecting variability changes for the cases when $p = 3$ and 4, $n = 10$, and $w = 0.2$, 0.4 and 0.6. The in-control process was assumed to have a standard multivariate normal distribution. The main results comparing the ARL performance of the EWMA V-chart with that of the $|S|$-chart

TABLE 4. Comparisons of ARL for changes in variability ($n = 8$, $w = 0.2$, $p = 2$)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\lvert S\rvert$-chart	$\lvert\Sigma\rvert/\lvert\Sigma_0\rvert$
(1.10,1.00,0)	125.54	77.29	113.66	114.14	1.210
(1.10,1.10,0)	93.19	28.13	86.49	64.20	1.464
(1.25,1.00,0)	76.30	21.50	70.53	50.68	1.563
(1.50,1.00,0)	32.01	8.24	32.07	15.76	2.250
(1.25,1.25,0)	31.25	7.29	39.83	12.85	2.441
(1.50,1.50,0)	12.42	3.74	14.84	2.99	5.063
(1.10,1.00,0.5)	125.57	131.28	83.24	142.61	0.908
(1.10,1.10,0.5)	106.07	123.70	65.78	132.41	1.098
(1.25,1.00,0.5)	89.37	91.11	52.59	122.66	1.172
(1.50,1.00,0.5)	39.82	16.54	28.80	40.21	1.688
(1.25,1.25,0.5)	40.99	12.90	32.83	29.52	1.831
(1.50,1.50,0.5)	14.75	4.41	14.71	4.68	3.797
(1.10,1.00, −0.2)	125.55	98.78	107.81	124.76	1.162
(1.10,1.10, −0.2)	95.69	32.90	82.65	71.20	1.406
(1.25,1.00, −0.2)	78.03	25.27	66.49	57.94	1.500
(1.50,1.00, −0.2)	32.83	9.01	31.19	17.57	2.160
(1.25,1.25, −0.2)	32.99	7.73	38.74	14.25	2.344
(1.50,1.50, −0.2)	12.81	3.81	14.87	3.15	4.860
(1.10,1.00,0.8)	93.11	8.57	61.09	58.71	0.436
(1.10,1.10,0.8)	92.61	13.21	48.42	88.59	0.527
(1.25,1.00,0.8)	87.80	15.89	41.06	97.51	0.563
(1.50,1.00,0.8)	51.17	79.57	24.06	140.89	0.810
(1.25,1.25,0.8)	55.99	118.50	26.89	145.57	0.879
(1.50,1.50,0.8)	20.73	13.06	13.28	30.13	1.823

are summarized in Table 5 ($p = 3$) and Table 6 ($p = 4$). In Tables 5 and 6, we only listed the parameters that have been changed. For instance, '$\sigma_{1,2} = 1.25$' indicates that the standard deviations of the first and second variables were both increased by 25%, and '$\rho_{12,13} = 0.5$' indicates that the correlations between the first and second, and between the first and third variables were both changed from 0 to 0.5. These different parameter changes were selected to represent a variety of possible changes to the variance–covariance matrix (e.g. changes in the standard deviations, or changes in correlations or both), as well as a variety of determinants of the corresponding variance–covariance matrices.

Tables 5 and 6 show that, similar to the bivariate case, the EWMA V-chart outperforms the $|S|$-chart, except when $|\Sigma|/|\Sigma_0|$ becomes large. Plotted on Fig. 9 ($p = 3$) and Fig. 10 ($p = 4$) are the ARLs versus $|\Sigma|/|\Sigma_0|$ for the EWMA V-chart ($w = 0.2$, 0.4 and 0.6) and the $|S|$-chart. Additional simulation results for the corresponding MEWMA control chart and the EWMA M-chart can be found in Tables B5–B10 in Appendix B.

There are two advantages for making the probability integral transformation to obtain m_t and v_t defined, respectively, in equation (6) and (1); one being that only one set of control limits needs to be developed since $S_m(t)$ and $S_v(t)$ have the same distribution when the process is in control. Another potentially appealing feature of the EWMA M-chart and EWMA V-chart is that since $S_m(t)$ and $S_v(t)$ have the same distribution, they can be combined into a single chart. Shown in Fig. 11 is the single control chart produced by combining the EWMA M-chart and the EWMA V-chart shown in Fig. 6. The $S_m(t)$ is represented by '+' and the $S_v(t)$ is represented

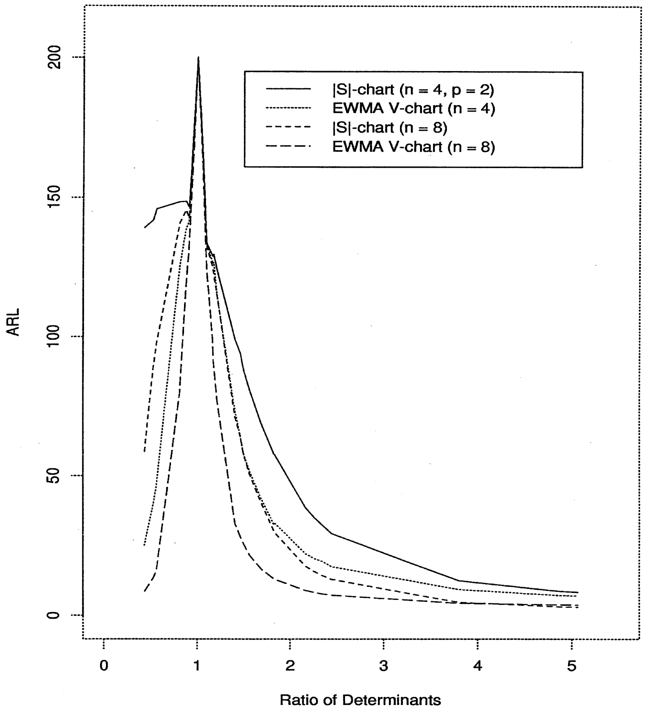

FIG. 7. The ARLs of EWMA V-chart and $|S|$-chart versus $|\Sigma|/|\Sigma_0|$ ($n=4$ and 8, and $p=2$).

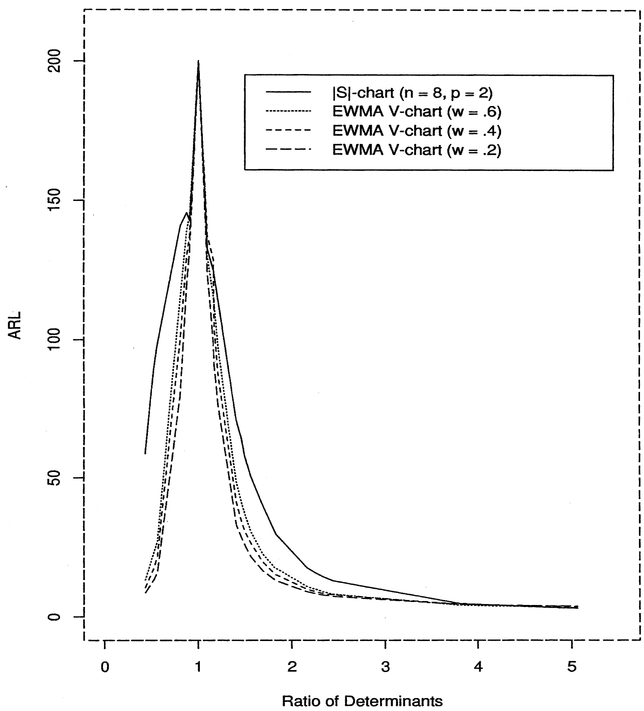

FIG. 8. The ARLs of EWMA V-chart versus $|\Sigma|/|\Sigma_0|$ ($w=0.2$, 0.4 and 0.6, $n=8$ and $p=2$).

TABLE 5. Comparisons of ARL for changes in variability, ($n = 10$, $p = 3$)

Changes	EWMA V-chart			$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
	$w = 0.2$	0.4	0.6		
$\sigma_1 = 1.1$	89.68	106.38	120.21	153.68	1.210
$\sigma_1 = 1.25$	26.07	32.77	39.31	72.28	1.563
$\sigma_1 = 1.5$	9.80	11.02	12.60	24.71	2.250
$\sigma_{1,2} = 1.1$	33.80	43.44	51.99	89.39	1.464
$\sigma_{1,2} = 1.25$	8.34	9.33	10.29	19.51	2.441
$\sigma_{1,2} = 1.5$	4.06	4.04	3.74	4.38	5.063
$\sigma_{1,2,3} = 1.1$	17.45	21.05	24.81	48.57	1.772
$\sigma_{1,2,3} = 1.25$	4.86	4.98	5.00	7.17	3.815
$\rho_{12} = 0.5$	58.32	77.34	93.40	184.01	0.750
$\rho_{12} = -0.2$	192.53	195.27	199.25	220.20	0.960
$\rho_{12} = 0.8$	7.09	8.25	9.73	39.67	0.360
$\rho_{12,13} = 0.5$	13.38	17.21	21.67	80.20	0.500
$\rho_{12,13} = -0.2$	167.29	178.61	186.24	220.35	0.920
$\rho_{12,13,23} = -0.2$	122.95	142.43	160.11	216.61	0.864
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = 0.5$	140.70	158.91	170.16	217.79	0.886
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = -0.2$	28.56	36.45	43.40	76.98	1.531
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = 0.8$	3.84	3.99	3.87	11.30	0.184
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = 0.5$	13.94	16.82	19.56	38.68	1.907
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = -0.2$	5.62	5.97	6.00	9.67	3.296
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = 0.08$	8.32	9.80	11.99	47.70	0.397
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.5$	3.80	3.76	3.34	3.64	5.695
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.8$	101.53	118.62	131.83	161.56	1.185

by '—'. In fact, as shown in Fig. 11, after proper probability integral trausformation, all performance measures (such as T^2 and $|S|$) lie in the range (0, 1). Consequently, their corresponding EWMAs will have the same distribution. Therefore, one can monitor more than two performance measures preferably using different colours on a single control chart, if so desired. This approach could potentially lead to a significant reduction in the number of control charts to be monitored if multiple processes are being investigated. On the other hand, it would not be possible to plot the MEWMA control chart and the EWMA V-chart on a single chart.

4.2 The inertia problem

Lowry *et al.* (1992) discussed the possible inertia problem that may occur and which could delay the MEWMA control chart in signalling an out-of-control signal (also see Lucas & Saccucci, 1990, and Crowder & Hamilton, 1992). The same phenomenon could also happen in both the EWMA M-chart and the EWMA V-chart. Here, we discuss how the inertia problem could be addressed in setting up the EWMA V-chart. Consider an example in which the process variability is in control, but at current time t the test statistic $S_v(t)$ is negative. Suppose that at

TABLE 6. Comparisons of ARL for changes in variability ($n = 10$, $p = 4$)

Changes	EWMA V-chart			$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
	$w = 0.2$	0.4	0.6		
$\sigma_1 = 1.1$	110.05	130.02	136.56	154.96	1.210
$\sigma_1 = 1.25$	35.70	47.75	54.32	88.66	1.563
$\sigma_1 = 1.5$	13.20	15.91	18.15	35.88	2.250
$\sigma_1 = 1.1$	45.99	59.43	68.03	103.38	1.464
$\sigma_{1,2} = 1.25$	11.21	13.32	14.89	29.15	2.441
$\sigma_{1,2} = 1.5$	4.88	5.13	4.93	7.05	5.063
$\sigma_{1,2,3} = 1.1$	23.91	30.95	36.29	64.60	1.772
$\sigma_{1,2,3} = 1.25$	6.22	6.49	6.83	11.55	3.815
$\sigma_{1,2,3,4} = 1.1$	14.49	17.77	20.40	40.37	2.144
$\sigma_{1,2,3,4} = 1.25$	4.39	4.49	4.14	5.61	5.960
$\rho_{12} = 0.5$	74.71	99.05	117.23	183.43	0.750
$\rho_{12} = -0.2$	190.87	199.81	206.02	203.98	0.960
$\rho_{12} = 0.8$	9.32	11.66	14.76	52.31	0.360
$\rho_{12,13} = 0.5$	17.84	25.56	33.05	94.06	0.500
$\rho_{12,13} = -0.2$	175.35	191.58	190.35	204.71	0.920
$\rho_{12,13,14} = 0.5$	5.94	6.84	7.78	27.73	0.250
$\rho_{12,13,14} = -0.2$	154.06	176.50	177.46	202.015	0.880
$\rho_{12,13,14,23} = 0.5$	7.74	9.11	11.24	41.05	0.313
$\rho_{12,13,14,23} = -0.2$	113.63	138.30	154.04	197.70	0.826
$\rho_{12,13,14,23,24} = -0.2$	81.39	108.37	124.10	187.33	0.768
$\sigma_{1,2,3,4} = 1.1$, $\rho_{12,13,14,23,24,34} = 0.5$	45.62	63.77	78.83	156.64	0.670
$\sigma_{1,2,3,4} = 1.1$, $\rho_{12,13,14,23,24,34} = -0.2$	44.61	56.95	64.32	100.81	1.482
$\sigma_{1,2,3,4} = 1.25$, $\rho_{12,13,14,23,24,34} = 0.5$	20.29	26.40	29.97	56.57	1.863
$\sigma_{1,2,3,4} = 1.25$, $\rho_{12,13,14,23,24,34} = -0.2$	5.72	6.14	6.12	10.00	4.120
$\sigma_{1,2,3,4} = 1.25$, $\rho_{12,13,14,23,24,34} = 0.8$	4.35	4.61	4.70	14.11	0.162
$\sigma_{1,2,3,4} = 1.5$, $\rho_{12,13,14,23,24,34} = 0.8$	53.53	75.28	91.66	165.51	0.697

time $t + 1$ the process variability changes in the sense that the determinant of the variance–covariance matrix increases. The fact that $S_v(t)$ is negative right before the change in process variability takes place may cause a delay in the EWMA V-chart in signalling such a change in the process. On the other hand, the inertia problem could also happen during process start-up, when the process variability is out of control at the process start-up and the process parameter was not reset to the target value prior to the process start-up.

The inertia problem can be remedied by setting the initial value $S_v(0)$ to a small constant k. This increases the sensitivity of the EWMA V-chart in reacting to changes in process variability. Specifically, setting $S_v(0)$ to a small positive constant k will increase the sensitivity of the EWMA V-chart in detecting an increase in the determinant of the variance–covariance matrix and, conversely, choosing a small negative k will increase the sensitivity in detecting a decrease in the determinant. The choice of k depends on the magnitude of change of interest in population parameters. In the present paper, we are not able to provide specific guidance on how k should be chosen. This is due to the fact that we are not able to convert the

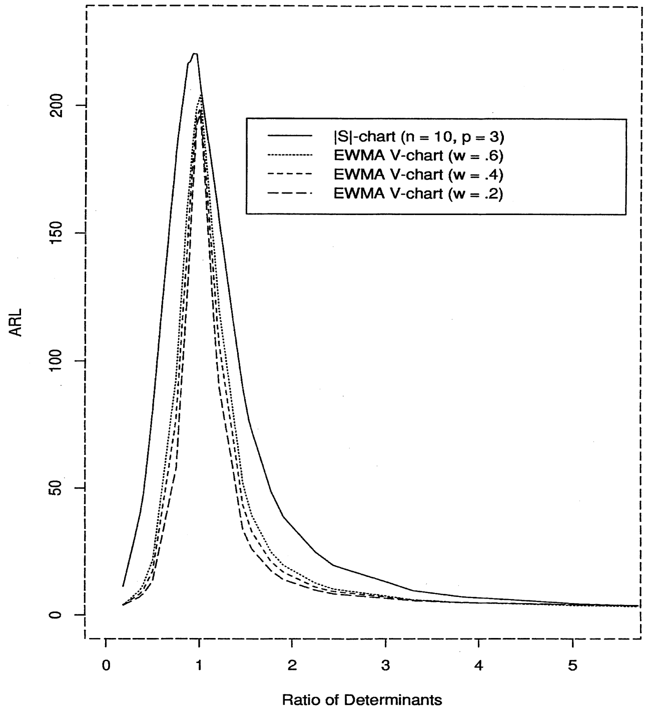

FIG. 9. The ARLs of EWMA V-chart versus $|\Sigma|/|\Sigma_0|$ ($w = 0.2$, 0.4 and 0.6, $n = 10$ and $p = 3$).

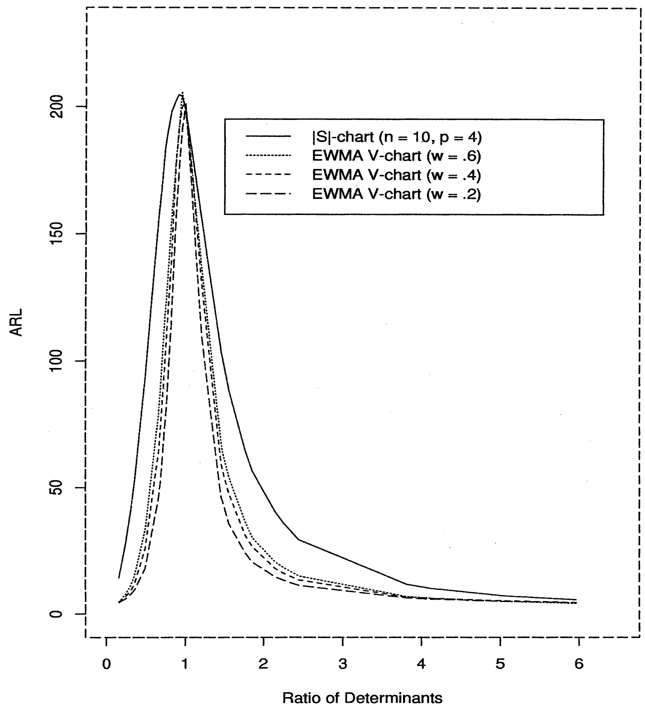

FIG. 10. The ARLs of EWMA V-chart versus $|\Sigma|/|\Sigma_0|$ ($w = 0.2$, 0.4 and 0.6, $n = 10$ and $p = 4$).

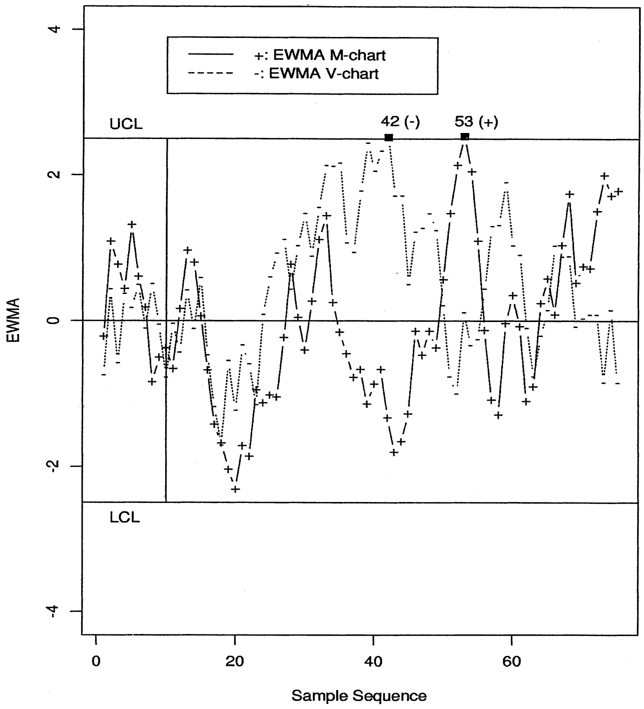

FIG. 11. The combined single control chart of $S_m(t)$ and $S_v(t)$ for the transmission assembly example.

changes of interest in population parameters into the corresponding changes in the transformed uniform distributions. As an alternative, we have run some simulations to look into how different values of k affect the sensitivity of the EWMA V-chart, as judged by the simulated ARL values. The results are summarized in Table 7. The case when $k=0$ refers to the originally proposed EWMA V-chart as defined in equation (2). In Table 7, we used the same combinations of $(\sigma_1, \sigma_2, \rho)$ as in Tables 3 and 4.

Note that when $|\Sigma|/|\Sigma_0|<1$ the EWMA V-chart with a positive k is expected to have a poorer performance than the one without a head start value. Similarly, when $|\Sigma|/|\Sigma_0|>1$ the EWMA V-chart with a negative k is expected to have a poorer performance than the one with $k=0$. It is worth noting that even for small values of k, the performance of the EWMA V-chart is reasonably improved when $|\Sigma|/|\Sigma_0|>1.5$ for positive ks and also when $|\Sigma|/|\Sigma_0|<0.8$ for negative ks, and that $k=0.1$ or -0.1 leads to a better performance of the EWMA V-chart.

In fact, the EWMA V-chart can be enhanced with an FIR feature to increase the sensitivity in detecting either an increase or a decrease in the determinant of the variance–covariance matrix. This is accomplished by simultaneously running two EWMA V-charts, one with a positive head start and the other with a negative head start. The simulated ARLs for the EWMA V-charts with $k=\pm 0.05$ and ± 0.10 are summarized in Table 7. The ARLs of the EWMA V-chart with $k=0$, $k=\pm 0.05$ and $k=\pm 0.10$ are plotted against $|\Sigma|/|\Sigma_0|$ in Fig. 12. Running two EWMA V-charts, each with a head start value increases the sensitivity of the chart in detecting either an increase or a decrease in the determinant of the variance–covariance matrix, except when $0.8 \leqslant |\Sigma|/|\Sigma_0| \leqslant 1.2$. Larger k values lead to smaller ARLs.

TABLE 7. Comparisons of ARL for EWMA V-chart with different head start values ($n=4$, $w=0.2$, $p=2$)

$k=-0.10$	$k=-0.05$	$k=0$	$k=0.05$	$k=0.10$	$k=\pm0.05$	$k=\pm0.10$	$\lvert\Sigma\rvert/\lvert\Sigma_0\rvert$
200.70	201.55	201.12	200.21	201.40	198.15	199.07	1.000
127.34	127.07	114.06	118.96	111.51	124.62	107.52	1.210
67.43	64.47	64.44	60.89	50.58	61.58	50.60	1.464
54.26	51.96	51.73	47.86	39.56	49.43	36.14	1.563
21.75	21.27	20.36	17.99	12.46	17.92	12.93	2.250
18.74	18.06	17.38	14.47	10.19	14.24	9.28	2.441
7.76	7.15	7.03	5.32	3.29	5.28	2.85	5.063
193.08	189.76	141.25	186.16	199.12	193.82	179.57	0.908
175.39	171.86	133.87	163.89	164.59	171.12	148.35	1.098
147.41	142.99	126.77	132.65	128.09	139.06	132.24	1.172
42.64	41.99	41.73	37.31	30.39	38.46	30.03	1.688
33.95	34.91	33.01	28.66	23.08	29.41	23.55	1.831
10.47	9.69	9.31	7.14	4.56	7.29	4.68	3.797
147.73	146.16	126.33	139.95	137.51	145.21	143.80	1.162
74.26	73.06	73.58	67.82	62.05	70.30	61.04	1.406
62.32	59.26	58.06	56.23	45.91	55.93	46.88	1.500
23.68	23.42	22.03	19.24	14.13	19.18	14.08	2.160
19.76	19.63	19.23	16.33	11.53	16.06	11.77	2.344
8.05	7.58	7.22	5.36	3.23	5.56	3.24	4.860
18.03	22.19	24.95	26.91	28.40	23.14	19.49	0.436
30.29	37.47	39.38	40.89	43.90	39.26	34.34	0.527
37.78	45.30	46.63	49.52	51.25	45.84	43.00	0.563
142.65	144.42	125.12	152.65	150.15	148.95	144.21	0.810
181.30	175.95	138.84	177.77	189.07	184.54	175.21	0.879
35.43	34.72	32.57	28.77	22.41	30.22	24.05	1.823

Note that an FIR feature will give the inertia problem caused at process start-up a bigger boost. As for the other inertia problem caused at some later time t, the help by an FIR feature in shortening the delay will depend on t. The boost becomes smaller as t becomes larger.

5 Conclusion

We have introduced and studied a new multivariate EWMA control chart, the EWMA V-chart, specifically designed to detect small changes in process variability. The EWMA V-chart is constructed essentially by first transforming, via the probability integral transformation, the test statistic used to test process variability into a $U(0,1)$ random variable, and then calculating the EWMA of the transformed $U(0,1)$s. Examples and simulations have demonstrated that the EWMA V-chart is more effective than the traditional $|S|$-chart in detecting small changes in process variability. The same idea in setting up the EWMA V-chart is also extended to defining the EWMA M-chart, the counterpart of the EWMA V-chart, for detecting small changes in the process mean. The combined EWMA M- and V-charts provide a faster and more accurate monitoring mechanism of signalling out-of-control process variability, while the combined MEWMA- and $|S|$-charts are either less sensitive or give misleading signals under the same process changes. The best control procedure for monitoring both process mean and process variability is the

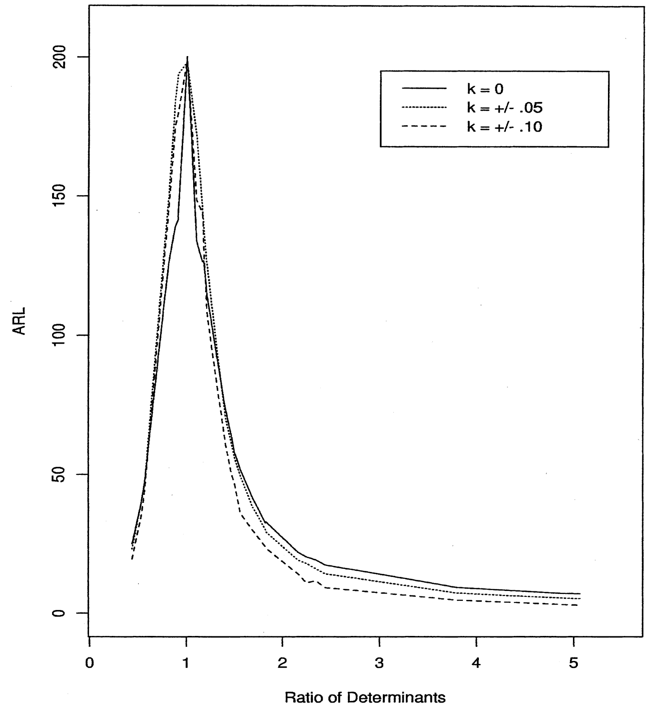

FIG. 12. The ARLs of EWMA V-charts with different head start values ($n = 4$, $w = 0.2$ and $p = 2$).

combined MEWMA and EWMA V-charts. Additionally, the inertia problem is also addressed.

From a technical perspective, it would be worthwhile studying the theoretical properties, such as the run length distribution, of the proposed EWMA V-chart, possibly in the same way as Lucas & Saccucci (1990) studied the theoretical properties of the univariate EWMA control chart. The Markov Chain method proposed by Brook & Evans (1972) and used in Lucas & Saccucci (1990) could possibly be used to derive the theoretical in-control run length distribution of the EWMA V-chart. However, when the process variability is out-of-control, it may be difficult to derive the corresponding run length distribution based on the Markov Chain method. The difficulty is due to the fact that when the process variability is out-of-control, the distribution of v_t, as defined in equation (1), is no longer $U(0, 1)$. Furthermore, the magnitude of change in population variability cannot be directly translated into a specific magnitude of change in v_t, which is needed if the Markov Chain method is to be used.

As explained earlier in Remark 2, in this paper the change in process variability is defined to be the change in the determinant of the variance–covariance matrix, the so-called generalized variance. Another multivariate control chart that can be used to monitor more general changes in process variability is the likelihood-ratio based S-chart (Alt, 1984). We are currently working on deriving a likelihood-ratio based multivariate EWMA control chart for monitoring general changes in process variability. The results of our investigation will be reported in a follow-up paper.

As pointed out by one referee, in the case when $n = 1$, most of the existing control charts for monitoring multivariate process variability, including the proposed EWMA

V-chart, will not work. This is an important problem for future investigation, for there are numerous industrial applications in which $n = 1$. One possible approach is to extend the exponentially weighted moving variance control chart (MacGregor & Harris, 1993) to multivariate cases. This approach can be done essentially by calculating the EWMA of the quadratic form $(X - \mu_0)(X - \mu_0)'$, $t \geqslant 1$, and evaluating the EWMA by some statistics, such as the determinant or the trace of the calculated EWMA. Another possible approach is to derive appropriate control charting schemes based on the testing procedures proposed by Hawkins (1992). Future research along these directions would be valuable. Moreover, since the probability integral transformation is not limited to normal distribution, it would be worthwhile studying whether the EWMA V-chart can be extended to other non-normal multivariate processes.

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Appendix A: Some technical derivations

First note that, when the process is in-control, $|nS_t|$ in equation (1) is distributed as (Anderson, 1984)

$$|\Sigma_0| \times \chi^2_{n-1} \times \chi^2_{n-2} \times \ldots \times \chi^2_{n-p}$$

and $|N\bar{S}|$ is distributed as

$$|\Sigma_0| \times \chi^2_{N-k} \times \chi^2_{N-k-1} \times \ldots \times \chi^2_{N-k-(p-1)}$$

where the chi-square distributions with various degrees of freedom which appear in the products are all independent. Furthermore, since S_t and $\bar{S}$ are independent,

$$\left(\prod_{i=1}^{p} \frac{N-k+1-i}{n-i}\right) + \frac{|nS_t|}{|N\bar{S}|} \sim \prod_{i=1}^{p} F_{n-i,N-k+1-i}$$

where $F_{n-i,N-k+1-i}, i = 1, 2, \ldots, p$, are independent. Therefore, for any given t, v_t defined in equation (1) is distributed as $U(0, 1)$ since it is nothing but a probability integral transformation.

For the special case when $p = 2$, the exact distribution of $|nS_t|/|N\bar{S}|$ can be determined. It is based on a simple result (Anderson, 1984) which states that if χ^2_{n-1} and χ^2_{n-2} are two independent chi-square distributions then $\chi^2_{n-1} \times \chi^2_{n-2}$ is distributed as $(\chi^2_{2n-4})^2/4$. Therefore, in the case when $p = 2$, the statistic v_t as defined in equation (1) can be computed as

$$v_t = P\left(F_{2n-4,2(N-k-1)} \leqslant \frac{N-k-1}{n-2} \times \frac{|nS_t|^{1/2}}{|N\bar{S}|^{1/2}}\right)$$

In the case when $p \geqslant 3$, as suggested in Gnanadesikan & Gupta (1970) and in Anderson (1984), one can use the normal distribution to approximate the distributions of

$$\log \frac{|S_t|}{|\Sigma_0|} \quad \text{and} \quad \log \frac{|\bar{S}|}{|\Sigma_0|}$$

More specifically, following the results in Muirhead (1982), it can easily be shown that both

$$\sqrt{\frac{n-1}{2p}}\log\frac{|S_t|}{|\Sigma_0|} \quad \text{and} \quad \sqrt{\frac{N-k}{2p}}\log\frac{|\bar{S}|}{|\Sigma_0|}$$

are asymptotically distributed as $N(0,1)$, the standard normal distribution. Furthermore, since S_t and S are independent, it follows that

$$\sqrt{\frac{N-k}{2p(k+1)}}\log\frac{|S_t|}{|\bar{S}|} \tag{A1}$$

is asymptotically distributed as $N(0,1)$. Therefore, when $p \geqslant 3$, we suggest that v_t as defined in equation (1) can be computed as

$$v_t = P\left(Z \leqslant \sqrt{\frac{N-k}{2p(k+1)}}\log\frac{|S_t|}{|\bar{S}|}\right)$$

It should be noted that when normal approximation is used, v_t is distributed as $U(0,1)$ asymptotically. For small n, the exact distribution of equation (A1) tends to have heavier tails than does $N(0,1)$. This in turn makes the statistic $S_v(t)$ tend to fall more likely in the rejection regions as determined based on $U(0,1)$.

Appendix B. Additional tables

TABLE B1. Comparisons of ARL for changes in variability ($n=4$, $w=0.4$, $p=2$)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
(1.10,1.00,0)	125.15	118.79	108.67	122.81	1.210
(1.10,1.10,0)	85.71	63.99	73.91	80.07	1.464
(1.25,1.00,0)	79.57	60.74	81.60	76.33	1.563
(1.50,1.00,0)	34.37	23.67	26.54	35.85	2.250
(1.25,1.25,0)	36.39	19.71	33.63	30.21	2.441
(1.50,1.50,0)	13.43	7.26	12.63	8.2 2	5.063
(1.10,1.00,0.5)	117.21	140.06	73.67	147.69	0.908
(1.10,1.10,0.5)	97.16	137.18	58.01	136.68	1.098
(1.25,1.00,0.5)	84.15	123.11	53.53	139.65	1.172
(1.50,1.00,0.5)	38.58	47.51	22.90	67.80	1.688
(1.25,1.25,0.5)	40.90	37.75	26.72	55.39	1.831
(1.50,1.50,0.5)	15.63	9.90	11.94	12.82	3.797
(1.10,1.00, −0.2)	124.95	125.81	104.76	130.42	1.162
(1.10,1.10, −0.2)	97.86	80.08	75.85	101.34	1.406
(1.25,1.00, −0.2)	78.89	67.52	58.65	87.56	1.500
(1.50,1.00, −0.2)	35.34	25.31	25.10	39.36	2.160
(1.25,1.25, −0.2)	36.49	21.60	37.78	31.94	2.344
(1.50,1.50, −0.2)	13.57	7.56	12.46	8.58	4.860
(1.10,1.00,0.8)	82.82	37.54	49.74	137.74	0.436
(1.10,1.10,0.8)	82.24	57.30	39.32	141.45	0.527
(1.25,1.00,0.8)	75.11	65.73	33.55	147.16	0.563
(1.50,1.00,0.8)	44.92	132.67	19.45	148.27	0.810
(1.25,1.25,0.8)	49.38	139.07	21.77	148.86	0.879
(1.50,1.50,0.8)	19.97	38.63	10.69	57.53	1.823

Table B2. Comparisons of ARL for changes in variability ($n=4$, $w=0.6$, $p=2$)

(σ_1, σ_2, ρ)	M-chart	V-chart	MEWMA	$\lvert S\rvert$-chart	$\lvert\Sigma\rvert/\lvert\Sigma_0\rvert$
(1.10,1.00,0)	129.97	123.93	80.33	126.38	1.210
(1.10,1.10,0)	92.24	72.39	51.70	81.02	1.464
(1.25,1.00,0)	83.83	64.47	43.14	79.63	1.563
(1.50,1.00,0)	37.07	26.92	19.43	35.15	2.250
(1.25,1.25,0)	38.86	22.37	24.41	29.52	2.441
(1.50,1.50,0)	14.09	7.39	9.36	8.16	5.063
(1.10,1.00,0.5)	117.10	148.96	53.16	146.71	0.908
(1.10,1.10,0.5)	96.09	138.51	40.37	135.95	1.098
(1.25,1.00,0.5)	80.69	127.75	32.51	126.61	1.172
(1.50,1.00,0.5)	38.80	55.94	16.69	69.13	1.688
(1.25,1.25,0.5)	43.21	43.42	20.56	57.02	1.831
(1.50,1.50,0.5)	16.53	10.39	8.91	12.70	3.797
(1.10,1.00,−0.2)	124.92	131.47	73.03	133.28	1.162
(1.10,1.10,−0.2)	99.78	91.01	52.42	100.18	1.406
(1.25,1.00,−0.2)	83.60	76.07	40.95	86.50	1.500
(1.50,1.00,−0.2)	36.30	28.83	19.15	38.99	2.160
(1.25,1.25,−0.2)	40.04	24.20	23.60	31.69	2.344
(1.50,1.50,−0.2)	14.13	7.70	9.37	8.68	4.860
(1.10,1.00,0.8)	83.33	53.53	37.01	138.90	0.436
(1.10,1.10,0.8)	76.51	79.08	29.02	145.65	0.527
(1.25,1.00,0.8)	68.22	93.92	24.45	145.46	0.563
(1.50,1.00,0.8)	40.66	141.25	14.49	144.93	0.810
(1.25,1.25,0.8)	44.54	141.94	16.21	147.37	0.879
(1.50,1.50,0.8)	19.10	43.42	8.15	58.10	1.823

Table B3. Comparisons of ARL for changes in variability ($n=8$, $w=0.4$, $p=2$)

(σ_1, σ_2, ρ)	M-chart	V-chart	MEWMA	$\lvert S\rvert$-chart	$\lvert\Sigma\rvert/\lvert\Sigma_0\rvert$
(1.10,1.00,0)	126.13	88.04	109.19	116.69	1.210
(1.10,1.10,0)	98.21	33.91	82.94	63.90	1.464
(1.25,1.00,0)	79.06	26.34	61.92	51.90	1.563
(1.50,1.00,0)	34.65	9.01	26.76	15.58	2.250
(1.25,1.25,0)	35.71	7.79	33.39	12.68	2.441
(1.50,1.50,0)	13.32	3.57	12.39	2.93	5.063
(1.10,1.00,0.5)	118.75	136.95	74.64	144.65	0.908
(1.10,1.10,0.5)	98.01	128.70	56.28	134.32	1.098
(1.25,1.00,0.5)	85.68	103.99	45.44	118.86	1.172
(1.50,1.00,0.5)	39.86	19.72	23.56	39.77	1.688
(1.25,1.25,0.5)	42.69	15.03	27.20	29.85	1.831
(1.50,1.50,0.5)	15.59	4.44	12.30	4.70	3.797
(1.10,1.00,−0.2)	125.59	105.85	103.66	122.26	1.162
(1.10,1.10,−0.2)	96.84	41.26	75.60	74.55	1.406
(1.25,1.00,−0.2)	81.29	30.56	57.55	58.23	1.500
(1.50,1.00,−0.2)	35.72	9.89	25.69	17.62	2.160
(1.25,1.25,−0.2)	37.40	8.27	33.03	13.81	2.344
(1.50,1.50,−0.2)	13.72	3.67	12.39	3.14	4.860
(1.10,1.00,0.8)	86.63	10.43	50.85	57.88	0.436
(1.10,1.10,0.8)	82.05	17.46	39.14	86.94	0.527
(1.25,1.00,0.8)	75.76	21.61	33.01	95.12	0.563
(1.50,1.00,0.8)	45.83	99.96	19.16	144.91	0.810
(1.25,1.25,0.8)	49.22	129.42	20.75	142.83	0.879
(1.50,1.50,0.8)	20.30	15.14	10.78	30.80	1.823

TABLE B4. Comparisons of ARL for changes in variability ($n = 8$, $w = 06$, $p = 2$)

$(\sigma_1, \sigma_2, \rho)$	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
(1.10,1.00,0)	127.94	97.62	80.54	112.08	1.210
(1.10,1.10,0)	103.43	40.72	58.06	63.02	1.464
(1.25,1.00,0)	83.13	30.49	43.31	50.04	1.563
(1.50,1.00,0)	36.44	9.88	19.91	15.87	2.250
(1.25,1.25,0)	39.31	8.07	23.72	12.76	2.441
(1.50,1.50,0)	13.97	3.21	9.38	2.89	5.063
(1.10,1.00,0.5)	118.98	144.93	53.03	148.39	0.908
(1.10,1.10,0.5)	97.63	129.74	40.14	133.24	1.098
(1.25,1.00,0.5)	81.69	112.20	32.02	121.63	1.172
(1.50,1.00,0.5)	38.49	22.59	16.89	38.79	1.688
(1.25,1.25,0.5)	42.93	17.34	19.63	29.65	1.831
(1.50,1.50,0.5)	15.94	4.11	9.22	4.59	3.797
(1.10,1.00, −0.2)	127.89	115.38	73.85	124.67	1.162
(1.10,1.10, −0.2)	101.20	48.64	53.48	73.63	1.406
(1.25,1.00, −0.2)	84.58	36.79	40.86	58.56	1.500
(1.50,1.00, −0.2)	36.19	10.81	18.98	17.82	2.160
(1.25,1.25, −0.2)	39.28	8.87	22.97	14.15	2.344
(1.50,1.50, −0.2)	14.41	3.32	9.39	3.16	4.860
(1.10,1.00,0.8)	83.64	13.18	36.37	56.63	0.436
(1.10,1.10,0.8)	76.82	23.04	29.49	86.96	0.527
(1.25,1.00,0.8)	66.62	26.93	25.29	100.32	0.563
(1.50,1.00,0.8)	39.67	114.92	14.35	141.06	0.810
(1.25,1.25,0.8)	44.58	138.89	16.30	143.29	0.879
(1.50,1.50,0.8)	19.56	17.33	8.15	29.75	1.823

TABLE B5. Comparisons of ARL for changes in variability, ($n = 10$, $w = 0.2$, $p = 3$)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\sigma_1 = 1.1$	161.66	89.68	139.69	154.03	1.210
$\sigma_1 = 1.25$	89.87	26.07	83.53	72.71	1.563
$\sigma_1 = 1.5$	38.88	9.80	38.09	24.58	2.250
$\sigma_{1,2} = 1.1$	111.23	33.80	106.33	89.75	1.464
$\sigma_{1,2} = 1.25$	39.45	8.34	47.03	19.38	2.441
$\sigma_{1,2} = 1.5$	14.91	4.06	18.71	4.38	5.063
$\sigma_{1,2,3} = 1.1$	79.39	17.45	81.95	48.77	1.772
$\sigma_{1,2,3} = 1.25$	22.55	4.86	32.42	7.10	3.815
$\rho_{12} = 0.5$	164.13	58.32	131.87	187.05	0.750
$\rho_{12} = -0.2$	193.51	192.53	181.10	220.10	0.960
$\rho_{12} = 0.8$	105.29	7.09	95.11	39.94	0.360
$\rho_{12,13} = 0.5$	127.19	13.38	102.06	80.32	0.500
$\rho_{12,13} = -0.2$	189.23	167.29	170.80	224.42	0.920
$\rho_{12,13,23} = -0.2$	182.50	122.95	158.11	217.25	0.864
$\sigma_{1,2,3} = 1.1$, $\rho_{12,13,23} = 0.5$	101.10	140.70	47.74	212.08	0.886
$\sigma_{1,2,3} = 1.1$, $\rho_{12,13,23} = -0.2$	81.53	28.56	71.66	77.89	1.531
$\sigma_{1,2,3} = 1.1$, $\rho_{12,13,23} = 0.8$	63.25	3.84	32.46	11.09	0.184

TABLE B5.—(*Continued*)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = 0.5$	32.61	13.94	24.50	38.68	1.907
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = -0.2$	23.73	5.62	29.84	9.69	3.296
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = -0.2$	51.32	8.32	18.90	47.05	0.397
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.5$	11.53	3.80	10.84	3.67	5.695
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.8$	19.52	101.53	10.07	160.22	1.185

TABLE B6. Comparisons of ARL for changes in variability, ($n = 10$, $w = 0.4$, $p = 3$)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\sigma_1 = 1.1$	159.93	106.38	135.19	153.89	1.210
$\sigma_1 = 1.25$	96.48	32.77	76.55	72.23	1.563
$\sigma_1 = 1.5$	42.99	11.02	32.68	24.73	2.250
$\sigma_{1,2} = 1.1$	118.61	43.44	102.52	89.58	1.464
$\sigma_{1,2} = 1.25$	45.99	9.33	40.99	19.73	2.441
$\sigma_{1,2} = 1.5$	16.22	4.04	15.24	4.46	5.063
$\sigma_{1,2,3} = 1.1$	87.42	21.05	76.80	48.37	1.772
$\sigma_{1,2,3} = 1.25$	26.40	4.98	28.04	7.20	3.815
$\rho_{12} = 0.5$	157.46	77.34	123.82	186.95	0.750
$\rho_{12} = 0.2$	197.34	195.27	185.39	220.46	0.960
$\rho_{12} = 0.8$	103.01	8.25	82.43	39.31	0.360
$\rho_{12,13} = 0.5$	121.58	17.21	94.21	81.39	0.500
$\rho_{12,13} = 0.2$	182.39	178.61	168.82	221.33	0.920
$\rho_{12,13,23} = -0.2$	180.15	142.43	156.35	214.97	0.864
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = 0.5$	89.26	158.91	38.99	216.34	0.886
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = -0.2$	87.22	36.45	63.81	76.79	1.531
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = 0.8$	54.47	3.99	25.60	11.68	0.184
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = 0.5$	34.17	16.82	20.33	39.02	1.907
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = -0.2$	26.57	5.97	25.81	9.76	3.296
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = -0.2$	42.28	9.80	14.82	46.98	0.397
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.5$	12.28	3.76	9.04	3.61	5.695
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.8$	18.42	118.62	8.13	161.89	1.185

TABLE B7 Comparisons of ARL for changes in variability, ($n = 10$, $w = 0.6$, $p = 3$)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\sigma_1 = 1.1$	164.08	120.21	134.72	15.12	1.210
$\sigma_1 = 1.25$	98.40	39.31	73.00	71.88	1.563
$\sigma_1 = 1.5$	44.41	12.60	29.42	24.80	2.250
$\sigma_{1,2} = 1.1$	126.06	51.99	99.10	88.84	1.464
$\sigma_{1,2} = 1.25$	48.80	10.29	38.28	19.44	2.441
$\sigma_{1,2} = 1.5$	17.09	3.74	13.96	4.31	5.063
$\sigma_{1,2,3} = 1.1$	92.31	24.81	75.81	48.58	1.772
$\sigma_{1,2,3} = 1.25$	28.32	5.00	25.65	7.21	3.815
$\rho_{12} = 0.5$	157.38	93.40	120.67	178.02	0.750
$\rho_{12} = -0.2$	199.33	199.25	183.46	220.05	0.960
$\rho_{12} = 0.8$	99.58	9.73	77.32	39.74	0.360
$\rho_{12,13} = 0.5$	119.87	21.67	88.20	78.88	0.500
$\rho_{12,13} = -0.2$	181.66	186.24	163.00	217.30	0.920
$\rho_{12,13} = -0.2$	178.08	160.11	151.12	217.60	0.864
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = 0.5$	79.28	170.16	35.57	223.15	0.886
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,23} = -0.2$	89.51	43.40	59.80	76.24	1.531
$\sigma_{1,2,3} = 1.1$ $\rho_{12,1323} = 0.8$	48.93	3.87	22.31	11.14	0.184
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = 0.5$	33.93	19.56	17.86	38.34	1.907
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = -0.2$	29.31	6.00	23.25	9.55	3.296
$\sigma_{1,2,3} = 1.25$ $\rho_{12,13,23} = -0.2$	34.39	11.99	13.03	49.07	0.397
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.5$	11.99	3.34	8.05	3.64	5.695
$\sigma_{1,2,3} = 1.5$ $\rho_{12,13,23} = 0.8$	16.37	131.83	7.09	162.57	1.185

TABLE B8. Comparisons of ARL for changes in variability, ($n = 10$, $w = 0.2$, $p = 4$)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\sigma_1 = 1.1$	167.95	110.05	148.60	154.93	1.210
$\sigma_1 = 1.25$	101.26	35.70	90.42	89.11	1.563
$\sigma_1 = 1.5$	43.78	13.20	43.90	35.75	2.250
$\sigma_{1,2} = 1.1$	121.72	45.99	111.69	103.91	1.464
$\sigma_{1,2} = 1.25$	46.46	11.21	54.85	29.20	2.441
$\sigma_{1,2} = 1.5$	16.91	4.88	21.15	7.09	5.063
$\sigma_{1,2,3} = 1.1$	89.30	23.91	91.80	64.81	1.772
$\sigma_{1,2,3} = 1.25$	25.78	6.22	36.54	11.82	3.815
$\sigma_{1,2,3,4} = 1.1$	66.38	14.49	75.42	40.41	2.144
$\sigma_{1,2,3,4} = 1.25$	17.51	4.39	26.69	5.67	5.960
$\rho_{12} = 0.5$	172.98	74.71	141.34	184.70	0.750
$\rho_{12} = -0.2$	196.25	190.87	183.28	204.44	0.960
$\rho_{12} = 0.8$	126.90	9.32	102.94	51.99	0.360
$\rho_{12,13} = 0.5$	142.04	17.84	108.73	95.01	0.500
$\rho_{12,13} = -0.2$	191.02	175.35	171.12	204.44	0.920
$\rho_{12,13,14} = 0.5$	113.50	5.94	94.08	27.72	0.250
$\rho_{12,13,14} = -0.2$	187.62	154.06	163.33	201.84	0.880
$\rho_{12,13,14,23} = 0.5$	105.28	7.74	82.08	40.85	0.313

TABLE B8.—(*Continued*)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\rho_{12,13,14,23} = -0.2$	185.75	113.63	156.48	195.13	0.826
$\rho_{12,13,14,23,24} = -0.2$	173.79	81.39	149.75	186.85	0.768
$\sigma_{1,2,3} = 1.1$ $\rho_{12,13,14,23,24,34} = 0.5$	90.62	45.62	37.76	158.82	0.670
$\sigma_{1,2,3,4} = 1.1$ $\rho_{12,13,14,23,24,34} = -0.2$	68.64	44.61	62.39	100.20	1.482
$\sigma_{1,2,3,4} = 1.1$ $\rho_{12,13,14,23,24,34} = 0.5$	29.39	20.29	20.13	55.34	1.863
$\sigma_{1,2,3,4} = 1.25$ $\rho_{12,13,14,23,24,34} = -0.2$	19.41	5.72	24.74	9.95	4.120
$\sigma_{1,2,3,4} = 1.25$ $\rho_{12,13,14,23,24,34} = 0.8$	46.87	4.35	15.10	14.09	0.162
$\sigma_{1,2,3,4} = 1.5$ $\rho_{12,13,14,23,24,34} = 0.8$	19.22	53.53	8.33	165.14	0.697

TABLE B9. Comparisons of ARL for changes in variability, ($n = 10$, $w = 0.4$, $p = 4$)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\sigma_1 = 1.1$	167.04	130.02	146.35	154.94	1.210
$\sigma_1 = 1.25$	107.55	47.75	81.91	89.44	1.563
$\sigma_1 = 1.5$	48.34	15.91	37.10	36.32	2.250
$\sigma_{1,2} = 1.1$	130.80	59.43	108.92	102.13	1.464
$\sigma_{1,2} = 1.25$	53.41	13.32	47.88	28.74	2.441
$\sigma_{1,2} = 1.5$	18.89	5.13	17.59	7.00	5.063
$\sigma_{1,2,3} = 1.1$	99.38	30.95	86.21	65.84	1.772
$\sigma_{1,2,3} = 1.25$	30.83	6.49	31.78	11.13	3.815
$\sigma_{1,2,3,4} = 1.1$	75.18	17.77	70.68	40.12	2.144
$\sigma_{1,2,3,4} = 1.25$	20.30	4.49	22.91	5.66	5.960
$\rho_{12} = 0.5$	161.03	99.05	134.85	184.93	0.750
$\rho_{12} = -0.2$	189.25	199.81	181.72	205.48	0.960
$\rho_{12} = 0.8$	120.71	11.66	95.20	52.00	0.360
$\rho_{12,13} = 0.5$	134.61	25.56	100.52	92.66	0.500
$\rho_{12,13} = -0.2$	189.48	191.59	172.05	204.79	0.920
$\rho_{12,13,14} = 0.5$	106.41	6.84	83.75	27.60	0.250
$\rho_{12,13,14} = -0.2$	180.71	176.50	160.30	201.13	0.880
$\rho_{12,13,14,23} = 0.5$	100.09	9.13	68.36	40.85	0.313
$\rho_{12,13,14,23} = -0.2$	178.49	138.30	149.66	197.67	0.826
$\rho_{12,13,14,23,24} = -0.2$	166.70	108.37	145.73	187.08	0.768
$\sigma_{1,2,3,4} = 1.1$ $\rho_{12,13,14,23,24,34} = 0.5$	79.93	63.77	29.78	156.07	0.670
$\sigma_{1,2,3,4} = 1.1$ $\rho_{12,13,14,23,24,34} = -0.2$	74.77	56.95	54.99	102.00	1.482
$\sigma_{1,2,3,4} = 1.1$ $\rho_{12,13,14,23,24,34} = 0.5$	30.00	26.40	15.78	56.35	1.863
$\sigma_{1,2,3,4} = 1.25$ $\rho_{12,13,14,23,24,34} = -0.2$	22.01	6.14	21.16	9.88	4.120
$\sigma_{1,2,3,4} = 1.25$ $\rho_{12,13,14,23,24,34} = 0.8$	34.31	4.61	11.70	14.17	0.162
$\sigma_{1,2,3,4} = 1.5$ $\rho_{12,13,14,23,24,34} = 0.8$	16.90	75.28	6.45	163.72	0.697

TABLE B10. Comparisons of ARL for changes in variability, ($n = 10$, $w = 0.6$, $p = 4$)

Changes	M-chart	V-chart	MEWMA	$\|S\|$-chart	$\|\Sigma\|/\|\Sigma_0\|$
$\sigma_1 = 1.1$	166.98	136.56	142.11	155.01	1.210
$\sigma_1 = 1.25$	106.46	54.32	80.28	87.44	1.563
$\sigma_1 = 1.5$	50.66	18.15	33.70	35.57	2.250
$\sigma_{1,2} = 1.1$	130.67	68.03	105.89	104.11	1.464
$\sigma_{1,2} = 1.25$	58.53	14.89	45.57	29.49	2.441
$\sigma_{1,2} = 1.5$	19.75	4.93	16.27	7.08	5.063
$\sigma_{1,2,3} = 1.1$	106.64	36.29	84.53	63.15	1.772
$\sigma_{1,2,3} = 1.25$	33.66	6.83	30.08	11.70	3.815
$\sigma_{1,2,3,4} = 1.1$	79.87	20.40	67.47	40.57	2.144
$\sigma_{1,2,3,4} = 1.25$	21.94	4.14	21.37	5.50	5.960
$\rho_{12} = 0.5$	158.54	117.23	128.36	180.65	0.750
$\rho_{12} = -0.2$	187.94	206.02	181.28	202.03	0.960
$\rho_{12} = 0.8$	112.77	14.76	89.47	52.95	0.360
$\rho_{12,13} = 0.5$	130.53	33.05	96.98	94.50	0.500
$\rho_{12,13} = -0.2$	186.73	190.35	171.67	204.89	0.920
$\rho_{12,13,14} = 0.5$	100.56	7.78	80.20	27.87	0.250
$\rho_{12,13,14} = -0.2$	179.64	177.46	159.11	203.48	0.880
$\rho_{12,13,14,23} = 0.5$	93.88	11.24	64.00	41.46	0.313
$\rho_{12,13,14,23} = -0.2$	174.70	154.04	150.42	200.30	0.826
$\rho_{12,13,14,23,24} = -0.2$	165.22	124.10	141.79	188.08	0.768
$\sigma_{1,2,3,4} = 1.1$ $\rho_{12,13,14,23,24,34} = 0.5$	68.00	78.83	27.56	155.03	0.670
$\sigma_{1,2,3,4} = 1.1$ $\rho_{12,13,14,23,24,34} = -0.2$	75.93	64.32	52.23	100.25	1.482
$\sigma_{1,2,3,4} = 1.25$ $\rho_{12,13,14,23,24,34} = 0.5$	28.31	29.97	14.12	58.02	1.863
$\sigma_{1,2,3,4} = 1.25$ $\rho_{12,13,14,23,24,34} = -0.2$	22.36	6.12	18.72	10.17	4.120
$\sigma_{1,2,3,4} = 1.25$ $\rho_{12,13,14,23,24,34} = 0.8$	29.11	4.70	10.52	14.08	0.162
$\sigma_{1,2,3,4} = 1.5$ $\rho_{12,13,14,23,24,34} = 0.8$	14.67	91.66	5.96	167.68	0.697