

# A multivariate Wald-Wolfowitz rank test against serial dependence

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## ABSTRACT

Rank-based cross-covariance matrices, extending to the case of multivariate observed series the (univariate) rank autocorrelation coefficients introduced by Wald and Wolfowitz (1943), are considered. A permutational central limit theorem is established for the joint distribution of such matrices, under the null hypothesis of (multivariate) randomness as well as under contiguous alternatives of (multivariate) ARMA dependence. A rank-based, permutationally distribution-free test of the portmanteau type is derived, and its asymptotic local power is investigated. Finally, a modified rank-based version of Tiao and Box's model specification procedure is proposed, which is likely to be more reliable under non-Gaussian conditions, and more robust against gross errors.

## RÉSUMÉ

Des matrices de covariances croisées fondées sur les rangs, généralisant au cas des séries multivariées les coefficients d'autocorrélation de rangs introduits par Wald et Wolfowitz (1943) sont considérées. Un théorème central-limite permutational est établi pour ces matrices, sous l'hypothèse que la série sous-jacente constitue la réalisation d'un bruit blanc multivarié, ainsi que sous des contre-hypothèses contiguës de dépendance ARMA. Un test de rangs du type portemanteau est également construit, et sa puissance asymptotique locale est explicitement calculée. Enfin, une version fondée sur les rangs de la procédure d'identification de Tiao et Box est proposée. Celle-ci est plus fiable que la procédure usuelle sous des conditions non gaussiennes, et plus robuste par rapport à la présence de valeurs aberrantes.

## 1. RANK-BASED CROSS-COVARIANCE MATRICES OF THE SPEARMAN-WALD-WOLFOWITZ TYPE

Denote by  $\mathbf{X}^{(n)} = (\mathbf{X}_1^{(n)}, \dots, \mathbf{X}_t^{(n)}, \dots, \mathbf{X}_n^{(n)})$  a series of  $m$ -variate observations  $\mathbf{X}_t^{(n)} = (X_{t,1}^{(n)}, \dots, X_{t,m}^{(n)})^T$ ,  $t = 1, \dots, n$ . We are interested in the null hypothesis  $\mathcal{H}^{(n)}$  of randomness under which the observations  $\mathbf{X}_t^{(n)}$  are independently and identically distributed, according to some unspecified density  $f(\mathbf{x})$ ,  $\mathbf{x} = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ . Whenever  $f$  needs to be specified, the resulting subhypothesis will be denoted by  $\mathcal{H}_f^{(n)}$ . The alternatives we are considering are serial-dependence alternatives, with emphasis on multivariate ARMA dependence.

Much attention has been devoted recently to the one-dimensional version of this nonparametric problem; see, e.g., Dufour (1981), Dufour and Roy (1985, 1986), Chan and Tran (1992) or Shaug and Tjøstheim (1993). In this univariate version [ $m = 1$ ; observed

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series  $\mathbf{X}^{(n)} = (X_1^{(n)}, \dots, X_n^{(n)})$ , usual invariance arguments lead to the consideration of rank-based testing procedures; see Hallin and Puri (1992). In this latter context, Wald and Wolfowitz (1943) proposed test statistics of the form (up to additive and multiplicative constants)  $S_i^{(n)} = n^{-1} \sum_{t=1}^n R_t^{(n)} R_{t-i}^{(n)}$ , where  $R_t^{(n)}$  denotes, as usual, the rank of  $X_t^{(n)}$  among  $X_1^{(n)}, \dots, X_n^{(n)}$ , and with the (convenient but rather arbitrary) convention that  $R_s^{(n)}$ ,  $s = 0, -1, \dots, -i+1$ , is set equal to  $R_{n+s}^{(n)}$ . Clearly,  $S_i^{(n)}$  is a (circular) serial version, of lag  $i$ , of Spearman's ordinary rank correlation coefficient (still, up to additive and multiplicative constants). The performance of such test statistics in a time series context have been investigated by Knoke (1977) and Bartels (1982), among others—see also Hallin and Mélard (1988). A systematic theoretical study of serial, rank-based statistics is provided in Hallin *et al.* (1985, 1987) and Hallin and Puri (1988, 1991, 1994).

Our objective here is to extend Wald and Wolfowitz's idea to the problem of testing for *multivariate* randomness. Due to the greater complexity inherent in the nature of multivariate variables, the development of multivariate theories—especially in the area of rank-based procedures—typically has been much slower than that of their univariate counterparts. Multivariate rank-based statistics for serial-dependence problems have been introduced in Hallin *et al.* (1989), from which several asymptotic results are used here.

The multivariate generalization of Wald and Wolfowitz's rank-based autocorrelation coefficients quite naturally takes the form of rank-based cross-covariance matrices. Letting  $R_{t,i}^{(n)}$  denote the rank of  $X_{t,i}^{(n)}$  among  $X_{1,i}^{(n)}, \dots, X_{n,i}^{(n)}$ , put  $\mathbf{R}_t^{(n)} = (R_{t,1}^{(n)}, \dots, R_{t,m}^{(n)})^\top$ , and  $\mathbf{R}^{(n)} = (\mathbf{R}_1^{(n)}, \dots, \mathbf{R}_n^{(n)})$ :  $\mathbf{R}^{(n)}$  is called the *rank-collection matrix* associated with the observed series  $\mathbf{X}^{(n)}$ .

It is well known that, unless  $m = 1$ , the rank vectors  $\mathbf{R}_t^{(n)}$  are not distribution-free, even under the null hypothesis of randomness. If however we denote by  $\mathbf{R}_*^{(n)}$  the rank-collection matrix resulting from  $\mathbf{R}^{(n)}$  on rearranging the columns in such a manner that the first row yields elements  $1, 2, \dots, n$  in ascending order, then the  $n!$  possible permutations of the columns of  $\mathbf{R}_*^{(n)}$  are conditionally equiprobable: the rank vectors  $\mathbf{R}_t^{(n)}$  are thus conditionally distribution-free, given the "ordered" rank-collection matrix  $\mathbf{R}_*^{(n)}$ .

The consideration of this conditional distribution actually also follows from unbiasedness and Neyman-structure arguments. The matrix  $\mathbf{X}_*^{(n)}$  resulting from  $\mathbf{X}^{(n)}$  on rearranging the columns as in  $\mathbf{R}_*^{(n)}$  indeed can be shown to be sufficient complete under  $\mathcal{H}^{(n)}$ . Conditioning, as usual, upon this sufficient complete statistic yields the permutational distribution of  $(\mathbf{X}_1^{(n)}, \dots, \mathbf{X}_n^{(n)})$ . Additional componentwise invariance arguments then lead to the permutational distribution of  $\mathbf{R}^{(n)}$ , which is precisely the conditional distribution of the rank vectors given in  $\mathbf{R}_*^{(n)}$ .

Define the *rank cross-covariance matrix* of lag  $i$  as

$$\Gamma_i^{(n)} = (n-i)^{-1} \sum_{t=i+1}^n (\mathbf{R}_t^{(n)})(\mathbf{R}_{t-i}^{(n)})^\top - \mathbf{m}_{i*}^{(n)}, \quad i \geq 1, \quad (1.1)$$

where  $\mathbf{m}_{i*}^{(n)}$  denotes the permutational mean of  $(n-i)^{-1} \sum (\mathbf{R}_t^{(n)})(\mathbf{R}_{t-i}^{(n)})^\top$ . Denote by  $\gamma_{i;k,l}^{(n)}$  the elements of  $\Gamma_i^{(n)}$ : for  $k \neq l$ ,  $\gamma_{i;k,l}^{(n)}$  is a rank *cross-covariance* coefficient of lag  $i$ , and for  $k = l$  it is a (univariate) rank *autocovariance* coefficient, of the type studied in Hallin *et al.* (1985, 1987). Note that a similar notation is adopted in Hallin *et al.* (1989) for a more general type of rank cross-covariance matrix, which we do not consider here.

Section 2 below is devoted to the asymptotic distribution theory of rank cross-covariance matrices:  $(n-i)^{\frac{1}{2}}(n+1)^{-2} \text{vec } \Gamma_i^{(n)}$  will be shown to be asymptotically multinormal, under the null hypothesis of randomness as well as under contiguous alternatives of ARMA dependence. Section 3 discusses statistical applications in time-series

analysis. Before proceeding to asymptotic results, however, we start with an explicit expression for the permutational mean matrices  $\mathbf{m}_{i*}^{(n)}$ . Here  $\mathbf{1}$  stands for a matrix of ones of appropriate dimensions;  $O_P$  and  $o_P$  are unconditional  $O_P$ 's and  $o_P$ 's.

PROPOSITION 1.1. *The permutational mean matrices  $\mathbf{m}_{i*}^{(n)}$  in (1.1) are of the form*

$$\mathbf{m}_{i*}^{(n)} = \frac{n(n+1)^2}{4(n-1)} \mathbf{1} - \frac{\mathbf{m}_{0*}^{(n)}}{n-1}, \quad (1.2)$$

with  $\mathbf{m}_{0*}^{(n)} = n^{-1} \sum_{t=1}^n (\mathbf{R}_t^{(n)})(\mathbf{R}_t^{(n)})^\top$ . Their elements uniformly satisfy

$$\frac{n^2}{4} < \frac{(n+1)(3n+2)}{12} \leq (\mathbf{m}_{i*}^{(n)})_{kl} < \frac{n(n+1)^2}{4(n-1)}. \quad (1.3)$$

Moreover, as  $n \rightarrow \infty$ ,

$$n^{-2} \mathbf{m}_{i*}^{(n)} = \left\{ \frac{1}{4} + O_P(n^{-1}) \right\} \mathbf{1}, \quad i \geq 1. \quad (1.4)$$

*Proof.* For  $i = 1, 2, \dots, n-1$ , we have

$$\begin{aligned} \mathbf{m}_{i*}^{(n)} &= \{n(n-1)\}^{-1} \sum_{1 \leq t_1 \neq t_2 \leq n} (\mathbf{R}_{t_1}^{(n)})(\mathbf{R}_{t_2}^{(n)})^\top \\ &= \{n(n-1)\}^{-1} \left( \sum_{t=1}^n (\mathbf{R}_t^{(n)}) \sum_{t=1}^n (\mathbf{R}_t^{(n)})^\top - n \mathbf{m}_{0*}^{(n)} \right) \\ &= \{n(n-1)\}^{-1} \left\{ \left( \sum_{r=1}^n r \right)^2 \mathbf{1} - n \mathbf{m}_{0*}^{(n)} \right\} \\ &= \frac{n(n-1)^2}{4(n-1)} \mathbf{1} - \frac{\mathbf{m}_{0*}^{(n)}}{n-1}. \end{aligned}$$

Now,  $(n \mathbf{m}_{0*}^{(n)})_{k,l} = \sum_{t=1}^n R_{t,k}^{(n)} R_{t,l}^{(n)} \leq \sum_{t=1}^n (t^2) = n(n+1)(2n+1)/6$ ; (1.3) and (1.4) straightforwardly follow.  $\square$

Note that  $\mathbf{m}_{i*}^{(n)}$ ,  $i \geq 1$ , does not depend on  $i$ .

## 2. ASYMPTOTIC DISTRIBUTION THEORY

Rank cross-covariance matrices can be used for testing  $\mathcal{H}^{(n)}$  against arbitrary alternatives of serial dependence. If however local asymptotic power results are desired, more specific alternatives are to be considered. Denote by  $\mathcal{K}_f^{(n)}(\mathbf{A}, \mathbf{B})$  the local alternative under which  $\mathbf{X}^{(n)}$  constitutes a finite realization of some solution of the stochastic difference equation (ARMA model)

$$\mathbf{X}_t - n^{-\frac{1}{2}} \sum_{i=1}^p \mathbf{A}_i \mathbf{X}_{t-i} = \boldsymbol{\epsilon}_t + n^{-\frac{1}{2}} \sum_{i=1}^q \mathbf{B}_i \boldsymbol{\epsilon}_{t-i}, \quad t \in \mathbb{Z}, \quad (2.1)$$

where  $\boldsymbol{\epsilon}_t$  is an independent  $m$ -variate white noise with density function  $f$ . Assume that  $f$  satisfies some mild technical conditions, such as being twice differentiable [put  $\boldsymbol{\Phi}(\mathbf{x}) = -\text{grad} \log f(\mathbf{x})$ ] and having finite Fisher information [ $\int \boldsymbol{\Phi}(\mathbf{x}) \boldsymbol{\Phi}^\top(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = \mathbf{I}(f) < \infty$ ].

Then the sequences  $\mathcal{K}_f^{(n)}(\mathbf{A}, \mathbf{B})$  and  $\mathcal{H}_f^{(n)}$  are contiguous [see Hallin *et al.* (1989) for details].

Denote by  $F$ ,  $f_i$  and  $F_i$  ( $i = 1, \dots, m$ ), respectively, the  $m$ -dimensional distribution function, marginal densities and marginal distribution functions associated with  $F$ . Assume that  $\int \mathbf{x}f(\mathbf{x}) d\mathbf{x} = \mathbf{0}$ , and that  $\int \mathbf{x}\mathbf{x}^T f(\mathbf{x}) d\mathbf{x} = \mathbf{\Sigma}$ , a finite, strictly positive definite covariance matrix. Finally, define  $\mathbf{F}(\mathbf{x}) = (F_1(x_1), \dots, F_m(x_m))^T$ ,  $\mathbf{x} = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ .

**PROPOSITION 2.1.** *Let  $i' \neq i''$ . As  $n \rightarrow \infty$ , the joint distribution of  $(n+1)^{-2}((n-i')^{\frac{1}{2}} \text{vec}^T \Gamma_{i'}^{(n)}, (n-i'')^{\frac{1}{2}} \text{vec}^T \Gamma_{i''}^{(n)})^T$  is asymptotically multinormal, with mean  $\mathbf{0}$  under  $\mathcal{H}^{(n)}$  and mean*

$$\begin{pmatrix} \{ \int \mathbf{F}(\mathbf{x})\mathbf{x}^T dF(\mathbf{x}) \} \otimes \{ \int \mathbf{F}(\mathbf{x})\boldsymbol{\Phi}^T(\mathbf{x}) dF(\mathbf{x}) \} \text{vec}(\mathbf{A}_{i'} + \mathbf{B}_{i'}) \\ \{ \int \mathbf{F}(\mathbf{x})\mathbf{x}^T dF(\mathbf{x}) \} \otimes \{ \int \mathbf{F}(\mathbf{x})\boldsymbol{\Phi}^T(\mathbf{x}) dF(\mathbf{x}) \} \text{vec}(\mathbf{A}_{i''} + \mathbf{B}_{i'') \end{pmatrix} \quad (2.2)$$

under  $\mathcal{K}_f^{(n)}(\mathbf{A}, \mathbf{B})$ , and with covariance matrix

$$\begin{pmatrix} [\int \mathbf{F}(\mathbf{x})\{\mathbf{F}(\mathbf{x})\}^T dF(\mathbf{x}) - \frac{1}{4}]^{\otimes 2} & \mathbf{0} \\ \mathbf{0} & [\int \mathbf{F}(\mathbf{x})\{\mathbf{F}(\mathbf{x})\}^T dF(\mathbf{x}) - \frac{1}{4}]^{\otimes 2} \end{pmatrix} \quad (2.3)$$

under both (all diagonal elements in this latter matrix are equal to  $\frac{1}{144}$ ).

*Proof.* Letting  $I = \max(i', i'')$ , we see that  $(n+1)^{-2}(\gamma_{j',k,l}^{(n)} + m_{j'',k,l}^{(n)})$  is of the form

$$(n-I)^{-1} \sum_{l=I+1}^n J_k \left( \frac{\mathbf{R}_l^{(n)}}{n+1} \right) J_l \left( \frac{\mathbf{R}_{l-j}^{(n)}}{n+1} \right) + o_P(n^{-\frac{1}{2}}),$$

where  $J_h(\mathbf{u}) = u_h$  ( $h = 1, \dots, m$ )—a nondecreasing function with respect to all the components of  $\mathbf{u} \in (0, 1)^m$ —satisfies  $\mathcal{E} J_h^2(\mathbf{F}(\mathbf{X}_1^{(n)})) = \int_0^1 u^2 du < \infty$ . The assumptions of Proposition 3.1 in Hallin *et al.* (1989) are thus fulfilled, and  $J(\mathbf{u}_1, \dots, \mathbf{u}_{I+1}) = u_{1,k} u_{j+1,l}$  is a score-generating function for  $(n+1)^{-2}(\gamma_{j',k,l}^{(n)} + m_{j'',k,l}^{(n)})$ . The corresponding “centered scores” (Proposition 2.1, same reference) are

$$J_{jkl}^*(\mathbf{u}_1, \dots, \mathbf{u}_{I+1}) = u_{1,k} u_{j+1,l} - \frac{1}{2}(u_{1,k} + u_{j+1,l}) - \frac{1}{4}.$$

**Proposition 2.2** (same reference as above) then entails the asymptotic joint binormality of  $(n+1)^{-2}((n-I)^{\frac{1}{2}} \gamma_{j',k',l'}^{(n)}, (n-I)^{\frac{1}{2}} \gamma_{j'',k'',l''}^{(n)})^T$ , with mean  $(\mathbf{0}, \mathbf{0})^T$  under  $\mathcal{H}_f^{(n)}$ , mean

$$\left( \sum_{i=1}^{\min(j', k')} \text{tr}\{\mathbf{C}_{j'k'l'}^i(\mathbf{A}_i + \mathbf{B}_i)\} \sum_{i=1}^{\min(j'', k'')} \text{tr}\{\mathbf{C}_{j''k''l''}^i(\mathbf{A}_i + \mathbf{B}_i)\} \right)^T$$

under  $\mathcal{K}_f^{(n)}(\mathbf{A}, \mathbf{B})$ , and covariance matrix

$$\begin{pmatrix} V_{j'k'l'}^2 & V_{12} \\ V_{12} & V_{j''k''l''}^2 \end{pmatrix}$$

under both, where  $\kappa = \max(p, q)$ ,  $\mathbf{A}_i = \mathbf{0}$  for  $i = p+1, \dots, \kappa$ ,  $\mathbf{B}_i = \mathbf{0}$  for  $i = q+1, \dots, \kappa$ , and with (all expectations below are taken under  $\mathcal{H}_f^{(n)}$ )

$$\begin{aligned}
 \mathbf{C}_{jkl}^i &= \sum_{h=0}^{\kappa-i} \mathcal{E} \{ J_{jkl}^* (\mathbf{F}(\mathbf{X}_1^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l-1}^{(n)})) \Phi(\mathbf{X}_{h+1}^{(n)}) (\mathbf{X}_{h+i+1}^{(n)})^\top \} \\
 &= \begin{cases} \mathcal{E} \{ F_k(X_{1,k}^{(n)}) F_l(X_{j+1,l}^{(n)}) \Phi(\mathbf{X}_1^{(n)}) (\mathbf{X}_{j+1}^{(n)})^\top \} & \text{if } i = j, \\ \mathbf{0} & \text{if } i \neq j \end{cases} \\
 &= \begin{cases} \int \int F_k(x_k) \Phi(\mathbf{x}) dF(\mathbf{x}) \int F_l(y_l) \mathbf{y}^\top dF(\mathbf{y}) & \text{if } i = j, \\ \mathbf{0} & \text{if } i \neq j, \end{cases} \\
 V_{jkl}^2 &= \mathcal{E} \left( J_{jkl}^{*2} (\mathbf{F}(\mathbf{X}_1^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1}^{(n)})) \right. \\
 &\quad \left. + 2 \sum_{h=1}^l J_{jkl}^* (\mathbf{F}(\mathbf{X}_1^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1}^{(n)})) J_{jkl}^* (\mathbf{F}(\mathbf{X}_{1+h}^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1+h}^{(n)})) \right) \\
 &= \mathcal{E} \left( [F_k(X_{1,k}^{(n)}) F_l(X_{j+1,l}^{(n)}) - \frac{1}{2} \{ F_k(X_{1,k}^{(n)}) + F_l(X_{j+1,l}^{(n)}) \} + \frac{1}{4}]^2 \right) \\
 &= \int \int \{ uv - \frac{1}{2}(u+v) + \frac{1}{4} \}^2 du dv = \frac{1}{144},
 \end{aligned}$$

and

$$\begin{aligned}
 V_{12} &= \mathcal{E} \left( J_{j'k'l'}^* (\mathbf{F}(\mathbf{X}_1^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1}^{(n)})) J_{j''k''l''}^* (\mathbf{F}(\mathbf{X}_1^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1}^{(n)})) \right. \\
 &\quad \left. + \sum_{h=1}^l \{ J_{j'k'l'}^* (\mathbf{F}(\mathbf{X}_1^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1}^{(n)})) J_{j''k''l''}^* (\mathbf{F}(\mathbf{X}_{1+h}^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1+h}^{(n)})) \right. \\
 &\quad \left. + J_{j'k'l'}^* (\mathbf{F}(\mathbf{X}_{1+h}^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1+h}^{(n)})) J_{j''k''l''}^* (\mathbf{F}(\mathbf{X}_1^{(n)}), \dots, \mathbf{F}(\mathbf{X}_{l+1}^{(n)})) \} \right) \\
 &= \begin{cases} \mathcal{E} \left( [F_{k'}(X_{1,k'}^{(n)}) F_{l'}(X_{j'+1,l'}^{(n)}) - \frac{1}{2} \{ F_{k'}(X_{1,k'}^{(n)}) + F_{l'}(X_{j'+1,l'}^{(n)}) \} + \frac{1}{4}] \right. \\ \quad \left. \times [F_{k''}(X_{1,k''}^{(n)}) F_{l''}(X_{j''+1,l''}^{(n)}) - \frac{1}{2} \{ F_{k''}(X_{1,k''}^{(n)}) + F_{l''}(X_{j''+1,l''}^{(n)}) \} + \frac{1}{4}] \right) & \text{if } j' = j'' = j, \\ 0 & \text{if } j' \neq j'' \end{cases} \\
 &= \begin{cases} \left\{ \int F_{k'}(x_{k'}) F_{k''}(x_{k''}) dF(\mathbf{x}) - \frac{1}{4} \right\} \left\{ \int F_{l'}(x_{l'}) F_{l''}(x_{l''}) dF(\mathbf{x}) - \frac{1}{4} \right\} & \text{if } j' = j'' = j, \\ 0 & \text{if } j' \neq j''. \end{cases}
 \end{aligned}$$

Proposition 2.1 follows.  $\square$

The asymptotic distribution of  $\Gamma_i^n$ , as provided by Proposition 2.1, unfortunately cannot be used for inferential purposes, since the asymptotic covariance matrix (2.3) is not distribution-free, and clearly depends on the underlying joint density  $f(\mathbf{x})$ . Genuinely permutationally distribution-free rank statistics with asymptotic distribution-free unconditional distributions, however, can be obtained if conditional (permutational) covariances

are substituted for the unconditional ones. The exact permutational covariance of  $\text{vec } \Gamma_i^{(n)}$  is (for  $n \geq 2i$ )

$$\begin{aligned} \mathbf{W}_i^* &= \{n(n-1)(n-i)\}^{-1} \sum_{t_1 \neq t_2} \{\mathbf{R}_{t_1}^{(n)}(\mathbf{R}_{t_2}^{(n)})^\top\}^{\otimes 2} \\ &+ 2(n-2i)\{n(n-1)(n-2)(n-i)^2\}^{-1} \sum_{t_1 \neq t_2 \neq t_3} \{\mathbf{R}_{t_1}^{(n)}(\mathbf{R}_{t_2}^{(n)})^\top\} \otimes \{\mathbf{R}_{t_3}^{(n)}(\mathbf{R}_{t_1}^{(n)})^\top\} \\ &+ \{n^2 - n(2i+3) + i^2 + 5i\}\{n(n-1)(n-2)(n-3)(n-i)^2\}^{-1} \\ &\times \sum_{t_1 \neq t_2 \neq t_3 \neq t_4} \{\mathbf{R}_{t_1}^{(n)}(\mathbf{R}_{t_2}^{(n)})^\top\} \otimes \{\mathbf{R}_{t_3}^{(n)}(\mathbf{R}_{t_4}^{(n)})^\top\} - (\text{vec } \mathbf{m}_{i*}^{(n)})(\text{vec } \mathbf{m}_{i*}^{(n)})^\top. \quad (2.4) \end{aligned}$$

This exact, somewhat cumbersome form results from combinatorial arguments [see Lemma 2.1 in Hallin *et al.* (1989)]. In practice, it can be replaced by the following much simpler one, without affecting asymptotic results.

PROPOSITION 2.2. *Let*

$$\begin{aligned} \tilde{\mathbf{W}}_i^* &= \{n(n-1)(n-i)\}^{-1} \\ &\times \sum_{t_1 \neq t_2} \left[ \mathbf{R}_{t_1}^{(n)}(\mathbf{R}_{t_1}^{(n)})^\top - \frac{(n+1)^2}{4} \mathbf{1} \right] \otimes \left[ \mathbf{R}_{t_2}^{(n)}(\mathbf{R}_{t_2}^{(n)})^\top - \frac{(n+1)^2}{4} \mathbf{1} \right]. \end{aligned}$$

Then  $\tilde{\mathbf{W}}_i^* - \mathbf{W}_i^*$  is  $o_P(n^3)$ , and  $(\text{vec } \Gamma_i^{(n)})^\top (\tilde{\mathbf{W}}_i^*)^{-1} (\text{vec } \Gamma_i^{(n)})$  is asymptotically chi-square (under  $\mathcal{H}^{(n)}$ ), with  $m^2$  degrees of freedom, provided that

$$\mathbf{V} = \int \mathbf{F}(\mathbf{x}) \{\mathbf{F}(\mathbf{x})\}^\top dF(\mathbf{x}) - \frac{1}{4}$$

is a full-rank matrix.

*Proof.* It follows from Proposition 2.4(ii) in Hallin *et al.* (1989) that the permutational covariance matrix of  $(n-i)^{\frac{1}{2}}(n+1)^{-2} \text{vec } \Gamma_i^{(n)}$ , viz.  $(n-i)(n+1)^{-4} \mathbf{W}_i^*$ , converges in probability to its asymptotical counterpart;  $(n-i)(n+1)^{-4} \mathbf{W}_i^* - \mathbf{V}^{\otimes 2} = o_P(1)$  in (2.3). On the other hand,  $(n-i)(n+1)^{-4} \tilde{\mathbf{W}}_i^*$  is the permutational mean of the (matrix of) linear serial multirank statistic(s) of order one

$$(n-1)^{-1} \sum_{t=2}^n \left\{ \frac{\mathbf{R}_t^{(n)}}{n+1} \left( \frac{\mathbf{R}_t^{(n)}}{n+1} \right)^\top - \frac{1}{4} \right\} \otimes \left\{ \frac{\mathbf{R}_{t-1}^{(n)}}{n+1} \left( \frac{\mathbf{R}_{t-1}^{(n)}}{n+1} \right)^\top - \frac{1}{4} \right\},$$

with a (matrix of) score-generating function(s)  $\mathbf{J}(\mathbf{u}, \mathbf{v}) = (\mathbf{u}\mathbf{u}^\top - \frac{1}{4}) \otimes (\mathbf{v}\mathbf{v}^\top - \frac{1}{4})$  [this is a consequence of Proposition 3.1 in Hallin *et al.* (1989)]. Applying again Proposition 2.4 (same reference), this permutational mean converges in probability to its unconditional asymptotical counterpart:

$$(n-i)(n+1)^{-4} \tilde{\mathbf{W}}_i^* - \mathbf{V}^{\otimes 2} = o_P(1).$$

It follows that  $\tilde{\mathbf{W}}_i^* - \mathbf{W}_i^*$  is  $o_P(n^3)$ . Now, from Proposition 2.1 above, and provided that  $\mathbf{V}$  (hence  $\mathbf{V}^{\otimes 2}$ ) is of full rank,  $Q_i^{(n)} = (n-i)(n+1)^{-4} (\text{vec } \Gamma_i^{(n)})^\top (\mathbf{V}^{\otimes 2})^{-1} \text{vec } \Gamma_i^{(n)}$  is asymptotically chi-square, under  $\mathcal{H}^{(n)}$ , with  $m^2$  degrees of freedom. Since

$(\text{vec } \Gamma_i^{(n)})^\top (\mathbf{W}_i^*)^{-1} (\text{vec } \Gamma_i^{(n)})$  and  $(\text{vec } \Gamma_i^{(n)})^\top (\tilde{\mathbf{W}}_i^*)^{-1} (\text{vec } \Gamma_i^{(n)})$  are asymptotically equivalent to  $Q_i^{(n)}$ , their asymptotic distribution is also chi-square—with however the additional property that they do not involve unknown quantities such as  $\mathbf{V}$ . This completes the proof.  $\square$

### 3. STATISTICAL APPLICATIONS

#### 3.1. Testing Randomness against Specified ARMA Dependence Alternatives.

The distributional results of Section 2 of course can be exploited in the problem of testing the null hypothesis  $\mathcal{H}^{(n)}$  of randomness. Two testing procedures are investigated here. The first one deals with the case of a specified alternative of ARMA dependence; the second one is a general, rank-based, portmanteau-type test, to be used against unspecified alternatives of serial dependence of order less than or equal to  $\kappa$ . Finally, a rank-based version of Tiao and Box's specification method is presented in Section 3.3.

If a particular local alternative of ARMA dependence, of the form (2.1), is considered, a "directional" test can be based on a linear combination of rank cross-covariances of the form

$$T_{\mathbf{W}\mathbf{W}}^{(n)} = \frac{\sum_{i=1}^{\kappa} \text{tr}\{\mathbf{W}_0^{*-1} \Gamma_i^{(n)} (\mathbf{A}_i + \mathbf{B}_i)^\top\}}{[\sum_{i=1}^{\kappa} \{\text{vec } \mathbf{W}_0^{*-1} (\mathbf{A}_i + \mathbf{B}_i)\}^\top \mathbf{W}_i^* \{\text{vec } \mathbf{W}_0^{*-1} (\mathbf{A}_i + \mathbf{B}_i)\}]^{\frac{1}{2}}}, \quad (3.1)$$

where

$$\mathbf{W}_0^* = n^{-1} \sum_{i=1}^n \mathbf{R}_i^{(n)} (\mathbf{R}_i^{(n)})^\top - \frac{(n+1)^2}{4} \mathbf{1}. \quad (3.2)$$

If necessary,  $\tilde{\mathbf{W}}_i^*$  can be substituted for  $\mathbf{W}_i^*$ ; as for  $\mathbf{W}_0^*$ , it is easy to see that  $(n+1)^{-2} \mathbf{W}_0^*$  is a consistent, permutationally invariant estimator of  $\mathbf{V}$ ; it is supposed here to be of full rank.

It follows from Proposition 2.1 that  $T_{\mathbf{W}\mathbf{W}}^{(n)}$  is exactly (i.e., permutationally) standardized, and hence asymptotically standard normal (under the assumptions of Proposition 2.1). Indeed, letting  $\mathcal{E}_*$  denote permutational expectations,

$$\begin{aligned} \mathcal{E}_* \{ \text{tr } \mathbf{W}_0^{*-1} \Gamma_i^{(n)} (\mathbf{A}_i + \mathbf{B}_i)^\top \}^2 &= \mathcal{E}_* \{ \text{vec}^\top (\mathbf{A}_i + \mathbf{B}_i) \text{vec} (\mathbf{W}_0^{*-1} \Gamma_i^{(n)}) \text{vec}^\top (\mathbf{W}_0^{*-1} \Gamma_i^{(n)}) \text{vec} (\mathbf{A}_i + \mathbf{B}_i) \} \\ &= \{ \text{vec}^\top (\mathbf{A}_i + \mathbf{B}_i) \} (\mathbf{I} \otimes \mathbf{W}_0^{*-1}) \mathbf{W}_i^* (\mathbf{I} \otimes \mathbf{W}_0^{*-1}) \{ \text{vec} (\mathbf{A}_i + \mathbf{B}_i) \} \\ &= \text{vec}^\top \{ \mathbf{W}_0^{*-1} (\mathbf{A}_i + \mathbf{B}_i) \} \mathbf{W}_i^* \text{vec} \{ \mathbf{W}_0^{*-1} (\mathbf{A}_i + \mathbf{B}_i) \}. \end{aligned}$$

Still on account of Proposition 2.1, the asymptotic power, under  $\mathcal{K}_f^{(n)}(\mathbf{A}, \mathbf{B})$ , of the test rejecting  $\mathcal{H}^{(n)}$  whenever  $T_{\mathbf{W}\mathbf{W}}^{(n)}$  exceeds the standard normal  $(1 - \alpha)$ -quantile  $k_{1-\alpha}$  is

$$\beta_{\mathbf{W}\mathbf{W}} = 1$$

$$- \Phi \left( k_{1-\alpha} - \frac{\sum_{i=1}^{\kappa} \text{tr} \{ \mathcal{E} \{ \mathbf{F}(\mathbf{X}) \Phi^\top(\mathbf{X}) \} (\mathbf{A}_i + \mathbf{B}_i) \mathcal{E} \{ \mathbf{F}(\mathbf{X}) \mathbf{X}^\top \} (\mathbf{A}_i + \mathbf{B}_i)^\top \mathbf{V}^{-1} \}}{(\sum_{i=1}^{\kappa} \text{tr} \{ (\mathbf{A}_i + \mathbf{B}_i) \mathbf{V} (\mathbf{A}_i + \mathbf{B}_i)^\top \mathbf{V}^{-1} \})^{\frac{1}{2}}} \right), \quad (3.3)$$

where the expectations are computed under  $\mathcal{H}_f^{(n)}$ , and  $\Phi$  denotes, as usual, the standard normal distribution function.

The corresponding normal-theory parametric procedure relies (Hallin *et al.* 1989, Section 4.1) on the usual sample cross-covariance matrices

$$\hat{\Gamma}_i^{(n)} = (n-i)^{-1} \sum_{t=i+1}^n \mathbf{X}_t^{(n)} (\mathbf{X}_{t-i}^{(n)})^T \quad (3.4)$$

[if the assumption that  $\mathcal{E} \mathbf{X}_t = \mathbf{0}$  is not to be made, the observations in (3.4) and in the subsequent formulas have to be centered about their mean] and rejects the null hypothesis of (supposedly full-rank Gaussian) randomness whenever

$$T_G^{(n)} = \frac{\sum_{i=1}^K (n-i)^{\frac{1}{2}} \text{tr}\{(\hat{\Gamma}_0^{(n)})^{-1} \hat{\Gamma}_i^{(n)} (\mathbf{A}_i + \mathbf{B}_i)^T\}}{\{\sum_{i=1}^K \{\text{vec}(\hat{\Gamma}_0^{(n)})^{-1} (\mathbf{A}_i + \mathbf{B}_i)\}^T (\hat{\Gamma}_0^{(n)})^{\otimes 2} \{\text{vec}(\hat{\Gamma}_0^{(n)})^{-1} (\mathbf{A}_i + \mathbf{B}_i)\}\}^{\frac{1}{2}}} > k_{1-\alpha}. \quad (3.5)$$

The resulting asymptotic power, still under  $\mathcal{K}_f^T(\mathbf{A}, \mathbf{B})$ , is

$$\beta_G = 1 - \Phi \left( k_{1-\alpha} - \left\{ \sum_{i=1}^K \text{tr}\{(\mathbf{A}_i + \mathbf{B}_i) \mathbf{\Sigma} (\mathbf{A}_i + \mathbf{B}_i)^T \mathbf{\Sigma}^{-1}\} \right\}^{\frac{1}{2}} \right). \quad (3.6)$$

The asymptotic relative efficiency of the Wald-Wolfowitz test based on  $T_{\text{WW}}^{(n)}$  with respect to the Gaussian one based on  $T_G^{(n)}$  accordingly is

$$\text{ARE}(T_{\text{WW}}^{(n)}; T_G^{(n)}) = \frac{(\sum_{i=1}^K \text{tr}\{\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{\Phi}^T(\mathbf{X})\}(\mathbf{A}_i + \mathbf{B}_i)\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{X}^T\}(\mathbf{A}_i + \mathbf{B}_i)^T \mathbf{V}^{-1}\})^2}{(\sum_{i=1}^K \text{tr}\{(\mathbf{A}_i + \mathbf{B}_i)\mathbf{V}(\mathbf{A}_i + \mathbf{B}_i)^T \mathbf{V}^{-1}\})(\sum_{i=1}^K \text{tr}\{(\mathbf{A}_i + \mathbf{B}_i) \mathbf{\Sigma} (\mathbf{A}_i + \mathbf{B}_i)^T \mathbf{\Sigma}^{-1}\})}. \quad (3.7)$$

The matrices  $\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{X}^T\}$ ,  $\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{\Phi}^T(\mathbf{X})\}$  and  $\mathbf{V} = \mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{F}^T(\mathbf{X})\} - \frac{1}{4}$  in general are not easy to compute. If however  $f$  itself is Gaussian, with covariance matrix  $\mathbf{\Sigma} = (\sigma_{ij})$ , then  $\mathbf{\Phi}(\mathbf{X}) = \mathbf{\Sigma}^{-1}\mathbf{X}$ ,

$$\mathcal{E}\{F_i(X_i)X_j\} = \frac{\sigma_{ij}}{\sigma_{ii}} \mathcal{E}\{F_i(X_i)X_i\} = \frac{\sigma_{ij}}{\sigma_{ii}^{\frac{1}{2}}} \int_0^1 u \Phi^{-1}(u) du = \frac{\sigma_{ij}}{\sigma_{ii}^{\frac{1}{2}}} (4\pi)^{-\frac{1}{2}},$$

which implies

$$\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{X}^T\} = \text{diag}(2\sqrt{\pi}\sigma)^{-1} \mathbf{\Sigma},$$

and

$$\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{\Phi}^T(\mathbf{X})\} = \mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{X}^T\} \mathbf{\Sigma}^{-1} = \text{diag}(2\sqrt{\pi}\sigma)^{-1},$$

where  $\text{diag}(2\sqrt{\pi}\sigma)^{-1}$  stands for the diagonal matrix with  $i$ th diagonal element  $(2\sqrt{\pi}\sigma_{ii}^{\frac{1}{2}})^{-1}$ . If moreover  $\mathbf{\Sigma} = \sigma^2 \mathbf{I}$ , then

$$\mathbf{V} = \mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{F}^T(\mathbf{X})\} - \frac{1}{4} = \left[ \int u^2 du - \frac{1}{4} \right] \mathbf{I} = \frac{\mathbf{I}}{12},$$

and (3.7) reduces to  $(3/\pi)^2$ , which is exactly the ARE, under Gaussian assumptions, of the univariate Wald-Wolfowitz test against serial dependence with respect to its parametric counterpart, based on usual sample autocorrelations (see, e.g., Hallin *et al.* 1985).



### 3.2. A Wald-Wolfowitz Multivariate Portmanteau Test.

Specified alternatives such as (2.1) seldom are considered in practice. In the case of an omnibus alternative of ARMA( $p, q$ ) dependence, with unspecified coefficients  $A_i$  and  $B_i$  but with orders  $p$  and  $q$  smaller than  $\kappa$ , one might like to consider the following rank-based *portmanteau* procedure, which consists in rejecting  $\mathcal{H}^{(n)}$  whenever

$$Q_{WW}^{(n)} = \sum_{i=1}^{\kappa} (\text{vec } \Gamma_i^{(n)})^T \{\bar{W}_i^*\}^{-1} (\text{vec } \Gamma_i^{(n)}) > \chi_{\kappa m^2; 1-\alpha}^2, \quad (3.8)$$

where  $\chi_{\kappa m^2; 1-\alpha}^2$  denotes the  $(1 - \alpha)$ -quantile of a chi-square variable with  $\kappa m^2$  degrees of freedom. The corresponding Gaussian portmanteau test relies on

$$Q_G^{(n)} = \sum_{i=1}^{\kappa} (n-i) (\text{vec } \hat{\Gamma}_i^{(n)})^T \{(\hat{\Gamma}_0^{(n)})^{\otimes 2}\}^{-1} (\text{vec } \hat{\Gamma}_i^{(n)}), \quad (3.9)$$

which is also (Chituri 1976) asymptotically chi-square, under the hypothesis of Gaussian, full-rank white noise, still with  $\kappa m^2$  degrees of freedom.

The asymptotic powers and relative efficiencies of these two portmanteau procedures under  $\mathcal{K}_f^{(n)}(\mathbf{A}, \mathbf{B})$  depend on the noncentrality parameters

$$\begin{aligned} \lambda_{WW}^2(\mathbf{A}, \mathbf{B}; f) &= \sum_{i=1}^{\kappa} \text{vec}^T(\mathbf{A}_i + \mathbf{B}_i) \\ &\quad \times [\mathcal{E}\{\mathbf{X}\mathbf{F}^T(\mathbf{X})\} \otimes \mathcal{E}\{\boldsymbol{\Phi}(\mathbf{X})\mathbf{F}^T(\mathbf{X})\}] (\mathbf{V}^{-1})^{\otimes 2} [\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{X}^T\} \otimes \mathcal{E}\{\mathbf{F}(\mathbf{X})\boldsymbol{\Phi}^T(\mathbf{X})\}] \\ &\quad \times \text{vec}(\mathbf{A}_i + \mathbf{B}_i) \\ &= \sum_{i=1}^{\kappa} \text{tr}[\mathbf{V}^{-1} \mathcal{E}\{\mathbf{F}(\mathbf{X})\boldsymbol{\Phi}^T(\mathbf{X})\}(\mathbf{A}_i + \mathbf{B}_i) \mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{X}^T\} \mathbf{V}^{-1} \\ &\quad \times \mathcal{E}\{\mathbf{X}\mathbf{F}^T(\mathbf{X})\}(\mathbf{A}_i + \mathbf{B}_i)^T \mathcal{E}\{\boldsymbol{\Phi}(\mathbf{X})\mathbf{F}^T(\mathbf{X})\}] \\ &= \sum_{i=1}^{\kappa} \text{vec}^T(\mathbf{A}_i + \mathbf{B}_i) [\mathcal{E}\{\mathbf{F}(\mathbf{X})\mathbf{X}^T\} \mathbf{V}^{-1} \mathcal{E}\{\mathbf{X}\mathbf{F}^T(\mathbf{X})\}] \\ &\quad \otimes [\mathcal{E}\{\boldsymbol{\Phi}(\mathbf{X})\mathbf{F}^T(\mathbf{X})\} \mathbf{V}^{-1} \mathcal{E}\{\mathbf{F}(\mathbf{X})\boldsymbol{\Phi}^T(\mathbf{X})\}] \text{vec}(\mathbf{A}_i + \mathbf{B}_i) \end{aligned}$$

and

$$\begin{aligned} \lambda_G^2(\mathbf{A}, \mathbf{B}; f) &= \sum_{i=1}^{\kappa} \text{vec}^T(\mathbf{A}_i + \mathbf{B}_i) (\boldsymbol{\Sigma} \otimes \mathbf{I})(\boldsymbol{\Sigma}^{-1})^{\otimes 2} (\boldsymbol{\Sigma} \otimes \mathbf{I}) \text{vec}(\mathbf{A}_i + \mathbf{B}_i) \\ &= \sum_{i=1}^{\kappa} \text{tr}\{\boldsymbol{\Sigma}^{-1}(\mathbf{A}_i + \mathbf{B}_i) \boldsymbol{\Sigma}(\mathbf{A}_i + \mathbf{B}_i)^T\} \\ &= \sum_{i=1}^{\kappa} \text{vec}^T(\mathbf{A}_i + \mathbf{B}_i) (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \text{vec}(\mathbf{A}_i + \mathbf{B}_i). \end{aligned}$$

The asymptotic relative efficiency of the Wald-Wolfowitz procedure with respect to the Gaussian one, under  $\mathcal{K}_f^{(n)}(\mathbf{A}, \mathbf{B})$ , accordingly is

$$\text{ARE}(Q_{\text{WW}}^{(n)}; Q_G^{(n)}) = \frac{\lambda_{\text{WW}}^2(\mathbf{A}, \mathbf{B}; f)}{\lambda_G^2(\mathbf{A}, \mathbf{B}; f)}. \quad (3.10)$$

Even under Gaussian assumptions, there is no guarantee that this ARE will be smaller than one; if, however,  $F$  is Gaussian with diagonal covariance matrix  $\sigma^2 \mathbf{I}$ , (3.10) again reduces to  $9/\pi^2$ .

### 3.3. A Rank-Based Version of Tiao and Box's Tentative Specification Method.

Tiao and Box (1981) suggest basing a tentative specification of multivariate ARMA models on the following device. Denoting by  $\hat{\rho}_{i;k,l}^{(n)}$  the sample cross-correlation

$$\sum_{t=i+1}^n X_{t,k}^{(n)} X_{t-i,l}^{(n)} \left( \sum_t (X_{t,k}^{(n)})^2 \sum_t (X_{t,l}^{(n)})^2 \right)^{-\frac{1}{2}},$$

they propose considering, for each lag  $i = 1, 2, \dots$ , a matrix where element  $(k, l)$  is replaced with

- a *plus* sign if  $\hat{\rho}_{i;k,l}^{(n)}$  is greater than  $2n^{-\frac{1}{2}}$ ,
- a *minus* sign if  $\hat{\rho}_{i;k,l}^{(n)}$  is less than  $-2n^{-\frac{1}{2}}$ ,
- a *dot* if  $\hat{\rho}_{i;k,l}^{(n)}$  lies in  $[\pm 2n^{-\frac{1}{2}}]$ .

These indicator symbols of course cannot be considered as resulting from formal (5%) significance tests, but (quoting the authors) "as a rather crude signal-to-noise ratio guide (...) providing useful and assimilable indicators of the general correlation pattern."

From Proposition 2.1, it follows that the rank-based cross-covariance coefficients  $\gamma_{i;k,l}^{(n)}$  are asymptotically normal under  $\mathcal{H}^{(n)}$ , with mean zero and variance  $(n+1)^4/144(n-i)$ . Accordingly, this widely used device of Tiao and Box can be modified in an obvious way, substituting rank-based cross-covariances for sample autocorrelations, and replacing  $\gamma_{i;k,l}^{(n)}$  in  $\Gamma_i^{(n)}$  with

- a *plus* sign if  $\gamma_{i;k,l}^{(n)}$  is greater than  $(n+1)^2/6(n-i)^{\frac{1}{2}}$ ,
- a *minus* sign if  $\gamma_{i;k,l}^{(n)}$  is less than  $-(n+1)^2/6(n-i)^{\frac{1}{2}}$ ,
- a *dot* if  $\gamma_{i;k,l}^{(n)}$  lies in  $[\pm(n+1)^2/6(n-i)^{\frac{1}{2}}]$ .

This Wald-Wolfowitz rank-based version of Tiao and Box's procedure is likely to be more reliable under non-Gaussian conditions, and more robust against gross error. Contrary to the methods described in Sections 3.1 and 3.2 above, its practical implementation is quite straightforward.

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