A Near Maximum Likelihood Decoding Algorithm for MIMO Systems Based on Semi-Definite Programming

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***Abstract*— In Multi-Input Multi-Output (MIMO) systems, Maximum-Likelihood (ML) decoding is equivalent to finding the closest lattice point in an N-dimensional complex space. In general, this problem is known to be NP hard. In this paper, we propose a quasi-maximum likelihood algorithm based on Semi-Definite Programming (SDP). We introduce several SDP relaxation models for MIMO systems, with increasing complexity. We use interior-point methods for solving the models and obtain a near-ML performance with polynomial computational complexity. Lattice basis reduction is applied to further reduce the computational complexity of solving these models[1].**

I. Introduction

Recently, there has been a considerable interest in Multi-Input Multi-Output (MIMO) antenna systems due to achieving a very high capacity as compared to single-antenna systems [1]. In MIMO systems, a vector is transmitted by the transmit antennas. In the receiver, a corrupted version of this vector affected by the channel noise and fading is received. Decoding concerns the operation of recovering the transmitted vector from the received signal. This problem is usually expressed in terms of "lattice decoding" which is known to be NP-hard.

In the last decade, Sphere Decoding (SD) is introduced as a Maximum Likelihood (ML) decoding method for MIMO systems with near-optimal performance [2]. The worst case complexity of the sphere decoder is exponential. In [3], the authors have claimed that the *average complexity* of this algorithm is polynomial time (almost cubic) over a wide range of SNR. However, recently, it has been shown that it is a misconception that the expected number of operations in SD asymptotically grows as a polynomial function of the problem size [4] (reference [4] derives an exponential lower bound on the average complexity of SD).

In [5], the authors introduce an efficient (polynominal computational complexity) quasi-maximum likelihood detection algorithm for lattice decoding based on a $\{-1, 1\}$ programming formulation and a semi-definite relaxation for rank one matrices. However, the method proposed in [5] is limited to Binary Phase Shift Keying (BPSK) modulation (as the input constellation per each transmit antenna).

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In this work, we develop an efficient approximate ML decoder for MIMO systems based on Semi-Definite Programming (SDP). In the proposed method, the transmitted vector is expanded as a linear combination (with zero-one coefficients) of all the possible constellation points in each dimension. Using this formulation, the distance minimization in Euclidean space is expressed in terms of a binary quadratic minimization problem. We derive a SDP relaxation using Lagrangian duality [6] for this problem. This results in a relaxation with many redundant constraints and with no strict interior for the feasible set. The feasible set is projected onto a face of the semi-definite cone, and, based on the identified redundant constraints, the final form of the relaxation is obtained. The relaxed problem has a polynomial time worst case complexity. Simulation results show that Bit Error Rate (BER) performance of the proposed algorithm is near optimal for M-ary QAM or PSK constellation (for an arbitrary binary labeling, say Gray labeling).

II. Problem Formulation

A MIMO system with $\tilde{N}$ transmit antenna and $\tilde{M}$ receive antenna is modeled as

$$\tilde{\mathbf{y}} = \sqrt{\frac{SNR}{\tilde{M}\tilde{E}_{s_{av}}}}\tilde{\mathbf{H}}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}, \tag{1}$$

where $\tilde{\mathbf{H}} = \left[\tilde{h}_{ij}\right]$ is the $\tilde{M} \times \tilde{N}$ channel matrix composed of independent, identically distributed complex Gaussian random elements with zero mean and unit variance, $\tilde{\mathbf{n}}$ is an $\tilde{M} \times 1$ complex additive white Gaussian noise vector with zero mean and unit variance, and $\tilde{\mathbf{x}}$ is an $\tilde{N} \times 1$ data vector whose components are selected from a complex set $\tilde{\mathcal{S}} = \{\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_K\}$ with an average energy of $\tilde{E}_{s_{av}}$. The parameter SNR in (1) is the Signal to Noise Ratio (SNR) per receive antenna.

To avoid using complex matrices, the system model (1) is represented by real matrices in (2).

$$\begin{bmatrix} \Re(\tilde{\mathbf{y}}) \\ \Im(\tilde{\mathbf{y}}) \end{bmatrix} = \sqrt{\frac{SNR}{\tilde{M}\tilde{E}_{s_{av}}}} \begin{bmatrix} \Re\left(\tilde{\mathbf{H}}\right) & \Im\left(\tilde{\mathbf{H}}\right) \\ -\Im\left(\tilde{\mathbf{H}}\right) & \Re\left(\tilde{\mathbf{H}}\right) \end{bmatrix} \begin{bmatrix} \Re(\tilde{\mathbf{x}}) \\ \Im(\tilde{\mathbf{x}}) \end{bmatrix} + \begin{bmatrix} \Re(\tilde{\mathbf{n}}) \\ \Im(\tilde{\mathbf{n}}) \end{bmatrix} \Rightarrow \mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{2}$$

where $\Re(.)$ and $\Im(.)$ denote the real and imaginary parts of a matrix, respectively, $\mathbf{y}$ is the *received vector*, and $\mathbf{x}$ is the *input vector*.

Consider the case that different components of $\tilde{\mathbf{x}}$ in (1), corresponding to the two-dimensional sub-constellations, are equal to the cartesian product of their underlying one-dimensional sub-constellations, e.g. QAM signalling. In this case, the components of $\mathbf{x}$ in (2) belong to the set $\mathcal{S} = \{s_1, \cdots, s_K\}$ with real elements, i.e.

$$x_i = u_i(1)s_1 + u_i(2)s_2 + \cdots + u_i(K)s_K, \tag{3}$$

where only one of the $u_i(j)$ is 1 and the rest are zero.

Let $N = 2\tilde{N}$, $\mathbf{S} = \mathbf{I}_N \otimes [s_1, \cdots, s_K]$. The vector $\mathbf{u} = [u_1(1)\cdots u_1(K)\cdots u_N(1)\cdots u_N(K)]^T$ is an $NK \times 1$ binary vector such that $\mathbf{Au} = \mathbf{e}_N$, where $\mathbf{A} = \mathbf{I}_N \otimes \mathbf{e}_K^T$. This constraint states that among each K components of the binary vector $\mathbf{u}$, i.e. $u_i(1), \cdots, u_i(K)$, there is only one element equal to "1". Then, the equation for the components of $\mathbf{x}$ in (3) reduces to $\mathbf{x} = \mathbf{Su}$ and the relationship for the MIMO system model is

$$\mathbf{y} = \mathbf{HSu} + \mathbf{n}. \tag{4}$$

At the receiver, the Maximum-Likelihood (ML) decoding rule is given by

$$\hat{\mathbf{x}} = \arg\min_{x_i \in \mathcal{S}} \|\hat{\mathbf{y}} - \mathbf{Hx}\|^2, \tag{5}$$

where $\hat{\mathbf{x}}$ is the most likely input vector and $\hat{\mathbf{y}}$ is the received vector. Noting $\mathbf{x} = \mathbf{Su}$, this problem is equivalent to

$$\min_{\mathbf{Au}=\mathbf{e}_N} \mathbf{u}^T\mathbf{S}^T\mathbf{H}^T\mathbf{HSu} - 2\hat{\mathbf{y}}^T\mathbf{HSu}, \tag{6}$$

where $\mathbf{u}$ is a binary vector. Let $\mathbf{Q} = \mathbf{S}^T\mathbf{H}^T\mathbf{HS}$, $\mathbf{c} = -\mathbf{S}^T\mathbf{H}^T\hat{\mathbf{y}}$. Therefore, the ML decoding problem is formulated as

$$\begin{aligned} \min\ & \mathbf{u}^T\mathbf{Qu} + 2\mathbf{c}^T\mathbf{u} \\ s.t.\ & \mathbf{Au} = \mathbf{e}_N \\ & u_i \in \{0,1\}^n, \end{aligned} \tag{7}$$

where $n = NK$. The formulation (7) is a quadratic minimization problem with binary variables [6]. Recent studies on solving binary quadratic minimization problems such as Graph Partitioning [7] and Quadratic Assignment Problem [8], show that semi-definite programming is a very promising approach to provide tight relaxations for such optimization problems. In the following, we derive several SDP relaxation models for the minimization problem in (7).

III. Semi-Definite Relaxation Solution

Consider the minimization problem in (7). Since $\mathbf{u}$ is a binary vector, the objective function is expressed as

$$\mathbf{u}^T\mathbf{Qu} + 2\mathbf{c}^T\mathbf{u} = \operatorname{trace}\left(\mathcal{L}_{\mathbf{Q}}\left[\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline \mathbf{u} & \mathbf{uu}^T \end{array}\right]\right),$$

where $\mathcal{L}_{\mathbf{Q}} := \left[\begin{array}{c|c} 0 & \mathbf{c}^T \\ \hline \mathbf{c} & \mathbf{Q} \end{array}\right]$.

Let $\mathcal{M}_{K\times N}$ denote the space of $K \times N$ real matrices and $\mathcal{E}_{K\times N}$ denote the set of all $K\times N$ binary matrices with column sums equal to one, i.e.

$$\mathcal{E}_{K\times N} = \left\{\mathbf{X} \in \mathcal{M}_{K\times N} : \mathbf{e}_K^T\mathbf{X} = \mathbf{e}_N^T,\ x_{ij} \in \{0,1\}\ \forall i,j\right\}. \tag{8}$$

Since the constraints $\mathbf{Au} = \mathbf{e}_N$, $u_i \in \{0,1\}^{NK}$ in (7) and $\mathbf{u} = \mathrm{vec}(\mathbf{U}), \mathbf{U} \in \mathcal{E}_{\mathrm{K\times N}}$ are equivalent, the minimization problem (7) can be written as

$$\begin{aligned} \min\quad & \operatorname{trace}\,\mathcal{L}_{\mathrm{Q}}\left[\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline \mathbf{u} & \mathbf{uu}^T \end{array}\right] \\ \text{s.t.}\quad & \mathbf{u} = \mathrm{vec}(\mathbf{U}),\ \mathbf{U} \in \mathcal{E}_{\mathrm{K\times N}}. \end{aligned} \tag{9}$$

For any $\mathbf{U} \in \mathcal{E}_{K\times N}$, $\mathbf{u} = \mathrm{vec}(\mathbf{U})$, the feasible points of (9) are expressed by

$$\mathbf{Y_u} = \left[\begin{array}{c} 1 \\ \hline \mathbf{u} \end{array}\right]\left[\begin{array}{cc} 1 & \mathbf{u}^T \end{array}\right] = \left[\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline \mathbf{u} & \mathbf{uu}^T \end{array}\right]. \tag{10}$$

The matrix $\mathbf{Y_u}$ is a rank-one and positive semi-definite matrix, i.e. $\mathbf{Y_u} \succeq 0$. Also, we have

$$\mathrm{diag}(\mathbf{Y_u}) = \mathbf{Y}_{\mathbf{u}_{0,:}} = \mathbf{Y}_{\mathbf{u}_{:,0}},$$

where $\mathbf{Y}_{\mathbf{u}_{0,:}}$ (resp. $\mathbf{Y}_{\mathbf{u}_{:,0}}$) denotes the first row (resp. the first column)[2] of $\mathbf{Y_u}$.

In order to obtain a tractable SDP relaxation of (9), we remove the rank-one restriction from the feasible set. In fact, the feasible set is approximated by another larger set , $\mathcal{F}$, defined as

$$\mathcal{F} := \mathrm{conv}\left\{\mathbf{Y_u} : \mathbf{u} = \mathrm{vec}(\mathbf{U}),\ \mathbf{U} \in \mathcal{E}_{\mathrm{K\times N}}\right\}, \tag{11}$$

where $\mathrm{conv}(.)$ denotes the convex hull of a set. This results in our first relaxation model:

$$\begin{array}{ll} \min & \operatorname{trace}\mathcal{L}_{\mathbf{Q}}\mathbf{Y} \\ \text{s.t.} & \mathbf{Y} \in \mathcal{F} \end{array} \tag{12}$$

It is clear that the matrices

$$\mathbf{Y_u} \text{ for } \mathbf{u} = \mathrm{vec}(\mathbf{U}),\ \mathbf{U} \in \mathcal{E}_{\mathrm{K\times N}}$$

are the feasible points of $\mathcal{F}$. Moreover, since these points are rank-one matrices, they are contained in the set of extreme points of $\mathcal{F}$. In other words, if the matrix $\mathbf{Y}$ is restricted to be rank-one in (12), i.e. $\mathbf{Y} = \left[\begin{array}{c} 1 \\ \hline \mathbf{u} \end{array}\right]\left[1\ \mathbf{u}^T\right]$, for some $\mathbf{u} \in \mathbb{R}^n$, then the optimal solution of (12) provides the optimal solution, $\mathbf{u}$, for (7).

It can be shown that any feasible point for the SDP relaxation (12) has to be singular, see [9]. This means that there are numerical difficulties in computing the solution for this problem. However, one can find a very simple structured matrix in the relative interior of the feasible set in order to project (and regularize) the problem into a smaller dimension. This results in the next relaxation model.

[2]Matrix $\mathbf{Y_u}$ is indexed from zero.

A. Geometry of the Relaxation

In order to approximate the feasible set $\mathcal{F}$ for solving the problem, we elaborate more on the geometrical structure of this set.

Theorem 1: Let

$$\mathbf{V}_K = \left[\begin{array}{c} \mathbf{I}_{K-1} \\ \hline -\mathbf{e}_{K-1}^T \end{array}\right] \in \mathcal{M}_{K\times(K-1)} \tag{13}$$

and

$$\hat{\mathbf{V}} = \left[\begin{array}{c|c} 1 & \mathbf{0}_{N(K-1)}^T \\ \hline \frac{1}{K}(\mathbf{e}_{NK} - (\mathbf{I}_N \otimes \mathbf{V}_K)\mathbf{e}_{(K-1)N}) & \mathbf{I}_N \otimes \mathbf{V}_K \end{array}\right], \tag{14}$$

where $\hat{\mathbf{V}} \in \mathcal{M}_{(NK+1)\times((K-1)N+1)}$. For any $\mathbf{Y} \in \mathcal{F}$ there exists a symmetric matrix $\mathbf{R}$ of order $N(K-1)+1$, indexed from 0 to $N(K-1)$, such that

$$\mathbf{Y} = \hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^T,\ \mathbf{R} \succeq 0, \quad \text{and} \quad r_{00} = 1,\ r_{ii} = r_{0i},\ \forall i. \tag{15}$$

Also, if $\mathbf{Y}$ is an extreme point of $\mathcal{F}$, then $r_{ij} \in \{0,1\}$, otherwise $r_{ij} \in [0,1]$ for $i,j \in \{0,\dots,N(K-1)\}$.

Proof: see [9]. ∎

Using Theorem 1, we can show that the set $\mathcal{F}_r$ contains $\mathcal{F}$:

$$\begin{aligned}\mathcal{F}_r = \big\{\mathbf{Y} \in \mathcal{S}_{NK+1} : \exists\, \mathbf{R} \in \mathcal{S}_{(K-1)N+1},\ \mathbf{R} \succeq \mathbf{0},\ R_{00} = 1,\\ \mathbf{Y} = \hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^T,\ \mathrm{diag}(\mathbf{Y}) = \mathbf{Y}_{0,:}\big\}.\end{aligned} \tag{16}$$

Therefore, the feasible set in (12) is approximated by $\mathcal{F}_r$ and the second relaxation model is expressed by

$$\begin{aligned}\min\ & \text{trace}\ (\hat{\mathbf{V}}^T\mathcal{L}_{\mathbf{Q}}\hat{\mathbf{V}})\mathbf{R} \\ \text{s.t.}\ & \mathrm{diag}(\hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^T) = (1, (\hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^T)_{0,1:n})^T \\ & \mathbf{R} \succeq 0.\end{aligned} \tag{17}$$

Solving the relaxation model in (12) over $\mathcal{F}$ results in the optimal solution of the original problem in (9), but this problem is NP-hard. Solving the relaxation model in (17) over $\mathcal{F}_r$ results in a weaker bound for the optimal solution of (9).

B. Tightening the Relaxation by Gangster Operator

In the following, we extract a condition which is implicit in the matrix $\mathbf{Y}_\mathbf{u}$ and explicitly add it to the relaxation model (17). Subsequently, some redundant constraint are removed and this results in an improved relaxation model.

Theorem 2: Let $\mathcal{U}$ denote the set of all binary vectors $\mathbf{u} = \mathrm{vec}(\mathbf{U}),\ \mathbf{U} \in \mathcal{E}_{\mathrm{K}\times\mathrm{N}}$. Define the *barycenter point*, $\hat{\mathbf{Y}}$, as the convex hull of all the feasible points in the minimization problem (9); therefore,

$$\hat{\mathbf{Y}} = \frac{1}{K^N}\sum_{\mathbf{u}\in\mathcal{U}} \mathbf{Y}_\mathbf{u} = \frac{1}{K^N}\sum_{\mathbf{u}\in\mathcal{U}} \left[\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline \mathbf{u} & \mathbf{u}\mathbf{u}^T \end{array}\right]. \tag{18}$$

Then $\hat{\mathbf{Y}}$ has (i) the value of 1 as its $(0,0)$ element, (ii) N blocks of dimension $K\times K$ which are diagonal matrices with elements $1/K$, and (iii) the first row and first column equal to the vector of its diagonal elements. The rest of the matrix is composed of $K\times K$ blocks with all elements equal to $1/K^2$:

$$\hat{\mathbf{Y}} = \left[\begin{array}{c|ccc} 1 & \multicolumn{3}{c}{\frac{1}{K}\mathbf{e}_n^T} \\ \hline & \frac{1}{K}\mathbf{I}_K & \frac{1}{K^2}\mathbf{E}_K & \cdots & \frac{1}{K^2}\mathbf{E}_K \\ \frac{1}{K}\mathbf{e}_n & \vdots & \vdots & \ddots & \vdots \\ & \vdots & \vdots & \ddots & \vdots \\ & \frac{1}{K^2}\mathbf{E}_K & \cdots & \frac{1}{K^2}\mathbf{E}_K & \frac{1}{K}\mathbf{I}_K \end{array}\right].$$

Proof: see [9]. ∎

Theorem 2 suggests a zero pattern for the elements of $\mathcal{F}$. We use a *Gangster Operator* [7] to represent these constraints more efficiently. Let J be a set of indices, then this operator is defined as

$$(\mathcal{G}_J(\mathbf{Y}))_{ij} = \begin{cases} Y_{ij} & \text{if } (i,j) \text{ or } (j,i) \in J \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Considering the barycenter point, we have $\mathcal{G}_J(\hat{\mathbf{Y}}) = 0$ for

$$\begin{aligned}J = \{(i,j) :\ & i = K(p-1)+q,\ j = K(p-1)+r, q < r, \\ & q,r \in \{1,\cdots,K\}, p \in \{1,\cdots,N\}\}.\end{aligned} \tag{20}$$

Since $\hat{\mathbf{Y}}$ is a convex combination of all matrices in $\mathcal{U}$ with entries either 0 or 1; hence, from (20), we have $\mathcal{G}_J(\mathbf{Y}_\mathbf{u}) = 0$. Also, all the points inside the feasible set $\mathcal{F}$ are the convex combination of $\mathbf{Y}_\mathbf{u}$. Therefore,

$$\mathcal{G}_J(\mathbf{Y}) = 0, \qquad \forall \mathbf{Y} \in \mathcal{F}. \tag{21}$$

The feasible set of the projected SDP in (17) is tightened by adding the constraints $\mathcal{G}_J(\mathbf{Y}) = 0$. By combining these constraints and (17), we note that there are some redundant constraints that can be removed to enhance the relaxation model. This is expressed in the following lemma.

Lemma 3: Let $\mathbf{R}$ be an arbitrary $(N(K-1)+1)\times(N(K-1)+1)$ symmetric matrix with

$$\mathbf{R} = \left[\begin{array}{c|ccc} r_{00} & \mathbf{R}_{01} & \cdots & \mathbf{R}_{0N} \\ \hline \mathbf{R}_{10} & \mathbf{R}_{11} & \cdots & \mathbf{R}_{1N} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{R}_{N0} & \mathbf{R}_{N1} & \cdots & \mathbf{R}_{NN} \end{array}\right], \tag{22}$$

where r_{00} is a scalar, $\mathbf{R}_{i0}$, for $i = 1,\cdots,N$ are $(K-1)\times 1$ vectors and $\mathbf{R}_{ij}$, for $i,j = 1,\cdots,N$, are $(K-1)\times(K-1)$ blocks of $\mathbf{R}$. Theorem 1 states that $\mathbf{Y} = \hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^T$. We can partition $\mathbf{Y}$ as

$$\mathbf{Y} = \left[\begin{array}{c|ccc} y_{00} & \mathbf{Y}_{01} & \cdots & \mathbf{Y}_{0N} \\ \hline \mathbf{Y}_{10} & \mathbf{Y}_{11} & \cdots & \mathbf{Y}_{1N} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{Y}_{N0} & \mathbf{Y}_{N1} & \cdots & \mathbf{Y}_{NN} \end{array}\right], \tag{23}$$

where y_{00} is a scalar, $\mathbf{Y}_{i0}$, for $i = 1,\cdots,N$ are $K\times 1$ vectors and $\mathbf{Y}_{ij}$, for $i,j = 1,\cdots,N$, are $K\times K$ blocks of $\mathbf{Y}$. Then,

1) $y_{00} = r_{00}$ and $\mathbf{Y}_{0i}\mathbf{e}_K = r_{00}$, for $i = 1,\cdots,N$.
2) $\mathbf{Y}_{0j} = \mathbf{e}_K^T\mathbf{Y}_{ij}$ for $i,j = 1,\cdots,N$.

Proof: see [9]. ∎

If the Gangster operator is applied to (17), it results in the following redundant constraint

$$\text{diag}(\hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^T) = (1, (\hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^T)_{0,1:n})^T.$$

Note that using Lemma 3, $\mathbf{Y}_{0j} = \mathbf{e}_K^T \mathbf{Y}_{jj}$ for $j = 1, \cdots, N$ and the off-diagonal entries of each $\mathbf{Y}_{jj}$ are zeros. Therefore, by defining a new set $\bar{J} = J \cup \{0, 0\}$ and eliminating the redundant constraints, we obtain a new SDP relaxation model:

$$\begin{aligned} \min \ & \text{trace}(\hat{\mathbf{V}}^T \mathcal{L}_{\mathbf{Q}} \hat{\mathbf{V}})\mathbf{R} \\ \text{s.t.} \ & \mathcal{G}_{\bar{J}}(\hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^{\mathbf{T}}) = \mathbf{E}_{00} \\ & \mathbf{R} \succeq 0, \end{aligned} \tag{24}$$

where $\mathbf{R}$ is an $(N(K-1)+1)\times(N(K-1)+1)$ matrix and $\mathbf{E}_{00}$ is an $(NK+1)\otimes(NK+1)$ all zero matrix except for a single element equal to 1 in its $(0,0)$th entry. With this new index set $\bar{J}$, we are able to remove all the redundant constraints while maintaining the SDP relaxation. The relaxation model in (24) corresponds to a tighter lower bound and has an interior point in its feasible set [9].

The relaxation in (24) is further tightened by considering the *non-negativity constraints* [8]. All the elements of the matrix $\mathbf{Y}$ which are not covered by the Gangster operator are greater than or equal to zero. These inequalities can be added to the set of constraints in (24), resulting in a stronger relaxation model:

$$\begin{aligned} \min \ & \text{trace}(\hat{\mathbf{V}}^T \mathcal{L}_{\mathbf{Q}} \hat{\mathbf{V}})\mathbf{R} \\ \text{s.t.} \ & \mathcal{G}_{\bar{J}}(\hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^{\mathbf{T}}) = E_{00} \\ & \mathcal{G}_{\hat{J}}(\hat{\mathbf{V}}\mathbf{R}\hat{\mathbf{V}}^{\mathbf{T}}) \geq 0 \\ & \mathbf{R} \succeq 0, \end{aligned} \tag{25}$$

where the set $\hat{J}$ indicates those indices which are not covered by $\bar{J}$.

Note that this model is considerably stronger than model (24) because non-negativity constraints are also imposed in the model. The advantage of this formulation is that the number of inequalities can be adjusted to provide a trade-off between the performance and complexity. The larger the model is, the better it approximates the optimization problem (9).

The most common methods for solving SDP problems of moderate sizes are IPMs, whose computational complexities are polynomial. There are a large number of IPM-based solvers to handle SDP problems, e.g., DSDP [10], SeDuMi [11], SDPA [12], etc. In our numerical experiments, we use DSDP and SDPA for solving (24), and SeDuMi is implemented for solving (25). Note that adding the non-negativity constraints increases the computational complexity of the model. Since the problem sizes of our interest are moderate, the complexity of solving (25) with IPM solvers is tractable.

IV. Complexity Reduction Using Lattice Basis Reduction

Lattice structures have been used frequently in different communication applications such as quantization or MIMO decoding. A real lattice Λ is a discrete set of M-dimensional vectors in the real Euclidean M-space, $\mathbb{R}^M$, that forms a group under the ordinary vector addition. Every lattice Λ is generated by the integer linear combinations of a set of linearly independent vectors $\{\mathbf{b}_1, \cdots, \mathbf{b}_N\}$, where $\mathbf{b}_i \in \Lambda$, and the integer $N(\leq M)$ is called the dimension of the lattice[3]. The set of vectors $\{\mathbf{b}_1, \cdots, \mathbf{b}_N\}$ is called a basis of Λ, and the $N \times M$ matrix $B = [\mathbf{b}_1, \cdots, \mathbf{b}_N]^T$ which has the basis vectors as its rows is called the basis matrix (or generator matrix) of Λ.

The basis for representing a lattice is not unique. Usually a basis consisting relatively short and nearly orthogonal vectors is desirable. The procedure of finding such a basis for a lattice is called *Lattice Basis Reduction*. Several distinct notions of reduction have been studied, including Lenstra-Lenstra and Lovasz (LLL) reduced basis [13], which can be computed in polynomial time.

An initial solution for the lattice decoding problem can be computed using one of the simple sub-optimal algorithms such as zero forcing decoder or channel inversion, e.g. $\mathbf{s}' = \left[\mathbf{H}^{-1}\mathbf{y}\right]$. If the channel is not ill-conditioned, i.e. the columns of the channel matrix are nearly orthogonal and short, it is most likely that the ML solution of the lattice decoding problem is around $\mathbf{s}'$. In the case of orthogonal basis for the channel, the ML solution is exactly $\mathbf{s}'$. Therefore, using a reduced basis for the lattice, each x_i in (3) can be expressed by a few points in $\mathcal{S}$ around s_i', not all the points in $\mathcal{S}$.

Let $\mathbf{L} = \mathbf{HQ}$ be the LLL reduced basis for the channel matrix $\mathbf{H}$, where $\mathbf{Q}$ is a unimodular matrix. The MIMO system model in (2) can be written as

$$\mathbf{y} = \mathbf{L}\mathbf{Q}^{-1}\mathbf{x} + \mathbf{n} \tag{26}$$

Consider the QAM signaling. Without loss of generality, we can assume coordinates of $\mathbf{x}$ are in the integer grid. Since $\mathbf{Q}$ is a unimodular matrix, the coordinates of a new variable defined as $\mathbf{x}' = \mathbf{Q}^{-1}\mathbf{x}$ are also in the integer grid. Therefore, the system in (26) is modelled by $\mathbf{y} = \mathbf{L}\mathbf{x}' + \mathbf{n}$. Note that by multiplying $\mathbf{x}$ by $\mathbf{Q}^{-1}$ the constellation boundary will change. However, it is shown that in the lattice decoding problem with finite constellations the best approach is to ignore the boundary and compute the solution [14]. If the solution is outside the region, it is considered as an error.

In order to implement the proposed method using LLL basis reduction, each component of $\mathbf{x}'$ is expressed by a linear combination (with zero-one coefficients) of L (usually much smaller than K) integers around s_i', where $\mathbf{s}' = \left[\mathbf{L}^{-1}\mathbf{y}\right]$. Then, the proposed algorithm can be applied to this new model. The importance of this new model is recognized with large constellations. It is worth to emphasis that the dimension of the semi-definite matrix $\mathbf{Y}$ is $N*(K-1)+1$. Therefore, the LLL reduction decreases the dimension of the matrix $\mathbf{Y}$, and consequently, decreases the computational complexity of the proposed algorithm. The performance of this method is shown in the simulation results.

[3]Without loss of generality, we assume that $N = M$.

V. Simulation Results

We simulate the two proposed model for decoding in MIMO systems with QAM and PSK constellations. Fig. 1 demonstrates that the proposed quasi-ML method using model (24) and the randomization procedure achieves near ML-BER performance with hard output in uncoded MIMO channels, which consist of $\tilde{N} = 4$ transmit antennas and $\tilde{M} = 4$ receive antennas and which are modulated with 16-QAM or 16-PSK constellations. It exhibits that the model in (25) is stronger than the model proposed in (24).

VI. Complexity Analysis

Semi-definite programs of reasonable size can be solved in polynomial time within any specified accuracy by IPMs. IPMs are iterative algorithms which use a Newton-like method to generate search directions to find an approximate solution to the nonlinear system. The IPMs converge vary fast and an approximately optimal solution is obtained within a polynomial number of iterations. In the sequel, we provide an analysis for the worst case complexity of solving models (24) and (25) by IPMs.

Since the SDP relaxation (24) contains $O(K^2N)$ equality constraints, it follows that a solution to (24) can be found in at most $O(N^{4.5}K^{6.5}\log(1/\epsilon))$ arithmetics operations where ϵ is a given accuracy [9]. SDP relaxation (25) contains $O(K^2N)$ equations and $O(K^2N^2)$ sign constraints. In order to solve relaxation (25) we formulate the SDP model as a standard linear cone program by adding slack variables, and solve the linear conic problem by using the optimization software SeDuMi [11]. The additional inequality constraints make the model in (25) considerably stronger than the model in (24) (see numerical results), but also more difficult to solve. An interior point method for solving SDP model (25) within a tolerance ϵ, requires $O(N^{6.5}K^{6.5}\log(1/\epsilon))$ arithmetics operations [9]. Since the problem sizes of our interest are moderate, the problem in (25) is tractable. However, a trade-off between the strength of the bounds and the computational effort for solving these two models is notable.

The randomization procedure that we perform in order to strengthen the bound obtained form the model (24) is negligible compared with that of solving the problem itself. Namely if we denote by N_{rand} the number of randomization, then the worst case complexity is $O(NKN_{rand})$.

The problems (24) and (25) are polynomially solvable. These problem have many variables, contain sparse low rank constraint matrices. However, exploiting the structure and sparsity characteristic of semi-definite programs can be critical to the efficient computation of their solution. All the constraint matrices in relaxation models (24) and (25) are rank-one [9]. In [15], it is shown that rank one matrices reduce the complexity of interior point algorithms for positive semi-definite programming by a factor of NK. Also, it converges linearly and has a saving in computation time and memory requirements.

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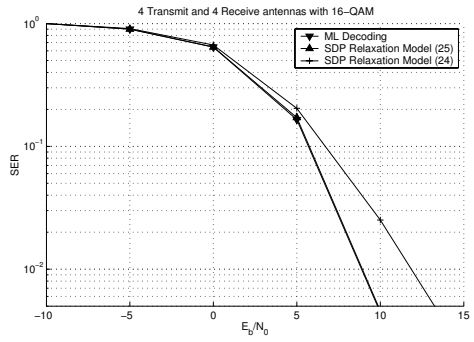

Fig. 1. Symbol Error Rates for the proposed algorithm and ML Decoding