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A necessary and sufficient condition for the representation of a function as a Hankel–Stieltjes transform

by

F. M. CHOLEWINSKI, D. T. HAIMO and A. E. NUSSBAUM

(St. Louis, Missouri) *

1. Introduction. Our goal is to establish for the Hankel–Stieltjes transform the analogue of Bochner's celebrated theorem on the integral representation of continuous positive definite functions. Surprisingly, the result does not appear in the literature although our method is entirely parallel to that used in standard proofs of the classical theorem. In a remark at the end of the paper, we show that our result strongly improves a theorem of S. Bochner.

2. Definitions and preliminary results. Let ν be a fixed positive number. We define

$$\mathscr{J}(x) = 2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})x^{\frac{1}{2}-\nu}J_{\nu-\frac{1}{2}}(x),$$

where $J_\gamma(x)$ is the ordinary Bessel function of order γ.

We set

$$d\mu(x) = [2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})]^{-1}x^{2\nu}dx,$$

and we define $L_\nu^1[0,\infty)$ as the linear space of Lebesgue measurable functions on $[0,\infty)$ for which

$$\|f\| = \Big[\int_0^\infty |f(x)|\,d\mu(x)\Big] < \infty.$$

By $L^\infty[0,\infty)$ we denote the ordinary space of essentially bounded Lebesgue measurable functions. Further, we let

$$D(x,y,z) = \frac{2^{2\nu-\frac{5}{2}}[\Gamma(\nu+\frac{1}{2})]^2}{\Gamma(\nu)\pi^{1/2}}(xyz)^{1-2\nu}[\Delta(x,y,z)]^{2\nu-2},$$

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where $\Delta(x, y, z)$ is the area of a triangle whose sides are x, y, z if there is such a triangle, and otherwise, $D(x, y, z) = 0$. We have the following basic formula

$$\int_0^\infty \mathcal{J}(zt) D(x, y, z)\, d\mu(z) = \mathcal{J}(xt)\, \mathcal{J}(yt),$$

valid for $0 < x, y < \infty$, $0 \leqslant t < \infty$; see [2]. Setting $t = 0$, we note that

$$\int_0^\infty D(x, y, z)\, d\mu(z) = 1.$$

If f and g are elements of $L_\nu^1[0, \infty)$, and if we set

$$(f \# g)(x) = \int_0^\infty \int_0^\infty f(y) g(z) D(x, y, z)\, d\mu(y)\, d\mu(z),$$

then $f \# g \in L_\nu^1[0, \infty)$ and $(f \# g)^\wedge = f^\wedge g^\wedge$, where

$$f^\wedge(t) = \int_0^\infty \mathcal{J}(xt) f(x)\, d\mu(x), \quad 0 \leqslant t < \infty,$$

is the Hankel transform of $f \in L_\nu^1[0, \infty)$. Moreover, if $\nu > 0$, then $L_\nu^1[0, \infty)$ is a self-adjoint, commutative, semi-simple Banach algebra (with involution defined by complex conjugation), and the space of closed maximal ideals is isomorphic to the set $[0, \infty)$ endowed with the usual topology; see [3]. The set of functions

$$G(x; t) = \frac{1}{(2t)^{\nu+\frac{1}{2}}} e^{-x^2/4t}, \quad t > 0,$$

is an approximate identity in $L_\nu^1[0, \infty)$; see [2].

Definition 2.1. The *associated function* or *Delsarte convolution kernel* corresponding to a function $f \in L^\infty[0, \infty)$ is given by

$$f(x, y) = \int_0^\infty f(z) D(x, y, z)\, d\mu(z), \quad 0 < x, y < \infty.$$

Definition 2.2. P_ν is the set of functions f in $L^\infty[0, \infty)$ such that

$$\sum_{k=1}^n \sum_{l=1}^n a_k \bar{a}_l f(x_k, x_l) \geqslant 0$$

for any $x_1, x_2, \ldots, x_n > 0$ and arbitrary complex numbers $a_1, a_2, \ldots, a_n$.

We note that the discrete condition of the definition implies its integrated counterpart

$$\int_0^\infty \int_0^\infty a(x) \bar{a}(y) f(x, y)\, d\mu(x)\, d\mu(y) \geqslant 0$$

for arbitrary $a \in L_\nu^1[0, \infty)$.

3. Main result. In what follows, a.e. denotes almost everywhere with respect to Lebesgue measure.

THEOREM. *A necessary and sufficient condition that a Lebesgue measurable function f on $[0, \infty)$ be representable as a Hankel–Stieltjes transform*

$$f(x) = \int_0^\infty \mathcal{J}(xt)\, d\gamma(t), \quad \text{a.e.},$$

of a bounded, positive Radon measure γ on $[0, \infty)$ is that $f \in P_\nu$. Furthermore, the measure γ is unique.

Proof. The necessity of the condition is verified at once by direct computation. If

$$f(x) = \int_0^\infty \mathcal{J}(xt)\, d\gamma(t), \quad \text{a.e.},$$

then

$$|f(x)| \leqslant \int_0^\infty d\gamma(t), \quad \text{a.e.},$$

and $f \in L^\infty[0, \infty)$. Further, by Fubini's theorem we have

$$f(x, y) = \int_0^\infty f(z) D(x, y, z)\, d\mu(z) = \int_0^\infty d\gamma(t) \int_0^\infty \mathcal{J}(zt) D(x, y, z)\, d\mu(z)$$

$$= \int_0^\infty \mathcal{J}(xt)\, \mathcal{J}(yt)\, d\gamma(t).$$

Hence

$$\sum_{i=1}^n \sum_{j=1}^n a_i \bar{a}_j f(x_i, x_j) = \int_0^\infty \Big| \sum_{i=1}^n a_i \mathcal{J}(x_i t) \Big|^2 d\gamma(t) \geqslant 0,$$

and $f \in P_\nu$.

Conversely, suppose that $f \in P_\nu$. We proceed in a way completely analogous to that used first by D. Raikov to prove Bochner's theorem on the integral representation of continuous, positive definite functions.

Consider the linear functional φ on the Banach algebra $L_\nu^1[0, \infty)$ defined by

$$\varphi(a) = \int_0^\infty f(x) a(x)\, d\mu(x).$$

Then, φ is continuous since

$$|\varphi(a)| \leqslant \|f\|_\infty \int_0^\infty |a(x)|\,d\mu(x) = \|f\|_\infty \|a\|.$$

Further, φ is positive definite; i.e., $\varphi(a \# a^*) \geqslant 0$ for all $a \in L_\nu^1[0, \infty)$. Indeed,

$$\begin{aligned}\varphi(a\# a^*) &= \int_0^\infty f(x)(a\# a^*)(x)\,d\mu(x)\\ &= \int_0^\infty f(x)\,d\mu(x) \int_0^\infty \int_0^\infty a(y)\bar{a}(z) D(x, y, z)\,d\mu(y)\,d\mu(z)\\ &= \int_0^\infty \int_0^\infty a(y)\bar{a}(z)\,d\mu(y)\,d\mu(z) \int_0^\infty f(x) D(x, y, z)\,d\mu(x)\\ &= \int_0^\infty \int_0^\infty a(y)\bar{a}(z) f(y, z)\,d\mu(y)\,d\mu(z) \geqslant 0\end{aligned}$$

since $f \in P_\nu$. Now, $L_\nu^1[0, \infty)$ has an approximate identity and hence φ is extendable ([5], p. 188). Therefore, by the Herglotz–Bochner–Weil–Raikov theorem ([4], p. 97), there exists a unique, bounded, positive Radon measure γ on the maximal ideal space $[0, \infty)$ of $L_\nu^1[0, \infty)$ such that

$$\varphi(a) = \int_0^\infty \hat{a}(t)\,d\gamma(t)$$

for all $a \in L_\nu^1[0, \infty)$. That is,

$$\int_0^\infty a(x) f(x)\,d\mu(x) = \int_0^\infty d\gamma(t) \int_0^\infty \mathscr{I}(xt) a(x)\,d\mu(x) = \int_0^\infty a(x)\,d\mu(x) \int_0^\infty \mathscr{I}(xt)\,d\gamma(t)$$

for all $a \in L_\nu^1[0, \infty)$. It follows that

$$f(x)\,d\mu(x) = \int_0^\infty \mathscr{I}(xt)\,d\gamma(t)\,d\mu(x),$$

which is equivalent to

$$f(x) = \int_0^\infty \mathscr{I}(xt)\,d\gamma(t), \qquad \mu\text{— a.e.}$$

and hence a.e. Finally, if ω is another bounded, positive measure such that

$$f(x) = \int_0^\infty \mathscr{I}(xt)\,d\omega(t), \qquad \text{a.e.},$$

then

$$\begin{aligned}\varphi(a) &= \int_0^\infty a(x)\,d\mu(x) \int_0^\infty \mathscr{I}(xt)\,d\omega(t) = \int_0^\infty d\omega(t) \int_0^\infty a(x)\mathscr{I}(xt)\,d\mu(x)\\ &= \int_0^\infty \hat{a}(t)\,d\omega(t)\end{aligned}$$

and so $\omega = \gamma$.

Corollary. *A bounded continuous function f on $[0, \infty)$ is the Hankel–Stieltjes transform of a bounded positive Radon measure γ on $[0, \infty)$ if and only if*

$$a \in L_\nu^1[0, \infty),\ \hat{a} \geqslant 0 \quad \textit{implies} \quad \int_0^\infty f(x) a(x)\,d\mu(x) \geqslant 0.$$

Proof. The "only if" part is trivial, for

$$\int_0^\infty f(x) a(x)\,d\mu(x) = \int_0^\infty \int_0^\infty a(x)\mathscr{I}(xt)\,d\gamma(t)\,d\mu(x) = \int_0^\infty \hat{a}(t)\,d\gamma(t) \geqslant 0.$$

To prove the "if" part, we note that if the hypothesis is satisfied, then $a \in L_\nu^1[0, \infty)$ implies that

$$\varphi(a\# a^*) = \int_0^\infty f(x)(a\# a^*)(x)\,d\mu(x) \geqslant 0.$$

Since $(a\# a^*)\hat{}\,(x) = |\hat{a}(x)|^2 \geqslant 0$. Thus, $f \in P_\nu$ and the result follows from the preceding theorem.

Remark. The corollary (and a fortiori the main theorem) vastly improves a theorem of S. Bochner ([1], p. 40, Theorem 3.6.1) which states that a bounded continuous function f on $[0, \infty)$ is the Hankel–Stieltjes transform of a bounded positive measure γ on $[0, \infty)$ if and only if

$$a \in L_\nu^1[0, \infty),\ \hat{a} \geqslant 0 \quad \text{implies} \quad \int_0^\infty \mathscr{I}(xt) f(t) a(t)\,d\mu(t) \geqslant 0$$

for all $0 \leqslant x < \infty$.

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CLEMSON UNIVERSITY
UNIVERSITY OF MISSOURI-ST. LOUIS
WASHINGTON UNIVERSITY

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