# A Network Economic Model for Supply Chain versus Supply Chain Competition

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**Abstract**

We study a supply chain economy that comprises heterogeneous supply chains involving multiple products and competing for multiple markets. The proposed network model is built upon operation links and interface links, representing, respectively, substantial supply chain operations and coordination functions between the operations. The paper presents a variational inequality formulation of the problem, the solution of which determines the winning supply chains and their market shares in the equilibrium of supply chain economy. We furnish qualitative properties such as existence and uniqueness of the equilibrium. Numerical examples are presented for illustrative purpose.

Keywords:
Supply chain economy, inter supply chain competition, operation and interface links, market pertinent chain, equilibrium, variational inequality problem.

## Introduction

To date, the major focus of the supply chain management literature has been on intra-chain subjects, such as distribution network design, inventory management, and production coordination (c.f., Beamon (1998), Bramel and Simchi-Levi (1997), Geoffrion and Power (1995) and references therein). Most of these are studied as an optimization problem in a centralized decision making scenario (c.f., Erenguc et al (1999), Federgruen (1993), and references therein). Lakhal et al (1999) and Lakhal et al (2001) provide an analytical framework for the study of competitive strategy formulation in which elementary resources, elementary methods, products, and activities are characterized in a bipartite graph model for a network company. Certain intra-chain competition problems are investigated through game theoretic models (e.g., Corbett and Karmarkar (2001), Nagurney, Dong and Zhang (2002), and Dong, Zhang and Nagurney (2002, 2004)). There is little analytic work however in the literature that studies the interaction of multiple supply chains and particularly, supply chain versus supply chain competition.

Most recognize that it is no longer a firm versus a firm but rather a supply chain versus a supply chain in the competition for many marketplaces today. Thus, the following questions naturally rise.
How do supply chains compete against each other?
Which supply chain will win the competition?
How much market share will a winning supply chain obtain?

In an attempt to answer the above questions, this paper proposes a modeling initiative to formalize both intra and inter supply chain cooperation and competition. We consider a supply chain economy, which can be perceived as a network of business agents and their activities involved in the production, distribution, vendition, and consumption of one or many related products. The agents, representing different stage supply chain entities, compete to form coalitions of supply chains, based on their specialized strengths and core competences. A supply chain economy may comprise several supply chains, which interact (cooperate and compete) in their procurement, production, distribution and retailing activities. In accordance with the spatial price equilibrium theory (c.f. Samuelson (1952), Takayama and Judge (1971), Nagurney, Takayama and Zhang (1995)), we define the equilibrium of supply chain economy that characterizes the winning chains as those with the lowest total marginal costs. This also reflects the philosophy that the firms' cooperation quest in supply chain management is to make the final product at overall lesser total cost than competing sets of supply chain firms (Cavinato (1992)). As the core contribution of this paper, we present a mathematical formulation for a supply chain economy as a variational inequality problem whose solution can be used to determine the answer to the above questions.

Our model features the following novel characteristics. (1) It comprises operation links and interface links with their orientation indicating the direction of the flow of material or information. We use an operation link to represent a business function performed by a firm in a supply chain network. To model the supply chain interactions, an operation link can participate in multiple chains, such as a third party logistic firm like UPS. We use an interface link to represent a coordination function within a supply chain. It may serve as a B2B bridge and can be used to measure the effectiveness of coordination and integration of different stage supply chain operations. (2) While most of the existing models for intra-chain competition consider a homogeneous product, the model presented in this paper is capable of studying a supply chain economy revolving inter chain cooperation and competition for multiple related products at multiple markets. (3) The proposed model accommodates the study of heterogeneous chains with different numbers of tiers, unlike most research that study problems limited to a fixed number of tiers/stages (e.g., Cachon and Zipkin (1999), Nagurney, Dong and Zhang (2002), and Dong, Zhang and Nagurney (2002, 2004)). In addition, the different tiers in our model may represent different functions not necessarily confined to a certain type, such as product assembly as in Corbett and Karmarkar (2001). (4) The cost functions represent generic costs for operation and coordination in supply chain management that measures not only the monetary cost, but also other ineffectiveness such as delivery time and quality issues. So, the winning chains determined by the model are actually the most "effective" chains that should demonstrate highest overall competence. (5) Furthermore, no artificial conditions are imposed on the cost functions for the operation and interface links. Therefore, the model can serve as a general framework, and the qualitative properties derived in this paper should hold for all application problems. Moreover, the main results can be sharpened under additional conditions such as linear cost functions as used in Corbett and Karmarkar (2001).

The rest of the paper is organized as follows. In Section 2, we propose the concepts of a supply chain economy and a market pertinent chain, which are essential to our study and fundamental for our model building. In Section 3, we present a network model for a supply chain economy that incorporates the process rate of bill of materials, accounts for link cost imputation, and addresses inter and intra chain cooperation and competition. Section 4 introduces the equilibrium of a supply chain economy, which is the core concept that characterizes the winning supply chains and their material flows. Variational inequality problem formulations in chain variables and in link variables are proposed for the equilibrium of a supply chain economy. Section 5 explores qualitative properties such as existence and uniqueness of the solution to the variational inequality problem formulations. We also give conditions under which there are equivalent optimization formulations. For illustrative purpose, we give numerical examples in Section 6, and shed light on the economic implications of the numerical solutions. Finally, Section 7 concludes the paper, suggests application prospects of the model, and proposes future research directions.

## 2. Supply Chain Economy

In this section, we introduce several new concepts that are essential for the study of supply chain (SC) formation and supply chain versus supply chain (SC vs. SC) competition. These concepts will be mathematically formulated into a network model in subsequent sections.

Although the existing definitions of SC are sufficiently clear for the subject of intra SC management, they usually do not define the boundary of a SC, or specify the border between two SCs. Hence, these definitions are inadequate to address inter chain problems, such as SC vs. SC competition. This entails the proposition of a market pertinent chain, defined below as a sub-chain in a network structure, which is analogous to the concept of a product-market chain in Lakhal et al (1999).

**Definition 1 (Market Pertinent Chain)**

*A market pertinent SC (in short, a M-chain) pertaining to an end product (or service) market, is a network of coordinated business activities involved in procurement, production, distribution, and vendition, that are associated with the delivery of the pertinent product (or service) to the pertinent market. Or, taking advantage of the commonly accepted notion of a SC, a M-chain is that part of its parent SC that is involved in delivery of the pertinent product (or service) to the pertinent market.*

Every SC is ultimately market driven and can be oriented towards more than one market. It is not uncommon that one SC wins over another SC in one market and loses to it in another. Therefore, the competition between SCs that are embroiled in many markets cannot be fully captured unless the competition of their individual products in individual markets are specified and investigated. From the above definition, a SC can be viewed as a family of coordinated M-chains, each of which is pulled by an end product (service) market. Viewing a SC as a composite of its M-chains offers a new dimension of inter-chain competition when studying intra chain issues such as vertical integration, inventory management and distribution. As depicted in Figure 1, Ford SC can be viewed as a composite of many M-chains including those for end markets of family cars, sport cars, and trucks. These are coordinated within the Ford SC in production and distribution, while competing with the corresponding GM M-chains in family car, sport car, and truck markets. It is from this perspective that one can see how intra-chain coordination would impact inter-chain competition and vice versa.

Today, no one would ignore the coexistence of multiple SCs and that many, such as Wal-Mart, Kmart, Target and Sears, compete in the same business. Of course, SC interactions are not limited to market competition but may include cooperation and other mutual influences as reflected in their material procurement, resource utilization, utility consumption, and production and distribution patterns. The network concept of a supply chain economy (SCE) is proposed to facilitate the development of analytical models for studying multi-dimensional SC interactions.

**Definition 2 (Supply Chain Economy)**

*A SCE is a network of interrelated activities of procurement, production, distribution, vendition, and consumption of one or many related products or services, conducted by several coalitions of business entities who act collectively within a coalition. Take as granted the definition of SC, we can state that a SCE is a network of interrelated SCs.*

A SCE describes a competitive environment of all the operation-related and market-related SCs under study. It allows one to examine any of its element SC internally, at the collective agents in terms of their operation and coordination, and externally, at the SC as a unified entity in terms of its productivity, competence, or market position. Figure 2 depicts the hierarchy of a SCE, its SCs, and their M-chains. Besides the subordinate relationship, it also shows the SC vs. SC competition. Particularly, M-chain 1A of SC 1 and M-chain 2A of SC 2 compete for market A, and M-chain 2B of SC 2 and M-chain 3B of SC 3 compete for market B.

## 3. Network Model

In this section, we present a network model for a supply chain economy (SCE). Let $G =[N, L]$ denote a connected network, where $N$ denotes the set of nodes and $L$ denotes the set of directed links in this network.

A link in our network model can be either an operation link or an interface link, with the orientation indicating the direction of flow of material or information. An operation link represents a substantial business function performed by a firm in a SC network and, as such, can be a manufacturing operation, a transportation operation, a storage operation, or a service operation. In general, our model allows an operation link to participate in multiple SCs. Examples would include a third party logistic (3PL) company, that provides transportation for several SCs, and a manufacturer, that participates in different and even competing SCs. For instance, Goodyear tire manufacturer makes tires for both Ford and GM. An interface link represents a coordination function between two successive operation links in a SC. In our model, an interface link is defined to be SC specific and thus, every interface link only belongs to one SC. Let $A$ denote the set of operation links and $B$ denote the set of interface links. Thus, we have $L = A \bigcup B$.

An M-chain in our model will be denoted by a connected subgraph of $G$, containing several origin node(s) and one destination node, which corresponds to its pertinent market. Thus, a SC, as a family of coordinated M-chains, is a connected subgraph with at least one origin node and at least one destination node. The entire $G$ represents the SCE under study, which usually comprises several interactive SCs. Suppose that, in a SCE, there are $m$ end products (or services), $i=1,\ldots,m$, which are sold at $n$ marketplaces, $j=1,\ldots,n$, that are spatially located in different geographic areas or being on the Internet. As different products can be sold at the same marketplace and the same product can be sold at different marketplaces, we have a total of $mn$ couples of $(i,j)$ to indicate all possible individual end consumer markets with product $i$ and marketplace $j$.

A potential M-chain pertaining to an end consumer market $(i,j)$ is represented by a connected subgraph of the network characterized by a destination node $(i,j)$. However, a potential M-chain may not gain realization (winning through competition) in practice unless it carries a positive flow of material, in which case it becomes active and will be called an active M-chain. One of the objectives of this paper is to find active M-chains through competition as well as their output to the markets. We will use $S_{ij}$ to denote the set of all potential M-chains pertaining to the end consumer market $(i, j)$, and let $S$ denote the set of all the potential M-chains. The chain flow of a M-chain is defined to be the volume of final product or service that the M-chain delivers to its pertinent end consumer market, or the output of the M-chain. Let $X_s$ denote the chain flow of chain $s$. Let $X = (X_s, s \in S)$ denote the vector of chain flows of all the M-chains in the SCE under study.

The output level of the end product of an M-chain determines the amount of work processed at each stage of its upstream material flow. Namely, tracking back the flow of material from the end product and exploding a series of bills of materials, one can determine the amount of work (or information processed or resources absorbed) in each participant link of an M-chain, in terms of labor hours, machine hours, utility consumed, or storage space taken, etc. Precisely, we present the following definition of process rate of an operation link with respect to an M-chain in which it participates (a similar term is called the absorption coefficient in Erenguc et al (1999)). For convenience, we adopt a convention that all the process rates are no less than one. In practice, a process rate can always be scaled to an appropriate unit so that it is greater or equal to one.

**Definition 3 (Process Rate)**

*For an M-chain $s$ and its participant operation link $a$, the process rate of $a$ with respect to $s$, denoted by, $\lambda_{as}$, is the amount of resources taken or work processed by link $a$, necessary to make one unit of end product of the M-chain $s$.*

For any link $a \in L$, we denote by $x_a$, the link flow of $a$. For an operation link $a \in A$, $x_a$ is the total amount of work processed on this link in the network of the SCE. For instance, the link flow of a transportation operation represented by a trucking company should be the total shipments it is handling for all the M-chains in which it participates. Units for measuring an operation link flow can vary according to the function of the operation, with examples such as a fabricated part, machine hour, labor hour, truckload, or cubic space (in a warehouse). We define the operational link-chain incidence matrix $\Delta = (\delta_{as})_{a \in A, s \in S}$ to be a $|A| \times |S|$ zero-one matrix, where

$$\delta_{as} = \begin{cases} 1, & \textit{if} \quad \textit{link } a \textit{ participates in } M-\textit{chain } s \\ 0, & \textit{if} \quad \qquad \textit{otherwise.} \end{cases} \tag{3.1}$$

Let $X_s$ be the chain flow of chain $s$, then the chain $s$ induced link flow on link $a$ is $\lambda_{as} X_s$, if $a$ is a participant of $s$. Mathematically, one has flow the following conservation equation for (operation) link flow variables and chain flow variables:

$$x_a = \sum_{s \in S} \delta_{as} \lambda_{as} X_s, \quad \forall a \in A. \tag{3.2}$$

In other words, the link flow of an operational link is the sum of chain-induced flows on this link of all the M-chains in which this link participates. For an interface link $b \in B$, the link flow $x_b$ indicates the amount of integration work, coordination effort or information processed on this link. Since interface links are chain specific, the family of all interface links, $B$, can be partitioned into subfamilies, $B_s$, according to their governing M-chains $s \in S$. Therefore, the link flow of an interface link is uniquely determined by the chain flow of its governing M-chain.

We define the link cost, $\bar{c}_a$ , by a generic cost function of its flow $x_a$ , $\bar{c}_a = \bar{c}_a(x_a), \forall a \in A$ . The generic cost function, $\bar{c}_a$ , measures an appropriate combination of monetary cost and ineffectiveness in performing the corresponding task. The ineffectiveness reflects other important factors besides the monetary value, such as time and quality in conforming the task. In a modeling effort, $\bar{c}_a$ can take a weighted majority form, with the monetary cost being assigned a weight of one, and the other factors of ineffectiveness being assigned relative weights that can convert to an equivalent dollar amount. Depending on the nature of an operation link, its cost can be linear or nonlinear. The cost of an interface link reflects the effectiveness of coordination and integration of the two operation links that it bridges. The geographic distance, past cooperation experience, level of information integration, and the compatibility between the two operation links, etc., could be the factors that account for the cost of an interface link. Since the effective coordination of two successive operation links may depend on other joints of the chain as well, we assume that the cost of an interface link may, in general, depend on the flows of all the interface links belonging to the same M-chain. As interface links are chain specific, however, flows on all interface links are determined by the chain flow of the governing M-chain. Therefore, one has $\bar{c}_b = \bar{c}_b(X_s), \forall b \in B_s, \forall s \in S$ .

In general, our model allows an operation link to participate in more than one M-chain, although this is not imperative in all instances. In lieu of a common operation link for multiple M-chains, there must be an imputation mechanism to impute the link cost of the common link to its participating M-chains.

**Definition 4 (chain-imputable link cost)**

*For an M-chain $s$ and its participant operation link $a$ , the chain-imputable link cost of link a with respect to M-chain s, denoted by $\bar{C}_{as}$ , is the cost accountable for M-chain s, attributed to the work processed for s on operation link a.*

The chain-imputable link cost $\bar{C}_{as}$ , may depend on the total link flow on link $x_a$ , and chain $s$ induced flow on link $a$ , $\lambda_{as} X_s$ , which is the amount of work contributed by chain $s$ to link $a$ (3.3). Also, the chain-imputable link costs, as the charges to the participating M-chains to an operation link, should aggregately cover the link cost (3.4). That is,

$$\bar{C}_{as} = \bar{C}_{as}(\lambda_{as} X_s, x_a), \quad \forall a \in A \tag{3.3}$$

$$\bar{c}_a = \sum_{s \in S} \delta_{as} \bar{C}_{as}, \quad \forall a \in A \tag{3.4}$$

Based on chain-imputable link costs for all the common links, we give the definition of an M-chain cost below.

**Definition 5 (M-chain cost)**

*The cost of an M-chain is the aggregated generalized cost (ineffectiveness) incurred in all the operation and interface links of this M-chain in delivering the final product to the pertinent market. The cost of an M-chain $s$ is the sum of s-imputable link costs of all the links it employs. In mathematics, this can be expressed as*

$$\overline{C}_s = \overline{C}_s(X) = \sum_{a \in A} \delta_{as} \overline{C}_{as} + \sum_{b \in B_s} \overline{c}_b \qquad (3.5)$$

Note in (3.5) that the SC cost may depend not only on its own flow but also on all the other SCs' flows in the SCE, which reflects the SC interactions on the common operation links. .

For the sake of simplicity, we restrict our presentation to a fixed demand model in this paper. Nevertheless, the model can be extended directly to incorporate market elasticity, in form of a demand function or a price function (cf. Takayama and Judge (1971)). Suppose that the market for product $i$ in marketplace $j$ has a fixed demand $d_{ij}$, for every $i = 1, \ldots, m;\ j = 1, \ldots, n.$ Therefore, we have

$$\sum_{s \in S_{ij}} X_s = d_{ij}, \quad \forall i = 1, \ldots, m;\ j = 1, \ldots, n. \qquad (3.6)$$

## 4. Mathematical Formulation

In this section, we present the equilibrium conditions of a supply chain economy that can be used to determine the answer to the central questions of our interest. Namely, (1) Which potential M-chains will win the competition and become active chains? (2) What outputs do they deliver to the end consumer markets? And, (3) Who among the multiple agents may enter the supply chain economy? The equilibrium conditions can be formulated as a finite dimensional variational inequality problem in chain variables, which serves as general analytical framework for modeling the SC interactions and particularly SC vs. SC competition.

We first give the definitions of link marginal cost and chain marginal cost as follows.

**Definition 6 (marginal cost)**

*For an operation or an interface link, the link marginal cost is defined to be the unit link processing cost, if the flow on the link is no less than one unit; otherwise, it is defined to be the link cost for one unit flow. The chain marginal cost of an M-chain s is defined to be the unit chain processing cost of chain s, if the chain flow of s is no less than one unit; otherwise, it is defined to be the corresponding chain cost with the chain flow of s being fixed at one unit.*

As any active link in a SCE must carry a minimum of one unit link flow in practice, the link marginal cost so defined represents the unit processing cost for an active link. If the link flow turns out to be less than one unit in the model, however, it would imply that the link is not actually being used. In this case, the link marginal cost should be seen as the triggering cost, i.e., the cost necessary to activate this link. The same rationale is behind he definition of chain marginal cost.

An end consumer market is a destination of a commodity, which pulls the flow of materials through its pertinent M-chains. Therefore, the M-chains, as alternative “paths” to the destination, are competing against each other in delivering the commodity to the destination. Covinato (1992) argued that the firms’ cooperation quest in SC management is to make the final product at overall lower total cost than competing sets of SC firms. Following this philosophy, the winning M-chains in this competition must be the “shortest paths,” who can deliver the commodity at the lowest marginal total cost.  There is an interesting analogue between this competition and that in a traffic network problem in which the travelers seek to determine the shortest paths (minimal cost paths) to travel from their origins to their destinations. The interaction of material flows among different M-chains on common operation links corresponds to interacting traffic on common road sections in a transportation network. This analogue inspires us to translate Wardrop’s first principle of traffic assignment (Wardrop 1952) into the context of a SC network economy, so we can characterize the winning chains, i.e., active chains, as those with lower marginal costs than their competing chains.

The equilibrium of SCE defined below now answers our central question (1): Which potential chains will win the competition and become active chains?

**Definition 7 (equilibrium of SCE)**

*In a SC network equilibrium, all the potential M-chains that have higher marginal total costs than their competing M-chains for a same market should have no market share. In other words, all the less cost-effective potential M-chains will be inactive in equilibrium. Mathematically, a feasible SCE $X^*$ constitutes a SC network equilibrium if and only if the following system of equalities and inequalities holds true:*

$$C_s(X^*)\begin{cases} = \min_{s\in S_{ij}} C_s(X^*) & if \quad X_s^* > 0, \\ \geq \min_{s\in S_{ij}} C_s(X^*) & if \quad X_s^* = 0, \end{cases} \qquad \forall s \in S_{ij} \tag{4.1}$$

*for all end consumer markets* $(i, j), \quad i = 1,\dots,m;\ j = 1,\dots,n.$

Economically, the above definition is in perfect line with the spatial price equilibrium (c.f. Samuelson (1952), Takayama and Judge (1971), Nagurney, Takayama and Zhang (1995), Nagurney and Zhang (1996)) in the context of a SCE. Note that, since the operation and interface costs in this model take the form of generic costs to account, not only for the monetary cost, but also for other ineffectiveness such as delivery time and quality issues, the active chains (winning chains) so defined in the equilibrium condition should identify those with highest competency in applications.

We are now ready to address the central question (3): Who among the multiple agents may enter this economy? The definition of the equilibrium of SCE articulates that the winning (active) chains are those M-chains with lowest marginal total costs for their pertinent markets, and they are the only ones that carry a positive flow of materials.

Cavinato (1992) identified the prevalent managerial concept towards SC management with the following rationale. "If the final customer does not select the OEM's product, then none of the firms in that SC will experience derived demand for their portion of the value-added that they provide to the product or service." This rationale is observed explicitly in our definition. As any M-chain with higher marginal cost than its competing M-chains delivers no product to its market, all its participating operation links get zero derived demand for their portion from this M-chain (cf. (3.2)). However, since we are considering a SCE, which generally consists of many SCs, some or all of these operation links may participate in other active M-chains and thus get positive derived demand for their portion from these active participant M-chains. Therefore, we can conclude with the following corollary as the answer to question (3)

**Corollary 1 (Active Agent)**

*An agent enters a SCE if and only if the operation links that it employs participate in at least one active M-Chain. A business agent is called an active agent in a SCE if it enters such SCE.*

Finally, in order to answer the central question (2), we present the following variational inequality problem that solves for the output from the active (winning) M-chains in a SC network equilibrium.

**Theorem 1 (VIP in chain variables)**

$X^*$ *is a SC network equilibrium if and only if* $X^*$ *solves the following variational inequality problem* (VIP(4.2)):

*Find* $X^* \in \Omega$ *such that*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{s\in S_{ij}} C_s\left(X^*\right)\left(X_s - X_s^*\right) \geq 0, \quad \forall X \in \Omega \tag{4.2}$$

*where* $\Omega = \left\{ X_s \geq 0, s \in S : \sum_{s\in S_{ij}} X_s = d_{ij}, \quad \forall i, \forall j \right\}$ *is the feasible set of chain variables for the SC network.*

**Proof:** In order to see that (4.1) implies (4.2), let $V_{ij} = \min_{s\in S_{ij}} C_s(X^*), \forall i, j$ and let $X \in \Omega$ be any feasible SCE. For any $i, j$, and $s \in S_{ij}$, if $X_s^* = 0$, then one has $\left(C_s\left(X^*\right) - V_{ij}\right)\left(X_s - X_s^*\right) = \left(C_s\left(X^*\right) - V_{ij}\right) X_s \geq 0$ from (4.1). If $X_s^* > 0$, then, $C_s\left(X^*\right) - V_{ij} = 0$, and hence $\left(C_s\left(X^*\right) - V_{ij}\right)\left(X_s - X_s^*\right) = 0$. Summing over for all $i, j$, and $s \in S_{ij}$, one has $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{s\in S_{ij}}\left(C_s\left(X_s^*\right) - V_{ij}\right)\left(X_s - X_s^*\right) \geq 0, \quad \forall X \in \Omega$, which is identical to (4.2) because all $V_{ij}$ will vanish when summing over all the chains in $S_{ij}$ under the fixed demand assumption. We now turn to prove that (4.2) implies (4.1). For any $i, j$, and $s \in S_{ij}$, if *s* is the only element in $S_{ij}$, then $X_s^* = d_{ij} > 0,$ and we have

$C_s(X^*) = V_{ij}$ trivially held. Suppose that $s$ is not the only element (chain) in $S_{ij}$ and that $s' \in S_{ij}$ is another element. We consider two separate cases. (i) If $X_s^* = 0,$ define $Y$ as:

$$\mathrm{Y}_{s''} = \begin{cases} \delta, & if \quad s'' = s \\ X_{s'}^* - \delta & if \quad s'' = s' \\ X_{s''} & if \quad s'' \neq s, s' \end{cases},$$

where $\delta$ is positive and sufficiently small. The $Y$ is feasible. Substituting $Y$ for $X$ in (4.2), one has $\left(C_s\left(X^*\right) - C_{s'}\left(X^*\right)\right)\delta \geq 0.$ Since $s' \in S_{ij}$ is arbitrarily chosen, this suggests that

$$C_s\left(X^*\right) \geq C_{s'}\left(X^*\right), \forall s' \in S_{ij}, \text{ or } C_s\left(X^*\right) \geq \min_{s' \in S_{ij}} C_{s'}\left(X^*\right) \tag{4.3}$$

which complies with (4.1). (ii) If $X_s^* > 0,$ define $Y$ as:

$$\mathrm{Y}_{s''} = \begin{cases} X_s^* - \delta, & if \quad s'' = s \\ X_{s'}^* + \delta & if \quad s'' = s' \\ X_{s''} & if \quad s'' \neq s, s' \end{cases},$$

where $\delta$ is positive and sufficiently small. Then obviously $Y \in \Omega$. Substituting $Y$ for $X$ in (4.2), one has $\left(C_s\left(X^*\right) - C_{s'}\left(X^*\right)\right)\delta \leq 0.$ Since $s' \in S_{ij}$ is arbitrarily chosen, this suggests that

$$C_s\left(X^*\right) \leq C_{s'}\left(X^*\right), \forall s' \in S_{ij}, \text{ or } C_s\left(X^*\right) = \min_{s' \in S_{ij}} C_{s'}\left(X^*\right) \tag{4.4}$$

which confirms the other part of (4.1). Therefore, we have proved that (4.2) implies (4.1). □

VIP (4.2) can be regarded as a general framework of mathematical formulation for the equilibrium of a SCE, since it is given in chain marginal cost functions and does not involve a link cost imputation, which may vary in applications. Once the link costs imputation $\{\bar{C}_{as}, \forall s \in S, \forall a \in A\}$ is established in the model, it sets forth the chain costs $\{\bar{C}_s, \forall s \in S\}$ by (3.5), which in turn sets forth the marginal chain costs $\{C_s, \forall s \in S\}$. VIP (4.2) can then be solved using an appropriate algorithm (see, e.g., Harker and Pang (1990) and Nagurney (1999) and the references therein). In the following, we proceed to investigate under a special link cost imputation, uniform link cost imputation.

**Definition 8 (Uniform Link Imputation)**

*For an operation link, $a$, we say its link cost imputation is uniform, if it charges all its participating chains at a uniform rate, i.e.,*

$$\bar{C}_{as} / \lambda_{as} X_s = \beta_a, \quad \forall s, \text{ with } \delta_{as} = 1 \tag{4.5}$$

One can directly observe the following corollary about the uniform imputation.

**Corollary 2**

*The necessary and sufficient condition of a uniform link cost imputation is that all the participating chains to the link are charged at the rate equal to the link marginal cost, i.e.,*

$$\overline{C}_{as} / \lambda_{as} X_s = c_a, \quad \forall s, \delta_{as} = 1 \tag{4.6}$$

**Proof:** The sufficient condition is trivial. For the necessary condition, we have, in light of the definition and (3.5),

$$\overline{c}_a = \sum_{s \in S} \delta_{as} \overline{C}_{as} = \sum_{s \in S} \delta_{as} \lambda_{as} X_s \beta_a \tag{4.7}$$

which, with notice to (3.2), implies that

$$c_a = \frac{\overline{c}_a}{x_a} = \frac{\sum_{s \in S} (\delta_{as} \lambda_{as} X_s) \beta_a}{\sum_{s \in S} (\delta_{as} \lambda_{as} X_s)} = \beta_a \tag{4.8}$$

If any of the participating chain, $s$, is active, i.e., $X_s \geq 1$, then by the previously adopted convention $\lambda_{as} \geq 1$, we have $x_a \geq 1$. The assertion of the proposition now follows directly from (4.8).

For any M-chain $s \in S$, let us denote the marginal chain cost for interface by

$$C_{sI} = C_{sI}(X_s) = \sum_{b \in B_s} c_b(X_s) \tag{4.9}$$

Under the uniform link cost imputation, the following theorem presents a separable variational inequality formulation for the equilibrium of SCE in link and chain variables.

**Theorem 2 (VIP in chain and link variables)**

*Suppose that the link costs imputation is uniform. Then, $X^*$ is a SC network equilibrium if and only if $X^*$ with its induced $x_a^*, \forall a \in A$ through (3.2) solves the following variational inequality problem* (VIP (4.10)-(4.13)):

*Find $X_s^*, \forall s \in S, x_a^*, \forall a \in A$ such that*

$$\sum_{a \in A} c_a(x_a^*)(x_a - x_a^*) + \sum_{s \in S} C_{sI}(X_s^*)(X_s - X_s^*) \geq 0 \tag{4.10}$$

*holds for all $x_a, \forall a \in A$, and $X_s, \forall s \in S$ subject to*

$$\sum_{s \in S_{ij}} X_s = d_{ij}, \quad \forall i = 1,..,m, \forall j = 1,...,n; \tag{4.11}$$

$$\sum_{s \in S} \delta_{as} \lambda_{as} X_s = x_a, \quad \forall a \in A \tag{4.12}$$

$$X_s \geq 0, \quad \forall s \in S; \qquad x_a \geq 0, \quad \forall a \in A \tag{4.13}$$

**Proof:** In light of Theorem 1, it suffices to show that VIP (4.2) is equivalent to VIP (4.10)-(4.13). It follows from the definition of chain marginal cost that, for $X_s \geq 1$,

$$\begin{aligned} C_s &= \overline{C}_s / X_s = (\sum_{a\in A}\delta_{as}\overline{C}_{as} + \sum_{b\in B_s}\overline{c}_b)/X_s \\ &= \sum_{a\in A}\delta_{as}\frac{\lambda_{as}X_s}{x_a}\overline{c}_a(x_a)/X_s + \sum_{b\in B_s}\overline{c}_b/X_s \\ &= \sum_{a\in A}\delta_{as}\lambda_{as}\overline{c}_a(x_a)/x_a + \sum_{b\in B_s}\overline{c}_b/X_s \\ &= \sum_{a\in A}\delta_{as}\lambda_{as}c_a(x_a) + \sum_{b\in B_s}c_b(X_s) \end{aligned} \tag{4.14}$$

where the last equality of (4.14) follows direct from the definition of link marginal cost with notice of $x_a \geq X_s \geq 1$ when $\delta_{as} = 1$. Note that the right-hand side of (4.14) coincides with $C_s$ for $X_s < 1$. Therefore, under the uniform link cost imputation, (4.14) holds true for all $X_s$, and one has

$$\begin{aligned} &\sum_{s\in S} C_s\left(X^*\right)(X_s - X_s^*) \\ &= \sum_{s\in S}\left(\sum_{a\in A}\delta_{as}\lambda_{as}c_a\left(x_a^*\right) + \sum_{b\in B_s}c_b\left(X_s^*\right)\right)\left(X_s - X_s^*\right) \\ &= \sum_{a\in A}c_a\left(x_a^*\right)\left(\sum_{s\in S}\left(\delta_{as}\lambda_{as}X_s - \delta_{as}\lambda_{as}X_s^*\right)\right) + \sum_{s\in S}C_{sI}\left(X_s^*\right)\left(X_s - X_s^*\right) \\ &= \sum_{a\in A}c_a\left(x_a^*\right)\left(x_a - x_a^*\right) + \sum_{s\in S}C_{sI}\left(X_s^*\right)\left(X_s - X_s^*\right). \end{aligned} \tag{4.15}$$

(4.15) shows that the left-hand side of VIP (4.2) is identical to the left-hand side of VIP (4.10)-(4.13). □

We note that VIP (4.10)-(4.13) has a separable cost structure while the cost structure of VIP (4.2) is nonseparable in that the marginal chain cost of chain $s$, $C_s\left(X^*\right)$, depends on all SC flows $X^*$ rather than its own chain flow, $X_s^*$. Therefore, VIP (4.10)-(4.13) can be solved in more efficient algorithms.

## 5. Qualitative Properties

In this section, we provide the fundamental qualitative properties of the mathematical formulation for equilibrium of a SCE presented in the previous section. We first show the existence of equilibrium of a SCE.

**Theorem 3 (existence)**

*Suppose that the link cost functions are continuous and that the link cost imputation is continuous. Then, there exists some equilibrium of a SCE by Definition 8.*

**Proof:** Under the condition of the theorem, the chain cost function of (3.5) is continuous for all M-chains. This implies that the marginal chain cost is continuous. It is easy to see that the constrain set, $\Omega$, of VIP (4.2), is convex, closed and bounded. In light of the standard theory of finite dimensional variational inequality problem (Harker and

Pang (1990), Nagurney (1993)), VIP (4.2) possesses at least one solution, which is equilibrium of SCE by Theorem 1. □

As strict monotonicity of the vector function of a variational inequality problem ensures the uniqueness of its solution (Nagurney 1999), the following theorem gives a general condition for the uniqueness of SC network equilibrium.

**Theorem 4 (uniqueness)**

*Suppose that the marginal chain costs* $C_s, \forall s \in S$ *, are strictly monotone. Then there exist a unique equilibrium of SCE.*

We proceed to explore further properties under the uniform link cost imputation. Under such a link cost imputation mechanism, the equilibrium of a SCE can be formulated as a solution to a separable variational inequality problem, VIP (4.10)-(4.13). Via the bridge of VIP (4.10)-(4.13)., we can establish some necessary and sufficient conditions for a SC network equilibrium as the solution to the optimization problem (5.1)-(5.4) below.

$$\text{Min} \sum_{a \in A} \int_0^{x_a} c_a(u)du + \sum_{s \in S} \int_0^{X_s} C_{sI}(u)du \tag{5.1}$$

subject to

$$\sum\nolimits_{s \in S_{ij}} X_s = d_{ij}, \quad \forall i = 1,..,m, \forall j = 1,...,n; \tag{5.2}$$

$$\sum\nolimits_{s \in S} \delta_{as} \lambda_{as} X_s = x_a, \quad \forall a \in A \tag{5.3}$$

$$X_s \geq 0, \quad \forall s \in S; \qquad x_a \geq 0, \quad \forall a \in A \tag{5.4}$$

**Theorem 5 (sufficient condition)**

*Under the uniform link cost imputation mechanism, an sufficient condition of* $X^*$ *being an equilibrium of a SCE is that* $X^*$ *with its induced* $x_a^*, \forall a \in A$ *through (3.2) solves the optimization problem (5.1)-(5.4).*

**Proof:** By Theorem 2, it suffices to show that any solution to optimization problem (5.1)-(5.4) is a solution to VIP (4.10)-(4.13). However, VIP (4.10)-(4.13) is merely a restatement of the first-order necessary condition of the optimization problem (5.1)-(5.4) by noticing that the vector function entering into VIP (4.10)-(4.13) is the gradient of the objective function in (5.1). □

**Theorem 6 (necessary and sufficient condition)**

*Suppose that the marginal link costs for operation and the marginal chain costs for interface are monotone increasing. Then, under the uniform link cost imputation,* $X^*$ *is an equilibrium of a SCE if and only if* $X^*$ *with its induced* $x_a^*, \forall a \in A$ *through (3.2) solves the optimization problem (5.1)-(5.4).*

**Proof:** Since the vector function entering into VIP (4.13)-(4.16) is the gradient of the objective function in (5.1), the assumption of monotone marginal link costs for operation and monotone marginal chain costs for interface implies that the objective function of (5.1) is convex. According to Bazzara, Sherali and Shetti (1993), VIP (4.13)-(4.16) is the necessary and sufficient condition of the solution to the convex programming (5.1)-(5.4). □

With a slightly stronger condition, the following theorem ensures the uniqueness of the equilibrium.

**Theorem 7**

*Suppose that the marginal link costs for operation and the marginal chain costs for interface are strictly monotone increasing. Then, under the uniform link cost imputation, there exists a unique equilibrium of a SCE.*

**Proof:** Following from Theorem 6, the equilibrium of a SCE is the unique solution of the strictly convex programming (5.1)-(5.4). □

Corbett and Karmarkar (2001) assume that the production cost functions are homogeneously linear and identical for all agents at each tier. Under this assumption, but relaxing the requirement for identical production cost rate at each tier, we can formulate the SC network equilibrium problem as a linear programming problem. Write explicitly the link cost functions as, $\bar{c}_a(x_a) = g_a x_a, \quad \forall a \in A$, for an operation link, and $\bar{c}_b(X_s) = q_b X_s, \quad \forall b \in B_s, \forall s \in S$, for an interface link. In line with Definition 6, the link marginal cost for an operational link and an interface link respectively becomes: $c_a = g_a, \quad \forall a \in A,\ c_b = q_b, \quad \forall b \in B_s, \forall s \in S.$

**Theorem 8**

*Suppose that the link costs for all the operation links and all the interface links are linear. Then, $X^*$ is an equilibrium of a SCE if and only if $X^*$ solves the following linear programming problem:*

$$\text{Min} \sum_{s \in S} \left( \sum_{a \in A} \delta_{as} \lambda_{as} g_a + \sum_{b \in B_s} q_b \right) X_s \tag{5.5}$$

subject to:

$$\sum\nolimits_{s \in S_{ij}} X_s = d_{ij}, \quad \forall i = 1,..,m, \forall j = 1,...,n; \tag{5.6}$$

$$X_s \geq 0, \quad \forall s \in S; \qquad x_a \geq 0, \quad \forall a \in A. \tag{5.7}$$

**Proof:** The linear link cost structures suggest that they are monotone increasing. Furthermore, they imply the uniform link cost imputation by Corollary 2. It follows from Theorem 6 that the necessary and sufficient condition for an equilibrium of a SCE is that it solves the optimization problem (5.1)-(5.4). The objective function (5.1) becomes (5.5) through a simple algebra when the flow conservation equation (5.3) is substituted for $x_a$. □

## 6. Numerical Examples

In this section, we present two numerical examples to illustrate the model. The solution of the variational inequality formulation (4.2) of these examples will determine the winning chains and their market shares as well as the output of all the participating firms. We also provide discussion of the results and highlight the economic interpretation. The computation of these examples is carried out by the projection method for variational inequality problem (cf. Nagurney 1999) as outlined below.

**Projection method for the solution of variational inequality (4.2)**

**Step 0. Initialization**

Prescribe a convergence tolerance $\varepsilon$ , choose any initial feasible point $X^0 \in \Omega$ , and set step size $\alpha < 1$. Let $k$=1.

**Step 1. Computation**

Compute $X^1 \in \Omega$ by solving the quadratic programming problem:

$$\min_{X \in \Omega} \sum_{s \in S} (\alpha / 2) X_s^2 + \left( C_s \left( X_s^{k-1} \right) - \alpha X_s^{k-1} \right) X_s$$

**Step 2. Convergence**

If $| X_s^k - X_s^{k-1} | \leq \varepsilon,$ for all $s \in S$ , stop and terminate with $X^k$ as the solution; otherwise, set k:=k+1, and go to Step 1.

The first example is proposed for a dairy product SCE, and is investigated in two different demand cases, symmetric demand markets and asymmetric demand markets. We bring in some explanations for the solution results of this example and their economic rationales. The second SCE example has a more complex nature of its network structure and its participants and their relationships. It comprises two manufacturers, two third-party logistic (3PL) companies, and three retailers, who compete in making, transporting and selling the product in two geographic markets.

**Example 1**

There are two markets, Mk-1 and Mk-2, for a dairy product in a geographic area. There are two producers (suppliers), a local producer (L.P.), and a regional producer (R.P.), both of whom supply this product to the two markets. We suppose that the L.P. is within the area and is close to the two market sites, and, therefore, it ships the product by itself. The R.P. is outside the area of the market sites, and needs to transport the product to a distribution center (DC) located in the area. The product then will be delivered from the DC to the two markets.

There are two SCs, one of L.P. and one of R.P., each comprising two M-chains, $s_{11}, s_{12}$ pertaining to L.P., and $s_{21,} s_{22}$ pertaining to R.P. Here $s_{11}$ and $s_{21}$ are competing for Mk-1, and $s_{12}$ and $s_{22}$ are competing for Mk-2.

Figure 3 gives a SCE model representation of this example, where the dotted arrows represent the interface links and the solid arrows represent the operation links. Particularly, the M-chains are defined by the following sequence of alternative operation links and interface links:

$$s_{11} = a_1 b_1 a_2 ,\ s_{12} = a_1 b_2 a_3 ,\ s_{21} = a_4 b_3 a_5 b_5 a_6 b_7 a_7 ,\ s_{22} = a_4 b_4 a_5 b_6 a_6 b_8 a_8 . \tag{6.1}$$

The two M-chains of L.P., $s_{11}$ and $s_{12}$ are represented by similar sequences of links: a common production link $a_1$, an interface link $b_1$ or $b_2$, and a transportation link $a_2$ or $a_3$, for transporting the dairy product from L.P. to Mk-1 or Mk-2, respectively. The two M-chains of R. P., $s_{21}$ and $s_{22}$ are represented by similar sequences of links: a common production link $a_4$, an interface link $b_3$ or $b_4$, a common transportation link $a_5$, for transporting the product from R.P. to DC, an interface link $b_5$, or $b_6$, a common distribution link $a_6$, for distribution/storage at DC, an interface link $b_7$ or $b_8$, and finally another transportation link $a_7$, or $a_8$, for transporting the product from DC to either Mk-1 or Mk-2. Here, $a_4, a_5, a_6$ are common operation links for both $s_{21}$ and $s_{22}$, while $a_7$ belongs to $s_{21}$, and $a_8$ belongs to $s_{22}$. The process rates for the operations links, $a_1, a_4,$ with respect to the M-chains, $s_{11}, s_{12}, s_{21}, s_{22},$ are 2, while the process rates for all the other operation links are 1.

Suppose that the operation link costs and interface link costs of the above SCE are given by:

$$\bar{c}_{a_1} = 5x_{a_1}, \quad \bar{c}_{a_2} = 2x_{a_2} + 1, \quad \bar{c}_{a_3} = 2x_{a_3} + 1, \quad \bar{c}_{a_4} = 3x_{a_4}, \tag{6.2}$$

$$\bar{c}_{a_5} = 2x_{a_5} + 10, \quad \bar{c}_{a_6} = x_{a_6}, \quad \bar{c}_{a_7} = x_{a_7} + 5, \quad \bar{c}_{a_8} = x_{a_8} + 5, \tag{6.3}$$

$$\bar{c}_{b_1} = X_{s_{11}}, \quad \bar{c}_{b_2} = X_{s_{12}}, \quad \bar{c}_{b_3} = X_{s_{21}} + 2, \quad \bar{c}_{b_4} = X_{s_{22}} + 2, \tag{6.4}$$

$$\bar{c}_{b_5} = 0.5X_{s_{21}} + 2, \quad \bar{c}_{b_6} = 0.5X_{s_{22}} + 2, \quad \bar{c}_{b_7} = 0.5X_{s_{21}} + 1, \quad \bar{c}_{b_8} = 0.5X_{s_{22}} + 1. \tag{6.5}$$

One may notice in (6.2)- (6.5) that the production cost $\bar{c}_{a_1}$ is higher than $\bar{c}_{a_4}$, suggesting that R.P.'s production is more efficient than that of L.P. due to a larger production scale per se. On the other hand, the transportation and distribution cost of R.P. as the sum of $\bar{c}_{a_5}$ and $\bar{c}_{a_7}$ or $\bar{c}_{a_8}$, in addition to $\bar{c}_{a_6}$, is higher than the transportation cost of L.P., $\bar{c}_{a_2}$ or $\bar{c}_{a_3}$. Therefore, the competition at these two markets characterizes the mass production advantage of R.P., who enjoys a lower production cost, and the geographic advantage of L.P. as a local supplier, who saves on transportation/distribution costs.

Note that the link costs in (6.2) - (6.5) are linear with nonzero constant terms in general, hence, the marginal link cost structure that enters the variational inequality formulation is nonlinear. In the following, we investigate the example separately in the case of symmetric markets and asymmetric markets.

**Case 1 (Symmetric Markets)**

For the network and cost structure of SCE given above, assume that the demand at Mk-1, $d_1 = 50,$ and the demand at Mk-2, $d_2 = 50$ , and assume a uniform link cost imputation. Then, the equilibrium is computed at

$$X_{s_{11}} = X_{s_{12}} = 35.44, \qquad X_{s_{21}} = X_{s_{22}} = 14.56 \tag{6.6}$$

$$x_{a_1} = 141.76,\ x_{a_2} = x_{a_3} = 35.44,\ \ x_{a_4} = 58.24,\ x_{a_5} = x_{a_6} = 29.12,\ x_{a_7} = x_{a_8} = 14.56. \tag{6.7}$$

with the marginal chain costs

$$C_{s_{11}} = C_{s_{12}} = 13.028, \quad C_{s_{21}} = C_{s_{22}} = 13.030. \tag{6.8}$$

**Case 2 (Asymmetric Markets)**

Assume now that the demand at Mk-1 is $d_1 = 20,$ and demand at Mk-2 is $d_2 = 100$ , and assume also the uniform link cost imputation. Then, the equilibrium is computed at

$$X_{s_{11}} = 20, \quad X_{s_{12}} = 0, \quad X_{s_{21}} = 0, \quad X_{s_{22}} = 100, \tag{6.9}$$

$$x_{a_1} = 40,\ x_{a_2} = 20,\ x_{a_3} = 0,\ x_{a_4} = 200,\ x_{a_5} = x_{a_6} = 100,\ x_{a_7} = 0,\ x_{a_8} = 100. \tag{6.10}$$

Namely, L.P. has one active M-chain $s_{11}$ , and one inactive M-chain $s_{12}$ , and R.P. has one active M-chain $s_{22}$ , and one inactive M-chain $s_{21}$ . The corresponding marginal chain costs are:

$$C_{s_{11}} = 13.05, \quad C_{s_{12}} = 14, \quad C_{s_{21}} = 22.1, \quad C_{s_{22}} = 12.2. \tag{6.11}$$

One notices that the marginal chain cost of the inactive chains are higher than that of their competing active M-chains, and hence at equilibrium.

Some economic implications of the solution can be observed in comparing the equilibrium solutions in the two cases above. Since the higher transportation costs of R.P. are due mainly to the longer distance of transport (notice the greater constant terms of $\bar{c}_{a_5}$ and $\bar{c}_{a_8}$ than those of $\bar{c}_{a_2}$ and $\bar{c}_{a_3}$ ), an increased demand helps to overcome this cost by spreading it over a larger shipment quantity. Therefore, when the demand at Mk-2 is increased to 100, the transportation and distribution disadvantages of R.P. are diminished and hence, are outweighed by its mass production strength. This drives L.P. out of the competition in the larger market, Mk-2. The opposite happens at Mk-1. When its demand shrinks to 20, the market size basically impedes R.P. digesting its transportation cost over shipment quantity, while the mass production advantage of R.P. cannot be realized in this small local market. This makes L.P. fully control Mk-1. Therefore, this example suggests that the market size may significantly impact the market shares of competing SCs with different characteristics. While a local producer finds it hard to enter and compete in a large market, it can secure a small local market by turning away any mass producer, who cannot justify its distribution costs in the small market size. Such a phenomenon is actually witnessed in various markets in small towns remote from population centroids.

**Example 2.**

Figure 4 depicts a certain supply chain economy involving two producers of a consumer product, In Figure 4, $P_1, P_2$ are the operation links representing the production functions respectively of Producer 1 and Producer 2. There are three retailers, denoted by $R_1, R_2, R_3$ in the figure. Retailer 1 is the only retailer in geographic market 1, and Retailers 2 and 3 are located in geographic market 2 and are competing for that market. There are two 3PL companies, *Y* and *Z*, who can provide the transportation to ship the product from the producers to the retailers. Therefore, we have two families of transportation operation links $y_{ij}, z_{ij}, i = 1,2; j = 1,2,3,$ respectively denoting the transportation operations performed by company *Y* and company *Z* from Producer *i* to Retailer *j*. That is, either of the two 3PL companies, or both of them, can be active in delivering the product from every producer to every retailer. To simulate the complexity of real world supply chains, we assume that there is some special supply relationship between Producer 2 and Retailer 2 that Producer 2 can choose to ship the product directly to Retailer 2. This alternative transportation mode is illustrated in our model by operation link $w_{22}$. For presentation simplicity, the interface links are not labeled in the figure, but they can be identified by the pair of successive operations links they connect and are thus denoted. Generally, an interface link connecting operation links $a_1$ and $a_2$ will be denoted by $(a_1, a_2)$. In Figure 4, all solid arrows denote the operation links and all the dotted lines denote interface links.

In this example, there are thirteen M-chains competing for the two geographic markets, as defined by their composition and pertinent markets as shown in Table 1 below.

| M-chains | $S_{1y1}$ | $S_{1y2}$ | $S_{1y3}$ | $S_{1z1}$ | $S_{1z2}$ | $S_{1z3}$ | $S_{2y1}$ | $S_{2y2}$ | $S_{2y3}$ | $S_{2z1}$ | $S_{2z2}$ | $S_{2z3}$ | $S_{2w2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Producer | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3PL co. | Y | Y | Y | Z | Z | Z | Y | Y | Y | Z | Z | Z | Direct |
| Retailer | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 2 |
| Market | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 2 |

Table 1. M-chains and their composition

Suppose that the costs of the operation and interface links of this SCE are given by:

$$\bar{c}_{p_1} = 19x_{p_1}, \bar{c}_{p_2} = 20x_{p_2}, \tag{6.12}$$

$$\bar{c}(p_1, y_{11}) = X_{S_{1y1}}, \bar{c}(p_1, y_{12}) = X_{S_{1y2}}, \bar{c}(p_1, y_{13}) = X_{S_{1y3}}, \tag{6.13}$$

$$\bar{c}(p_1, z_{11}) = X_{S_{1z1}}, \bar{c}(p_1, z_{12}) = X_{S_{1z2}}, \bar{c}(p_1, z_{13}) = X_{S_{1z3}}, \tag{6.14}$$

$$\bar{c}(p_2, y_{21}) = X_{S_{2y1}}, \bar{c}(p_2, y_{22}) = X_{S_{2y2}}, \bar{c}(p_2, y_{23}) = X_{S_{2y3}}, \tag{6.15}$$

$$\bar{c}(p_2, z_{21}) = X_{S_{2z1}}, \bar{c}(p_2, z_{22}) = X_{S_{2z2}}, \bar{c}(p_2, z_{23}) = X_{S_{2z3}}, \bar{c}(p_2, w_{22}) = X_{S_{2w2}}, \tag{6.16}$$

$$\bar{c}_{y_{11}} = 3x_{y_{11}} + 30, \bar{c}_{y_{12}} = 3x_{y_{12}} + 30, \bar{c}_{y_{13}} = 3x_{y_{13}} + 30, \tag{6.17}$$

$$\bar{c}_{y_{21}} = 3x_{y_{21}} + 30,\ \bar{c}_{y_{22}} = 3x_{y_{22}} + 30,\ \bar{c}_{y_{23}} = 3x_{y_{23}} + 30, \tag{6.18}$$

$$\bar{c}_{z_{11}} = 7x_{z_{11}} + 30,\ \bar{c}_{z_{12}} = 2x_{z_{12}} + 30,\ \bar{c}_{z_{13}} = x_{z_{13}} + 30, \tag{6.19}$$

$$\bar{c}_{z_{21}} = 7x_{z_{21}} + 20,\ \bar{c}_{z_{22}} = 3x_{z_{22}} + 20,\ \bar{c}_{z_{23}} = 2x_{z_{11}} + 20,\ \bar{c}_{w_{22}} = 2x_{w_{22}}, \tag{6.20}$$

$$\bar{c}(y_{11}, r_1) = X_{S_{1y1}} + 20,\ \bar{c}(y_{12}, r_2) = X_{S_{1y2}} + 20,\ \bar{c}(y_{13}, r_3) = X_{S_{1y3}} + 20, \tag{6.21}$$

$$\bar{c}(y_{21}, r_1) = 30,\ \bar{c}(y_{22}, r_2) = 30,\ \bar{c}(y_{23}, r_3) = 30, \tag{6.22}$$

$$\bar{c}(z_{11}, r_1) = X_{S_{1z1}} + 10,\ \bar{c}(z_{12}, r_2) = X_{S_{1z2}} + 10,\ \bar{c}(z_{13}, r_3) = X_{S_{1z3}} + 10, \tag{6.23}$$

$$\bar{c}(z_{21}, r_1) = 30,\ \bar{c}(z_{22}, r_2) = 30,\ \bar{c}(z_{23}, r_3) = 30,\ \bar{c}(w_{22}, r_2) = 0, \tag{6.24}$$

$$\bar{c}_{r_1} = x_{r_1} + 10,\ \bar{c}_{r_2} = 2x_{r_2},\ \bar{c}_{r_3} = 1.5x_{r_3} + 5. \tag{6.25}$$

The link-chain flow conservation equations are:

$$x_{p_1} = \sum_{j=1}^{3} X_{S_{1yj}} + \sum_{j=1}^{3} X_{S_{1zj}} \tag{6.26}$$

$$x_{p_2} = \sum_{j=1}^{3} X_{S_{2yj}} + \sum_{j=1}^{3} X_{S_{2zj}} + X_{S_{2w2}} \tag{6.27}$$

$$x_{w_{22}} = X_{S_{2w2}},\ \ x_{y_{ij}} = X_{S_{iyj}},\ \ x_{z_{ij}} = X_{S_{izj}},\ i = 1,2,3,\ j = 1,2,3. \tag{6.28}$$

$$x_{r_j} = \sum_{i=1,2} X_{S_{iyj}} + \sum_{i=1,2} X_{S_{izj}},\ \ j = 1,3;\ \ x_{r_2} = \sum_{i=1,2} X_{S_{iy2}} + \sum_{i=1,2} X_{S_{iz2}} + X_{S_{2w2}}. \tag{6.29}$$

Suppose that the demand is 200 at each of the two demand markets. Therefore, we have the following two constraints for the chain flows to meet the demand at the demand markets.

$$\sum_{i=1,2} X_{S_{iy1}} + \sum_{i=1,2} X_{S_{iz1}} = 200,\ \ \sum_{i=1,2} X_{S_{iy2}} + \sum_{i=1,2} X_{S_{iz2}} + X_{S_{2w2}} + \sum_{i=1,2} X_{S_{iy3}} + \sum_{i=1,2} X_{S_{iz3}} = 200 \tag{6.30}$$

The feasible set of the chain flows is then given by

$$\Omega = \{X_S \geq 0, s \in S : (6.39) - (6.43)\ \text{hold}\} \tag{6.31}$$

The equilibrium as the solution to the variational inequality formulation of this problem is computed using the projection method with $\alpha = 0.05$ and $\varepsilon = 0.01$. Table 2 below summarizes the equilibrium chain flows, their marginal costs, and their corresponding retailers and pertinent markets. The winning chain columns are highlighted with underlines.

| M-chains | <u>$S_{1y1}$</u> | <u>$S_{2y1}$</u> | $S_{1z1}$ | $S_{2z1}$ | $S_{1y2}$ | $S_{2y2}$ | $S_{1z2}$ | $S_{2z2}$ | <u>$S_{2w2}$</u> | $S_{1y3}$ | $S_{2y3}$ | <u>$S_{1z3}$</u> | <u>$S_{2z3}$</u> |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Chain flow | <u>97.17</u> | <u>107.83</u> | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | <u>60.02</u> | 0.00 | 0.00 | <u>30.00</u> | <u>109.98</u> |
| Marg. Cost | <u>25.65</u> | <u>25.65</u> | 79.01 | 88.98 | 67.05 | 85.98 | 65.03 | 76.01 | <u>25.01</u> | 71.50 | 90.50 | <u>25.01</u> | <u>25.00</u> |
| Retailer | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| Market | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

Table 2. Equilibrium chain flows and their marginal costs

In the above table, one can see that all the winning chains for a pertinent market have equal and minimum marginal costs, and the inactive chains with zero flows and higher than minimum marginal costs. The equilibrium solution determines the link flows on all the operation links through the link-chain flow conservation equations (6.39)-(6.42). Particularly, the production output of Producer 1 and Producer 2 is respectively 127.17 and 277.83. The 3PL company Y will obtain a contract of shipment of 97.17 from Producer 1 to Retailer 1, and a contract of shipment of 107.83 from Producer 2 to Retailer 1, but it will not be able to get any other shipping contract. On the other hand, the 3PL company Z will only ship from Producer 1 and Producer 2 to Retailer 3, with a shipment of 30 from Producer 1 to Retailer 3 and a shipment of 109.98 from Producer 2 to Retailer 3. In this example, Retailer 2 will procure the product exclusively from Producer 2 via the direct transport, with the advantage of the special supply relationship, and the shipment will be 60.02. Finally, the sales volume at Retailer 1, Retailer 2, and Retailer 3 is respectively 200, 60.02, and 139.98.

## 7. Conclusion

In this paper, we present a general framework for the formulation of SC interactions, particularly of SC vs. SC competition. The model is proposed from a macroscopic perspective. It takes such inputs as (i) a set of related business entities (as the network topology and the operation links), with (ii) their business functions, strengths and weaknesses (as given by the link costs), and (iii) their connection and relationship (as the network topology and the interface links with their cost functions). Based on the inputs, the model identifies those agents who will form viable (winning) SCs, determine the structure of the viable chains, and predict the output they deliver to the end markets under study.

The VIP formulation in chain variables proposed in this paper is a general framework for an arbitrary link cost imputation mechanism. In a particular application, however, there might be no common operation links in a SCE under study. In most cases, when there are common links, their link cost imputations are chain-independent, such as the uniform link cost imputation investigated in this paper. Under these circumstances, the VIP formulation will be reduced to a simpler one, such as the separable VIP in chain and link variables presented in the paper, by exploiting the special structure of specific link cost imputation mechanism. We also give conditions of the link cost function under which there is equivalent optimization problem formulation. Particularly, there is an equivalent linear programming formulation for equilibrium of a SCE under the linear link cost structure of Corbett and Karmarkar (2001). Qualitative properties regarding to the existence and uniqueness of equilibrium have also been established in this paper.

The model has a broad application potential. First, it can be used for economic studies towards policy making, intervention and leverage. For instance, it should be of a high interest to predict the prevailing SCs under the influence of a new policy implement, taking into account of the inter-chain competition. In this scenario, the model serves to suggest the structure and the scale of the prevailing SCs, their market shares, and what business agents will be involved in which SC(s). The results then can be fed back to the policy makers to analyze the impact of such prospective policy and to make necessary changes towards a certain economic or environmental goal. Second, it can be used for strategic SC planning with reflection of the competing SCs' reaction, which is novel to most existing SC management study that is not embedded in an inter-chain competitive environment. For instance, one can run the model as a simulation process to determine the return or market gain in equilibrium for a SC that will result from building a new distribution center, or establishing a long term contract with a third party logistic firm, in an attempt to reach a regional market. Such a "simulation" process facilitates a mechanism for, not only selecting the best strategy, but also understanding the reaction of the competing SCs. Analogously, it conducts a what-if analysis for a hypothetic move of a competitor. Third, it can be used in a coalition formation study. For instance, to suggest the consequence of an individual firm joining a SC, we can lower the cost on the interface links that connect the operation link representing the firm to the operation links already participating in this SC. The costs on other interface links remain unchanged. We then solve the model to find the new equilibrium for the viable chains and their market shares. Following these directions, more application prospects can be proposed.

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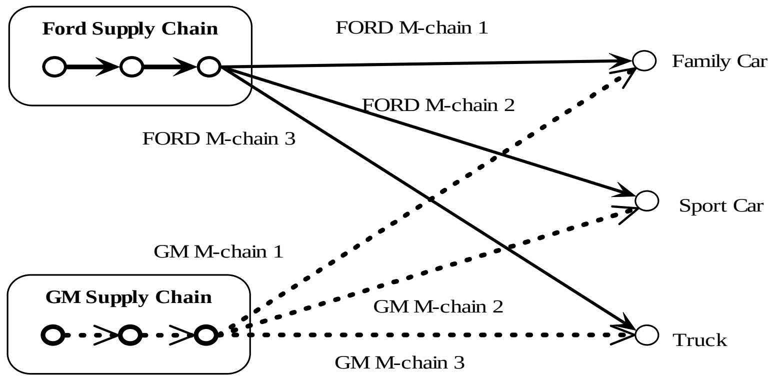


Figure 1
Ford M-chains against GM M-chains

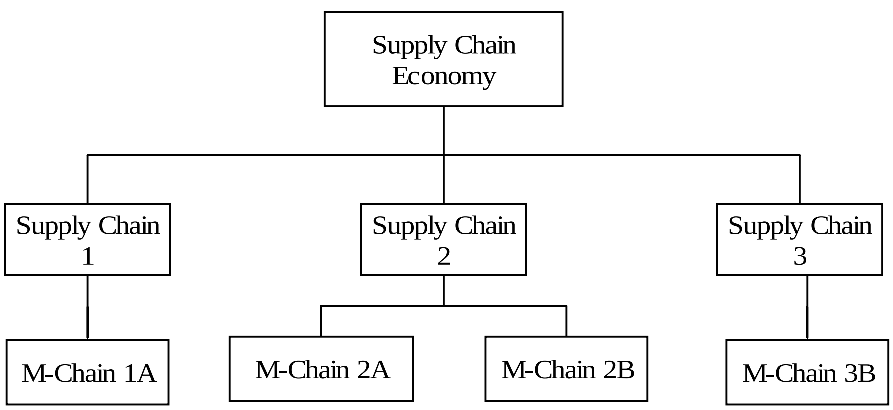


Figure 2
A hierarchy of supply chain economy

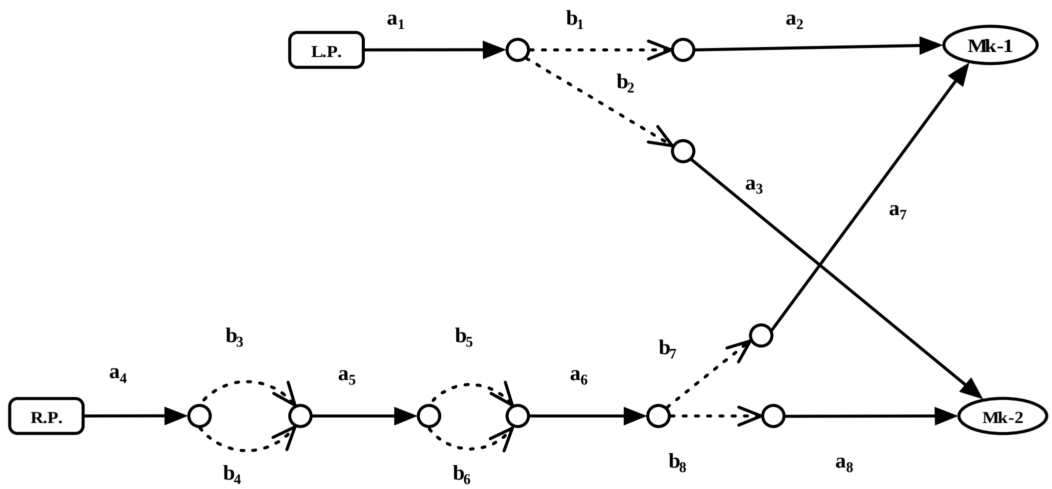


Figure 3
Competition between a local producer and a regional producer over a dairy product

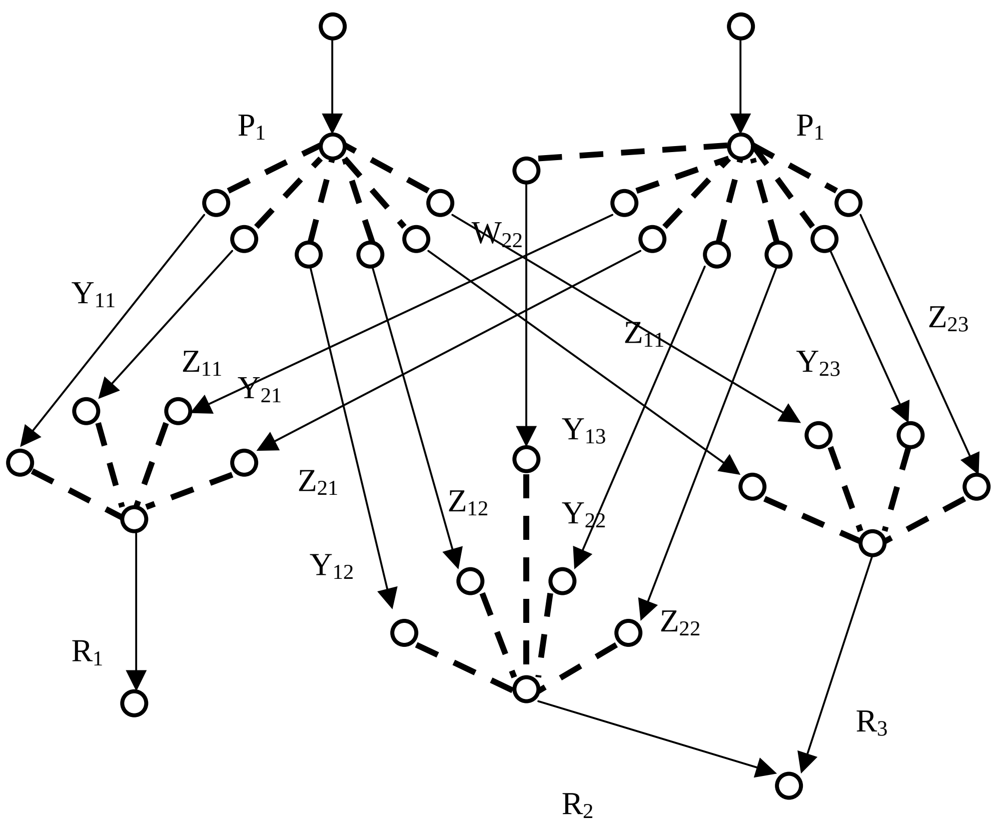


Figure 4
A supply chain economy with producers, retailers and 3PL companies.