A NEUMANN SERIES REPRESENTATION FOR SOLUTIONS TO BOUNDARY-VALUE PROBLEMS IN DYNAMIC ELASTICITY*

By

JOHN F. AHNER** AND GEORGE C. HSIAO

University of Delaware

Abstract. A regularized integral equation formulation for two exterior fundamental boundary-value problems in elastodynamics is presented. In either case, the displacement vector is assumed to be harmonic in time with a small frequency. It is shown that the solution can be expressed as a Neumann series in terms of the prescribed function; moreover, a sufficient condition for the convergence of the series is established.

1. Introduction. In recent years, increasing attention has been given to solving scattering problems by the method of integral equations directly in terms of the unknown functions in contrast with the corresponding density of the usual simple (or double) layer potential. The idea of the underlying method is based on a functional interpretation of the differential equations analogous to Green's formula. The integral equation that results may be regularized in the sense that the unknown function appears in such a way as to vanish at the weak singularity of the kernel. This regularization enables one to obtain a Neumann series solution for the problem under consideration. Complete details of this approach can be found in the paper by Ahner and Kleinman [1].

In elasticity, the method of integral equations has been extensively explored for various boundary-value problems. Most of them concern the integral equation formulation in terms of the density of the potential, with the exception of those considered by Kupradze [5] where the unknown function itself is used. The aforementioned approach in scattering problems, however, has not been adapted there. The purpose of this paper is to discuss the feasibility of applying this approach to problems in elasticity.

We consider two problems of small harmonic vibrations of an elastic body. The first resembles a Neumann scattering problem in acoustical theory, which does not fall into the standard boundary value problems in elasticity. The motivation for considering such a special problem is, of course, obvious from the success of the technique used in scattering problems. It is interesting to see the resemblance between the development of these analyses. We shall, hereafter, refer to the first problem as the scattering problem in elasticity (or simply the scattering problem). The second one is the classical second fundamental boundary-value problem, the stress being prescribed on the internal boundary. For simplicity, from now on we shall simply refer to the second problem as the second fundamental problem. The precise statement of the scattering problem will be given in Sec. 2. In Sec. 3, the problem is reformulated as an integral representation which

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** Present address: Vanderbilt University, Nashville, Tennessee.

is continuous as the field point approaches the boundary, unlike the Betti formula representation (or the Green's formula representation) where a jump discontinuity occurs at the boundary. In Sec. 4, the solution of the scattering problem is expressed as a Neumann series, each term of which involves the formal inverse of the corresponding integral representation for the case where the frequency of the vibrations is zero. It is demonstrated that this infinite series converges for small, but nonzero, values of frequency. In the last section, we will simply present the results for the second fundamental problem and omit the details, since the analysis is similar to that of the scattering problem presented. For ease of reading, we present some of the proofs in the Appendix.

2. The statement of the scattering problem. Let S be a closed surface in R^3 filled with homogeneous isotropic elastic material. We assume that S is a Lyapunoff surface, in the sense of Sobolev [6], on which a Hölder continuous normal exists everywhere. Let B_i and B_a denote the regions interior and exterior to S respectively. With respect to an orthogonal Cartesian coordinate system whose origin, $\mathbf{0}$, is in B_i, a point (x_1, x_2, x_3) will be denoted by $\mathbf{x}$. The distance between two points $\mathbf{x}$ and $\mathbf{y}$ will be denoted by $r(\mathbf{x}, \mathbf{y})$ (or simply r) while $r(\mathbf{x}, \mathbf{0}) = \rho$; $\hat{n}_\mathbf{y}$ represents an outward unit normal vector to S at $\mathbf{y}$ and $\hat{e}_i$, $i = 1, 2, 3$, designates the unit vector along the x_i-axis.

We wish to find the displacement vector field* $\mathbf{u}(\mathbf{x})$ which satisfies the scattering problem defined by

i) $\mathbf{u}(\mathbf{x}) = \mathbf{u}^i(\mathbf{x}) + \mathbf{u}^s(\mathbf{x}), \qquad \mathbf{x} \in S \cup B_a,$

ii) $(\Delta^* + \omega^2)\mathbf{u}^s(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in B_a,$

iii) $T\mathbf{u}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in S,$

iv) $\mathbf{u}^s(\mathbf{x})$ satisfies an elastic radiation condition at infinity. Here

$$\Delta^* = \mu\Delta + (\lambda + \mu) \text{ grad div}, \tag{2.1}$$

$$T = 2\mu \frac{\partial}{\partial \hat{n}} + \lambda \hat{n} \text{ div} + \mu(\hat{n} \times \text{curl}), \tag{2.2}$$

where μ and λ are the Lamé constants. Δ is the Laplacian operator and T is the stress operator. $\mathbf{u}^i(\mathbf{x})$ is a given vector field defined in $B_a \cup S$ and satisfying $(\Delta^* + \omega^2)\mathbf{u}^i(\mathbf{x}) = \mathbf{0}$ in B_i; while $\mathbf{u}^s$ is an unknown vector field. ω designates the frequency of vibration. In order to give a precise definition of the elastic radiation condition at infinity, we introduce the notations:

$$\mathbf{u}_1(\mathbf{x}) = -\frac{1}{k_1^2} \text{ grad div } \mathbf{u}^s(\mathbf{x}), \qquad \mathbf{u}_2(\mathbf{x}) = \frac{1}{k_1^2} \text{ grad div } \mathbf{u}^s(\mathbf{x}) + \mathbf{u}^s(\mathbf{x}). \tag{2.3}$$

These are the potential and solenoidal parts of $\mathbf{u}^s(\mathbf{x})$ respectively which satisfy the equations

$$\begin{aligned} &(\Delta + k_1^2)\mathbf{u}_1(\mathbf{x}) = \mathbf{0}, \qquad \text{curl } \mathbf{u}_1(\mathbf{x}) = \mathbf{0}, \\ &(\Delta + k_2^2)\mathbf{u}_2(\mathbf{x}) = \mathbf{0}, \qquad \text{div } \mathbf{u}_2(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{u}_1(\mathbf{x}) + \mathbf{u}_2(\mathbf{x}) = \mathbf{u}^s(\mathbf{x}), \end{aligned} \tag{2.4}$$

with $k_1^2 = \omega^2/(\lambda + 2\mu)$ and $k_2^2 = \omega^2/\mu$. In terms of $\mathbf{u}_1$ and $\mathbf{u}_2$, the elastic radiation condition at infinity is given by

* With the understanding that the harmonic time factor exp $(i\omega t)$ is omitted.

$$\lim_{\rho\to\infty} \rho\left(\frac{\partial \mathbf{u}_1}{\partial \rho} - ik_1\mathbf{u}_1\right) = \mathbf{0}, \qquad \lim_{\rho\to\infty} \rho\left(\frac{\partial}{\partial \rho}\mathbf{u}_2 - ik_2\mathbf{u}_2\right) = \mathbf{0} \tag{2.5}$$

(see Kupradze [5]).

We shall also consider the case for $\omega = 0$, i.e. the boundary-value problem (p_0) defined by

i) $\mathbf{u}_0(\mathbf{x}) = \mathbf{u}_0^{\,i}(\mathbf{x}) + \mathbf{u}_0^{\,s}(\mathbf{x})$,
ii) $\Delta^*\mathbf{u}_0^{\,s}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in B_a$,
iii) $T\mathbf{u}_0(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in S$,
iv) $\mathbf{u}_0^{\,s}(\mathbf{x})$ is regular at infinity.

Notice that a real-valued function $f\colon B_a \to$ reals is defined to be regular at infinity, if

$$\lim_{\rho\to\infty} |\rho f(\mathbf{x})| < \infty, \qquad \lim_{\rho\to\infty} \left|\rho^2 \frac{\partial}{\partial x_i} f(\mathbf{x})\right| < \infty, \qquad \forall\ i = 1, 2, 3 \tag{2.6}$$

(see Kellogg [4], p. 217). It is understood in this definition of regularity that $f(\mathbf{x})$ is differentiable for ρ sufficiently large. A complex-valued function is regular at infinity if both its real and imaginary parts are regular. A vector function $\mathbf{u}(\mathbf{x})$ is regular at infinity if all of its components are regular.

3. An integral representation. In this section, we shall reformulate the scattering problem as a continuous integral representation which is valid everywhere in $S \cup B_a$. We begin by applying the Betti formulas to $\mathbf{u}^s(\mathbf{x})$. It can be shown that

$$\begin{aligned}\frac{1}{4\pi}\int_S \{\Gamma_1(\mathbf{x}, \mathbf{y})\mathbf{u}^s(\mathbf{y}) - \Gamma(\mathbf{x}, \mathbf{y})T\mathbf{u}^s(\mathbf{y})\}\, dS_{\mathbf{y}} &= \mathbf{u}^s(\mathbf{x}), \qquad \mathbf{x} \in B_a,\\ &= \tfrac{1}{2}\mathbf{u}^s(\mathbf{x}), \qquad \mathbf{x} \in S,\\ &= \mathbf{0}, \qquad \mathbf{x} \in B_i\,.\end{aligned} \tag{3.1}$$

Here $\Gamma(\mathbf{x}, \mathbf{y})$ is the matrix of fundamental solutions of $(\Delta^* + \omega^2)\mathbf{u}(\mathbf{x}) = \mathbf{0}$, which takes the form:

$$\Gamma(\mathbf{x}, \mathbf{y}) = [\Gamma_j^{(k)}(\mathbf{x}, \mathbf{y})] \tag{3.2}$$

where

$$\Gamma_j^{(k)}(\mathbf{x}, \mathbf{y}) = \frac{1}{\mu}\,\delta_{kj}\,\frac{\exp(ik_2 r)}{r} - \frac{1}{\omega^2}\frac{\partial^2}{\partial x_k\,\partial x_j}\left(\frac{\exp(ik_1 r)}{r} - \frac{\exp(ik_2 r)}{r}\right),$$

$$\begin{aligned}\delta_{kj} &= 1, \qquad k = j \\ &= 0, \qquad k \neq j\end{aligned} \quad \text{the Kronecker symbol,}$$

and $\Gamma_1(\mathbf{x}, \mathbf{y})$ is defined by

$$\Gamma_1(\mathbf{x}, \mathbf{y}) = [T_j^{(\mathbf{y})}\Gamma^{(k)}(\mathbf{x}, \mathbf{y})] \tag{3.3}$$

where

$$\begin{aligned}T_j^{(\mathbf{y})}\Gamma^{(k)}(\mathbf{x}, \mathbf{y}) = {}& 2\mu\,\frac{\partial\Gamma_j^{(k)}(\mathbf{x}, \mathbf{y})}{\partial\hat{n}_{\mathbf{y}}} + \frac{\lambda}{\lambda + 2\mu}(\hat{n}_{\mathbf{y}}\cdot\hat{e}_j)\frac{\partial}{\partial y_k}\frac{\exp(ik_1 r)}{r}\\ &+ (\hat{n}_{\mathbf{y}}\cdot\hat{e}_k)\frac{\partial}{\partial y_j}\frac{\exp(ik_2 r)}{r} - \delta_{jk}\frac{\partial}{\partial\hat{n}_{\mathbf{y}}}\frac{\exp(ik_2 r)}{r}.\end{aligned}$$

A similar result can be obtained for $\mathbf{u}^i(\mathbf{x})$, if one requires that $\mathbf{u}^i(\mathbf{x})$ satisfy $(\Delta^* + \omega^2)\mathbf{u}^i(\mathbf{x}) = \mathbf{0}$ in B_i

$$\begin{aligned}\frac{1}{4\pi}\int_S \{\Gamma_1(\mathbf{x}, \mathbf{y})\mathbf{u}^i(\mathbf{y}) - \Gamma(\mathbf{x}, \mathbf{y})T\mathbf{u}^i(\mathbf{y})\}\, dS_{\mathbf{y}} &= \mathbf{0}, && \mathbf{x} \in B_a, \\ &= -\tfrac{1}{2}\mathbf{u}^i(\mathbf{x}), && \mathbf{x} \in S, \\ &= -\mathbf{u}^i(\mathbf{x}), && \mathbf{x} \in B_i, \end{aligned} \tag{3.4}$$

By adding (3.1) and (3.4), we obtain an integral representation $\mathbf{u}(\mathbf{x})$,

$$\begin{aligned}\mathbf{u}^i(\mathbf{x}) + \frac{1}{4\pi}\int_S \Gamma_1(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y})\, dS_{\mathbf{y}} &= \mathbf{u}(\mathbf{x}), && \mathbf{x} \in B_a, \\ &= \tfrac{1}{2}\mathbf{u}(\mathbf{x}), && \mathbf{x} \in S, \\ &= \mathbf{0}, && \mathbf{x} \in B_i. \end{aligned} \tag{3.5}$$

Similarly, we have, in the particular case $\omega = 0$,

$$\begin{aligned}\frac{1}{4\pi}\int_S \{\mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})\mathbf{u}_0{}^i(\mathbf{y}) - \mathring{\Gamma}(\mathbf{x}, \mathbf{y})T\mathbf{u}_0{}^i(\mathbf{y})\}\, dS_{\mathbf{y}} &= \mathbf{0}, && \mathbf{x} \in B_a, \\ &= -\tfrac{1}{2}\mathbf{u}_0{}^i(\mathbf{x}), && \mathbf{x} \in S, \\ &= -\mathbf{u}_0{}^i(\mathbf{x}), && \mathbf{x} \in B_i, \end{aligned} \tag{3.6}$$

where $\mathring{\Gamma}_1$ and $\mathring{\Gamma}$ are the corresponding Γ_1 and Γ in the limiting case. Note that unit vectors $\hat{\mathbf{e}}_k$ satisfy the equation $\Delta^*\mathbf{u}_0{}^i = \mathbf{0}$ in B_i. Hence from (3.6)

$$\begin{aligned}\frac{1}{4\pi}\int_S \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})\hat{e}_k\, dS_{\mathbf{y}} &= \mathbf{0}, && \mathbf{x} \in B_a, \\ &= -\tfrac{1}{2}\hat{e}_k, && \mathbf{x} \in S, \quad k = 1, 2, 3 \\ &= \hat{e}_k, && \mathbf{x} \in B_i. \end{aligned} \tag{3.7}$$

Consequently, we also have an integral representation for $\mathbf{u}(\mathbf{x})$ in the form:

$$\begin{aligned}\frac{1}{4\pi}\int_S \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{x})\, dS_{\mathbf{y}} &= \mathbf{0}, && \mathbf{x} \in B_a, \\ &= -\tfrac{1}{2}\mathbf{u}(\mathbf{x}), && \mathbf{x} \in S, \end{aligned} \tag{3.8}$$

Subtracting (3.8) from (3.5), we finally obtain the following integral representation for $\mathbf{x} \in B_a \cup S$:

$$\mathbf{u}^i(\mathbf{x}) + \frac{1}{4\pi}\int_S \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})[\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})]\, dS_{\mathbf{y}} + \frac{1}{4\pi}\int_S [\Gamma_1(\mathbf{x}, \mathbf{y}) - \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})]\mathbf{u}(\mathbf{y})\, dS_{\mathbf{y}} = \mathbf{u}(\mathbf{x}). \tag{3.9}$$

This integral representation (3.9) is continuous as the field point $\mathbf{x}$ approaches the boundary and is valid everywhere on S.

Now, let us introduce the notation

$$\begin{aligned} L_0\mathbf{u} &= \frac{1}{4\pi}\int_S \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})[\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})]\, dS_{\mathbf{y}}, \\ L\mathbf{u} &= L_0\mathbf{u} + \frac{1}{4\pi}\int_S [\mathring{\Gamma}_1(\mathbf{x}, \mathbf{y}) - \Gamma_1(\mathbf{x}, \mathbf{y})]\mathbf{u}(\mathbf{y})\, dS_{\mathbf{y}} \end{aligned} \tag{3.10}$$

and $M(\omega)\mathbf{u} = L\mathbf{u} - L_0\mathbf{u}$. Thus, the scattering problem may be expressed as the following integral representation

$$(I - L)\mathbf{u} = \mathbf{u}^i(\mathbf{x}) \tag{3.11}$$

where I is the identity matrix.

4. A Neumann series. We now seek a solution to the integral equation (3.11) in the form of a Neumann series, and we shall show that this series converges for ω sufficiently small. In view of definition (3.10), the integral equation under consideration for $\mathbf{x} \in S$ is

$$(I - L)\mathbf{u} = (I - L_0 - M(\omega))\mathbf{u} = \mathbf{u}^i(\mathbf{x}). \tag{4.1}$$

Observe that the integral equation $(I - L_0)\mathbf{u}_0 = \mathbf{u}_0^i(\mathbf{x})$ corresponds to the boundary-value problem (p_0) in Sec. 2. The inverse operator $(I - L_0)^{-1}$ is known to exist for Lyapunoff surfaces [5]. Hence Eq. (4.1) can be rewritten in the form:

$$[I - (I - L_0)^{-1}M(\omega)]\mathbf{u} = (I - L_0)^{-1}\mathbf{u}^i(\mathbf{x}). \tag{4.2}$$

To this end, it remains to show that $(I - L_0)^{-1}M(\omega)$ is bounded by one, for small ω. One recalls that if A is a bounded linear operator in a Banach space and if $||A|| < 1$, then the solution to the operator equation $(I - A)\phi = f$ may be expressed by $\phi = \sum_{n=0}^{\infty} A^n f$. (See Kantorovich and Akilov [3], p. 173.)

We consider here the Banach space $(C(S), ||\cdot||)$, where $C(S)$ is the vector space of vector functions $\mathbf{u}$ in R^3, the components of which are continuous complex functions defined on S, and the norm of $\mathbf{u}(\mathbf{x})$, $||\mathbf{u}||$, is defined by

$$||\mathbf{u}|| = \sup_{\mathbf{x} \in S} \{|u_j(\mathbf{x})| : j = 1, 2, 3\}. \tag{4.3}$$

For Lyapunoff surfaces, we have that $L_0 : C(S) \to C(S)$ and from the bounded inverse theorem [2, p. 271], $(I - L_0)^{-1}$ is a bounded linear operator mapping $C(S)$ into $C(S)$. The components of the kernel of $M(\omega)$ are continuous and hence $M(\omega)$ maps $C(S)$ into $C(S)$ (see (A.2)). Thus $(I - L_0)^{-1}M(\omega)$ is a bounded linear operator mapping $C(S)$ into $C(S)$. Hence, if

$$||M(\omega)|| < \frac{1}{||(I - L_0)^{-1}||} \tag{4.4}$$

then the solution to (4.2) may be expressed by

$$\mathbf{u}(\mathbf{x}) = \sum_{n=0}^{\infty} [(I - L_0)^{-1}M(\omega)]^n (I - L_0)^{-1}\mathbf{u}^i(\mathbf{x}). \tag{4.5}$$

In the Appendix, it is shown that the inequality (4.4) is valid for small values of ω and it is further shown that a more explicit condition for ω sufficient to guarantee the convergence of the Neumann series is

$$\omega^2 H(\omega) < \frac{1}{||(I - L_0)^{-1}||}, \tag{4.6}$$

where

$$H(\omega) \equiv A(S)\left\{\frac{1}{\pi(\lambda + 2\mu)} + \frac{\lambda}{\pi(\lambda + 2\mu)^2} + \frac{1}{\pi\mu} + \frac{3}{2\pi}\left[\frac{\exp(k_1 d)}{(\lambda\mu + 2)^2} + \frac{\exp(k_2 d)}{\mu^2}\right]\right\}.$$

Here $A(S)$ is the surface area of S and d is the diameter of S.

5. The second fundamental problem. As in Sec. 2, we denote by B_i and B_a respectively the regions interior and exterior to a closed Lyapunoff surface in R^3. The region B_a is filled with homogeneous isotropic elastic medium. The second fundamental problem states: determine the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x})$ that satisfies the equation $\Delta^*\mathbf{u} + \omega^2\mathbf{u} = \mathbf{0}$ in B_a, together with the boundary condition, $T\mathbf{u} = \mathbf{f}(\mathbf{x})$ in S, and the elastic radiation condition at infinity. Here $\mathbf{f}(\mathbf{x})$ is the given stress vector which is assumed to be smooth enough for whatever the existence of the following integral involves.

This problem again can be reformulated as a continuous integral representation which is valid everywhere in $S \cup B_a$. The integral representation in this case takes the form, for $\mathbf{x} \in S \cup B_a$:

$$\mathbf{u}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \frac{1}{4\pi}\int_S \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})[\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})]\, dS_\mathbf{y} + \frac{1}{4\pi}\int_S [\Gamma_1(\mathbf{x}, \mathbf{y}) - \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})]\mathbf{u}(\mathbf{y})\, dS_\mathbf{y} \tag{5.1}$$

where

$$\mathbf{F}(\mathbf{x}) \equiv -\frac{1}{4\pi}\int_S \Gamma(\mathbf{x}, \mathbf{y})\mathbf{f}(\mathbf{y})\, dS_\mathbf{y}\,.$$

This result follows immediately from (3.7) and the Betti's formula for an infinite region (see Eq. (3.1)). A comparison between (5.1) and (3.9) indicates that under the condition (4.4), the solution of the second problem may be expressed by the same series (4.5) with $\mathbf{u}^i(\mathbf{x})$ replaced by $\mathbf{F}(\mathbf{x})$.

Appendix: establishing a bound for $\|M(\omega)\|$. In the following, we first give an analytical representation for each of the elements in the kernel matrix of $M(\omega)$, which is defined in (3.10), and then using this expansion establish an upper bound for the norm of $M(\omega)$. We recall that the kth component of the vector $M(\omega)\mathbf{u}$, denoted by $[M(\omega)\mathbf{u}]_k$, is defined by

$$[M(\omega)\mathbf{u}]_k = \frac{1}{4\pi}\int_S [\Gamma_1(\mathbf{x}, \mathbf{y}) - \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})]_{kj} u_j(\mathbf{y})\, dS_\mathbf{y}\,. \tag{A.1}$$

Here $[\Gamma_1(\mathbf{x}, \mathbf{y}) - \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})]_{kj}$ denotes the kjth elements of the tensor $[\Gamma_1(\mathbf{x}, \mathbf{u}) - \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})]$ which takes the form:

$$[\Gamma_1(\mathbf{x}, \mathbf{y}) - \mathring{\Gamma}_1(\mathbf{x}, \mathbf{y})]_{kj} = 2\mu \frac{\partial}{\partial \hat{n}_\mathbf{y}}(\Gamma_j^{(k)} - \mathring{\Gamma}_j^{(k)}) + (\hat{n}_\mathbf{y} \cdot \hat{e}_k)\frac{\partial}{\partial y_j}\left(\frac{e^{ik_2 r} - 1}{r}\right)$$
$$+ \frac{\lambda}{\lambda + 2\mu}(\hat{n}_\mathbf{y} \cdot \hat{e}_j)\frac{\partial}{\partial y_k}\left(\frac{e^{ik_1 r} - 1}{r}\right) - \delta_{jk}\frac{\partial}{\partial \hat{n}_\mathbf{y}}\left(\frac{e^{ik_2 r} - 1}{r}\right)$$

with $k_1^2 = \omega^2/(\lambda + 2\mu)$ and $k_2^2 = \omega^2/\mu$. The term $\Gamma_j^{(k)}$ (and $\mathring{\Gamma}_j^{(k)}$) is defined in (3.2) and can be simplified so that

$$\Gamma_i^{(k)} - \mathring{\Gamma}_i^{(k)} = \delta_{ik}\left(\frac{1}{b^2}\frac{\exp(ik_2r)}{r} - \frac{a^2+b^2}{2a^2b^2}\frac{1}{r}\right) - \frac{1}{\omega^2}\frac{\partial^2}{\partial x_k\,\partial x_i}\left(\frac{\exp(ik_1r)}{r} - \frac{\exp(ik_2r)}{r}\right)$$
$$- \frac{(a^2-b^2)}{2a^2b^2}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k}\frac{1}{r}, \qquad \text{(A.2)}$$

where $a^2 = \lambda + 2\mu$ and $b^2 = \mu$. Using the series expansion

$$\frac{\exp(ik_1r)}{r} - \frac{\exp(ik_2r)}{r} = \sum_{n=0}^{\infty}\frac{i^{n+1}[k_1^{n+1} - k_2^{n+2}]}{(n+1)!}r^n,$$

we obtain

$$\Gamma_i^{(k)} - \mathring{\Gamma}_i^{(k)} = \delta_{ik}\frac{1}{b^2}\frac{\exp(ik_2r)-1}{r}$$
$$- \omega\sum_{n=0}^{\infty}\frac{i^{n+3}[b^3k_1^n - a^3k_2^n](n+2)}{a^3b^3(n+3)!}\left\{nr^n\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_i} + r^n\,\delta_{ik}\right\}$$

and hence

$$[\Gamma_1(\mathbf{x},\mathbf{y}) - \mathring{\Gamma}_1(\mathbf{x},\mathbf{y})]_{ki} = \delta_{ik}\frac{\partial}{\partial n_y}\left(\frac{\exp(ik_2r)-1}{r}\right) + (\hat{n}_y\cdot\hat{e}_k)\frac{\partial}{\partial y_i}\left(\frac{\exp(ik_2r)-1}{r}\right)$$
$$+ \frac{\lambda}{\lambda+2\mu}(\hat{n}_y\cdot\hat{e}_i)\frac{\partial}{\partial y_k}\left(\frac{\exp(ik_1r)-1}{r}\right) \qquad \text{(A.3)}$$
$$- \omega\frac{\partial}{\partial n_y}\sum_{n=0}^{\infty}\frac{i^{n+3}[b^3k_1^n - a^3k_2^n](n+2)}{a^3b^3(n+3)!}\left\{nr^r\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_i} + r^n\,\delta_{ik}\right\}.$$

Using this representation for the kjth element of the kernel matrix, we can establish a bound for the norm of $M(\omega)$. We shall show that there exists an $H = H(\omega)$ such that for all $k = 1, 2, 3$, $|[M(\omega)\mathbf{u}]_k| \leq \omega^2H(\omega)\,||\mathbf{u}||$ and hence $||M(\omega)|| \leq \omega^2H(\omega)$. For simplicity, we present here only the case for $k = 1$, since the analysis proceeds in the same manner for $k = 2, 3$.

By the definition of $[M(\omega)\mathbf{u}]_1$ in (A.1) and the formula (A.2), it is easy to see that

$$|[M(\omega)\mathbf{u}]_1| \leq \frac{1}{4\pi}\int_S |u_1(\mathbf{y}) - u_1(\mathbf{x})|\left|\frac{\partial}{\partial n_y}\left(\frac{\exp(ik_2r)-1}{r}\right)\right| dS_y$$
$$+ \frac{\lambda}{4\pi(\lambda+2\mu)}\int_S |\hat{n}_y\cdot[\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})]|\left|\frac{\partial}{\partial y_1}\left(\frac{\exp(ik_1r)-1}{r}\right)\right| dS_y$$
$$+ \frac{1}{4\pi}\int_S |\hat{n}_y\cdot\hat{e}_1|\left|ik_2\frac{\exp(ik_2r)}{r} - \frac{\exp(ik_2r)-1}{r}\right| |\nabla r\cdot[\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})]|\,dS_y \qquad \text{(A.4)}$$
$$+ \omega\frac{1}{4\pi}\int_S\left|\sum_{m=0}^{\infty}\frac{i^{m+1}[b^3k_1^m - a^3k_2^m](m+2)}{a^3b^3(m+3)!}\right.$$
$$\left.\cdot\left\{\sum_{j=1}^{3}\frac{\partial}{\partial\hat{n}_y}\left[mr^m\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_j}\right](u_j(\mathbf{y}) - u_j(\mathbf{x})) - \frac{\partial}{\partial n_y}r^m[u_1(\mathbf{y}) - u_1(\mathbf{x})]\right\}\right| dS_y\,.$$

Then by making use of the relation

$$\left|\hat{\alpha}\cdot\nabla_y\left(\frac{\exp(ilr)-1}{r}\right)\right| = \left|\left(ik\frac{\exp(ilr)}{r} - \frac{\exp(ilr)-1}{r_2}\right)\hat{\alpha}\cdot\mathbf{r}\right| \leq 2l^2$$

for any constant $l > 0$ and unit vector $\hat{\alpha}$, one obtains the estimates for the first three terms on the right-hand side of (A.4):

$$\frac{1}{4\pi}\int_S |u_1(\mathbf{y}) - u_1(\mathbf{x})| \left|\frac{\partial}{\partial \hat{n}_{\mathbf{y}}}\left(\frac{\exp(ik_2r) - 1}{r}\right)\right| dS_{\mathbf{y}} \leq \frac{\|\mathbf{u}\|\, k_2^{\,2}}{\pi} A(S),$$

$$\frac{\lambda}{4\pi(\lambda + 2\mu)}\int_S |\hat{n}_{\mathbf{y}} \cdot [\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})]| \left|\frac{\partial}{\partial y_1}\left(\frac{\exp(ik_1r) - 1}{r}\right)\right| dS_{\mathbf{y}} \leq \frac{\|\mathbf{u}\|\, \lambda k_1^{\,2}}{(\lambda + 2\mu)\pi} A(S), \tag{A.5}$$

$$\frac{1}{4\pi}\int_S |\hat{n}_{\mathbf{y}} \cdot \hat{e}_1| \left| ik_2 \frac{\exp(ik_2r)}{r} - \frac{\exp(ik_2r) - 1}{r^2}\right| |\nabla r \cdot [\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})]|\, dS_{\mathbf{y}} \leq \frac{\|\mathbf{u}\|\, k_2^{\,2}}{\pi} A(S),$$

where $A(S)$ is the surface area of S. Also, from the inequalities:

$$\left| m \frac{\partial}{\partial \hat{n}_{\mathbf{y}}} r^m \frac{(y_1 - x_1)^2}{r^2} - mr^{m-1} \frac{\partial r}{\partial \hat{n}_{\mathbf{y}}}\right| \leq m(m + 1)r^{m-1}$$

and

$$m \left|\frac{\partial}{\partial \hat{n}_{\mathbf{y}}} r^m \frac{\partial r}{\partial x_1} \frac{\partial r}{\partial x_k}\right| \leq m(m + 1)r^{m-1} \quad \text{for} \quad k = 2, 3;$$

$m = 1, 2, 3, \cdots$, it is not difficult to see that the last term of (A.3) is dominated by

$$\frac{6\omega\, \|\mathbf{u}\|}{4\pi} \sum_{m=1}^{\infty} \frac{b^3k_1^{\,m} + a^3k_2^{\,m}}{a^3b^3(m + 3)!}\, m(m + 1)(m + 2) \int_S r^{m-1}\, dS_{\mathbf{y}}$$

$$\leq \frac{3}{2\pi}\omega^2\, \|\mathbf{u}\|\, A(S)\left\{\frac{\exp(k_1d)}{(\lambda + 2\mu)^2} + \frac{\exp(k_2d)}{\mu^2}\right\} \tag{A.6}$$

where $d = \sup_{\mathbf{x},\mathbf{y}\in S} r(\mathbf{x}, \mathbf{y})$. Hence, it follows from (A.5) and (A.6) that

$$|[M(\omega)\mathbf{u}]_1| \leq \omega^2 H(\omega)\, \|\mathbf{u}\| \tag{A.7}$$

where $H(\omega)$ is defined by

$$H(\omega) = A(S)\left\{\frac{1}{\pi(\lambda + 2\mu)} + \frac{\lambda}{\pi(\lambda + 2\mu)^2} + \frac{1}{\pi\mu} + \frac{3}{2\pi}\left[\frac{\exp(k_1d)}{(\lambda + 2\mu)^2} + \frac{\exp(k_2d)}{\mu^2}\right]\right\}. \tag{A.8}$$

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