A New 2d/4d Duality via Integrability

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A New 2d/4d Duality

- **Theory I**: Four dim $\mathcal{N} = 2$ SQCD with $G = SU(L)$, plus L fundamental hypermultiplets of masses $\vec{m}_{\mathrm{F}} = (m_1, \ldots, m_L)$ and L anti-fundamental hypermultiplets of masses $\vec{m}_{\mathrm{AF}} = (\tilde{m}_1, \ldots, \tilde{m}_L)$. The complex gauge coupling is $\tau = \frac{4\pi i}{g^2} + \frac{\Theta_{\mathrm{4D}}}{2\pi}$
- **Theory I** is subjected to Ω-deformation with $(\epsilon_1, \epsilon_2) = (\epsilon, 0)$, which preserves $\mathcal{N} = (2,2)$ SUSY in $x^0 - x^1$ plane. The coulomb branch of undeformed theory is lifted, only discrete points remain:

$$\vec{a} = \vec{m}_F - \vec{n}\epsilon\,, \qquad \vec{n} = (n_1, \ldots, n_L) \in \mathbb{Z}^L\,. \tag{1}$$

- The low energy dynamics are governed by twisted superpotential $\mathcal{W}^{(I)}(\vec{a}, \epsilon)$, which is inherited from Nekrasov partition function $\mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2)$ as:

$$\mathcal{W}^{(I)}(\vec{a}, \epsilon) = \lim_{\epsilon_1 \to \epsilon, \epsilon_2 \to 0} [\epsilon_2 \mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2)] + \text{quantized fluxes} \tag{2}$$

- **Theory II**: Two dim $\mathcal{N}=(2,2)$ SYM with $G=U(N)$, plus L fundamental chiral multiplet of twisted masses $\vec{M}_{\mathrm{F}}=(M_1,\ldots,M_L)$, L anti-fundamental chirals of twisted masses $\vec{M}_{\mathrm{AF}}=(\tilde{M}_1,\ldots,\tilde{M}_L)$; and an adjoint chiral multiplet of twisted mass ϵ. The complex gauge coupling is $\hat{\tau}=ir+\frac{\Theta_{\mathrm{2D}}}{2\pi}$.
- This is the world volume theory of 4 dim "vortex/surface operator". Its low energy dynamics is also governed by effective twisted superpotential $\mathcal{W}^{(II)}(\{\lambda_k\})$ from an one-loop computation.
- $\mathcal{W}^{(II)}(\{\lambda_k\})$ is a "Yang-Yang" potential so that the F-term equation $d_{\lambda_j}\mathcal{W}^{(II)}=0$ coincides with the Bethe Ansatz Equation (BAE) of $SL(2,\mathbb{R})$ spin chain:

$$\prod_{l=1}^{L}\frac{\lambda_j-M_l}{\lambda_j-\tilde{M}_l}=-q\prod_{k=1}^{N}\frac{\lambda_j-\lambda_k-\epsilon}{\lambda_j-\lambda_k+\epsilon}\,,\qquad q=(-1)^{N+1}e^{2\pi\hat{\tau}}. \tag{3}$$

- The solution "Bethe Roots" $\{\lambda_j\equiv\lambda_{(ls)}\}$ are given by:

$$\lambda_{(ls)}=M_l-(s-1)\epsilon+\mathcal{O}(q)\,,\quad s=1,\ldots,\hat{n}_l\,,\quad N=\sum_{l=1}^{L}\hat{n}_l\,. \tag{4}$$

- The Conjectured Duality states that, the on-shell values of the twisted superpotentials for **Theory I/II** coincide:

$$\mathcal{W}^{(I)}(a_I = m_I - n_I\epsilon) - \mathcal{W}^{(I)}(a_I = m_I - \epsilon) = \mathcal{W}^{(II)}(\{\hat{n}_I\}) \,, \qquad (5)$$

  if we make following identification of parameters:

$$\hat{\tau} = \tau + \frac{1}{2}(N+1), \;\; \hat{n}_I = n_I - 1, \;\; M_I = m_I - \frac{3}{2}\epsilon, \;\; \tilde{M}_I = \tilde{m}_I - \frac{1}{2}\epsilon. \quad (6)$$

- The VEVs of Chiral ring of **Theory I** $\mathcal{O}_k = \mathrm{Tr}\varphi^k$ are also mapped the conserved charges of $SL(2,\mathbb{R})$ spin chain arising from **Theory II**.
- The exact perturbative matching, and first few instanton checks were performed earlier. Here we shall prove the duality exactly, by saddle point analysis of $\mathcal{Z}(\vec{a}, \epsilon_{1,2})$, such that $SL(2,\mathbb{R})$ BAE appears and $\mathcal{W}^{(I)}$ and $\mathcal{W}^{(II)}$ match on-shell. The steps can be easily generalized for proving the duality in wide range of set-ups.

BAE from Nekrasov Instanton Partition Function

- We begin with the Gamma-function representation of Nekrasov Partition function [Nekrasov-Okounkov]:

$$\mathcal{Z}_{\text{inst}} = \sum_{\{\vec{Y}\}} q^{|\vec{Y}|} \mathcal{Z}_{\text{vec}}(\vec{Y}) \prod_{n=1}^{2L} \mathcal{Z}_{\text{hyp}}(\vec{Y}, \mu_n) \ , \qquad q = e^{2\pi i \tau} \tag{7}$$

where $\mathcal{Z}_{\text{vec}}(\vec{Y})$ and $\mathcal{Z}_{\text{hyp}}(\vec{Y}, \mu_n)$ are:

$$\mathcal{Z}_{\text{vec}}(\vec{Y}) = \prod_{(li)\neq(nj)} \frac{\Gamma\left(\epsilon_2^{-1}(x_{li} - x_{nj} - \epsilon_1)\right)}{\Gamma\left(\epsilon_2^{-1}(x_{li} - x_{nj})\right)} \cdot \frac{\Gamma\left(\epsilon_2^{-1}(x_{li}^{(0)} - x_{nj}^{(0)})\right)}{\Gamma\left(\epsilon_2^{-1}(x_{li}^{(0)} - x_{nj}^{(0)} - \epsilon_1)\right)} \ ,$$

$$\mathcal{Z}_{\text{hyp}}(\vec{Y}, \mu_n) = \prod_{li} \frac{\Gamma\left(\epsilon_2^{-1}(x_{li} + \mu_n)\right)}{\Gamma\left(\epsilon_2^{-1}(x_{li}^{(0)} + \mu_n)\right)} \ .$$

$$x_{li} = a_l + (i-1)\epsilon_1 + \epsilon_2 k_{li} \ , \qquad x_{li}^{(0)} = a_l + (i-1)\epsilon_1 \ . \tag{8}$$

with k_{li} being the length of i-th row in the Young Tableau Y_l.

- Now if we take the limit $(\epsilon_1, \epsilon_2) \to (\epsilon, 0)$ [Nekrasov-Shatashvili], Stirling's approximation for $\Gamma(x)$ yields:

$$\mathcal{Z}_{\text{inst}} = \int \prod_{li} dx_{li} \; \exp[\epsilon_2^{-1} \mathcal{H}_{\text{inst}}(x_{li}, x_{li}^{(0)})] \, , \qquad \mathcal{H}_{\text{inst}}(x_{li}) = \mathcal{Y}(x_{li}) - \mathcal{Y}(x_{li}^{(0)}) \, , \tag{9}$$

where

$$\begin{aligned} \mathcal{Y}(x_{li}) &= \log q \sum_{(li)} x_{li} + \sum_{(li),n} \big(f(x_{li} + \tilde{m}_n) + f(x_{li} - m_n + \epsilon) \big) \\ &+ \frac{1}{2} \sum_{(li) \neq (kj)} \big(f(x_{li} - x_{kj} - \epsilon) - f(x_{li} - x_{kj} + \epsilon) \big) \, , \end{aligned} \tag{10}$$

with $f(x) = x \log x - x$ and $\mathcal{Y}(x_{li}^{(0)}) = \mathcal{Y}(x_{li} \to x_{li}^{(0)})$.

- As $\epsilon_2 \to 0$, the instanton positions condense and become constant on the intervals:

$$\mathcal{I} = \bigcup_{li} [x_{li}^{(0)}, x_{li}] \, . \tag{11}$$

- We can re-express $\mathcal{H}_{\text{inst}}$ in terms of instanton density $\rho(x)$:

$$\mathcal{H}_{\mathrm{inst}}[\rho] = -\frac{1}{2}\int_{\mathcal{I}\times\mathcal{I}} dx\, dy\, \rho(x)\mathfrak{G}(x-y)\rho(y) + \int_{\mathcal{I}} dx\, \rho(x) \log\left(q\,\mathfrak{R}(x)\right) , \tag{12}$$

where the kernels are:

$$\mathfrak{G}(x) = \frac{d}{dx}\log\left(\frac{x-\epsilon}{x+\epsilon}\right), \qquad \mathfrak{R}(x) = \frac{A(x)D(x+\epsilon)}{P(x)P(x+\epsilon)} ,$$

$$A(x) = \prod_{l=1}^{L}(x-\tilde{m}_l) , \; D(x) = \prod_{l=1}^{L}(x-m_l) , \; P(x) = \prod_{l=1}^{L}(x-a_l) .$$

- In the $\epsilon_2 \to 0$ limit, the functional integral is dominated by "saddle point equation":

$$\frac{\delta\mathcal{H}_{\mathrm{inst}}[\rho]}{\delta x_j} = -\int_{\mathcal{I}} dy\, \mathfrak{G}(x_j-y)\rho(y) + \log\left(q\,\mathfrak{R}(x_j)\right) = 0 , \tag{13}$$

- Integrating and exponetiating the saddle point equation, we obtained infinite set of equations for $\{x_{li}\}$:

$$\frac{\mathfrak{Q}(x_{li}+\epsilon)\mathfrak{Q}^{(0)}(x_{li}-\epsilon)}{\mathfrak{Q}(x_{li}-\epsilon)\mathfrak{Q}^{(0)}(x_{li}+\epsilon)} = -q\,\mathfrak{R}(x_{li})\,, \tag{14}$$

$$\mathfrak{Q}(x) = \prod_{k=1}^{L}\prod_{j=1}^{\infty}(x-x_{kj})\ , \qquad \mathfrak{Q}^{(0)}(x) = \prod_{k=1}^{L}\prod_{j=1}^{\infty}(x-x_{kj}^{(0)})\ .$$

- To see $SL(2,\mathbb{R})$ spin chain appearing, the infinite equations (14) can be truncated to finite set, if we impose the "quantization condition":

$$a_l = m_l - n_l\epsilon\,,\ n_l\in\mathbb{Z}>0\,,\ \longrightarrow\ x_{li} = x_{li}^{(0)} = a_l + (i-1)\epsilon\,,\ \text{for } i\geq n_l\,. \tag{15}$$

One Slide Proof for (15)

- We can consider the following equality:

$$\mathfrak{W}(x+\epsilon) - \frac{(1+q)}{2}\mathfrak{W}(x)\frac{T(x)}{P(x+\epsilon)} = -q\mathfrak{R}(x)\mathfrak{W}(x-\epsilon)\ , \qquad (16)$$

  where

$$\mathfrak{W}(x) = \frac{\mathfrak{Q}(x)}{\mathfrak{Q}^{(0)}(x)}\ ,\ T(x) = \frac{2}{(1+q)}\left(\frac{\mathfrak{Q}(x+\epsilon)}{\mathfrak{Q}(x)} + qA(x)D(x)\frac{\mathfrak{Q}(x-\epsilon)}{\mathfrak{Q}(x)}\right)\ ,$$

  $T(x)$ is a degree L polynomial in x.

- Now is the quantization condition $a_I = m_I - n_I\epsilon$ is imposed, the simple pole at $x = a_I + (n_I - 1)\epsilon$ in $\mathfrak{W}(x-\epsilon)$ on RHS of (16) coincides with a zero of $\mathfrak{R}(x)$, this implies $\mathfrak{W}(x)$ cannot have simple pole at $x = a_I + (n_I - 1)\epsilon$ either. The argument can be repeated continuously for $i \geq n_I$, and only possible if $x_{Ii} = x_{Ii}^{(0)},\ i \geq n_I$, hence we obtain (15).

- Having truncated the infinite set of equations by quantization condition, we arrive at:

$$\frac{D(x_{li}+2\epsilon)}{A(x_{li})} = -q\frac{\hat{\mathfrak{Q}}(x_{li}-\epsilon)}{\hat{\mathfrak{Q}}(x_{li}+\epsilon)} \ , \qquad \hat{\mathfrak{Q}}(x) = \prod_{l=1}^{L}\prod_{i=1}^{n_l-1}(x-x_{li})\,, \quad (17)$$

  substituting in the identifications of parameters (6) and $x_{li} = \lambda_{li} - \frac{\epsilon}{2}$, we finally see that (17) precisely coincides with the $SL(2,\mathbb{R})$ BAE (3).

- To complete the proof, we can now evaluate $\mathcal{H}_{\text{inst}}[\rho]$ with the truncation/quantization condition imposed, and obtain:

$$\mathcal{W}^{(\mathrm{I})}_{\text{inst}}(m_l - n_l\epsilon) - \mathcal{W}^{(\mathrm{I})}_{\text{inst}}(m_l - \epsilon) = \hat{\mathcal{Y}}(x_{li}) - \hat{\mathcal{Y}}(x^{(0)}_{li}) \ , \qquad (18)$$

$$\hat{\mathcal{Y}}(x_{li}) = \log q \sum_{(li)=1}^{N} x_{li} + \sum_{(li)=1}^{N}\sum_{n=1}^{L}\Big(f(x_{li}-\tilde{m}_n) - f(x_{li}-m_n+2\epsilon)\Big)$$

$$+\frac{1}{2}\sum_{(li)\neq(mj)=1}^{N}\Big(f(x_{li}-x_{mj}-\epsilon) - f(x_{li}-x_{mj}+\epsilon)\Big) \ . \qquad (19)$$

  Again after matching the parameters, this precisely matches with the $\hat{q}$/instanton-dependent part of $\mathcal{W}^{(II)}(\{\hat{n}_l\})$ and completes our proof.

Simple Generalization: Linear Quiver Gauge Theories

Here we provide a simple generalization to A_p-linear quiver gauge theories and their associated spin chains.

- **Theory I:** Four dim $\mathcal{N} = 2$ with $G = SU(L)^p$, plus bi-fundamental hypermultiplets between adjacent nodes of mass μ_I $I = 1, \ldots, p-1$, the last (first) node has L (anti)-fundamental hypermultiplets of masses $-m_k + \epsilon$ $(-\tilde{m}_l)$. Each $SU(L_I)$ has $\tau_I = \frac{4\pi i}{g_I^2} + \frac{\Theta_I^{4D}}{2\pi}$.
- **Theory II:** Two dim $\mathcal{N} = (2,2)$ SYM with $G = \prod_{I=1}^{p} U(N_I)$, with matter content of one adjoint of twisted mass ϵ for each $U(N_I)$, bi-fundamentals of twisted mass $\epsilon/2$ under $U(N_I) \times U(N_{I+1})$. The $U(N_1)$ node also has L fundamentals of $\vec{M}_{\rm F} = (M_1, \ldots, M_L)$ and L anti-fundamentals of $\vec{M}_{\rm AF} = (\tilde{M}_1, \ldots, \tilde{M}_L)$. The complex gauge couplings are $\hat{\tau}_I = ir_I + \frac{\Theta_I^{2D}}{2\pi}$ $I = 1, \ldots, p$.

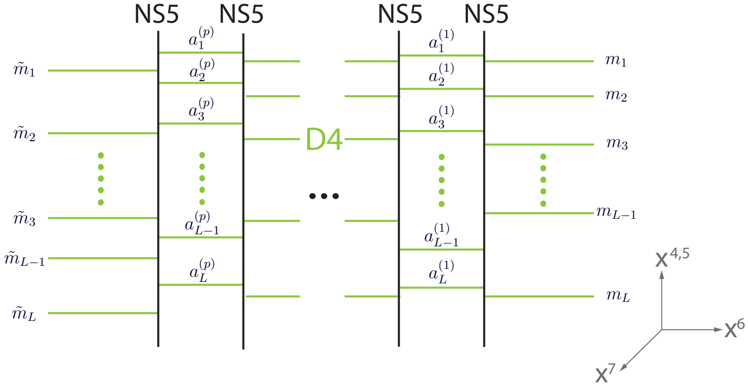

Figure: The IIA-brane construction for Theory I in the linear quiver case.

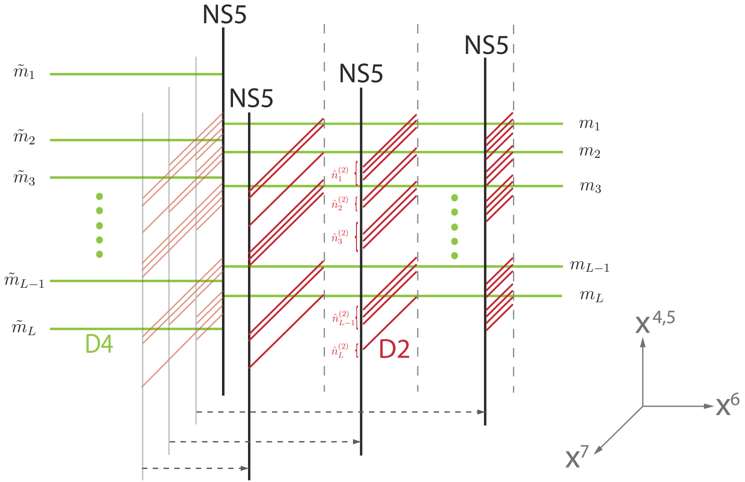

Figure: The IIA-brane brane construction for Theory II in the linear quiver case.

- From the explicit brane set-up, we see that $N_I = \sum_{J=I}^{p} \sum_{l=1}^{L} \hat{n}_l^{(J)}$, where $\hat{n}_l^{(J)}$ is the number of D2s between l-th D4 and J-th NS5.
- The F-term equation of **Theory II** is identified with the BAE of $SL(p+1,\mathbb{R})$ spin chain (C_{IJ} = Cartan matrix of $SL(p+1,\mathbb{R})$):

$$-q_I \prod_{J=1}^{p} \frac{Q_J(\lambda_j^{(I)} - \frac{1}{2}\epsilon C_{IJ})}{Q_J(\lambda_j^{(I)} + \frac{1}{2}\epsilon C_{IJ})} = \begin{cases} \frac{d(\lambda_j^{(1)})}{a(\lambda_j^{(1)})} & I = 1 \\ 1 & I > 1 \ , \end{cases} \tag{20}$$

- The duality in this generalization states that:

$$\mathcal{W}^{(\mathrm{I})}\Big(m_I - n_I^{(I)}\epsilon - \sum_{J=I}^{p-1}\mu_J\Big) - \mathcal{W}^{(\mathrm{I})}\Big(m_I - \epsilon - \sum_{J=I}^{p-1}\mu_J\Big) = \mathcal{W}^{(\mathrm{II})}\big(\{n_I^{(I)}\}\big) \ , \tag{21}$$

with the following identification of parameters:

$$x^{(I)} = \lambda^{(I)} - \sum_{J=I}^{p-1}(\mu_J - \frac{1}{2}\epsilon) - \frac{1}{2}\epsilon \ , \qquad \hat{q}_I = (-1)^{N_I+1} q_I$$

$$M_I = m_I - \frac{p+2}{2}\epsilon \ , \qquad \tilde{M}_I = \tilde{m}_I + \sum_{J=1}^{p-1}(\mu_J - \frac{1}{2}\epsilon) + \frac{1}{2}\epsilon \ .$$

Future Directions

- Generalization of duality to other gauge groups $SO(N)$ etc., or to other dimensions, compactifications from higher dimensions.
- Quantizing other more interesting integrable systems, such elliptic Calogero-Moser, Toda, Hitchin, Ruijsenaar-Schneider systems etc.?
- How do electromagnetic duality and mirror symmetry affect our duality/correspondence?
- Connections with matrix models and topological strings from instanton partition functions.
- Connections with wall-crossing phenomena in both 2 dim and 4 dim supersymmetric gauge theories?