# A new 3-component Novikov hierarchy 

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#### Abstract

We study the bi-Hamiltonian structure of the hierarchy of a 3-component Novikov system. We show that Hamiltonian functionals of the 3-Novikov hierarchy in negative direction are local, and in both directions are homogenous. We construct a reciprocal transformation to connect the 3 -Novikov system to a reduction of the first negative flow in a modified Yajima-Oikawa hierarchy, which is shown to pass the standard Painlevé test. Besides we discuss bi-Hamiltonian structures of the 3-Novikov hierarchy under the reciprocal transformation. Moreover, we consider a limit for the 3 -Novikov system.


## 1. Introduction

The Camassa-Holm (CH) equation

$$
\begin{equation*}
m_{t}+u m_{x}+2 u_{x} m=0, \quad m=u-u_{x x}, \tag{1}
\end{equation*}
$$

has attracted much attention since it is derived as the governing equation for dispersive shallow-water motion in 1993 [1]. It is remarkable that the CH equation has peakon solutions which are interesting in general analysis of PDEs [2]. The CH equation is integrable from the point of view of Lax pair and bi-Hamiltonian structure [1, 3]. It is linked to the negative KdV equation by a reciprocal transformation [4, 5, 6]. In Ref. [7] and its references, many other algebraic and geometric properties of the CH equation are introduced.

By applying asymptotic integrability method to a family of third order dispersive PDE, Degasperis and Procesi [8] found another equation possessing peakon solutions

$$
\begin{equation*}
m_{t}+u m_{x}+3 u_{x} m=0, \quad m=u-u_{x x} . \tag{2}
\end{equation*}
$$

The DP equation has a Lax pair and a bi-Hamiltonian structure [9]. An infinite sequence of conservation laws for the equation are also obtained. Besides a reciprocal transformation is constructed to connect it with a negative flow in the KaupKupershmidt hierarchy. Hereafter, many other equations of CH type were proposed and studied. For example, the Novikov equation, the modified CH equation, a 2-component CH equation and the Geng-Xue equation (see e.g. [10, 11, 12, 13, 14, 15]).

Recently, Geng and Xue [16] presented a 3-component CH type hierarchy by consider the following $3 \times 3$ matrix spectral problem

$$
\varphi_{x}=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{3}\\
1+\lambda u & 0 & v \\
\lambda w & 0 & 0
\end{array}\right) \varphi .
$$

The spectral problem (3) may reduce to that of the CH equation, the DP equation, the Novikov equation and the Geng-Xue equation. The corresponding hierarchy was derived by choosing the trivial flow as $(u, v, w)_{t}^{T}=(u, v, w)_{x}^{T}$. The first negative flow in the hierarchy reads

$$
\begin{align*}
& u_{t}=-v p_{x}+u_{x} q+\frac{3}{2} u q_{x}-\frac{3}{2} u\left(p_{x} r_{x}-p r\right) \\
& v_{t}=2 v q_{x}+v_{x} q, \\
& w_{t}=v r_{x}+w_{x} q+\frac{3}{2} w q_{x}+\frac{3}{2} w\left(p_{x} r_{x}-p r\right)  \tag{4}\\
& u=p-p_{x x}, \\
& v=\frac{1}{2}\left(q_{x x}-4 q+p_{x x} r_{x}-r\right. \\
& \left.v r_{x x} p_{x}+3 p_{x} r-3 p r_{x}\right)
\end{align*}
$$

This system can be reduced to the CH equation as $p=r=0$. It admits a bi-Hamiltonian structure and an infinite sequence of conserved quantities [16, 17]. However, it is hard to construct some exact solutions for this system.

Subsequently, by considering reductions of a 4-component CH type system, we proposed another 3-component CH type system

$$
\begin{align*}
& m_{1 t}+u_{2} g m_{1 x}-m_{3}\left(u_{2 x} f-u_{2} g\right)-m_{1}\left(3 u_{2} f-m_{3} u_{2}\right)=0, \\
& m_{2 t}+u_{2} g m_{2 x}+m_{2}\left(3 u_{2 x} g+m_{3} u_{2}\right)=0, \\
& m_{3 t}+u_{2} g m_{3 x}-m_{3}\left(2 u_{2} f+u_{2 x} g-m_{3} u_{2}\right)=0,  \tag{5}\\
& m_{i}=u_{i}-u_{i x x}, \quad i=1 . .3, \quad f=u_{3}-u_{1 x}, \quad g=u_{1}-u_{3 x}
\end{align*}
$$

associated with the spectral problem [18]

$$
\phi_{x}=\left(\begin{array}{ccc}
0 & 0 & 1  \tag{6}\\
\lambda m_{1} & 0 & \lambda m_{3} \\
1 & \lambda m_{2} & 0
\end{array}\right) \phi .
$$

It is shown to possess a bi-Hamiltonian structure and infinitely many conserved quantities. The system (5) is found to connect with a negative generalized MKdV system (a modified Yajima-Oikawa (mYJ) system[19, 20]) via a reciprocal transformation, and the associated system is shown to pass the standard Painlevé test of WTC [21.

In this paper, we will study a new 3-Novikov hierarchy associated with the following spectral problem

$$
\varphi_{x}=U \varphi, \quad U=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{7}\\
1+\lambda u^{2} & 0 & v \\
\lambda w & 0 & 0
\end{array}\right), \quad \varphi=\left(\begin{array}{c}
\varphi_{1} \\
\varphi_{2} \\
\varphi_{3}
\end{array}\right)
$$

which is obtained by replacing $u$ in (3) with $u^{2}$ for convenience. The new hierarchy is different from the hierarchy found by Geng and Xue, because we take the trivial flow
as $(u, v, w)_{t}^{T}=(0, v,-w)^{T}$. The first typical member in the 3-Novikov hierarchy is the 3-Novikov system

$$
\begin{align*}
& u_{t}+(u p r)_{x}=0, \\
& v_{t}+3 v p_{x} r+v_{x} p r+u^{2} p=0, \\
& w_{t}+3 w p r_{x}+w_{x} p r-u^{2} r=0,  \tag{8}\\
& v=p-p_{x x}, \quad w=r-r_{x x} .
\end{align*}
$$

It can be reduced to the DP equation, the Novikov equation and the Geng-Xue equation as $u=0, r=1$, as $u=0, p=r$ and as $u=0$ respectively. We will construct infinitely many conserved quantities and study the bi-Hamiltonian structure of the 3-Novikov hierarchy, and construct a reciprocal transformation for the system (8).

The outline of this paper is as follows. In Section 2, we construct infinitely many conserved quantities for the 3-Novikov equation with the aid of the spectral problem (77). We also analyze the homogeneous and local properties of the Hamiltonian functionals in the 3-Novikov hierarchy. In Section 3, we find the relationship between the two systems (5) and (8). In Section 4, we construct a reciprocal transformation to connect the 3 -Novikov system with the first negative flow in a mYJ hierarchy, and analyse the bi-Hamiltonian structure under this transformation. In Section 5, we present a limit of the 3-Novikov system.

## 2. Conserved quantities and bi-Hamiltonian structure of the 3 -Novikov hierarchy

### 2.1. Conserved quantities

The 3-Novikov system (8) arises as the compatibility condition for the linear system

$$
\begin{equation*}
\varphi_{x}=U \varphi, \quad \varphi_{t}=V \varphi \tag{9}
\end{equation*}
$$

where

$$
V=\left(\begin{array}{ccc}
\frac{1}{3 \lambda}+p r_{x} & -p r & \frac{p}{\lambda} \\
p_{x} r_{x}-\lambda u^{2} p r & \frac{1}{3 \lambda}-p_{x} r & \frac{p_{x}}{\lambda}-v p r \\
-\lambda w p r-r_{x} & r & p_{x} r-p r_{x}-\frac{2}{3 \lambda}
\end{array}\right) .
$$

With the aid of the Lax pair (9), infinitely many conserved quantities or conservation laws for the 3 -Novikov system can be constructed. For example, setting $\rho=\left(\ln \varphi_{3}\right)_{x}$ and expanding it in powers of $\lambda$, as pointed out in [16], one may able to obtain an infinite sequence of conserved densities for (8) from coefficients of $\rho$ by solving

$$
\begin{equation*}
(\partial+\rho)\left[\left(\frac{\rho}{w}\right)_{x}+\frac{\rho^{2}}{w}\right]-\left(1+\lambda u^{2}\right) \frac{\rho}{w}-\lambda v=0 . \tag{10}
\end{equation*}
$$

However, it is not easy to solve (10) and the expansion of $\rho$ in [16] can be generalized. Therefore we will consider a better formulation for computations and get more exact conserved quantities, which may be useful to generalize flows of the 3-Novikov hierarchy and to construct reciprocal transformations.

Let $a=\frac{\varphi_{1}}{\varphi_{3}}, b=\frac{\varphi_{2}}{\varphi_{3}}$. It follows that $\rho=\lambda w a$ with $a$ and $b$ satisfying

$$
\begin{align*}
& a_{x}=b-\lambda w a^{2},  \tag{11}\\
& b_{x}=\left(1+\lambda u^{2}\right) a+v-\lambda w a b . \tag{12}
\end{align*}
$$

Solving the above system by expanding $a, b$ as $a=\sum_{j \geq 0} a_{j} \lambda^{j}, b=\sum_{j \geq 0} b_{j} \lambda^{j}$ yields

$$
\begin{array}{ll}
a_{0 x}=b_{0}, & b_{0 x}=a_{0}+v \\
a_{1 x}=b_{1}-w a_{0}^{2}, & b_{1 x}=a_{1}+u^{2} a_{0}-w a_{0} b_{0} \\
a_{i x}=b_{i}-w \sum_{k=0}^{i-1} a_{k} a_{i-k-1}, & b_{i x}=a_{i}+u^{2} a_{i-1}-w \sum_{k=0}^{i-1} a_{k} b_{i-k-1},(i \geq 2) .
\end{array}
$$

We obtain, after some calculations, that

$$
\begin{array}{ll}
a_{0}=-p, & b_{0}=-p_{x} \\
a_{1}=\left(1-\partial^{2}\right)^{-1}\left(u^{2} p+3 w p p_{x}+w_{x} p^{2}\right), & b_{1}=w p^{2}+a_{1 x} \\
a_{i}=\left(1-\partial^{2}\right)^{-1}\left[w \sum_{k=0}^{i-1} a_{k} b_{i-k-1}+\left(w \sum_{k=0}^{i-1} a_{k} a_{i-k-1}\right)_{x}-u^{2} a_{i-1}\right], \\
b_{i}=a_{i x}+w \sum_{k=0}^{i-1} a_{k} a_{i-k-1}, \quad(i \geq 2) .
\end{array}
$$

Then an infinite sequences of conserved quantities are gotten. The first three are

$$
\begin{aligned}
& \Gamma_{1}=-\int p w d x \\
& \Gamma_{2}=\int\left[u^{2} p r+w p p_{x} r-w p^{2} r_{x}\right] d x \\
& \Gamma_{3}=\int\left[a_{1}\left(3 w p r_{x}+w_{x} p r-u^{2} r\right)-w^{2} p^{3} r\right] d x
\end{aligned}
$$

Furthermore, we can also expanding $a, b$ as

$$
a=\sum_{j \geq 1} a_{j} \lambda^{-\frac{1}{2} j}, \quad b=\lambda^{\frac{1}{2}} \sum_{j \geq 1} b_{j} \lambda^{-\frac{1}{2} j},
$$

which are different from the expansions in [16]. Taking the similar procedure as the previous, we have

$$
\begin{array}{ll}
a_{1}=u w^{-1}, & b_{1}=u^{2} w^{-1}, \\
a_{2}=\frac{1}{2} u^{-2} v-\frac{3}{2}(u w)^{-1} u_{x}+w^{-2} w_{x}, \quad b_{2}=u^{-1} v-u^{-1}\left(u^{2} w^{-1}\right)_{x}, \\
b_{i+1}=-u^{-1}\left(b_{i x}-a_{i-1}+w \sum_{k=2}^{i} a_{k} b_{i+2-k}\right), \\
a_{i+1}=\frac{1}{2} u^{-1}\left(b_{i+1}-a_{i x}-w \sum_{k=2}^{i} a_{k} a_{i+2-k}\right) .
\end{array}
$$

Then the first four conserved quantities may be obtained, which are

$$
\begin{aligned}
& \Upsilon_{1}=\int u d x \\
& \Upsilon_{2}=\frac{1}{2} \int u^{-2} v w d x
\end{aligned}
$$

$$
\begin{aligned}
& \Upsilon_{3}=\frac{1}{4} \int u^{-5}\left(\frac{1}{2} u^{2} u_{x}^{2}+u^{2} w v_{x}+2 u^{4}-\frac{3}{2} v^{2} w^{2}-u^{2} v w_{x}\right) d x \\
& \Upsilon_{4}=\frac{1}{2} \int\left[-u^{-4}\left(v w+v_{x} w_{x}\right)+2 u^{-6}\left(u u_{x}(v w)_{x}-w^{2} v v_{x}-2 v w u_{x}^{2}\right)\right. \\
& \\
& \left.\quad+u^{-8}\left(3 w^{2} v^{2} u u_{x}+w^{3} v^{3}\right)\right] d x
\end{aligned}
$$

### 2.2. Hamiltonian structure

In this part, we will study the 3-Novikov hierarchy in the view of bi-Hamiltonian structure. Notice that the 3-Novikov system (8) is generated by the two conserved quantities $\Gamma_{1}, \Gamma_{2}$, we have the following result.

Theorem 1 The 3-Novikov equation (8) is a bi-Hamiltonian system, namely, it may be written as

$$
\left(\begin{array}{c}
u  \tag{13}\\
v \\
w
\end{array}\right)_{t}=\mathcal{J}\left(\begin{array}{c}
\frac{\delta H_{2}}{\delta u} \\
\frac{\delta H_{2}}{\delta v} \\
\frac{\delta H_{2}}{\delta w}
\end{array}\right)=\mathcal{K}\left(\begin{array}{c}
\frac{\delta H_{1}}{\delta u} \\
\frac{\delta H_{1}}{\delta v} \\
\frac{\delta H_{1}}{\delta w}
\end{array}\right),
$$

where

$$
\begin{aligned}
\mathcal{J} & =\left(\begin{array}{ccc}
\frac{1}{2} \partial & 0 & 0 \\
0 & 0 & 1-\partial^{2} \\
0 & \partial^{2}-1 & 0
\end{array}\right) \\
\mathcal{K} & =\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \frac{3}{2} v \partial^{-1} v & -u^{2}-\frac{3}{2} v \partial^{-1} w \\
0 & u^{2}-\frac{3}{2} w \partial^{-1} v & \frac{3}{2} w \partial^{-1} w
\end{array}\right)-2 \Omega\left(\partial^{3}-4 \partial\right)^{-1} \Omega^{*},
\end{aligned}
$$

herein

$$
\begin{aligned}
& \Omega=\left(\partial u, \frac{1}{2} v \partial+\partial v, \frac{1}{2} w \partial+\partial w\right)^{T} \\
& H_{1}=-\Gamma_{1}, \\
& H_{2}=-\Gamma_{2}
\end{aligned}
$$

Since $\mathcal{J}, \mathcal{K}$ forms a Hamiltonian pair [16], one can prove the theorem easily. Hence a recursion operator for 3 -Novikov hierarchy is gotten as $\mathcal{R}=\mathcal{K} \mathcal{J}^{-1}$, and we can derive a new 3-Novikov hierarchy by taking the trivial flow as $(u, v, w)_{t}^{T}=\mathcal{R}(0, v,-w)^{T}$. Then the positive flows in the hierarchy may be obtained as

$$
\left(\begin{array}{c}
u  \tag{14}\\
v \\
w
\end{array}\right)_{t_{n}}=\mathcal{J}\left(\begin{array}{c}
\frac{\delta H_{n+1}}{\delta u} \\
\frac{\delta H_{n+1}}{\delta v} \\
\frac{\delta H_{n+1}}{\delta w}
\end{array}\right)=\mathcal{K}\left(\begin{array}{c}
\frac{\delta H_{n}}{\delta u} \\
\frac{\delta H_{n}}{\delta v} \\
\frac{\delta H_{n}}{\delta w}
\end{array}\right), \quad n=1,2, \ldots,
$$

and infinitely many negative flows read as

$$
\left(\begin{array}{c}
u  \tag{15}\\
v \\
w
\end{array}\right)_{t_{-n}}=\mathcal{K}\left(\begin{array}{l}
\frac{\delta H_{-(n+1)}}{\delta u} \\
\frac{\delta H_{-(n+1)}}{\delta v} \\
\frac{\delta H_{-(n+1)}}{\delta w}
\end{array}\right)=\mathcal{J}\left(\begin{array}{c}
\frac{\delta H_{-n}}{\delta u} \\
\frac{\delta H_{-n}}{\delta v} \\
\frac{\delta H_{-n}}{\delta w}
\end{array}\right), \quad n=1,2, \ldots
$$

with the first two Hamiltonian functionals giving by $H_{-1}=-2 \Upsilon_{2}, H_{-2}=-2 \Upsilon_{4}$. In particular, the first negative flow in the hierarchy is obtained by using the Hamiltonian functionals $H_{-1}, H_{-2}$, that is

$$
\begin{align*}
& u_{t}-\left(\frac{v w}{u^{3}}\right)_{x}=0 \\
& v_{t}-\left(\frac{v}{u^{2}}\right)_{x x}+\frac{v}{u^{2}}=0  \tag{16}\\
& w_{t}-\frac{w}{u^{2}}+\left(\frac{w}{u^{2}}\right)_{x x}=0 .
\end{align*}
$$

It is worth to note that $\Upsilon_{1}$ and $\Upsilon_{3}$ are the Casimir functionals of the Hamiltonian operators $\mathcal{J}$ and $\mathcal{K}$ respectively.

Since the structure of Hamiltonian functionals $H_{n} s$ in the 3-Novikov hierarchy is largely unknown, like the cases in [22, 23], we will consider the homogeneous and local properties of them. Introducing $\theta=(u, v, w)^{T}$ and $X_{n}[\theta]=\frac{\delta H_{n}}{\delta \theta}$, then recursive relation in the positive direction

$$
\mathcal{J} \frac{\delta H_{n+1}}{\delta \theta}=\mathcal{K} \frac{\delta H_{n}}{\delta \theta}, \quad n=1,2, \ldots
$$

yields an infinite sequence of variational derivatives for the Hamiltonian functionals $H_{n}$ s

$$
\begin{equation*}
X_{n+1}[\theta]=\mathcal{J}^{-1} \mathcal{K} X_{n}[\theta], \quad n=1,2, \ldots \tag{17}
\end{equation*}
$$

Similarly, the variational derivatives for the Hamiltonian functionals $H_{-n} \mathrm{~s}$ in the negative direction are given by

$$
X_{-(n+1)}[\theta]=\mathcal{K}^{-1} \mathcal{J} X_{-n}[\theta], \quad n=1,2, \ldots
$$

Proposition 1 The variational derivatives $X_{n}[\theta]$ are homogeneous in the sense that

$$
\begin{equation*}
X_{n}[\epsilon \theta]=\epsilon^{2 n-1} X_{n}[\theta], \quad n \geq 1 \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{n}[\varepsilon \theta]=\frac{1}{2 n} \int X_{n}[\theta] \cdot \theta d x, \quad n \geq 1 \tag{19}
\end{equation*}
$$

Proof: When $n=1$, the formulate (18) holds clearly. Now suppose (18) also holds for $n=k$, that is

$$
X_{k}[\epsilon \theta]=\epsilon^{2 k-1} X_{k}[\theta] .
$$

Then for $n=k+1$, we have

$$
X_{k+1}[\epsilon \theta]=\mathcal{J}^{-1}[\epsilon \theta] \mathcal{K}[\epsilon \theta] X_{k}[\epsilon \theta]=\epsilon^{2} \mathcal{J}^{-1}[\theta] \mathcal{K}[\theta] X_{k}[\epsilon \theta],
$$

which implies that

$$
X_{k+1}[\epsilon \theta]=\epsilon^{2 k+1}[\theta] X_{k+1}[\theta] .
$$

In addition, for any $n \geq 1$, we have

$$
H_{n}[\theta]=\int_{0}^{1} \int X_{n}[\varepsilon \theta] \cdot \theta d x d \varepsilon=\frac{1}{2 n} \int X_{n}[\theta] \cdot \theta d x
$$

then the Hamiltonian functionals $H_{n}$ s are also homogeneous with

$$
H_{n}[\varepsilon \theta]=\varepsilon^{2 n} H_{n}[\theta], \quad n=1,2, \ldots
$$

The recursive formula for $H_{n}$ s yields infinitely many Hamiltonian functionals in the positive direction, and $H_{1}$ and $H_{2}$ are local. However, $H_{n}, n \geq 3$ becomes nonlocal. For example $H_{3}=-\Gamma_{3}$, which is shown to be nonlocal.

Proposition 2 The variational derivatives $X_{-n}[\theta]$ s satisfy

$$
\begin{equation*}
X_{-n}[\epsilon \theta]=\epsilon^{1-2 n} X_{-n}[\theta], \quad n=1,2, \ldots \tag{20}
\end{equation*}
$$

while

$$
\begin{equation*}
H_{-n}[\theta]=\frac{1}{2-2 n} \int X_{-n}[\theta] \cdot \theta d x \tag{21}
\end{equation*}
$$

and $H_{-n} \mathrm{~s}$ are all local.
The formulae (20) and (21) may be proven by taking the process before, and we will prove the local property of $H_{-n}$ s below.

Lemma 1 ([22, 233, 24]) If a differential function $M[\theta]$ satisfies

$$
\int M[\theta] d x=0
$$

for all $\theta$, then there exists a unique differential function $N[\theta]$ up to addition of a constant such that $M[\theta]$ is the total $x$-derivative $M[\theta]=(N[\theta])_{x}$.

Introducing

$$
\begin{aligned}
& X_{-k}[\theta]=\left(A_{k}, B_{k}, C_{k}\right)^{T} \\
& E_{k}=\left(\partial^{3}-4 \partial\right)^{-1}\left(u \partial, \frac{3}{2} v \partial+\frac{1}{2} v_{x}, \frac{3}{2} w \partial+\frac{1}{2} w_{x}\right) X_{-k}[\theta], \quad k \geq 1 .
\end{aligned}
$$

When $n=1, X_{-1}[\theta]$ is local since

$$
X_{-1}[\theta]=\left(2 \frac{v w}{u^{3}},-\frac{w}{u^{2}},-\frac{v}{u^{2}}\right)^{T}
$$

Now suppose $X_{-k}[\theta]$ is local for $n=k$. Then for $n=k+1$, we have

$$
X_{-(k+1)}[\theta]=\mathcal{K}^{-1} \mathcal{J} X_{-k}[\theta]=\left(\mathcal{K}^{-1} \mathcal{J}\right)^{k} X_{-1}[\theta]
$$

which is equal to

$$
\begin{equation*}
\mathcal{K} X_{-(k+1)}[\theta]=\mathcal{J} X_{-k}[\theta] . \tag{22}
\end{equation*}
$$

This shows that

$$
\begin{align*}
& E_{k+1}=\frac{1}{4 u} A_{k}  \tag{23}\\
& \frac{3}{2} v \partial^{-1}\left(v B_{k+1}-w C_{k+1}\right)-u^{2} C_{k+1}+\left(3 v \partial+2 v_{x}\right) E_{k+1}=\left(1-\partial^{2}\right) C_{k}  \tag{24}\\
& u^{2} B_{k+1}-\frac{3}{2} w \partial^{-1}\left(v B_{k+1}-w C_{k+1}\right)+\left(3 w \partial+2 w_{x}\right) E_{k+1}=\left(\partial^{2}-1\right) B_{k} . \tag{25}
\end{align*}
$$

Then we will prove the local property of $X_{-(k+1)}$ in two steps. The first step is to prove that $B_{k+1}$ and $C_{k+1}$ are local. Since $A_{k}, B_{k}, C_{k}$ are all local, we can obtain
immediately from (24) and (25) that $B_{k+1}$ and $C_{k+1}$ are local, if there exist a differential function $M_{k}$ such that

$$
\begin{aligned}
v B_{k+1}-w C_{k+1} & =\frac{w}{u^{2}}\left(1-\partial^{2}\right) C_{k}+\frac{v}{u^{2}}\left(\partial^{2}-1\right) B_{k}-\frac{3 v w}{2 u^{2}} \partial \frac{A_{k}}{u}-\frac{(v w)_{x}}{2 u^{3}} A_{k} \\
& =M_{k x} .
\end{aligned}
$$

Then according to the Lemma 1, we only need to prove

$$
Y_{1}=\int\left[\frac{w}{u^{2}}\left(1-\partial^{2}\right) C_{k}+\frac{v}{u^{2}}\left(\partial^{2}-1\right) B_{k}-\frac{3 v w}{2 u^{2}} \partial \frac{A_{k}}{u}-\frac{(v w)_{x}}{2 u^{3}} A_{k}\right] d x=0
$$

In fact

$$
\begin{aligned}
Y_{1} & =\int\left[\frac{w}{u^{2}}\left(1-\partial^{2}\right) C_{k}+\frac{v}{u^{2}}\left(\partial^{2}-1\right) B_{k}-\frac{3 v w}{2 u^{2}} \partial \frac{A_{k}}{u}-\frac{(v w)_{x}}{2 u^{3}} A_{k}\right] d x \\
& =\int\left[C_{k}\left(1-\partial^{2}\right) \frac{w}{u^{2}}+B_{k}\left(\partial^{2}-1\right) \frac{v}{u^{2}}+A_{k}\left(\frac{v w}{u^{3}}\right)_{x}\right] d x \\
& =\int\left(\begin{array}{c}
A_{k} \\
B_{k} \\
C_{k}
\end{array}\right) \cdot \mathcal{J}\left(\begin{array}{c}
2 \frac{v w}{u^{2}} \\
-\frac{W}{u^{2}} \\
-\frac{v}{u^{2}}
\end{array}\right) d x \\
& =\int X_{-k}[\theta] \cdot \mathcal{J} X_{-1}[\theta] d x .
\end{aligned}
$$

On the other hand, using the recursion relation, we have

$$
\begin{aligned}
Y_{1} & =\int\left(\mathcal{K}^{-1} \mathcal{J}\right)^{k-1} X_{-1}[\theta] \cdot \mathcal{J} X_{-1}[\theta] d x \\
& =-\int X_{-1}[\theta] \cdot \mathcal{J}\left(\mathcal{K}^{-1} \mathcal{J}\right)^{k-1} X_{-1}[\theta] d x \\
& =-\int X_{-1}[\theta] \cdot\left(\mathcal{J} \mathcal{K}^{-1}\right)^{k-1} \mathcal{J} X_{-1}[\theta] d x \\
& =-\int\left(\mathcal{K}^{-1} \mathcal{J}\right)^{k-1} X_{-1}[\theta] \cdot \mathcal{J} X_{-1}[\theta] d x \\
& =-\int X_{-k}[\theta] \cdot \mathcal{J} X_{-1}[\theta] d x .
\end{aligned}
$$

Therefore $Y_{1}=0$, and hence $B_{k+1}$ and $C_{k+1}$ are local.
The next step is to prove that $A_{k+1}$ is local. From (23), we infer that

$$
A_{k+1 x}=\frac{1}{u}\left[\left(\partial^{3}-4 \partial\right) \frac{A_{k}}{4 u}-\left(3 v \partial+2 v_{x}\right) B_{k+1}-\left(3 w \partial+2 w_{x}\right) C_{k+1}\right] .
$$

Notice that $B_{k+1}$ and $C_{k+1}$ are all local, so $A_{k+1}$ is local if the right part of the above equality is a total $x$-derivative $N_{k x}$ for a differential function $N_{k}$. That is to say, $A_{k+1}$ is local if

$$
Y_{2}=\int\left(\frac{1}{u}\left[\left(\partial^{3}-4 \partial\right) \frac{A_{k}}{4 u}-\left(3 v \partial+2 v_{x}\right) B_{k+1}-\left(3 w \partial+2 w_{x}\right) C_{k+1}\right]\right) d x=0 .
$$

Lemma 2 Define

$$
\mathcal{D}=\left(\begin{array}{cc}
\frac{3}{2} v \partial^{-1} v & -u^{2}-\frac{3}{2} v \partial^{-1} w \\
u^{2}-\frac{3}{2} w \partial^{-1} v & \frac{3}{2} w \partial^{-1} w
\end{array}\right)
$$

we have

$$
\mathcal{D}^{-1}=\frac{1}{u^{2}}\left(\begin{array}{cc}
\frac{3}{2} w \partial^{-1} w & u^{2}+\frac{3}{2} w \partial^{-1} v \\
\frac{3}{2} v \partial^{-1} w-u^{2} & \frac{3}{2} v \partial^{-1} v
\end{array}\right) \frac{1}{u^{2}}
$$

To make the expressions compact, we introduce some new notations as:

$$
\begin{aligned}
Z_{1} & =\left(1-\partial^{2}\right) C_{k}-\left(3 v \partial+2 v_{x}\right) \frac{A_{k}}{4 u}, Z_{2}
\end{aligned}=\left(\partial^{2}-1\right) B_{k}-\left(3 v w \partial+2 w_{x}\right) \frac{A_{k}}{4 u}, ~=Z_{4}=-\frac{w_{x}}{u^{3}}+\frac{3}{2} \frac{u_{x} w}{u^{4}}-\frac{3}{2} \frac{v w^{2}}{u^{5}} .
$$

Using the Lemma 2 to solve $B_{k+1}$ and $C_{k+1}$ from (24) and (25), we arrive at

$$
\begin{aligned}
Y_{2}= & \int\left(\frac{1}{u}\left[\left(\partial^{3}-4 \partial\right) \frac{A_{k}}{4 u}-\left(3 v \partial+2 v_{x}\right) B_{k+1}-\left(3 w \partial+2 w_{x}\right) C_{k+1}\right]\right) d x \\
= & \int\left[-\frac{A_{k}}{4 u}\left(\partial^{3}-4 \partial\right) \frac{1}{u}+B_{k+1}\left(\frac{v_{x}}{u}-\frac{3 u_{x} v}{u^{2}}\right)+C_{k+1}\left(\frac{w_{x}}{u}-\frac{3 u_{x} w}{u^{2}}\right)\right] d x \\
= & \int\left[-\frac{A_{k}}{4 u}\left(\partial^{3}-4 \partial\right) \frac{1}{u}+\left(\frac{v_{x}}{u^{3}}-\frac{3 u_{x} v}{2 u^{4}}\right)\left[\frac{3}{2} w \partial^{-1}\left(\frac{w Z_{1}+v Z_{2}}{u^{2}}\right)+Z_{2}\right]\right. \\
& \left.+\left(\frac{w_{x}}{u^{3}}-\frac{3 u_{x} w}{2 u^{4}}\right)\left[\frac{3}{2} v \partial^{-1}\left(\frac{w Z_{1}+v Z_{2}}{u^{2}}\right)-Z_{1}\right]\right] d x \\
= & \int\left[-\frac{A_{k}}{4 u}\left(\partial^{3}-4 \partial\right) \frac{1}{u}+Z_{2} Z_{3}+Z_{1} Z_{4}\right] d x \\
= & \int\left(\frac{A_{k}}{4 u}\left[-\left(\partial^{3}-4 \partial\right) \frac{1}{u}+\left(3 w \partial+2 w_{x}\right) Z_{3}+\left(3 v \partial+2 v_{x}\right) Z_{4}\right]\right. \\
& \left.+B_{k}\left(\partial^{2}-1\right) Z_{1}+C_{k}\left(1-\partial^{2}\right) Z_{2}\right) d x \\
= & \int\left(\begin{array}{l}
A_{k} \\
B_{k} \\
=
\end{array}\right) \cdot \mathcal{J}\binom{\frac{3 w v_{x}-3 v w_{x}}{2 u^{4}}-\frac{15 v^{2} w^{2}}{4 u^{6}}+\frac{u_{x x}}{2 u^{3}}-\frac{3 u_{x}^{2}}{4 u^{4}}+\frac{1}{u^{2}}}{2 u^{3}} d x \\
= & -2 X_{-k}[\theta] \cdot \mathcal{J} \frac{v_{x}}{u^{3}}+\frac{3 v u_{x}}{2 u^{4}}+\frac{3 v^{5} w}{2 u^{5}} \\
= & \int \mathcal{J}\left(\mathcal{K}^{-1} \mathcal{J}\right)^{k-1} X_{-1}[\theta] \cdot \frac{\delta \Upsilon_{3}}{\delta \theta} d x \\
= & \int-2\left(\mathcal{K}^{-1} \mathcal{J}\right)^{k} X_{-1}[\theta] \cdot \mathcal{K} \frac{\delta \Upsilon_{3}}{\delta \theta} d x \\
= & \int-2\left(\mathcal{K}^{-1} \mathcal{J}\right)^{k} X_{-1}[\theta] \cdot(0,0,0)^{T} d x \\
= & 0 .
\end{aligned}
$$

Therefore $A_{k+1}$ is local. Consequently, we prove $X_{-n} \mathrm{~s}$ are all local. Then using the Lemma 4.4 in [23] (see also [22, 24]), $H_{-n}$ s are found to be local.

## 3. Relationship with a 3-component CH type system

As pointed out in [18], the 3-CH type system (5) is reciprocal linked to the first negative flow in a mYJ hierarchy. Since the spectral problem (6) is gauge linked to the spectral problem (77), it would seem to be a reasonable guess that the 3-CH type system (5) is equal to the 3 -Novikov system (8).

In fact the 3 - CH type system (5) may be rewritten as

$$
\begin{align*}
& m_{1 t}+u_{2} g m_{1 x}-m_{3}\left(u_{2 x} f-u_{2} g\right)-m_{1}\left(3 u_{2} f-m_{3} u_{2}\right)=0, \\
& m_{2 t}+u_{2} g m_{2 x}+m_{2}\left(3 u_{2 x} g+m_{3} u_{2}\right)=0 \\
& m_{3 t}+u_{2} g m_{3 x}-m_{3}\left(2 u_{2} f+u_{2 x} g-m_{3} u_{2}\right)=0,  \tag{26}\\
& m_{2}=u_{2}-u_{2 x x}, \quad m_{1}-m_{3 x}=g-g_{x x}, \quad f=m_{3}-g_{x}
\end{align*}
$$

then a directly calculation shows that (26) is connected to (8) via

$$
\begin{equation*}
u=\left(m_{2} m_{3}\right)^{\frac{1}{2}}, \quad v=m_{2}, \quad w=m_{1}-m_{3 x}, \quad p=u_{2}, \quad r=g . \tag{27}
\end{equation*}
$$

## 4. A reciprocal transformation for the 3-Novikov system

### 4.1. A reciprocal transformation

Although many CH type systems are completely integrable, they have some nonstandard features such as the DT, the Bäcklund transformation and the weak Painlevé property [9, 25]. To study the Painlevé behaviour of the 3 -Novikov system (8), we can relate it with a equation displaying the standard (strong) Painlevé test of WTC [26], which is easy to construct the DT and the Bäcklund transformation. Our strategy is to use the steps in [21], and we will connect the 3-Novikov system (8) with a negative flow in a mYJ hierarchy.

The 3-Novikov system (8) has a conserved density $u$, and the correspondence conservation law reads

$$
u_{t}=(-u p r)_{x},
$$

which allows a reciprocal transformation

$$
\begin{equation*}
d y=u d x-u p r d t, \quad \quad d \tau=d t \tag{28}
\end{equation*}
$$

Set $\mu=\lambda^{\frac{1}{2}}$ and define $\psi_{2}=\mu \frac{u^{2}}{v} \varphi_{1}+\frac{1}{\mu} \varphi_{3}$. Then, under change of variables, we may rewrite the spectral problem (7) as

$$
\begin{align*}
& \varphi_{1 y y}+\frac{u_{y}}{u} \varphi_{1 y}-\frac{1}{u^{2}} \varphi_{1}-\mu \frac{v}{u^{2}} \psi_{2}=0  \tag{29}\\
& \psi_{2 y}-\mu \frac{u^{2}}{v} \varphi_{1 y}-\mu\left[\left(\frac{u^{2}}{v}\right)_{y}+\frac{w}{u}\right] \varphi_{1}=0
\end{align*}
$$

Now, introducing the gauge transformation

$$
\varphi_{1}=\frac{v}{u^{2}} e^{-\partial_{y}^{-1}\left(\frac{v w}{u^{3}}\right)} \phi_{1}, \quad \psi_{2}=e^{-\partial_{y}^{-1}\left(\frac{v w}{u^{3}}\right)} \phi_{2}
$$

the spectral problem (29) may be converted to

$$
\begin{align*}
& \phi_{1 y y}-Q_{2} \phi_{1 y}-Q_{1} \phi_{1}=\mu \phi_{2},  \tag{30}\\
& \phi_{2 y}-Q_{3} \phi_{2}=\mu \phi_{1 y},
\end{align*}
$$

where

$$
\begin{aligned}
Q_{1} & =\left(3 \frac{v_{y}}{v}-6 \frac{u_{y}}{u}+\frac{w_{y}}{w}-\frac{v w}{u^{3}}\right) \frac{v w}{u^{3}}+3 \frac{u_{y} v_{y}}{u v}+\frac{1+2 u u_{y y}-4 u_{y}^{2}}{u^{2}}-\frac{v_{y y}}{v} \\
Q_{2} & =2 \frac{v w}{u^{3}}-2 \frac{v_{y}}{v}+3 \frac{u_{y}}{u} \\
Q_{3} & =\frac{v w}{u^{3}}
\end{aligned}
$$

N. $L i$

It is easy to check that the auxiliary problem in (9) is transformed to

$$
\begin{align*}
\phi_{1 \tau} & =\frac{1}{\mu} q_{1} \phi_{2}+\frac{1}{3 \mu^{2}} \phi_{1},  \tag{31}\\
\phi_{2 \tau} & =\frac{1}{\mu}\left(q_{2} \phi_{1 y}+\left[1-q_{2 y}+\left(Q_{3}-Q_{2}\right) q_{2}\right] \phi_{1}\right)+\left(q_{1}-\frac{2}{3 \mu^{2}}\right) \phi_{2},
\end{align*}
$$

where

$$
\begin{equation*}
q_{1}=p \frac{u^{2}}{v}, \quad \quad q_{2}=\frac{r v}{u} \tag{32}
\end{equation*}
$$

For the convenience of constructing exact solutions of the 3-Novikov equation, let us rewrite the above Lax pair in scalar form. Eliminating $\phi_{2}$ from the systems (30) and (31), we obtain

$$
\begin{align*}
& \phi_{1 y y y}+u_{1} \phi_{1 y y}+\left(v_{1}+u_{1 y}\right) \phi_{1}+\left(w_{1}+v_{1 y}\right) \phi_{1}=\lambda \phi_{1 y},  \tag{33}\\
& \phi_{1 \tau}-\frac{1}{\lambda}\left(q_{1} \phi_{1 y y}+\left(u_{1}+Q_{3}\right) q_{1} \phi_{1 y}-\chi \phi_{1}\right)=0, \tag{34}
\end{align*}
$$

where

$$
\left(\begin{array}{c}
u_{1}  \tag{35}\\
v_{1} \\
w_{1}
\end{array}\right)=\left(\begin{array}{c}
-Q_{2}-Q_{3} \\
Q_{2} Q_{3}-Q_{1}+Q_{3 y} \\
Q_{1} Q_{3}-\left(Q_{2} Q_{3}\right)_{y}-Q_{3 y y}
\end{array}\right)
$$

with

$$
\chi=q_{1 y y}+\left(u_{1}+3 Q_{3}\right) q_{1 y}+\left[\left(u_{1}+2 Q_{3}\right) Q_{3}+Q_{3 y}\right] q_{1}+\frac{2}{3} .
$$

Then the compatibility condition for the Lax representation (33+34) yields the associated 3-Novikov equation

$$
\left(\begin{array}{c}
u_{1}  \tag{36}\\
v_{1} \\
w_{1}
\end{array}\right)_{\tau}=\left(\begin{array}{c}
-2 q_{1 y} \\
-q_{1 y y}-u_{1} q_{1 y}-2\left(Q_{3} q_{1}\right)_{y}, \\
-\left[Q_{3} q_{1 y}+q_{1}\left(u_{1} Q_{3}+2 Q_{3}^{2}-Q_{3 y}\right)\right]_{y}
\end{array}\right),\binom{s_{1}}{s_{2}}=0
$$

where

$$
\begin{aligned}
& s_{1}=q_{1 y y}+q_{1}\left(2 u_{1} Q_{3}+3 Q_{3}^{2}+v_{1}\right)+q_{1 y}\left(3 Q_{3}+u_{1}\right)+1, \\
& s_{2}=w_{1}+Q_{3}\left(v_{1}+Q_{3}^{2}-3 Q_{3 y}-u_{1 y}\right)+Q_{3 y y}+\left(Q_{3}^{2}-Q_{3 y}\right) u_{1} .
\end{aligned}
$$

Furthermore, one can also gain the associated 3 -Novikov system (36) by applying the reciprocal transformation (28) to the 3-Novikov system (8) directly. Now, we claim that the 3 -Novikov equation (8) and the Lax pair (9) is reciprocal transformed to the associated equation (36) and the Lax pair (33F(34) respectively. More precisely, we have:

Proposition 3 The 3-Novikov equation (36) may be changed to the associated equation (36) by the Liouville transformation

$$
\left\{\begin{array}{c}
y=I\left(x, \theta^{(n)}\right)=\int_{-\infty}^{x} u(\nu) d \nu  \tag{37}\\
\left(\begin{array}{c}
u_{1}(y) \\
v_{1}(y) \\
w_{1}(y)
\end{array}\right)=\left(\begin{array}{c}
P_{1}\left(x, \theta^{(n)}\right) \\
P_{2}\left(x, \theta^{(n)}\right) \\
P_{3}\left(x, \theta^{(n)}\right),
\end{array}\right)
\end{array}\right.
$$

N. $L i$
where

$$
\begin{aligned}
P_{1}= & 2 \frac{v_{x}}{v u}-3 \frac{u_{x}}{u^{2}}-3 \frac{v w}{u^{3}} \\
P_{2}= & \frac{v_{x x}-v}{u^{2} v}-4 \frac{u_{x} v_{x}}{u^{3} v}-\frac{4 w v_{x}+2 u u_{x x}-6 u_{x}^{2}}{u^{4}}+6 \frac{v w u_{x}}{u^{5}}+3 \frac{v^{2} w^{2}}{u^{6}}, \\
P_{3}= & \frac{v\left(w-w_{x x}\right)}{u^{5}}+\frac{4 v w_{x} u_{x}+2 v w u_{x x}}{u^{6}}-\frac{6 v w u_{x}^{2}+3 v^{2} w w_{x}+v w^{2} v_{x}}{u^{7}} \\
& +\frac{6 v^{2} w^{2} u_{x}}{u^{8}}-\frac{v^{3} w^{3}}{u^{9}},
\end{aligned}
$$

with $v=p-p_{x x}, w=r-r_{x x}$.
It is worth to note that the associated 3-Novikov system passes the Painlevé test. Powers of the leader terms for $u_{1}, v_{1}, w_{1}, q_{1}, Q_{3}$ are $-1,-2,-2,-1,-1$ respectively, and the resonances are $j=-2,-1,1,2,3,4,5$.

The spectral problem (33) may be rewritten as the Lax operator for a mYJ hierarchy

$$
\begin{equation*}
L \phi_{1}=\lambda \phi_{1}, \quad L=\partial_{y}^{2}+u_{1} \partial_{y}+v_{1}+\partial_{y}^{-1} w_{1} \tag{38}
\end{equation*}
$$

which is just a member in the constrained modified KP hierarchy [27, 28]. It can reduce to that of the mKdV equation and the KdV hierarchy as $Q_{1}=Q_{3}=0$ and $Q_{2}=Q_{3}=0$ respectively. We claim that the associated 3 -Novikov equation is a reduction of the first negative flow in the mYJ hierarchy.

Notice that the mYJ hierarchy admits a Hamiltonian pair

$$
\begin{aligned}
\mathcal{J}_{1} & =\left(\begin{array}{ccc}
0 & 0 & 2 \partial_{y} \\
0 & 2 \partial_{y} & \partial_{y}^{2}+u_{1} \partial_{y} \\
2 \partial_{y} & -\partial_{y}^{2}+\partial_{y} u_{1} & 0
\end{array}\right), \\
\mathcal{K}_{1} & =\left(\begin{array}{ccc}
6 \partial_{y} & * & * \\
4 u_{1} \partial_{y} & 2 \partial_{y}^{3}+2 u_{1} \partial_{y} u_{1}+\partial_{y} v_{1}+v_{1} \partial_{y} & * \\
2 \partial_{y}^{3}-2 \partial_{y} u_{1} \partial_{y}+2 v_{1} \partial_{y} & \chi_{1} & \chi_{2}
\end{array}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& \chi_{1}=2 w_{1} \partial_{y}+\partial_{y} w_{1}-\left(\partial_{y}^{3}-\partial_{y} u_{1} \partial_{y}+v_{1} \partial_{y}\right)\left(\partial_{y}-u_{1}\right), \\
& \chi_{2}=\partial_{y} u_{1} w_{1}+u_{1} w_{1} \partial_{y}+w_{1} \partial_{y}^{2}-\partial_{y}^{2} w_{1}
\end{aligned}
$$

and the omitted terms are determined by skew-symmetry. Then a recursion operator for the mYJ hierarchy is obtained as $\mathcal{R}=\mathcal{K}_{1} \mathcal{J}_{1}^{-1}$, and the first negative flow in the correspondence hierarchy are obtained as

$$
\left(\begin{array}{c}
u_{1}  \tag{39}\\
v_{1} \\
w_{1}
\end{array}\right)_{\tau}=\mathcal{J}_{1}\left(\begin{array}{c}
A \\
B \\
C
\end{array}\right), \quad \mathcal{K}_{1}\left(\begin{array}{c}
A \\
B \\
C
\end{array}\right)=0
$$

where $A=A(y, \tau), B=B(y, \tau), C=C(y, \tau)$. To find the relation between the associated 3 -Novikov system (36) and the negative flow (39), we can take

$$
\begin{equation*}
A=-Q_{3} q_{1 y}-Q_{3}^{2} q_{1}, \quad B=-Q_{3} q_{1}, \quad C=-q_{1} . \tag{40}
\end{equation*}
$$

N. $L i$

Then the negative flow (39) is changed to

$$
\left(\begin{array}{c}
u_{1}  \tag{41}\\
v_{1} \\
w_{1}
\end{array}\right)_{\tau}=\left(\begin{array}{c}
-2 q_{1 y} \\
-q_{1 y y}-u_{1} q_{1 y}-2\left(Q_{3} q_{1}\right)_{y} \\
-\left[Q_{3} q_{1 y}+q_{1}\left(u_{1} Q_{3}+2 Q_{3}^{2}-Q_{3 y}\right)\right]_{y}
\end{array}\right), \quad\left(\begin{array}{c}
z_{1} \\
z_{2} \\
z_{3}
\end{array}\right)=0
$$

where

$$
\begin{aligned}
z_{1}= & -2 s_{1}, \\
z_{2}= & -s_{1 y y}+\left(Q_{3}-u_{1}\right) s_{1 y}-2 p_{1} s_{2 y}-3 p_{1 y} s_{2}, \\
z_{3}= & \left(Q_{3}^{2}-2 Q_{3 y}+Q_{3} u_{1}\right) s_{1 y}-Q_{3} s_{1 y y}+p_{1} s_{2 y y}+\left(2 p_{1 y}-Q_{3} p_{1}-u_{1} p_{1}\right) s_{2 y} \\
& -\left(p_{1} u_{1 y}+2 p_{1 y} u_{1}+3\left(p_{1} Q_{3}\right)_{y}\right) s_{2} .
\end{aligned}
$$

Thus the associated 3-Novikov system (36) is a reduction of the first negative flow (39) in the mYJ hierarchy, since $s_{1}=0, s_{2}=0$ yields $z_{1}=z_{2}=z_{3}=0$.

### 4.2. Hamiltonian structure behavior under the Liouville transformation

According to [29], if two soliton equations are linked by a Liouville transformation, Hamiltonian structures and conserved quantities of them can be related. In this part we will consider the Hamiltonian structures of the 3 -Novikov system (8) under the Liouville transformation (37). To this end, let $\vartheta=\left(u_{1}, v_{1}, w_{1}\right)^{T}$. Then from the point of view of Hamiltonian structures, we have

$$
\begin{align*}
& \theta_{t}=\mathcal{B}(\theta) \frac{\delta H}{\delta \theta}=\mathcal{B}(\theta) E_{\theta} h,  \tag{42}\\
& \vartheta_{t}=\tilde{\mathcal{B}}(\vartheta) \frac{\delta \tilde{H}}{\delta \vartheta}=\tilde{\mathcal{B}}(\vartheta) E_{\vartheta} \tilde{h}, \tag{43}
\end{align*}
$$

where

$$
H=\int h\left(x, \theta^{(n)}\right) d x, \quad \tilde{H}=\int \tilde{h}\left(y, \vartheta^{(n)}\right) d y
$$

Herein $E_{\theta}, E_{\vartheta}$ are the corresponding Euler operators, and $H\left[\theta^{(n)}\right]=\tilde{H}\left[\vartheta^{(n)}\right]$. Defining $\Lambda(\vartheta, \theta)=\vartheta-\left(P_{1}, P_{2}, P_{3}\right)^{T}$, hence it is easy to see that

$$
\begin{equation*}
\vartheta_{t}=-T_{1} \theta_{t}, \quad T_{1}=\Lambda_{\theta}, \tag{44}
\end{equation*}
$$

where $\Lambda_{\theta}$ is Frechét derivative for the vector variable. Then a direct computation shows that

$$
T_{1}=\left(\begin{array}{ccc}
u_{1 y} I^{\prime}[u]-P_{1}^{\prime}[u] & u_{1 y} I^{\prime}[v]-P_{1}^{\prime}[v] & u_{1 y} I^{\prime}[w]-P_{1}^{\prime}[w] \\
v_{1 y} I^{\prime}[u]-P_{2}^{\prime}[u] & v_{1 y} I^{\prime}[v]-P_{2}^{\prime}[v] & v_{1 y} I^{\prime}[w]-P_{2}^{\prime}[w] \\
w_{1 y} I^{\prime}[u]-P_{3}^{\prime}[u] & w_{1 y} I^{\prime}[v]-P_{3}^{\prime}[v] & w_{1 y} I^{\prime}[w]-P_{3}^{\prime}[w]
\end{array}\right) .
$$

Furthermore, the action of Euler operator under a change of variables is given by

$$
\begin{equation*}
E_{\theta} h=T_{2} E_{\vartheta} \tilde{h}, \tag{45}
\end{equation*}
$$

where

$$
T_{2}=\left(\begin{array}{ccc}
P_{1, u}^{\prime \dagger}\left(I_{x}\right)-I_{u}^{\prime \dagger}\left(P_{1 x}\right) & P_{2, u}^{\prime \dagger}\left(I_{x}\right)-I_{u}^{\prime \dagger}\left(P_{2 x}\right) & P_{3, u}^{\prime \dagger}\left(I_{x}\right)-I_{u}^{\prime \dagger}\left(P_{3 x}\right) \\
P_{1, v}^{\prime \dagger}\left(I_{x}\right)-I_{v}^{\prime \dagger}\left(P_{1 x}\right) & P_{2, v}^{\dagger}\left(I_{x}\right)-I_{v}^{\prime \dagger}\left(P_{2 x}\right) & P_{3, v}^{\prime \dagger}\left(I_{x}\right)-I_{v}^{\prime \dagger}\left(P_{3 x}\right) \\
P_{1, w}^{\prime \dagger}\left(I_{x}\right)-I_{w}^{\prime \dagger}\left(P_{1 x}\right) & P_{2, w}^{\prime}\left(I_{x}\right)-I_{w}^{\prime \dagger}\left(P_{2 x}\right) & P_{3, w}^{\prime \dagger}\left(I_{x}\right)-I_{w}^{\prime \dagger}\left(P_{3 x}\right)
\end{array}\right)
$$

Lemma 3 Under the transformation (37), we have the following formulaes:

$$
T_{1}=\mathcal{O} \operatorname{diag}\left(u^{-1}, v^{-1}, v u^{-3}\right), \quad T_{2}=-\operatorname{diag}\left(1, u v^{-1}, v u^{-2}\right) \mathcal{O}^{\dagger}
$$

where

$$
\mathcal{O}=\left(\begin{array}{ccc}
u_{1 y} \partial_{y}^{-1}+3 \partial_{y}+u_{1}-6 Q_{3} & -2 \partial_{y}+3 Q_{3} & 3 \\
v_{1 y} \partial_{y}^{-1}+2 \partial_{y}^{2}+2 u_{1} \partial_{y}+2 v_{1}-4 Q_{3} u_{1} & \left(Q_{3}-u_{1}\right) \partial_{y}+2 Q_{3} u_{1}-\partial_{y}^{2} & 2 u_{1} \\
w_{1 y} \partial_{y}^{-1}+3 w_{1}-2 \chi_{3} Q_{3} & \partial_{y} Q_{3}^{2}-w_{1}+Q_{2} Q_{3}^{2}+Q_{3} Q_{3 y} & \chi_{3}
\end{array}\right)
$$

with

$$
\chi_{3}=\partial_{y}^{2}-\partial_{y} u_{1}+v_{1} .
$$

Lemma 4 Under the reciprocal transformation (28), the following identities hold:

$$
\begin{equation*}
\frac{1}{v}\left(1-\partial_{x}^{2}\right) \frac{v}{u^{2}}=\Theta_{1} \equiv Q_{1}-\left(\partial_{y}-Q_{2}+Q_{3}\right)\left(\partial_{y}+Q_{3}\right) \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{u^{2}}\left(\partial_{x}^{3}-4 \partial_{x}\right) \frac{1}{u}=\Theta_{2} \equiv\left(\partial_{y}-Q_{2}\right) \partial_{y}\left(\partial_{y}+Q_{2}\right)-2 Q_{1} \partial_{y}-2 \partial_{y} Q_{1} \tag{47}
\end{equation*}
$$

The two Lemmas above can be proved through a straightforward computation. Hence the main results can be summarized as:

Theorem 2 The associated 3-Novikov system is a bi-Hamiltonian system, namely, it can be written as

$$
\left(\begin{array}{c}
u_{1}  \tag{48}\\
v_{1} \\
w_{1}
\end{array}\right)_{t}=\mathcal{K}_{1} \mathcal{J}_{1}^{-1} \mathcal{K}_{1}\left(\begin{array}{c}
\frac{\delta \tilde{H}_{2}}{\delta u_{1}} \\
\frac{\delta H_{2}}{\delta v_{1}} \\
\frac{\delta H_{2}}{\delta w_{1}}
\end{array}\right)=\mathcal{K}_{1}\left(\begin{array}{c}
\frac{\delta \tilde{H}_{1}}{\delta u_{1}} \\
\frac{\delta H_{1}}{\delta v_{1}} \\
\frac{\delta H_{1}}{\delta w_{1}}
\end{array}\right),
$$

where

$$
\begin{aligned}
\tilde{H}_{1} & =\int Q_{3} q_{1} d y \\
\tilde{H}_{2} & =\int\left[Q_{3} q_{1}\left(q_{1} q_{2 y}-q_{2} q_{1 y}+q_{1} q_{2}\left(Q_{2}-2 Q_{3}\right)\right)-q_{1} q_{2}\right] d y
\end{aligned}
$$

Proof: Substituting (44-45) into (42-43), a Hamiltonian pair for the associated 3Novikov system is obtained as

$$
\begin{equation*}
\tilde{\mathcal{J}}=-T_{1} \mathcal{J} T_{2}, \quad \tilde{\mathcal{K}}=-T_{1} \mathcal{K} T_{2} . \tag{49}
\end{equation*}
$$

Hamiltonian functionals of the 3-Novikov system and the associated 3-Novikov system connected by the formula (45).

To obtain bi-Hamiltonian structure of the associated 3-Novikov system, we should calculate $\tilde{\mathcal{J}}$ and $\tilde{\mathcal{K}}$ in the new variable $y$. Using conjugation of operator to the identity (46), we can easily check that

$$
\begin{equation*}
\frac{v}{u^{3}}\left(\partial_{x}^{2}-1\right) \frac{u}{v}=\left(\partial_{y}-Q_{3}\right)\left(\partial_{y}+Q_{2}-Q_{3}\right)-Q_{1} . \tag{50}
\end{equation*}
$$

Let us substitute the equalities (46) and (50) into the first equality in (49). Then, through tedious calculations, we get

$$
\begin{aligned}
\tilde{\mathcal{J}} & =\mathcal{O} \operatorname{diag}\left(u^{-1}, v^{-1}, v u^{-3}\right) \mathcal{J} \operatorname{diag}\left(1, u v^{-1}, v u^{-2}\right) \mathcal{O}^{\dagger} \\
& =\mathcal{O}\left(\begin{array}{ccc}
\frac{1}{2} \partial_{y} & 0 & 0 \\
0 & 0 & \Theta_{1} \\
0 & -\Theta_{1}^{\dagger} & 0
\end{array}\right) \mathcal{O}^{\dagger} \\
& =\mathcal{K}_{1} \mathcal{J}_{1}^{-1} \mathcal{K}_{1} .
\end{aligned}
$$

On the other hand, introducing

$$
\mathcal{P}=\left(\partial_{y}, \frac{3}{2} \partial_{y}-\frac{1}{2} Q_{2}+Q_{3}, \frac{3}{2} Q_{3} \partial_{y}+\frac{1}{2} Q_{2} Q_{3}+Q_{3 y}-Q_{3}^{2}\right)^{T}
$$

Then the second equality in (49) may be changed to

$$
\begin{aligned}
\tilde{\mathcal{K}} & =\mathcal{O} \operatorname{diag}\left(u^{-1}, v^{-1}, v u^{-3}\right) \mathcal{K} \operatorname{diag}\left(1, u v^{-1}, v u^{-2}\right) \mathcal{O}^{\dagger} \\
& =\mathcal{O}\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \frac{3}{2} \partial_{y}^{-1} & -1-\frac{3}{2} \partial_{y}^{-1} Q_{3} \\
0 & 1-\frac{3}{2} Q_{3} \partial_{y}^{-1} & \frac{3}{2} Q_{3} \partial_{y}^{-1} Q_{3}
\end{array}\right) \mathcal{O}^{\dagger}-2 \mathcal{O} \mathcal{P} \Theta_{2}^{-1} \mathcal{P}^{\dagger} \mathcal{O}^{\dagger} \\
& =\mathcal{O}\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \frac{3}{2} \partial_{y}^{-1} & -1-\frac{3}{2} \partial_{y}^{-1} Q_{3} \\
0 & 1-\frac{3}{2} Q_{3} \partial_{y}^{-1} & \frac{3}{2} Q_{3} \partial_{y}^{-1} Q_{3}
\end{array}\right) \mathcal{O}^{\dagger}-\left(\begin{array}{c}
0 \\
1 \\
-Q_{3}
\end{array}\right) \frac{\Theta_{2}^{\dagger}}{2}\left(\begin{array}{c}
0 \\
1 \\
-Q_{3}
\end{array}\right)^{T} \\
& =\mathcal{K}_{1}
\end{aligned}
$$

by using the identity (47) and $\mathcal{O P}=\frac{1}{2}\left(0, \Theta_{2},-Q_{3} \Theta_{2}\right)^{T}$.
Furthermore, since the Hamiltonian functionals of the two hierarchy are connected by the relation $H\left[\theta^{(n)}\right]=\tilde{H}\left[\vartheta^{(n)}\right]$, we can easy to find the relationship between the two hierarchies.

## 5. A limit system

The limits of the CH type equations might also contain some important models. For example, the Hunter-Saxton equation, which can describe wave motion in a nematic liquid crystal [30], may be consider as a limit of the CH equation [31]. The Ostrovsky equation, which appears as the description of high-frequency waves in a relaxing medium [32], can be obtained as a short wave limit of the DP equation [11]. In this section, we will consider a limit of the 3-Novikov system (8)).

Let us consider the transformation

$$
\begin{equation*}
x \rightarrow \epsilon x, \quad t \rightarrow \epsilon t, \quad u \rightarrow \epsilon^{\frac{3}{2}} u . \tag{51}
\end{equation*}
$$

Then a limit for the 3 -Novikov system may be obtained in the limit $\epsilon \rightarrow 0$, that is

$$
\begin{align*}
& u_{t}+(u p r)_{x}=0, \\
& v_{t}+3 v p_{x} r+v_{x} p r+u^{2} p=0,  \tag{52}\\
& w_{t}+3 w p r_{x}+w_{x} p r-u^{2} r=0, \\
& v=-p_{x x}, \quad w=-r_{x x} .
\end{align*}
$$

The short wave model (52) is also integrable in the sense of admitting bi-Hamiltonian structure and a Lax pair. The bi-Hamiltonian structure can be obtained by applying the transformation (51) to that of the 3-Novikov system, that is

$$
\left(\begin{array}{c}
u  \tag{53}\\
v \\
w
\end{array}\right)_{t}=\overline{\mathcal{J}}_{1}\left(\begin{array}{c}
\frac{\delta \bar{H}_{2}}{\delta u} \\
\frac{\delta H_{2}}{\delta v} \\
\frac{\delta H_{2}}{\delta w}
\end{array}\right)=\overline{\mathcal{K}}_{1}\left(\begin{array}{c}
\frac{\delta \bar{H}_{1}}{\delta u} \\
\frac{\delta H_{1}}{\delta v} \\
\frac{\delta H_{1}}{\delta w}
\end{array}\right),
$$

where

$$
\begin{aligned}
& \overline{\mathcal{J}}_{1}=\left(\begin{array}{ccc}
\frac{1}{2} \partial & 0 & 0 \\
0 & 0 & -\partial^{2} \\
0 & \partial^{2} & 0
\end{array}\right), \\
& \overline{\mathcal{K}}_{1}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \frac{3}{2} v \partial^{-1} v & -u^{2}-\frac{3}{2} v \partial^{-1} w \\
0 & u^{2}-\frac{3}{2} w \partial^{-1} v & \frac{3}{2} w \partial^{-1} w
\end{array}\right)-2 \Omega \partial^{-3} \Omega^{*}
\end{aligned}
$$

with the functionals given by

$$
\begin{aligned}
\bar{H}_{1} & =\int p_{x} r_{x} d x \\
\bar{H}_{2} & =\int\left(p p_{x} r r_{x x}+p p_{x} r_{x}^{2}-u^{2} p r\right) d x
\end{aligned}
$$

Taking the transformation (51) to the Lax pair (9) with $\lambda \rightarrow \epsilon \lambda$, a Lax pair for the limit system (52) is obtained as

$$
\begin{aligned}
\varphi_{x} & =\left(\begin{array}{ccc}
0 & 1 & 0 \\
\lambda u^{2} & 0 & v \\
\lambda w & 0 & 0
\end{array}\right) \varphi, \\
\varphi_{t} & =\left(\begin{array}{ccc}
\frac{1}{3 \lambda}+p r_{x} & -p r & \frac{p}{\lambda} \\
p_{x} r_{x}-\lambda u^{2} p r & \frac{1}{3 \lambda}-p_{x} r & \frac{p_{x}}{\lambda}-v p r \\
-\lambda w p r-r_{x} & r & p_{x} r-p r_{x}-\frac{2}{3 \lambda}
\end{array}\right) \varphi .
\end{aligned}
$$

Moreover, the short wave model (52) is also reciprocal connected to the first negative flow in the mYJ hierarchy by taking the similar process before. It may reduce to that of the Geng-Xue, the Novikov and the DP as $u=0$ and $u=0, p=r$ as well as $u=0, r=1$ respectively.

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