A new 3-component Novikov hierarchy

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Abstract. We study the bi-Hamiltonian structure of the hierarchy of a 3-component Novikov system. We show that Hamiltonian functionals of the 3-Novikov hierarchy in negative direction are local, and in both directions are homogenous. We construct a reciprocal transformation to connect the 3-Novikov system to a reduction of the first negative flow in a modified Yajima-Oikawa hierarchy, which is shown to pass the standard Painlevé test. Besides we discuss bi-Hamiltonian structures of the 3-Novikov hierarchy under the reciprocal transformation. Moreover, we consider a limit for the 3-Novikov system.

1. Introduction

The Camassa-Holm (CH) equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx},$$
 (1)

has attracted much attention since it is derived as the governing equation for dispersive shallow-water motion in 1993 [1]. It is remarkable that the CH equation has peakon solutions which are interesting in general analysis of PDEs [2]. The CH equation is integrable from the point of view of Lax pair and bi-Hamiltonian structure [1, 3]. It is linked to the negative KdV equation by a reciprocal transformation [4, 5, 6]. In Ref. [7] and its references, many other algebraic and geometric properties of the CH equation are introduced.

By applying asymptotic integrability method to a family of third order dispersive PDE, Degasperis and Procesi [8] found another equation possessing peakon solutions

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}.$$
 (2)

The DP equation has a Lax pair and a bi-Hamiltonian structure [9]. An infinite sequence of conservation laws for the equation are also obtained. Besides a reciprocal transformation is constructed to connect it with a negative flow in the Kaup-Kupershmidt hierarchy. Hereafter, many other equations of CH type were proposed and studied. For example, the Novikov equation, the modified CH equation, a 2-component CH equation and the Geng-Xue equation (see e.g. [10, 11, 12, 13, 14, 15]).

Recently, Geng and Xue [16] presented a 3-component CH type hierarchy by consider the following 3×3 matrix spectral problem

$$\varphi_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda u & 0 & v \\ \lambda w & 0 & 0 \end{pmatrix} \varphi.$$
(3)

The spectral problem (3) may reduce to that of the CH equation, the DP equation, the Novikov equation and the Geng-Xue equation. The corresponding hierarchy was derived by choosing the trivial flow as $(u, v, w)_t^T = (u, v, w)_x^T$. The first negative flow in the hierarchy reads

$$u_{t} = -vp_{x} + u_{x}q + \frac{3}{2}uq_{x} - \frac{3}{2}u(p_{x}r_{x} - pr),$$

$$v_{t} = 2vq_{x} + v_{x}q,$$

$$w_{t} = vr_{x} + w_{x}q + \frac{3}{2}wq_{x} + \frac{3}{2}w(p_{x}r_{x} - pr),$$

$$u = p - p_{xx}, \qquad w = r_{xx} - r,$$

$$v = \frac{1}{2}(q_{xx} - 4q + p_{xx}r_{x} - r_{xx}p_{x} + 3p_{x}r - 3pr_{x}).$$
(4)

This system can be reduced to the CH equation as p = r = 0. It admits a bi-Hamiltonian structure and an infinite sequence of conserved quantities [16, 17]. However, it is hard to construct some exact solutions for this system.

Subsequently, by considering reductions of a 4-component CH type system, we proposed another 3-component CH type system

$$m_{1t} + u_2 g m_{1x} - m_3 (u_{2x} f - u_2 g) - m_1 (3u_2 f - m_3 u_2) = 0,$$

$$m_{2t} + u_2 g m_{2x} + m_2 (3u_{2x} g + m_3 u_2) = 0,$$

$$m_{3t} + u_2 g m_{3x} - m_3 (2u_2 f + u_{2x} g - m_3 u_2) = 0,$$

$$m_i = u_i - u_{ixx}, \ i = 1..3, \quad f = u_3 - u_{1x}, \quad g = u_1 - u_{3x},$$

(5)

associated with the spectral problem [18]

$$\phi_x = \begin{pmatrix} 0 & 0 & 1\\ \lambda m_1 & 0 & \lambda m_3\\ 1 & \lambda m_2 & 0 \end{pmatrix} \phi.$$
(6)

It is shown to possess a bi-Hamiltonian structure and infinitely many conserved quantities. The system (5) is found to connect with a negative generalized MKdV system (a modified Yajima-Oikawa (mYJ) system[19, 20]) via a reciprocal transformation, and the associated system is shown to pass the standard Painlevé test of WTC [21].

In this paper, we will study a new 3-Novikov hierarchy associated with the following spectral problem

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda u^2 & 0 & v \\ \lambda w & 0 & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \tag{7}$$

which is obtained by replacing u in (3) with u^2 for convenience. The new hierarchy is different from the hierarchy found by Geng and Xue, because we take the trivial flow

as $(u, v, w)_t^T = (0, v, -w)^T$. The first typical member in the 3-Novikov hierarchy is the 3-Novikov system

$$u_{t} + (upr)_{x} = 0,$$

$$v_{t} + 3vp_{x}r + v_{x}pr + u^{2}p = 0,$$

$$w_{t} + 3wpr_{x} + w_{x}pr - u^{2}r = 0,$$

$$v = p - p_{xx}, \qquad w = r - r_{xx}.$$

(8)

It can be reduced to the DP equation, the Novikov equation and the Geng-Xue equation as u = 0, r = 1, as u = 0, p = r and as u = 0 respectively. We will construct infinitely many conserved quantities and study the bi-Hamiltonian structure of the 3-Novikov hierarchy, and construct a reciprocal transformation for the system (8).

The outline of this paper is as follows. In Section 2, we construct infinitely many conserved quantities for the 3-Novikov equation with the aid of the spectral problem (7). We also analyze the homogeneous and local properties of the Hamiltonian functionals in the 3-Novikov hierarchy. In Section 3, we find the relationship between the two systems (5) and (8). In Section 4, we construct a reciprocal transformation to connect the 3-Novikov system with the first negative flow in a mYJ hierarchy, and analyse the bi-Hamiltonian structure under this transformation. In Section 5, we present a limit of the 3-Novikov system.

2. Conserved quantities and bi-Hamiltonian structure of the 3-Novikov hierarchy

2.1. Conserved quantities

The 3-Novikov system (8) arises as the compatibility condition for the linear system

$$\varphi_x = U\varphi, \qquad \varphi_t = V\varphi, \tag{9}$$

where

$$V = \begin{pmatrix} \frac{1}{3\lambda} + pr_x & -pr & \frac{p}{\lambda} \\ p_x r_x - \lambda u^2 pr & \frac{1}{3\lambda} - p_x r & \frac{p_x}{\lambda} - vpr \\ -\lambda w pr - r_x & r & p_x r - pr_x - \frac{2}{3\lambda} \end{pmatrix}.$$

With the aid of the Lax pair (9), infinitely many conserved quantities or conservation laws for the 3-Novikov system can be constructed. For example, setting $\rho = (\ln \varphi_3)_x$ and expanding it in powers of λ , as pointed out in [16], one may able to obtain an infinite sequence of conserved densities for (8) from coefficients of ρ by solving

$$(\partial + \rho)\left[\left(\frac{\rho}{w}\right)_x + \frac{\rho^2}{w}\right] - (1 + \lambda u^2)\frac{\rho}{w} - \lambda v = 0.$$
(10)

However, it is not easy to solve (10) and the expansion of ρ in [16] can be generalized. Therefore we will consider a better formulation for computations and get more exact conserved quantities, which may be useful to generalize flows of the 3-Novikov hierarchy and to construct reciprocal transformations. N. Li

$$a_x = b - \lambda w a^2,\tag{11}$$

$$b_x = (1 + \lambda u^2)a + v - \lambda wab.$$
⁽¹²⁾

Solving the above system by expanding a, b as $a = \sum_{j \ge 0} a_j \lambda^j, b = \sum_{j \ge 0} b_j \lambda^j$ yields

$$\begin{aligned} a_{0x} &= b_0, & b_{0x} &= a_0 + v, \\ a_{1x} &= b_1 - w a_0^2, & b_{1x} &= a_1 + u^2 a_0 - w a_0 b_0, \\ a_{ix} &= b_i - w \sum_{k=0}^{i-1} a_k a_{i-k-1}, & b_{ix} &= a_i + u^2 a_{i-1} - w \sum_{k=0}^{i-1} a_k b_{i-k-1}, (i \ge 2). \end{aligned}$$

We obtain, after some calculations, that

$$a_{0} = -p, \qquad b_{0} = -p_{x},$$

$$a_{1} = (1 - \partial^{2})^{-1} (u^{2}p + 3wpp_{x} + w_{x}p^{2}), \qquad b_{1} = wp^{2} + a_{1x},$$

$$a_{i} = (1 - \partial^{2})^{-1} [w \sum_{k=0}^{i-1} a_{k}b_{i-k-1} + (w \sum_{k=0}^{i-1} a_{k}a_{i-k-1})_{x} - u^{2}a_{i-1}],$$

$$b_{i} = a_{ix} + w \sum_{k=0}^{i-1} a_{k}a_{i-k-1}, \quad (i \ge 2).$$

Then an infinite sequences of conserved quantities are gotten. The first three are

$$\Gamma_1 = -\int pwdx,$$

$$\Gamma_2 = \int [u^2 pr + wpp_x r - wp^2 r_x]dx,$$

$$\Gamma_3 = \int [a_1(3wpr_x + w_x pr - u^2 r) - w^2 p^3 r]dx.$$

Furthermore, we can also expanding a, b as

$$a = \sum_{j \ge 1} a_j \lambda^{-\frac{1}{2}j}, \qquad b = \lambda^{\frac{1}{2}} \sum_{j \ge 1} b_j \lambda^{-\frac{1}{2}j},$$

which are different from the expansions in [16]. Taking the similar procedure as the previous, we have

$$a_{1} = uw^{-1}, \qquad b_{1} = u^{2}w^{-1},$$

$$a_{2} = \frac{1}{2}u^{-2}v - \frac{3}{2}(uw)^{-1}u_{x} + w^{-2}w_{x}, \quad b_{2} = u^{-1}v - u^{-1}(u^{2}w^{-1})_{x},$$

$$b_{i+1} = -u^{-1}(b_{ix} - a_{i-1} + w\sum_{k=2}^{i}a_{k}b_{i+2-k}),$$

$$a_{i+1} = \frac{1}{2}u^{-1}(b_{i+1} - a_{ix} - w\sum_{k=2}^{i}a_{k}a_{i+2-k}).$$

Then the first four conserved quantities may be obtained, which are

$$\begin{split} \Upsilon_1 &= \int u dx, \\ \Upsilon_2 &= \frac{1}{2} \int u^{-2} v w dx, \end{split}$$

$$\begin{split} \Upsilon_3 &= \frac{1}{4} \int u^{-5} (\frac{1}{2} u^2 u_x^2 + u^2 w v_x + 2u^4 - \frac{3}{2} v^2 w^2 - u^2 v w_x) dx, \\ \Upsilon_4 &= \frac{1}{2} \int [-u^{-4} (vw + v_x w_x) + 2u^{-6} (uu_x (vw)_x - w^2 v v_x - 2vw u_x^2) \\ &\quad + u^{-8} (3w^2 v^2 u u_x + w^3 v^3)] dx. \end{split}$$

2.2. Hamiltonian structure

In this part, we will study the 3-Novikov hierarchy in the view of bi-Hamiltonian structure. Notice that the 3-Novikov system (8) is generated by the two conserved quantities Γ_1, Γ_2 , we have the following result.

Theorem 1 The 3-Novikov equation (8) is a bi-Hamiltonian system, namely, it may be written as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t} = \mathcal{J} \begin{pmatrix} \frac{\delta H_{2}}{\delta u} \\ \frac{\delta H_{2}}{\delta v} \\ \frac{\delta H_{2}}{\delta w} \end{pmatrix} = \mathcal{K} \begin{pmatrix} \frac{\delta H_{1}}{\delta u} \\ \frac{\delta H_{1}}{\delta v} \\ \frac{\delta H_{1}}{\delta w} \end{pmatrix},$$
(13)

where

$$\begin{aligned} \mathcal{J} &= \begin{pmatrix} \frac{1}{2}\partial & 0 & 0 \\ 0 & 0 & 1 - \partial^2 \\ 0 & \partial^2 - 1 & 0 \end{pmatrix}, \\ \mathcal{K} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2}v\partial^{-1}v & -u^2 - \frac{3}{2}v\partial^{-1}w \\ 0 & u^2 - \frac{3}{2}w\partial^{-1}v & \frac{3}{2}w\partial^{-1}w \end{pmatrix} - 2\Omega(\partial^3 - 4\partial)^{-1}\Omega^*, \end{aligned}$$

here in

$$\Omega = (\partial u, \frac{1}{2}v\partial + \partial v, \frac{1}{2}w\partial + \partial w)^T,$$

$$H_1 = -\Gamma_1, \qquad H_2 = -\Gamma_2.$$

Since \mathcal{J}, \mathcal{K} forms a Hamiltonian pair [16], one can prove the theorem easily. Hence a recursion operator for 3-Novikov hierarchy is gotten as $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$, and we can derive a new 3-Novikov hierarchy by taking the trivial flow as $(u, v, w)_t^T = \mathcal{R}(0, v, -w)^T$. Then the positive flows in the hierarchy may be obtained as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_n} = \mathcal{J}\begin{pmatrix} \frac{\delta H_{n+1}}{\delta u} \\ \frac{\delta H_{n+1}}{\delta v} \\ \frac{\delta H_{n+1}}{\delta w} \end{pmatrix} = \mathcal{K}\begin{pmatrix} \frac{\delta H_n}{\delta u} \\ \frac{\delta H_n}{\delta v} \\ \frac{\delta H_n}{\delta w} \end{pmatrix}, \quad n = 1, 2, ...,$$
(14)

and infinitely many negative flows read as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t-n} = \mathcal{K} \begin{pmatrix} \frac{\delta H_{-(n+1)}}{\delta u} \\ \frac{\delta H_{-(n+1)}}{\delta v} \\ \frac{\delta H_{-(n+1)}}{\delta w} \end{pmatrix} = \mathcal{J} \begin{pmatrix} \frac{\delta H_{-n}}{\delta u} \\ \frac{\delta H_{-n}}{\delta w} \\ \frac{\delta H_{-n}}{\delta w} \end{pmatrix}, \quad n = 1, 2, \dots$$
(15)

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with the first two Hamiltonian functionals giving by $H_{-1} = -2\Upsilon_2, H_{-2} = -2\Upsilon_4$. In particular, the first negative flow in the hierarchy is obtained by using the Hamiltonian functionals H_{-1}, H_{-2} , that is

$$u_{t} - (\frac{vw}{u^{3}})_{x} = 0,$$

$$v_{t} - (\frac{v}{u^{2}})_{xx} + \frac{v}{u^{2}} = 0,$$

$$w_{t} - \frac{w}{u^{2}} + (\frac{w}{u^{2}})_{xx} = 0.$$
(16)

It is worth to note that Υ_1 and Υ_3 are the Casimir functionals of the Hamiltonian operators \mathcal{J} and \mathcal{K} respectively.

Since the structure of Hamiltonian functionals $H_n s$ in the 3-Novikov hierarchy is largely unknown, like the cases in [22, 23], we will consider the homogeneous and local properties of them. Introducing $\theta = (u, v, w)^T$ and $X_n[\theta] = \frac{\delta H_n}{\delta \theta}$, then recursive relation in the positive direction

$$\mathcal{J}\frac{\delta H_{n+1}}{\delta \theta} = \mathcal{K}\frac{\delta H_n}{\delta \theta}, \qquad n = 1, 2, ...,$$

yields an infinite sequence of variational derivatives for the Hamiltonian functionals H_n s

$$X_{n+1}[\theta] = \mathcal{J}^{-1} \mathcal{K} X_n[\theta], \qquad n = 1, 2, \dots$$
 (17)

Similarly, the variational derivatives for the Hamiltonian functionals H_{-n} s in the negative direction are given by

$$X_{-(n+1)}[\theta] = \mathcal{K}^{-1}\mathcal{J}X_{-n}[\theta], \qquad n = 1, 2, \dots$$

Proposition 1 The variational derivatives $X_n[\theta]$ are homogeneous in the sense that

$$X_n[\epsilon\theta] = \epsilon^{2n-1} X_n[\theta], \quad n \ge 1,$$
(18)

and

$$H_n[\varepsilon\theta] = \frac{1}{2n} \int X_n[\theta] \cdot \theta dx, \quad n \ge 1.$$
(19)

Proof: When n = 1, the formulate (18) holds clearly. Now suppose (18) also holds for n = k, that is

$$X_k[\epsilon\theta] = \epsilon^{2k-1} X_k[\theta].$$

Then for n = k + 1, we have

$$X_{k+1}[\epsilon\theta] = \mathcal{J}^{-1}[\epsilon\theta]\mathcal{K}[\epsilon\theta]X_k[\epsilon\theta] = \epsilon^2 \mathcal{J}^{-1}[\theta]\mathcal{K}[\theta]X_k[\epsilon\theta],$$

which implies that

$$X_{k+1}[\epsilon\theta] = \epsilon^{2k+1}[\theta] X_{k+1}[\theta].$$

In addition, for any $n \ge 1$, we have

$$H_n[\theta] = \int_0^1 \int X_n[\varepsilon\theta] \cdot \theta dx d\varepsilon = \frac{1}{2n} \int X_n[\theta] \cdot \theta dx,$$

then the Hamiltonian functionals H_n s are also homogeneous with

$$H_n[\varepsilon\theta] = \varepsilon^{2n} H_n[\theta], \quad n = 1, 2, \dots$$

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The recursive formula for H_n s yields infinitely many Hamiltonian functionals in the positive direction, and H_1 and H_2 are local. However, $H_n, n \ge 3$ becomes nonlocal. For example $H_3 = -\Gamma_3$, which is shown to be nonlocal.

Proposition 2 The variational derivatives $X_{-n}[\theta]$ s satisfy

$$X_{-n}[\epsilon\theta] = \epsilon^{1-2n} X_{-n}[\theta], \quad n = 1, 2, \dots$$
(20)

while

$$H_{-n}[\theta] = \frac{1}{2-2n} \int X_{-n}[\theta] \cdot \theta dx, \qquad (21)$$

and H_{-n} s are all local.

The formulae (20) and (21) may be proven by taking the process before, and we will prove the local property of H_{-n} s below.

Lemma 1 ([22, 23, 24]) If a differential function $M[\theta]$ satisfies

$$\int M[\theta] dx = 0$$

for all θ , then there exists a unique differential function $N[\theta]$ up to addition of a constant such that $M[\theta]$ is the total x-derivative $M[\theta] = (N[\theta])_x$.

Introducing

$$X_{-k}[\theta] = (A_k, B_k, C_k)^T$$

$$E_k = (\partial^3 - 4\partial)^{-1} (u\partial, \frac{3}{2}v\partial + \frac{1}{2}v_x, \frac{3}{2}w\partial + \frac{1}{2}w_x)X_{-k}[\theta], \quad k \ge 1$$

When $n = 1, X_{-1}[\theta]$ is local since

$$X_{-1}[\theta] = (2\frac{vw}{u^3}, -\frac{w}{u^2}, -\frac{v}{u^2})^T.$$

Now suppose $X_{-k}[\theta]$ is local for n = k. Then for n = k + 1, we have

$$X_{-(k+1)}[\theta] = \mathcal{K}^{-1}\mathcal{J}X_{-k}[\theta] = (\mathcal{K}^{-1}\mathcal{J})^k X_{-1}[\theta],$$

which is equal to

$$\mathcal{K}X_{-(k+1)}[\theta] = \mathcal{J}X_{-k}[\theta]. \tag{22}$$

This shows that

$$E_{k+1} = \frac{1}{4u} A_k,\tag{23}$$

$$\frac{3}{2}v\partial^{-1}(vB_{k+1} - wC_{k+1}) - u^2C_{k+1} + (3v\partial + 2v_x)E_{k+1} = (1 - \partial^2)C_k, \quad (24)$$

$$u^{2}B_{k+1} - \frac{3}{2}w\partial^{-1}(vB_{k+1} - wC_{k+1}) + (3w\partial + 2w_{x})E_{k+1} = (\partial^{2} - 1)B_{k}.$$
 (25)

Then we will prove the local property of $X_{-(k+1)}$ in two steps. The first step is to prove that B_{k+1} and C_{k+1} are local. Since A_k, B_k, C_k are all local, we can obtain

immediately from (24) and (25) that B_{k+1} and C_{k+1} are local, if there exist a differential function M_k such that

$$vB_{k+1} - wC_{k+1} = \frac{w}{u^2}(1 - \partial^2)C_k + \frac{v}{u^2}(\partial^2 - 1)B_k - \frac{3vw}{2u^2}\partial\frac{A_k}{u} - \frac{(vw)_x}{2u^3}A_k$$

= M_{kx} .

Then according to the Lemma 1, we only need to prove

$$Y_1 = \int \left[\frac{w}{u^2}(1-\partial^2)C_k + \frac{v}{u^2}(\partial^2 - 1)B_k - \frac{3vw}{2u^2}\partial\frac{A_k}{u} - \frac{(vw)_x}{2u^3}A_k\right]dx = 0.$$

In fact

$$Y_{1} = \int \left[\frac{w}{u^{2}}(1-\partial^{2})C_{k} + \frac{v}{u^{2}}(\partial^{2}-1)B_{k} - \frac{3vw}{2u^{2}}\partial\frac{A_{k}}{u} - \frac{(vw)_{x}}{2u^{3}}A_{k}\right]dx$$
$$= \int \left[C_{k}(1-\partial^{2})\frac{w}{u^{2}} + B_{k}(\partial^{2}-1)\frac{v}{u^{2}} + A_{k}(\frac{vw}{u^{3}})_{x}\right]dx$$
$$= \int \begin{pmatrix}A_{k}\\B_{k}\\C_{k}\end{pmatrix} \cdot \mathcal{J}\begin{pmatrix}\frac{2\frac{vw}{u^{2}}}{-\frac{w}{u^{2}}}\\-\frac{w}{u^{2}}\end{pmatrix}dx$$
$$= \int X_{-k}[\theta] \cdot \mathcal{J}X_{-1}[\theta]dx.$$

On the other hand, using the recursion relation, we have

$$Y_{1} = \int (\mathcal{K}^{-1}\mathcal{J})^{k-1}X_{-1}[\theta] \cdot \mathcal{J}X_{-1}[\theta]dx$$

$$= -\int X_{-1}[\theta] \cdot \mathcal{J}(\mathcal{K}^{-1}\mathcal{J})^{k-1}X_{-1}[\theta]dx$$

$$= -\int X_{-1}[\theta] \cdot (\mathcal{J}\mathcal{K}^{-1})^{k-1}\mathcal{J}X_{-1}[\theta]dx$$

$$= -\int (\mathcal{K}^{-1}\mathcal{J})^{k-1}X_{-1}[\theta] \cdot \mathcal{J}X_{-1}[\theta]dx$$

$$= -\int X_{-k}[\theta] \cdot \mathcal{J}X_{-1}[\theta]dx.$$

Therefore $Y_1 = 0$, and hence B_{k+1} and C_{k+1} are local.

The next step is to prove that A_{k+1} is local. From (23), we infer that

$$A_{k+1x} = \frac{1}{u} [(\partial^3 - 4\partial) \frac{A_k}{4u} - (3v\partial + 2v_x)B_{k+1} - (3w\partial + 2w_x)C_{k+1}].$$

Notice that B_{k+1} and C_{k+1} are all local, so A_{k+1} is local if the right part of the above equality is a total x-derivative N_{kx} for a differential function N_k . That is to say, A_{k+1} is local if

$$Y_2 = \int (\frac{1}{u} [(\partial^3 - 4\partial) \frac{A_k}{4u} - (3v\partial + 2v_x)B_{k+1} - (3w\partial + 2w_x)C_{k+1}])dx = 0.$$

Lemma 2 Define

$$\mathcal{D} = \begin{pmatrix} \frac{3}{2}v\partial^{-1}v & -u^2 - \frac{3}{2}v\partial^{-1}w \\ u^2 - \frac{3}{2}w\partial^{-1}v & \frac{3}{2}w\partial^{-1}w \end{pmatrix},$$

we have

$$\mathcal{D}^{-1} = \frac{1}{u^2} \begin{pmatrix} \frac{3}{2}w\partial^{-1}w & u^2 + \frac{3}{2}w\partial^{-1}v \\ \frac{3}{2}v\partial^{-1}w - u^2 & \frac{3}{2}v\partial^{-1}v \end{pmatrix} \frac{1}{u^2}.$$

To make the expressions compact, we introduce some new notations as:

$$Z_{1} = (1 - \partial^{2})C_{k} - (3v\partial + 2v_{x})\frac{A_{k}}{4u}, Z_{2} = (\partial^{2} - 1)B_{k} - (3vw\partial + 2w_{x})\frac{A_{k}}{4u}$$
$$Z_{3} = \frac{v_{x}}{u^{3}} - \frac{3}{2}\frac{u_{x}v}{u^{4}} - \frac{3}{2}\frac{wv^{2}}{u^{4}}, \qquad Z_{4} = -\frac{w_{x}}{u^{3}} + \frac{3}{2}\frac{u_{x}w}{u^{4}} - \frac{3}{2}\frac{vw^{2}}{u^{5}}.$$

Using the Lemma 2 to solve B_{k+1} and C_{k+1} from (24) and (25), we arrive at

$$\begin{split} Y_{2} &= \int (\frac{1}{u} [(\partial^{3} - 4\partial) \frac{A_{k}}{4u} - (3v\partial + 2v_{x})B_{k+1} - (3w\partial + 2w_{x})C_{k+1}])dx \\ &= \int [-\frac{A_{k}}{4u} (\partial^{3} - 4\partial) \frac{1}{u} + B_{k+1} (\frac{v_{x}}{u} - \frac{3u_{x}v}{u^{2}}) + C_{k+1} (\frac{w_{x}}{u} - \frac{3u_{x}w}{u^{2}})]dx \\ &= \int [-\frac{A_{k}}{4u} (\partial^{3} - 4\partial) \frac{1}{u} + (\frac{v_{x}}{u^{3}} - \frac{3u_{x}v}{2u^{4}})] \frac{2}{2} w \partial^{-1} (\frac{wZ_{1} + vZ_{2}}{u^{2}}) + Z_{2}] \\ &+ (\frac{w_{x}}{u^{3}} - \frac{3u_{x}w}{2u^{4}})[\frac{3}{2}v\partial^{-1} (\frac{wZ_{1} + vZ_{2}}{u^{2}}) - Z_{1}]]dx \\ &= \int [-\frac{A_{k}}{4u} (\partial^{3} - 4\partial) \frac{1}{u} + Z_{2}Z_{3} + Z_{1}Z_{4}]dx \\ &= \int (\frac{A_{k}}{4u} [-(\partial^{3} - 4\partial) \frac{1}{u} + (3w\partial + 2w_{x})Z_{3} + (3v\partial + 2v_{x})Z_{4}] \\ &+ B_{k} (\partial^{2} - 1)Z_{1} + C_{k} (1 - \partial^{2})Z_{2})dx \\ &= \int \left(\frac{A_{k}}{B_{k}}\right) \cdot \mathcal{J} \left(\frac{\frac{3wv_{x} - 3vw_{x}}{2u^{4}} - \frac{15v^{2}w^{2}}{4u^{6}} + \frac{u_{xx}}{2u^{5}} - \frac{3u^{2}}{4u^{4}} + \frac{1}{u^{2}}}{-\frac{v_{x}}{u^{3}} + \frac{3wu_{x}}{2u^{5}}} \right) dx \\ &= \int [-2X_{-k}[\theta] \cdot \mathcal{J} \frac{\delta\Upsilon_{3}}{\delta\theta}]dx \\ &= \int 2\mathcal{J}(\mathcal{K}^{-1}\mathcal{J})^{k-1}X_{-1}[\theta] \cdot \mathcal{K} \frac{\delta\Upsilon_{3}}{\delta\theta}dx \\ &= \int -2(\mathcal{K}^{-1}\mathcal{J})^{k}X_{-1}[\theta] \cdot (0,0,0)^{T}dx \\ &= 0. \end{split}$$

Therefore A_{k+1} is local. Consequently, we prove X_{-n} s are all local. Then using the Lemma 4.4 in [23] (see also [22, 24]), H_{-n} s are found to be local.

3. Relationship with a 3-component CH type system

As pointed out in [18], the 3-CH type system (5) is reciprocal linked to the first negative flow in a mYJ hierarchy. Since the spectral problem (6) is gauge linked to the spectral problem (7), it would seem to be a reasonable guess that the 3-CH type system (5) is equal to the 3-Novikov system (8).

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In fact the 3-CH type system (5) may be rewritten as

$$m_{1t} + u_2 g m_{1x} - m_3 (u_{2x} f - u_2 g) - m_1 (3u_2 f - m_3 u_2) = 0,$$

$$m_{2t} + u_2 g m_{2x} + m_2 (3u_{2x} g + m_3 u_2) = 0,$$

$$m_{3t} + u_2 g m_{3x} - m_3 (2u_2 f + u_{2x} g - m_3 u_2) = 0,$$

$$m_2 = u_2 - u_{2xx}, \quad m_1 - m_{3x} = g - g_{xx}, \quad f = m_3 - g_x,$$

(26)

then a directly calculation shows that (26) is connected to (8) via

$$u = (m_2 m_3)^{\frac{1}{2}}, \quad v = m_2, \quad w = m_1 - m_{3x}, \quad p = u_2, \quad r = g.$$
 (27)

4. A reciprocal transformation for the 3-Novikov system

4.1. A reciprocal transformation

Although many CH type systems are completely integrable, they have some nonstandard features such as the DT, the Bäcklund transformation and the weak Painlevé property [9, 25]. To study the Painlevé behaviour of the 3-Novikov system (8), we can relate it with a equation displaying the standard (strong) Painlevé test of WTC [26], which is easy to construct the DT and the Bäcklund transformation. Our strategy is to use the steps in [21], and we will connect the 3-Novikov system (8) with a negative flow in a mYJ hierarchy.

The 3-Novikov system (8) has a conserved density u, and the correspondence conservation law reads

$$u_t = (-upr)_x,$$

which allows a reciprocal transformation

$$dy = udx - uprdt, \qquad d\tau = dt. \tag{28}$$

Set $\mu = \lambda^{\frac{1}{2}}$ and define $\psi_2 = \mu \frac{u^2}{v} \varphi_1 + \frac{1}{\mu} \varphi_3$. Then, under change of variables, we may rewrite the spectral problem (7) as

$$\begin{aligned} \varphi_{1yy} &+ \frac{u_y}{u} \varphi_{1y} - \frac{1}{u^2} \varphi_1 - \mu \frac{v}{u^2} \psi_2 = 0, \\ \psi_{2y} &- \mu \frac{u^2}{v} \varphi_{1y} - \mu [(\frac{u^2}{v})_y + \frac{w}{u}] \varphi_1 = 0. \end{aligned}$$
(29)

Now, introducing the gauge transformation

$$\varphi_1 = \frac{v}{u^2} e^{-\partial_y^{-1}(\frac{vw}{u^3})} \phi_1, \qquad \psi_2 = e^{-\partial_y^{-1}(\frac{vw}{u^3})} \phi_2,$$

the spectral problem (29) may be converted to

$$\begin{aligned}
\phi_{1yy} - Q_2 \phi_{1y} - Q_1 \phi_1 &= \mu \phi_2, \\
\phi_{2y} - Q_3 \phi_2 &= \mu \phi_{1y},
\end{aligned}$$
(30)

where

$$\begin{aligned} Q_1 &= (3\frac{v_y}{v} - 6\frac{u_y}{u} + \frac{w_y}{w} - \frac{vw}{u^3})\frac{vw}{u^3} + 3\frac{u_yv_y}{uv} + \frac{1 + 2uu_{yy} - 4u_y^2}{u^2} - \frac{v_{yy}}{v}, \\ Q_2 &= 2\frac{vw}{u^3} - 2\frac{v_y}{v} + 3\frac{u_y}{u}, \\ Q_3 &= \frac{vw}{u^3}. \end{aligned}$$

It is easy to check that the auxiliary problem in (9) is transformed to

$$\phi_{1\tau} = \frac{1}{\mu} q_1 \phi_2 + \frac{1}{3\mu^2} \phi_1,$$

$$\phi_{2\tau} = \frac{1}{\mu} (q_2 \phi_{1y} + [1 - q_{2y} + (Q_3 - Q_2)q_2]\phi_1) + (q_1 - \frac{2}{3\mu^2})\phi_2,$$
(31)

where

$$q_1 = p \frac{u^2}{v}, \qquad \qquad q_2 = \frac{rv}{u}. \tag{32}$$

For the convenience of constructing exact solutions of the 3-Novikov equation, let us rewrite the above Lax pair in scalar form. Eliminating ϕ_2 from the systems (30) and (31), we obtain

$$\phi_{1yyy} + u_1\phi_{1yy} + (v_1 + u_{1y})\phi_1 + (w_1 + v_{1y})\phi_1 = \lambda\phi_{1y}, \tag{33}$$

$$\phi_{1\tau} - \frac{1}{\lambda}(q_1\phi_{1yy} + (u_1 + Q_3)q_1\phi_{1y} - \chi\phi_1) = 0, \qquad (34)$$

where

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} -Q_2 - Q_3 \\ Q_2 Q_3 - Q_1 + Q_{3y} \\ Q_1 Q_3 - (Q_2 Q_3)_y - Q_{3yy} \end{pmatrix}$$
(35)

with

$$\chi = q_{1yy} + (u_1 + 3Q_3)q_{1y} + [(u_1 + 2Q_3)Q_3 + Q_{3y}]q_1 + \frac{2}{3}$$

Then the compatibility condition for the Lax representation (33-34) yields the associated 3-Novikov equation

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}_{\tau} = \begin{pmatrix} -2q_{1y}, \\ -q_{1yy} - u_1q_{1y} - 2(Q_3q_1)_y, \\ -[Q_3q_{1y} + q_1(u_1Q_3 + 2Q_3^2 - Q_{3y})]_y, \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0, \quad (36)$$

where

$$s_1 = q_{1yy} + q_1(2u_1Q_3 + 3Q_3^2 + v_1) + q_{1y}(3Q_3 + u_1) + 1,$$

$$s_2 = w_1 + Q_3(v_1 + Q_3^2 - 3Q_{3y} - u_{1y}) + Q_{3yy} + (Q_3^2 - Q_{3y})u_1.$$

Furthermore, one can also gain the associated 3-Novikov system (36) by applying the reciprocal transformation (28) to the 3-Novikov system (8) directly. Now, we claim that the 3-Novikov equation (8) and the Lax pair (9) is reciprocal transformed to the associated equation (36) and the Lax pair (33-34) respectively. More precisely, we have:

Proposition 3 The 3-Novikov equation (36) may be changed to the associated equation (36) by the Liouville transformation

$$\begin{cases} y = I(x, \theta^{(n)}) = \int_{-\infty}^{x} u(\nu) d\nu, \\ \left(\begin{array}{c} u_{1}(y) \\ v_{1}(y) \\ w_{1}(y) \end{array} \right) = \left(\begin{array}{c} P_{1}(x, \theta^{(n)}), \\ P_{2}(x, \theta^{(n)}), \\ P_{3}(x, \theta^{(n)}), \end{array} \right), \end{cases}$$
(37)

where

$$P_{1} = 2\frac{v_{x}}{vu} - 3\frac{u_{x}}{u^{2}} - 3\frac{vw}{u^{3}},$$

$$P_{2} = \frac{v_{xx} - v}{u^{2}v} - 4\frac{u_{x}v_{x}}{u^{3}v} - \frac{4wv_{x} + 2uu_{xx} - 6u_{x}^{2}}{u^{4}} + 6\frac{vwu_{x}}{u^{5}} + 3\frac{v^{2}w^{2}}{u^{6}},$$

$$P_{3} = \frac{v(w - w_{xx})}{u^{5}} + \frac{4vw_{x}u_{x} + 2vwu_{xx}}{u^{6}} - \frac{6vwu_{x}^{2} + 3v^{2}ww_{x} + vw^{2}v_{x}}{u^{7}} + \frac{6v^{2}w^{2}u_{x}}{u^{8}} - \frac{v^{3}w^{3}}{u^{9}},$$

with $v = p - p_{xx}, w = r - r_{xx}$.

It is worth to note that the associated 3-Novikov system passes the Painlevé test. Powers of the leader terms for u_1, v_1, w_1, q_1, Q_3 are -1, -2, -2, -1, -1 respectively, and the resonances are j = -2, -1, 1, 2, 3, 4, 5.

The spectral problem (33) may be rewritten as the Lax operator for a mYJ hierarchy

$$L\phi_1 = \lambda\phi_1, \qquad L = \partial_y^2 + u_1\partial_y + v_1 + \partial_y^{-1}w_1, \qquad (38)$$

which is just a member in the constrained modified KP hierarchy [27, 28]. It can reduce to that of the mKdV equation and the KdV hierarchy as $Q_1 = Q_3 = 0$ and $Q_2 = Q_3 = 0$ respectively. We claim that the associated 3-Novikov equation is a reduction of the first negative flow in the mYJ hierarchy.

Notice that the mYJ hierarchy admits a Hamiltonian pair

$$\mathcal{J}_{1} = \begin{pmatrix} 0 & 0 & 2\partial_{y} \\ 0 & 2\partial_{y} & \partial_{y}^{2} + u_{1}\partial_{y} \\ 2\partial_{y} & -\partial_{y}^{2} + \partial_{y}u_{1} & 0 \end{pmatrix},$$

$$\mathcal{K}_{1} = \begin{pmatrix} 6\partial_{y} & * & * \\ 4u_{1}\partial_{y} & 2\partial_{y}^{3} + 2u_{1}\partial_{y}u_{1} + \partial_{y}v_{1} + v_{1}\partial_{y} & * \\ 2\partial_{y}^{3} - 2\partial_{y}u_{1}\partial_{y} + 2v_{1}\partial_{y} & \chi_{1} & \chi_{2} \end{pmatrix},$$

where

$$\chi_1 = 2w_1\partial_y + \partial_y w_1 - (\partial_y^3 - \partial_y u_1\partial_y + v_1\partial_y)(\partial_y - u_1),$$

$$\chi_2 = \partial_y u_1w_1 + u_1w_1\partial_y + w_1\partial_y^2 - \partial_y^2w_1$$

and the omitted terms are determined by skew-symmetry. Then a recursion operator for the mYJ hierarchy is obtained as $\mathcal{R} = \mathcal{K}_1 \mathcal{J}_1^{-1}$, and the first negative flow in the correspondence hierarchy are obtained as

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}_{\tau} = \mathcal{J}_1 \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \qquad \mathcal{K}_1 \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0, \qquad (39)$$

where $A = A(y,\tau), B = B(y,\tau), C = C(y,\tau)$. To find the relation between the associated 3-Novikov system (36) and the negative flow (39), we can take

$$A = -Q_3 q_{1y} - Q_3^2 q_1, \qquad B = -Q_3 q_1, \qquad C = -q_1.$$
(40)

Then the negative flow (39) is changed to

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}_{\tau} = \begin{pmatrix} -2q_{1y}, \\ -q_{1yy} - u_1q_{1y} - 2(Q_3q_1)_y, \\ -[Q_3q_{1y} + q_1(u_1Q_3 + 2Q_3^2 - Q_{3y})]_y, \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = 0, (41)$$

where

$$z_{1} = -2s_{1},$$

$$z_{2} = -s_{1yy} + (Q_{3} - u_{1})s_{1y} - 2p_{1}s_{2y} - 3p_{1y}s_{2},$$

$$z_{3} = (Q_{3}^{2} - 2Q_{3y} + Q_{3}u_{1})s_{1y} - Q_{3}s_{1yy} + p_{1}s_{2yy} + (2p_{1y} - Q_{3}p_{1} - u_{1}p_{1})s_{2y}$$

$$-(p_{1}u_{1y} + 2p_{1y}u_{1} + 3(p_{1}Q_{3})_{y})s_{2}.$$

Thus the associated 3-Novikov system (36) is a reduction of the first negative flow (39) in the mYJ hierarchy, since $s_1 = 0$, $s_2 = 0$ yields $z_1 = z_2 = z_3 = 0$.

4.2. Hamiltonian structure behavior under the Liouville transformation

According to [29], if two soliton equations are linked by a Liouville transformation, Hamiltonian structures and conserved quantities of them can be related. In this part we will consider the Hamiltonian structures of the 3-Novikov system (8) under the Liouville transformation (37). To this end, let $\vartheta = (u_1, v_1, w_1)^T$. Then from the point of view of Hamiltonian structures, we have

$$\theta_t = \mathcal{B}(\theta) \frac{\delta H}{\delta \theta} = \mathcal{B}(\theta) E_{\theta} h, \tag{42}$$

$$\vartheta_t = \tilde{\mathcal{B}}(\vartheta) \frac{\delta H}{\delta \vartheta} = \tilde{\mathcal{B}}(\vartheta) E_\vartheta \tilde{h},\tag{43}$$

where

$$H = \int h(x, \theta^{(n)}) dx, \quad \tilde{H} = \int \tilde{h}(y, \vartheta^{(n)}) dy.$$

Herein $E_{\theta}, E_{\vartheta}$ are the corresponding Euler operators, and $H[\theta^{(n)}] = \tilde{H}[\vartheta^{(n)}]$. Defining $\Lambda(\vartheta, \theta) = \vartheta - (P_1, P_2, P_3)^T$, hence it is easy to see that

$$\vartheta_t = -T_1 \theta_t, \quad T_1 = \Lambda_\theta, \tag{44}$$

where Λ_{θ} is Frechét derivative for the vector variable. Then a direct computation shows that

$$T_{1} = \begin{pmatrix} u_{1y}I'[u] - P'_{1}[u] & u_{1y}I'[v] - P'_{1}[v] & u_{1y}I'[w] - P'_{1}[w] \\ v_{1y}I'[u] - P'_{2}[u] & v_{1y}I'[v] - P'_{2}[v] & v_{1y}I'[w] - P'_{2}[w] \\ w_{1y}I'[u] - P'_{3}[u] & w_{1y}I'[v] - P'_{3}[v] & w_{1y}I'[w] - P'_{3}[w] \end{pmatrix}.$$

Furthermore, the action of Euler operator under a change of variables is given by

$$E_{\theta}h = T_2 E_{\vartheta}\tilde{h},\tag{45}$$

where

$$T_{2} = \begin{pmatrix} P_{1,u}^{\dagger}(I_{x}) - I_{u}^{\dagger}(P_{1x}) & P_{2,u}^{\dagger}(I_{x}) - I_{u}^{\dagger}(P_{2x}) & P_{3,u}^{\dagger}(I_{x}) - I_{u}^{\dagger}(P_{3x}) \\ P_{1,v}^{\dagger}(I_{x}) - I_{v}^{\prime\dagger}(P_{1x}) & P_{2,v}^{\dagger\dagger}(I_{x}) - I_{v}^{\prime\dagger}(P_{2x}) & P_{3,v}^{\dagger\dagger}(I_{x}) - I_{v}^{\prime\dagger}(P_{3x}) \\ P_{1,w}^{\dagger\dagger}(I_{x}) - I_{w}^{\prime\dagger}(P_{1x}) & P_{2,w}^{\prime\dagger}(I_{x}) - I_{w}^{\prime\dagger}(P_{2x}) & P_{3,w}^{\prime\dagger}(I_{x}) - I_{w}^{\prime\dagger}(P_{3x}) \end{pmatrix}.$$

Lemma 3 Under the transformation (37), we have the following formulaes:

$$T_1 = \mathcal{O}$$
diag $(u^{-1}, v^{-1}, vu^{-3}), \quad T_2 = -$ diag $(1, uv^{-1}, vu^{-2})\mathcal{O}^{\dagger},$

where

$$\mathcal{O} = \begin{pmatrix} u_{1y}\partial_y^{-1} + 3\partial_y + u_1 - 6Q_3 & -2\partial_y + 3Q_3 & 3\\ v_{1y}\partial_y^{-1} + 2\partial_y^2 + 2u_1\partial_y + 2v_1 - 4Q_3u_1 & (Q_3 - u_1)\partial_y + 2Q_3u_1 - \partial_y^2 & 2u_1\\ w_{1y}\partial_y^{-1} + 3w_1 - 2\chi_3Q_3 & \partial_yQ_3^2 - w_1 + Q_2Q_3^2 + Q_3Q_{3y} & \chi_3 \end{pmatrix}$$

with

$$\chi_3 = \partial_y^2 - \partial_y u_1 + v_1.$$

Lemma 4 Under the reciprocal transformation (28), the following identities hold:

$$\frac{1}{v}(1-\partial_x^2)\frac{v}{u^2} = \Theta_1 \equiv Q_1 - (\partial_y - Q_2 + Q_3)(\partial_y + Q_3), \tag{46}$$

and

$$\frac{1}{u^2}(\partial_x^3 - 4\partial_x)\frac{1}{u} = \Theta_2 \equiv (\partial_y - Q_2)\partial_y(\partial_y + Q_2) - 2Q_1\partial_y - 2\partial_yQ_1.$$
(47)

The two Lemmas above can be proved through a straightforward computation. Hence the main results can be summarized as:

Theorem 2 The associated 3-Novikov system is a bi-Hamiltonian system, namely, it can be written as

$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}_t = \mathcal{K}_1 \mathcal{J}_1^{-1} \mathcal{K}_1 \begin{pmatrix} \frac{\delta \tilde{H}_2}{\delta u_1} \\ \frac{\delta H_2}{\delta v_1} \\ \frac{\delta H_2}{\delta w_1} \end{pmatrix} = \mathcal{K}_1 \begin{pmatrix} \frac{\delta \tilde{H}_1}{\delta u_1} \\ \frac{\delta H_1}{\delta v_1} \\ \frac{\delta H_1}{\delta w_1} \end{pmatrix},$$
(48)

where

$$\tilde{H}_1 = \int Q_3 q_1 dy,$$

$$\tilde{H}_2 = \int [Q_3 q_1 (q_1 q_{2y} - q_2 q_{1y} + q_1 q_2 (Q_2 - 2Q_3)) - q_1 q_2] dy.$$

Proof: Substituting (44-45) into (42-43), a Hamiltonian pair for the associated 3-Novikov system is obtained as

$$\tilde{\mathcal{J}} = -T_1 \mathcal{J} T_2, \quad \tilde{\mathcal{K}} = -T_1 \mathcal{K} T_2.$$
(49)

Hamiltonian functionals of the 3-Novikov system and the associated 3-Novikov system connected by the formula (45).

To obtain bi-Hamiltonian structure of the associated 3-Novikov system, we should calculate $\tilde{\mathcal{J}}$ and $\tilde{\mathcal{K}}$ in the new variable y. Using conjugation of operator to the identity (46), we can easily check that

$$\frac{v}{u^3}(\partial_x^2 - 1)\frac{u}{v} = (\partial_y - Q_3)(\partial_y + Q_2 - Q_3) - Q_1.$$
(50)

Let us substitute the equalities (46) and (50) into the first equality in (49). Then, through tedious calculations, we get

$$\begin{split} \tilde{\mathcal{J}} &= \mathcal{O}\mathrm{diag}(u^{-1}, v^{-1}, vu^{-3})\mathcal{J}\mathrm{diag}(1, uv^{-1}, vu^{-2})\mathcal{O}^{\dagger} \\ &= \mathcal{O}\begin{pmatrix} \frac{1}{2}\partial_{y} & 0 & 0\\ 0 & 0 & \Theta_{1}\\ 0 & -\Theta_{1}^{\dagger} & 0 \end{pmatrix}\mathcal{O}^{\dagger} \\ &= \mathcal{K}_{1}\mathcal{J}_{1}^{-1}\mathcal{K}_{1}. \end{split}$$

On the other hand, introducing

$$\mathcal{P} = (\partial_y, \frac{3}{2}\partial_y - \frac{1}{2}Q_2 + Q_3, \frac{3}{2}Q_3\partial_y + \frac{1}{2}Q_2Q_3 + Q_{3y} - Q_3^2)^T.$$

Then the second equality in (49) may be changed to

$$\begin{split} \tilde{\mathcal{K}} &= \mathcal{O} \text{diag}(u^{-1}, v^{-1}, vu^{-3}) \mathcal{K} \text{diag}(1, uv^{-1}, vu^{-2}) \mathcal{O}^{\dagger} \\ &= \mathcal{O} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} \partial_{y}^{-1} & -1 - \frac{3}{2} \partial_{y}^{-1} Q_{3} \\ 0 & 1 - \frac{3}{2} Q_{3} \partial_{y}^{-1} & \frac{3}{2} Q_{3} \partial_{y}^{-1} Q_{3} \end{pmatrix} \mathcal{O}^{\dagger} - 2 \mathcal{O} \mathcal{P} \Theta_{2}^{-1} \mathcal{P}^{\dagger} \mathcal{O}^{\dagger} \\ &= \mathcal{O} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} \partial_{y}^{-1} & -1 - \frac{3}{2} \partial_{y}^{-1} Q_{3} \\ 0 & 1 - \frac{3}{2} Q_{3} \partial_{y}^{-1} & \frac{3}{2} Q_{3} \partial_{y}^{-1} Q_{3} \end{pmatrix} \mathcal{O}^{\dagger} - \begin{pmatrix} 0 \\ 1 \\ -Q_{3} \end{pmatrix} \frac{\Theta_{2}^{\dagger}}{2} \begin{pmatrix} 0 \\ 1 \\ -Q_{3} \end{pmatrix}^{T} \\ &= \mathcal{K}_{1} \end{split}$$

by using the identity (47) and $\mathcal{OP} = \frac{1}{2}(0, \Theta_2, -Q_3\Theta_2)^T$.

Furthermore, since the Hamiltonian functionals of the two hierarchy are connected by the relation $H[\theta^{(n)}] = \tilde{H}[\vartheta^{(n)}]$, we can easy to find the relationship between the two hierarchies.

5. A limit system

The limits of the CH type equations might also contain some important models. For example, the Hunter-Saxton equation, which can describe wave motion in a nematic liquid crystal [30], may be consider as a limit of the CH equation [31]. The Ostrovsky equation, which appears as the description of high-frequency waves in a relaxing medium [32], can be obtained as a short wave limit of the DP equation [11]. In this section, we will consider a limit of the 3-Novikov system (8).

Let us consider the transformation

$$x \to \epsilon x, \quad t \to \epsilon t, \quad u \to \epsilon^{\frac{3}{2}} u.$$
 (51)

Then a limit for the 3-Novikov system may be obtained in the limit $\epsilon \to 0$, that is

$$u_{t} + (upr)_{x} = 0,$$

$$v_{t} + 3vp_{x}r + v_{x}pr + u^{2}p = 0,$$

$$w_{t} + 3wpr_{x} + w_{x}pr - u^{2}r = 0,$$

$$v = -p_{xx}, \qquad w = -r_{xx}.$$
(52)

The short wave model (52) is also integrable in the sense of admitting bi-Hamiltonian structure and a Lax pair. The bi-Hamiltonian structure can be obtained by applying the transformation (51) to that of the 3-Novikov system, that is

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t} = \bar{\mathcal{J}}_{1} \begin{pmatrix} \frac{\delta \bar{H}_{2}}{\delta u} \\ \frac{\delta \bar{H}_{2}}{\delta v} \\ \frac{\delta \bar{H}_{2}}{\delta w} \end{pmatrix} = \bar{\mathcal{K}}_{1} \begin{pmatrix} \frac{\delta \bar{H}_{1}}{\delta u} \\ \frac{\delta \bar{H}_{1}}{\delta v} \\ \frac{\delta \bar{H}_{1}}{\delta w} \end{pmatrix},$$
(53)

where

$$\begin{split} \bar{\mathcal{J}}_1 &= \begin{pmatrix} \frac{1}{2}\partial & 0 & 0\\ 0 & 0 & -\partial^2\\ 0 & \partial^2 & 0 \end{pmatrix}, \\ \bar{\mathcal{K}}_1 &= \begin{pmatrix} 0 & 0 & 0\\ 0 & \frac{3}{2}v\partial^{-1}v & -u^2 - \frac{3}{2}v\partial^{-1}w\\ 0 & u^2 - \frac{3}{2}w\partial^{-1}v & \frac{3}{2}w\partial^{-1}w \end{pmatrix} - 2\Omega\partial^{-3}\Omega^*, \end{split}$$

with the functionals given by

$$\bar{H}_1 = \int p_x r_x dx,$$

$$\bar{H}_2 = \int (p p_x r r_{xx} + p p_x r_x^2 - u^2 p r) dx$$

Taking the transformation (51) to the Lax pair (9) with $\lambda \to \epsilon \lambda$, a Lax pair for the limit system (52) is obtained as

$$\varphi_x = \begin{pmatrix} 0 & 1 & 0 \\ \lambda u^2 & 0 & v \\ \lambda w & 0 & 0 \end{pmatrix} \varphi,$$
$$\varphi_t = \begin{pmatrix} \frac{1}{3\lambda} + pr_x & -pr & \frac{p}{\lambda} \\ p_x r_x - \lambda u^2 pr & \frac{1}{3\lambda} - p_x r & \frac{p_x}{\lambda} - vpr \\ -\lambda w pr - r_x & r & p_x r - pr_x - \frac{2}{3\lambda} \end{pmatrix} \varphi.$$

Moreover, the short wave model (52) is also reciprocal connected to the first negative flow in the mYJ hierarchy by taking the similar process before. It may reduce to that of the Geng-Xue, the Novikov and the DP as u = 0 and u = 0, p = r as well as u = 0, r = 1respectively.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (Grant Nos. 11805071 and 11505064) and Promotion Program for Young and Middle-aged Teacher in Science and Technology Research of Huaqiao University (Project No. ZQN-PY301).

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